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K_2 and the K-Theory of Automorphisms

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Let A be a ring with 1. Let $\mathscr{P}(A) = \mathscr{P}_A$ denote the exact category of finitely-generated projective A-modules. Finally, let $\mathscr{Aut}_A$ denote the exact category whose objects are all pairs (P, f) with $P \in \mathscr{P}(A)$ and f an automorphism of P, whose arrows from (P, f) to (P', f') are all maps $g: P \to P'$ such that $gf = f'g$, and whose exact sequences are those lying over exact sequences in $\mathscr{P}_A$. In this paper we examine the higher algebraic K-theory of these categories, defined by Quillen [5].

Let $\mathrm{Aut}_i\, A$ be the kernel of the forgetful map $K_i(\mathscr{Aut}_A) \to K_i(A)$, where $K_i(A) = K_i(\mathscr{P}_A)$. For commutative rings A, we defined in [3] a map

$$c: \mathrm{Aut}_i\, A \to K_{i+1}A, \tag{1}$$

and showed that for $i = 0$, this map is a surjection which expresses K_1A as the group with one generator $[P, f]$ for each object (P, f) of $\mathscr{Aut}_A$, and the relations:

(i) $[P, f] = [P', f'] + [P'', f'']$ for each eact sequence $0 \to (P', f') \to (P, f) \to (P'', f'') \to 0$ in $\mathscr{Aut}_A$, and

(ii) $[P, f][P, f'] = [P, ff']$.

The first goal of this paper is to formulate a definition of the map (1) when A is not necessarily commutative. This is done in Section 1 using an idea of Waldhausen [6] which he calls Mayer–Vietoris representations. Using them, we may replace the category of vector bundles on the projective line by a more tractable category without altering the K-theory. This category has the advantage that every object has a projective resolution of length 1, and thus the techniques of [2] may be applied to it.

The proof that our new definition agrees with the old in the commutative case occupies Section 2.

The remainder of the paper is devoted to explicit calculations of the map for $i = 1$:

$$c: \mathrm{Aut}_1\, A \to K_2A.$$

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Milnor [4, p. 63] defines a bimultiplicative skew-symmetric pairing $f \star g$ in K_2A; here f and g are commuting elements of Gl_nA. Notice that g gives an automorphism of the object $(A^n, -f)$ in $\mathscr{Aut}_A$. Given an automorphism of an object in an exact category, one can construct a certain element of K_1. In our case, because $\mathrm{Aut}_1\, A \hookrightarrow K_1\, \mathscr{Aut}_A$ is split, we obtain an element $f * g$ in $\mathrm{Aut}_1\, A$. One immediately suspects that $c(f * g)$ should be $f \star g$, at least up to sign. We prove that this is so.

In Section 3 we compute $c(f * g)$ in terms of a certain exact sequence

$$0 \to K_2A \to G(A) \to Gl(A) \times Gl(A) \to K_1A \to 0.$$

The group $Gl(A) \times Gl(A)$ acts on each group in the sequence; it acts by conjugation on itself and trivially on K_2A and K_1A. The key to the computations is to describe the group $G(A)$ and the action of $Gl(A) \times Gl(A)$ on it, fully, in terms of the Steinberg group $St(A)$.

In Section 4 we define an action of $Gl(A) \times Gl(A)$ on the semi-direct product $H(A) = Gl(A) \ltimes St(A)$, and show that it fits into the same context as $G(A)$. The technique involves the extension of Milnor's pairing $f \star g \in K_2A$ to a pairing $\star: Gl(A) \times Gl(A) \to St(A)$ which lifts the commutator pairing $Gl(A) \times Gl(A) \to Gl(A)$, and satisfies certain familiar identities. The reader should notice that Section 4 makes no appeal to higher algebraic K-theory.

In Section 5 we show that $G(A) \cong H(A)$ and that the actions agree, and complete the computation $c(f * g) = f^{-1} \star g$. This computation should be compared with those in Proposition 2.2.3 of [7], which are analogous.

Bass has asked the following question: Is K_2A generated by the $f \star g$? Clearly, K_2A is generated by elements

$$(f_1 \star g_1)(f_2 \star g_2) \cdots (f_n \star g_n)$$

where $[f_1, g_1][f_2, g_2] \cdots [f_n, g_n] = 1 \in Gl(A)$, but there may be no way to shorten these commutator identities. One can, however, ask the weaker question: Is $\mathrm{Aut}_1\, A \to K_2A$ surjective?

1. CONSTRUCTING THE MAP

In this section we define the map (1) by modifying a construction of Waldhausen's [6] which he calls Mayer–Vietoris resolutions.

Let $\mathscr{N}_A$ be the exact category whose objects are all pairs of arrows $(P \rightrightarrows Q)$ from $\mathscr{P}_A$. There are two types of standard projective objects in $\mathscr{N}_A$, namely, $(0 \rightrightarrows P)$ and $(P \rightrightarrows P \oplus P)$. Given any object $(P \rightrightarrows Q)$ of $\mathscr{N}_A$, there is an exact sequence

$$0 \to (0 \rightrightarrows K) \to (P \rightrightarrows P \oplus P) \oplus (0 \rightrightarrows Q) \to (P \rightrightarrows Q) \to 0. \tag{2}$$

Thus $\mathcal{N}_A$ is a hereditary category (i.e. has global projective dimension $\leqslant 1$). We may identify $\mathcal{A}ut_A$ with the full subcategory of $\mathcal{N}_A$ whose objects are pairs of isomorphisms: $(P \overset{\cong}{\rightrightarrows} Q)$. Using the results on hereditary categories contained in [2], we obtain a homotopy cartesian square

$$\begin{array}{ccc} S^{-1}E\,\mathcal{A}ut_A & \longrightarrow & S^{-1}E\mathcal{N}_A \simeq pt \\ \downarrow & & \downarrow \\ Q\,\mathcal{A}ut_A & \longrightarrow & Q\mathcal{N}_A . \end{array}$$

Here $E\,\mathcal{A}ut_A$ (resp. $E\mathcal{N}_A$) denotes a category whose objects are short exact sequences

$$0 \to B \to C \to D \to 0 \tag{4}$$

in $\mathcal{N}_A$ with B and C projective, and $D \in \mathcal{A}ut_A$ (resp. $D \in \mathcal{N}_A$); S denotes the category whose arrows are all isomorphisms of projective objects in $\mathcal{N}_A$; S^{-1} denotes a certain construction of Quillen [1]. (It should cause no confusion to use the same letter S in association with categories other than $\mathcal{A}ut_A$; it will always take its meaning from the context. For instance, $S^{-1}S(\mathcal{P}_A)$ denotes the loop space $S^{-1}S$ of $Q\mathcal{P}_A$ discussed in [1].)

It is an easy exercise, using the exact sequence (2), to characterize the projective objects of $\mathcal{N}_A$ as those pairs $(P \rightrightarrows Q)$ for which $P \oplus P \to Q$ is an admissible monomorphism (i.e. has a cokernel in $\mathcal{P}_A$); let $\mathcal{P}\mathcal{N}_A$ denote the full subcategory of all projective objects of $\mathcal{N}_A$. We define two exact functors $r_0, r_\infty\colon \mathcal{P}\mathcal{N}_A \rightrightarrows \mathcal{P}_A$ so that r_0 (resp. r_∞) sends an object $(f, g\colon P \rightrightarrows Q)$ to $\operatorname{ckr} g$ (resp. $\operatorname{ckr} f$). Let

$$c\colon S^{-1}E\,\mathcal{A}ut_A \to S^{-1}S(\mathcal{P}_A) \tag{5}$$

be the functor which sends $(E, 0 \to B \to C \to D \to 0)$ to $(r_0E \oplus r_\infty C, r_0C \oplus r_\infty E)$. This definition extends to arrows because (i) if $C' \to C$ is an admissible monomorphism with cokernel in $\mathcal{A}ut_A$, then $r_0C' = r_0C$ and $r_\infty C' = r_\infty C$, and (ii) if E' is an object of $\mathcal{P}\mathcal{N}_A$ then $r_0E' \oplus r_\infty E' = r_\infty E' \oplus r_0E'$, so an arrow in $S^{-1}E\,\mathcal{A}ut_A$ resulting from adding E' maps to an arrow in $S^{-1}S(\mathcal{P}_A)$ given by adding $r_0E' \oplus r_\infty E'$.

The K-groups of $\mathcal{P}\mathcal{N}_A$ are easily calculated. Any projective $(P \rightrightarrows Q)$ is, in a natural way, an extension of standard projective objects

$$0 \to (P \rightrightarrows P \oplus P) \to (P \rightrightarrows Q) \to (0 \rightrightarrows Q/P \oplus P) \to 0.$$

Making use of the additivity of K-theory [5; sect. 3, Cor. 1], we see that the map $Q\mathcal{P}\mathcal{N}_A \to Q\mathcal{P}_A \times Q\mathcal{P}_A$ given by $(P \rightrightarrows Q) \to (P, Q/P \oplus P)$ is a homotopy equivalence, with homotopy inverse given by $(P, Q) \to (P \rightrightarrows P \oplus P \oplus Q)$.

We will now show that $\pi_{i+1}S^{-1}E\,\mathcal{A}ut_A = \operatorname{Aut}_i A \oplus K_{i+1}A$ by splitting up the long exact sequence resulting from (3) as in [3]. For this purpose we use the isomorphism

$$(h_1 - h_0, h_0)\colon K_iA \oplus K_iA \to K_i\mathcal{N}_A \tag{6}$$

where h_0 (resp. h_1): $Q\mathcal{P}_A \to Q\mathcal{N}_A$ is the exact functor $P \mapsto (0 \rightrightarrows P)$ (resp. $P \mapsto (P \rightrightarrows P \oplus P)$). The exact sequence (2) in which $K = P \oplus P$ and the exactness theorem [5] show that

$$\begin{array}{ccc} K_i\,\mathcal{A}ut_A & \to & K_i\mathcal{N}_A \\ \downarrow & \nearrow_{h_1 - h_0} & \\ K_iA & & \end{array}$$

is a commutative diagram, where the left-hand map is the forgetful map. It follows from (3) that we have an exact sequence

$$0 \to K_{i+1}A \xrightarrow{\lambda} \pi_{i+1}S^{-1}E\,\mathcal{A}ut_A \to \operatorname{Aut}_iA \to 0.$$

We can see that the injection λ comes from the functor $S^{-1}S(\mathcal{P}_A) \to S^{-1}E\,\mathcal{A}ut_A$ given by $(P, Q) \mapsto (h_0P, 0 \to h_0Q \to h_0Q \to 0 \to 0)$ by using the fibration up to homotopy

$$S^{-1}S(\mathcal{P}\mathcal{N}_A) \to S^{-1}E\,\mathcal{A}ut_A \to Q\,\mathcal{A}ut_A .$$

Since $r_0h_0 = r_\infty h_0 = 1$, the map c kills the image of λ and factors through $\operatorname{Aut}_i A$ to yield the desired map

$$c\colon \operatorname{Aut}_i A \to K_{i+1}A \tag{7}$$

We shall now modify the definition of c a bit to make computations in Section 3 easier. The two maps $r_0, r_\infty\colon S(\mathcal{P}\mathcal{N}_A) \rightrightarrows S(\mathcal{P}_A)$ are not naturally isomorphic, but it is true that for an object of $\mathcal{P}\mathcal{N}_A$ of one of the standard types, $(0 \rightrightarrows P)$ or $(P \rightrightarrows P \oplus P)$, the two cokernels are isomorphic; this isomorphism is natural and compatible with direct sum, provided we restrict attention to one type or the other. So, let S' denote the category whose objects are all objects $(P \rightrightarrows Q)$ of $\mathcal{P}\mathcal{N}_A$ together with a splitting of $P \oplus P \rightarrowtail Q$, and whose arrows are all isomorphisms of such data. The point is that the splitting determines a direct sum decomposition of $(P \rightrightarrows Q)$ into the two standard types. The map $S' \to S(\mathcal{P}\mathcal{N}_A)$ of monoidal categories is cofinal, so

$$S'^{-1}E\,\mathcal{A}ut_A \overset{\sim}{\to} S^{-1}E\,\mathcal{A}ut_A$$

is a homotopy equivalence [1]. We claim that the composite map $S'^{-1}E\,\mathcal{A}ut_A \to S^{-1}S(\mathcal{P}_A)$ is homotopic to the functor

$$c'\colon S'^{-1}E\,\mathcal{A}ut_A \to S^{-1}S(\mathcal{P}_A)$$

given by $(E, 0 \to B \to C \to D \to 0) \mapsto (r_\infty C, r_0 C)$. We can define c' on arrows because $r_0 = r_\infty$ on S', and for the same reason we obtain a natural transformtion $c' \to c$. Let h_0', h_1': $S(\mathscr{P}_A) \to S'$ denote the maps h_0 and h_1 with splittings added.

2. Agreement for Commutative A

Let A be commutative. We must see that the map (7) agrees with the map defined in [3]; for the duration of this section we adopt all terminology and notation introduced there.

Recall that $\mathscr{P}_X$ denotes the category of vector bundles on the projective line over A. We begin by defining an exact functor $b\colon \mathscr{P}\mathscr{N}_A \to \mathscr{P}_X$. Given $(f, g\colon P \rightrightarrows Q)$ in $\mathscr{P}\mathscr{N}_A$, consider the map of vector bundles

$$h = fT + gU\colon P(-1) \to Q(0).$$

It is easy to see that h is an admissible monomorphism; for instance, if p splits the monomorphism $P \oplus P \to Q$, then $pr_1 \cdot p \cdot f = 1_p$ and $pr_1 \cdot p \cdot g = 0$, so $pr_1 \cdot p$ splits $f + gU\colon P[U] \to Q[U]$. We define $b\colon \mathscr{P}\mathscr{N}_A \to \mathscr{P}_X$ to be the functor which sends $(P \rightrightarrows Q)$ to $(ckr\ h)$.

Now $b(0 \rightrightarrows P) = P(0)$, and $b(P \rightrightarrows P \oplus P) = P(1)$, so Quillen's computation $Q\mathscr{P}_X \simeq Q\mathscr{P}_A \times Q\mathscr{P}_A$ says that $b\colon Q\mathscr{P}\mathscr{N}_A \to Q\mathscr{P}_X$ is a homotopy equivalence.

For technical reaons, it is convenient to introduce the full subcategory $\mathscr{N}'_A$ of $\mathscr{N}_A$ consisting of all pairs of admissible monomorphisms $(f, g\colon P \rightrightarrows Q)$. The point is that b extends to a map $b'\colon \mathscr{N}'_A \to \mathscr{P}_X{}^1$ and that $\mathscr{N}'_A$ contains $\mathscr{A}ut_A$. (Here $\mathscr{P}_X{}^1$ denotes the category of quasi-coherent sheaves on X which have resolutions of length 1 by vector bunles. It is enough to show that $h = fT + gU$ is injective.) The resolution theorem says that the vertical maps in the square:

$$\begin{array}{ccc} Q\mathscr{P}\mathscr{N}_A & \xrightarrow{b} & Q\mathscr{P}_X \\ \downarrow & & \downarrow \\ Q\mathscr{N}'_A & \xrightarrow{b'} & Q\mathscr{P}_X{}^1, \end{array}$$

are homotopy equivalences; thus b' is a homotopy equivalence.

Consider the following diagram:

$$\begin{array}{ccccc} Q\,\mathscr{A}ut_A & \longrightarrow & Q\mathscr{N}'_A & \longrightarrow & Q\mathscr{P}^1(R_1) \\ \| & & \downarrow{\scriptstyle b'} & & \| \\ Q\,\mathscr{A}ut_A & \longrightarrow & Q\mathscr{P}_X{}^1 & \longrightarrow & Q\mathscr{P}^1(R_1) \end{array} \tag{8}$$

Recall that the bottom row is the fibraction (up to homotopy) used in [3], and that R_1 denotes the coordinate ring of the intersection, in the projective line X, of the neighborhoods of the union of the 0-section and the ∞-section.

We recall now the definition of the map (1) given in [3], rephrasing it, as we may, in terms of the top fibration in (8). The long exact sequence of the fibration is:

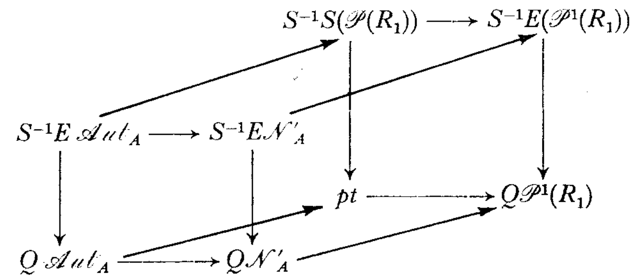
$$\cdots \to K_i\,\mathscr{A}ut_A \to K_iA \oplus K_iA \to K_i(R_1) \to K_{i-1}\,\mathscr{A}ut_A \to \cdots$$

which split up into shorter ones:

$$0 \to K_iA \to K_iR_1 \to K_{i-1}\,\mathscr{A}ut_A \to K_{i-1}A \to 0.$$

The map $s_0 - s_\infty\colon K_iR_1 \to K_iA$ kills K_iA, and thus factors through the cokernel $\mathrm{Aut}_{i-1} A$. Here s_0, $s_\infty\colon R_1 \rightrightarrows A$ are the two augmentations of R_1.

Consider the following cube:



The techniques developed in [2] for hereditary categories show that the front and back squares are homotopy cartesian, and we know the bottom square is homotopy cartesian. Thus the top square is homotopy cartesian; since $S^{-1}E\mathscr{N}'_A$ and $S^{-1}E(\mathscr{P}(R_1))$ are contractible, this means that $S^{-1}E\,\mathscr{A}ut_A \to S^{-1}S(\mathscr{P}(R_1))$ is a homotopy equivalence. If we compare the long exact sequence for the diagonal square:

$$\begin{array}{ccc} S^{-1}E\,\mathscr{A}ut_A & \longrightarrow & S^{-1}E\mathscr{N}'_A \\ \downarrow & & \downarrow \\ pt & \longrightarrow & Q\mathscr{P}(R_1) \end{array}$$

with those for the back and bottom, we see that

$$\begin{array}{ccc} \pi_{i+1}S^{-1}E\,\mathscr{A}ut_A & \rightleftarrows & K_{i+1}R_1 \\ & \searrow & \downarrow \\ & & K_i\,\mathscr{A}ut_A \end{array}$$

commutes.

The map $s_0 - s_\infty: K_{i+1}R_1 \to K_{i+1}A$ is represented by the functor

$$d: S^{-1}S(\mathscr{P}(R_1)) \to S^{-1}S(\mathscr{P}(A))$$
$$(E, F) \mapsto (s_0^*E \oplus s_\infty^*F,\ s_0^*F \oplus s_\infty^*E).$$

(To see this, show the map $S^{-1}S \to S^{-1}S$ given by $(P, Q) \mapsto (P \oplus Q, Q \oplus P)$ is zero on homotopy groups by restricting to BGl_n^+ and then to BGl_n, at which level there is a natural transformation giving a null-homotopy.) Thus the triangle

$$\begin{array}{ccc} S^{-1}E\,\mathscr{A}ut_A & \longrightarrow & S^{-1}S(\mathscr{P}(R_1)) \\ & {\scriptstyle c}\searrow & \downarrow{\scriptstyle d} \\ & & S^{-1}S(\mathscr{P}_A) \end{array}$$

commutes, and we have shown the two definitions of $\mathrm{Aut}_{i-1} A \to K_iA$ agree when A is commutative.

3. The Pairing

In this section we define the pairing $f * g \in \mathrm{Aut}_1 A$ where f and g are commuting automorphisms of some $P \in \mathscr{P}_A$. We also compute the image of $f * g$ in K_2A under the map defined in Section 1, in terms of a certain group $G(A)$.

Let f and g be commuting endomorphims of $P \in \mathscr{P}_A$. Then the diagram in $Q\,\mathscr{A}ut_A$:

$$\begin{array}{ccccc} & & (P, -f) & & \\ & \nearrow & \wr\downarrow{\scriptstyle g} & \searrow & \\ 0 & & & & 0 \\ & \searrow & & \nearrow & \\ & & (P, -f) & & \end{array}$$

commutes, and defines a homotopy H_1 of the loop

$$\gamma_1(f): 0 \rightarrowtail (P, -f) \twoheadrightarrow 0$$

to itself. (This loop, itself, represents $[P, -f] \in K_0\,\mathscr{A}ut_A = \pi_1 Q\,\mathscr{A}ut_A$.) We obtain an element of $\pi_2 Q\,\mathscr{A}ut_A$ which may be represented by the diagram

$$\gamma_1H_1(f, g):\ \boxed{\overset{\downarrow}{0} \rightarrowtail (P, \overset{\downarrow}{-f}) \twoheadrightarrow 0 \quad g}$$

We define $f * g$ as the class in $K_1\,\mathscr{A}ut_A = \pi_2 Q\,\mathscr{A}ut_A$ of $\gamma_1H_1(f, g) - \gamma_1H_1(1, g)$. This difference maps to zero in K_1A, thus lies in $\mathrm{Aut}_1 A$. In a similar fashion, given any loop γ, in a pointed topological space X, and a homotopy H

of γ to itself, we obtain a map $(D^2, \partial D^2) \to (X, *)$ which we call γH. (Here D^2 denotes the 2-ball.)

Consider the map $c': S'^{-1}E\,\mathscr{A}ut_A \to Q\,\mathscr{A}ut_A$; we want to lift $\gamma_1H_1(f, g) - \gamma_1H_1(1, g)$ along it. The path $\gamma_1(f) \cdot \gamma_1(1)^{-1}$ lifts to a path

$$\gamma_2: (0, 0) \to (h_0' P, G_1)$$
$$\xrightarrow{\sigma(f)} (h_0'P, G_2(f)) \xleftarrow{\tau(f)} (h_0'P, G_3) \xrightarrow{\tau(1)} (h_0'P, G_2(1))$$
$$\xleftarrow{\sigma(1)} (h_0'P, G_1) \longleftarrow (0, 0) \text{ in } S'^{-1}E\,\mathscr{A}ut_A,$$

where

$$G_1 = \begin{pmatrix} 0 & \rightarrowtail & 0 & \twoheadrightarrow & 0 \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ P & \rightarrowtail & P & \twoheadrightarrow & 0 \end{pmatrix},$$

$$G_2(f) = \begin{pmatrix} 0 & \rightarrowtail & P & \twoheadrightarrow & P \\ \downdownarrows & & \downdownarrows & & {\scriptstyle 1}\downarrow\downarrow{\scriptstyle -f} \\ P & \xrightarrow[\binom{f}{1}]{} & P \oplus P & \xrightarrow[(1,-f)]{} & P \end{pmatrix},$$

and

$$G_3 = \begin{pmatrix} P & \rightarrowtail & P & \twoheadrightarrow & 0 \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ P \oplus P & \rightarrowtail & P \oplus P & \twoheadrightarrow & 0 \end{pmatrix}.$$

The arrows $0 \to G_1$ and $0 \to G_3$ are those given by the action of S'. The arrow $\sigma(f): G_1 \to G_2(f)$ in $E\,\mathscr{A}ut_A$ and the arrow $\tau(f): G_3 \to G_2(f)$ are given by the inclusion $\binom{f}{1}: P \to P \oplus P$ and by the identity or zero on other parts of the diagrams. The automorphism g defines automorphisms of G_1, G_2, G_3 which commute with the arrows in γ_2, and thus defines a homotopy H_2 of the path γ_2 to itself. Thus γ_2H_2 is a lifting of $\gamma_1H_1(f, g) - \gamma_1H_1(1, g)$.

Now we apply the functor c' to γ_2H_2 and obtain γ_3H_3, where γ_3 is the loop

$$\gamma_3: (0, 0) \longrightarrow (P, P) \xrightarrow{(1,f)} (P, P) \longleftarrow 0,$$

and H_3 is the homotopy defined by (g, g). Notice that γ_3 represents the class of f in $K_1A = \pi_1S^{-1}S(\mathscr{P}_A)$.

The final result is that $c(f * g)$ is represented by the diagram in Figure 1.

The representation just found for $c(f * g)$ is fairly explicit, but what we really want is an expression for $c(f * g)$ as an element of the Steinberg group $St(A)$ in terms of its standard generators.

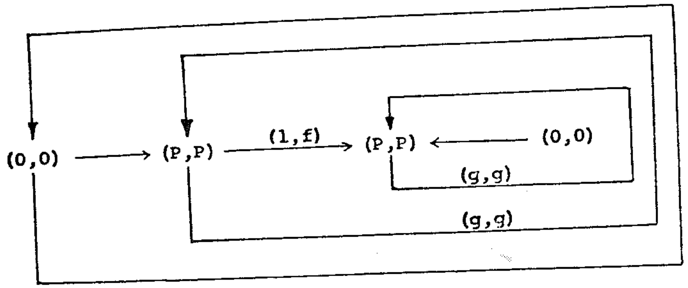


FIGURE 1

Recall that the exact sequence:

$$0 \to K_2A \to St(A) \to Gl(A) \to K_1A \to 0$$

is part of the long exact sequence for the fibration

$$F \to B\,Gl(A) \to B\,Gl(A)^+.$$

An examination of the map $B\,Gl_n(A) \to S^{-1}S(\mathscr{P}_A)$, [1], reveals the arrows in the image of this functor are pairs $(1, f)$, with $1, f \in Gl_n(A)$. Naturally, the fact that arrows (g, g) occur in Figure 1 leads us instantly into difficulties when computing $c(f * g)$ as an element of $\pi_1 F$. We would like to be able to lift at least one 2-simplex of Figure 1 to $B\,Gl(A)$ from $S^{-1}S(\mathscr{P}_A)$, but we can't.

The solution to this difficulty is obvious: construct the homotopy fibration

$$F' \to B(Gl(A) \times Gl(A)) \xrightarrow{\gamma} BGl(A)^+,$$

where γ is the subtraction map. Part of the resulting long exact sequence is:

$$0 \to K_2A \to G(A) \to Gl(A) \times Gl(A) \to K_1A \to 0, \tag{9}$$

where $G(A) = \pi_1 F'$ is the group mentioned in the introduction. The group $Gl(A) \times Gl(A)$ acts on the fibration (in the homotopy category), and thus acts on the groups in (9). For the reader's convenience, we recall the following general facts about this action, which we will use repeatedly:

Facts. Let $F \to E \xrightarrow{p} B$ be a fibration of pointed topological spaces. Then $\pi_1 E$ acts on $\pi_i F$, $\pi_i E$, and $\pi_i B$.

(i) $\pi_1 E$ acts on $\pi_i B$ through $\pi_1 B$.

(ii) The image of $\pi_2 F \to \pi_1 F$ is central.

(iii) $\pi_1 E$ acts on itself by conjugation.

(iv) The action of $\pi_1 F$ on itself induced by $\pi_1 F \to \pi_1 E$ is conjugation.

(v) $\pi_1 F$ is given by homotopy classes of pairs of base-point preserving maps:

$$\begin{array}{ccc} S^1 & \xrightarrow{y} & E \\ \downarrow & & \downarrow \\ S^1 \wedge I & \xrightarrow{z} & B. \end{array}$$

They may be represented pictorially as in Figure 2a, where e and b denote the base-points in E and B.

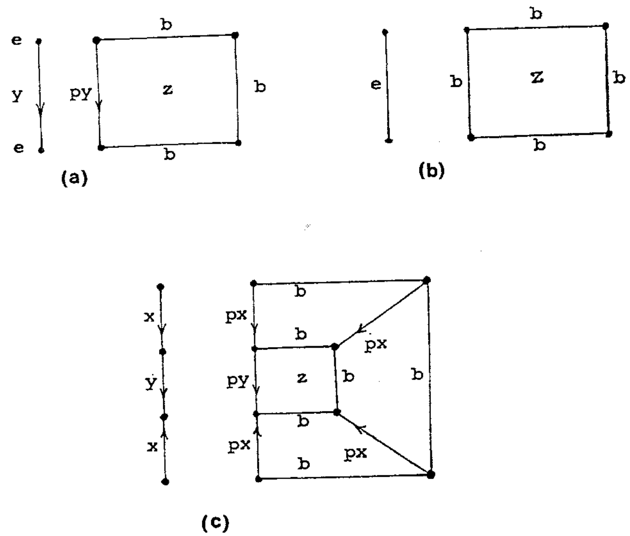


FIGURE 2

(vi) In terms of (v), $\pi_2 B \to \pi_1 F$ is the map induced by

$$\begin{array}{ccc} S^1 & \longrightarrow & pt \\ \downarrow & & \downarrow \\ S^1 \wedge I & \longrightarrow & S^1 \wedge S^1 = S^2, \end{array}$$

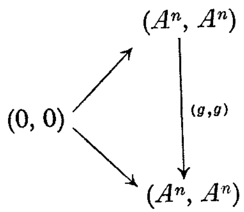
with the evident pictorial representation in Figure 2b.

(vii) The action of $\pi_1 E$ on $\pi_1 F$ is that represented pictorially in Figure 2c — x denotes a loop in E.

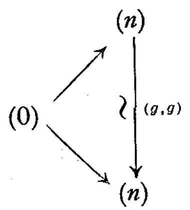
LEMMA 3.1. *Assume $P = A^n$, and f, g are commuting elements of $Gl_nA = \mathrm{Aut}(P)$. Let $h \in G(A)$ be any element which maps onto (g, g) in $Gl(A) \times Gl(A)$. Then $c(f * g) \in K_2A \subset G(A)$ is equal to $h^{-1}h^{(1,f)}$.*

Proof. We may represent $B(Gl(A) \times Gl(A))$ as the classifying space of the following category, modelled after the telescope construction used in [1]: There is one object (n) for each $n \geqslant 0$, and $\mathrm{Hom}((m), (n)) = Gl_n(A) \times Gl_n(A)$ if $n \geqslant m$, but $= \varnothing$ if $n < m$. Composition is defined by the formula $(f, g) \cdot (f', g') = (f \cdot (f' \oplus 1_{n-m}), g(g' \oplus 1_{n-m}))$. Presumably, no confusion will result if we denote this category by $Gl(A) \times Gl(A)$. The base-point is the object (0). The functor $Gl \times Gl \to S^{-1}S$ is given by $(n) \to (A^n, A^n)$ on objects, whereas the arrow $(f, g): (m) \to (n)$ is sent to the composite $(A^m, A^m) \to (A^m \oplus A^{n-m}, A^m \oplus A^{n-m}) \xrightarrow[\sim]{(f,g)} (A^n, A^n)$.

The key to the arguments seems to be the observation that



is a commutative diagram in $S^{-1}S(\mathscr{P}_A)$, but that



does not commute in $Gl(A) \times Gl(A)$.

Let us describe an element h which lifts (g, g) to $G(A) = \pi_1 F'$; see Figure 3.

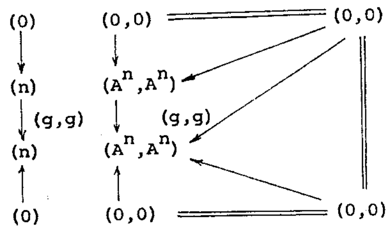


FIGURE 3

In a similar fashion, we may depict the element $h^{-1}h^{(1,f)}$, where we let $(0) \to (n) \to^{(1,f)} (n) \leftarrow 0$ be the loop which represents $(1, f) \in \pi_1(Gl_A \times Gl_A)$; see Figure 4.

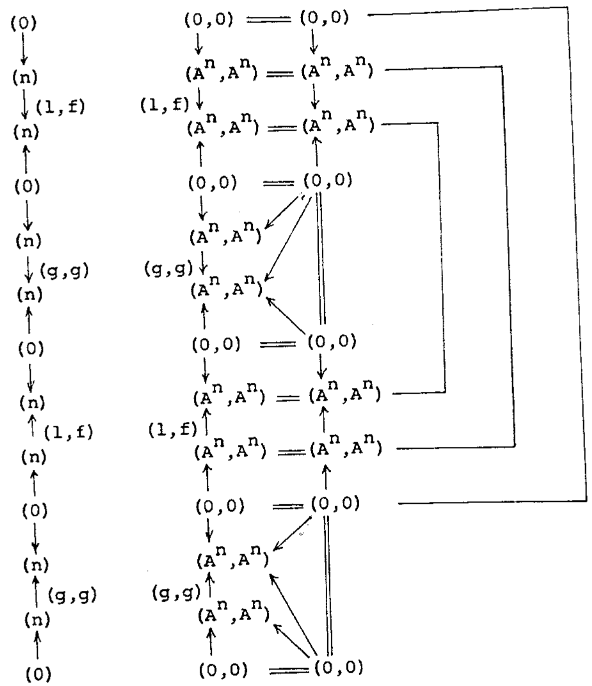
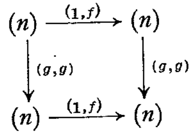


FIGURE 4

We may make use of the commutative diagram:



to deform Figure 4 into the representative of $h^{-1}h(1, f)$ depicted in Figure 5.

The diagram in Figure 5 represents an element in the image of $K_2A \to G(A)$, and a little rearrangement yields Figure 1, which represents $c(f * g)$. Q.E.D.

4. The Pairing in the Steinberg Group

This section is devoted exclusively to those computations which can be performed without any reference to higher K-theory. We define an element $f \star g$ of the Steinberg group, $St(A)$, for any elements $f, g \in Gl(A)$. When f and g commute, this agrees with Milnor's pairing. We use this pairing to define an action of $Gl(A) \times Gl(A)$ on $H(A) = St(A) \rtimes Gl(A)$.

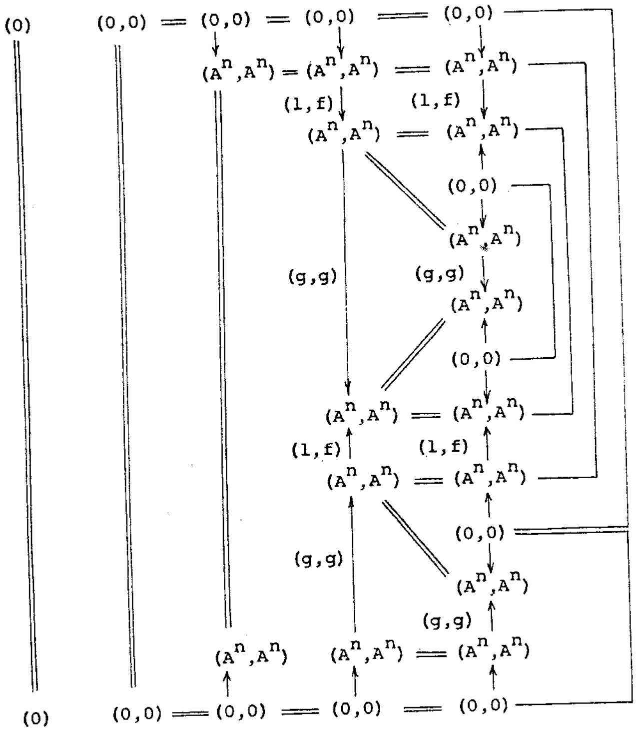


FIGURE 5

We observe first that there is a natural map $\varphi_n\colon St(M_n(A)) \to St(A)$ compatible with the identification $Gl(M_n(A)) = Gl(A)$ obtained by splitting a matrix up into $n \times n$-blocks. Here $M_n(A)$ denotes the ring of $n \times n$-matrices with coefficients in A. In terms of Milnor's generators, we may define $x^n_{ij}(\alpha) = \varphi_n(x_{ij}(\alpha)) = \prod x_{r+n(i-1),s+n(j-1)}(\alpha_{rs}) \in St(A)$, where $\alpha \in M_nA$, and the product runs over $1 \leqslant i, j \leqslant n$. The order of terms in this product does not matter, because all the terms commute. It is a simple exercise to check that φ_n satisfies the relations, and thus is well-defined. Following Milnor's notation now, we may also define elements $w^n_{ij}(\alpha) = \varphi_n(w_{ij}(\alpha))$ and $h^n_{ij}(\alpha) = \varphi_n(h_{ij}(\alpha))$. The point is that these elements x^n_{ij}, w^n_{ij} and h^n_{ij} satisfy the same identities that their counterparts x_{ij}, w_{ij}, h_{ij} satisfy over a not-necessarily-commutative ring. In particular we have the following [4, Lemma 9.6]:

LEMMA 4.1. *Given $f, g \in Gl_n(A)$, the commutator $[h^n_{12}(f), h^n_{13}(g)]$ is equal to $h^n_{13}(fg)h^n_{13}(f)^{-1}h^n_{13}(g)^{-1}$.*

We also have:

LEMMA 4.1bis. $[h^n_{12}(f), h^n_{13}(g)] = h^n_{12}(f)\, h^n_{12}(g)\, h^n_{12}(gf)^{-1}$.

Thus we may define $f \star g = [h^n_{1i}(f), h^n_{1j}(g)]$ for any i and j distinct from each other and 1. The image of $f \star g$ in $Gl(A)$ is $[f \oplus f^{-1} \oplus 1,\ g \oplus 1 \oplus g^{-1}] = [f, g]$, and if $[f, g] = 1$ then $f \star g \in K_2A$.

LEMMA 4.2. *The definition of $f \star g$ for $f, g \in Gl(A)$ is independent of the number n chosen so that $f, g \in Gl_n(A)$.*

Proof. This follows from the fact that $h^n_{1,m+1}(f) = h^m_{1,n+1}(f)$ for $m, n \geqslant 1$. Q.E.D.

LEMMA 4.3. *If f, g are elementary matrices which commute, (i.e. $f, g \in E(A)$) then our definition of $f \star g$ agrees with Milnor's.*

Proof. Our $f \star g$ is actually Milnor's $(f \oplus f^{-1} \oplus 1) \star (g \oplus 1 \oplus g^{-1})$, which is just $f \star g + (f^{-1} \star 1) + (1 \star g^{-1}) = f \star g$, according to [4, Lemmas 8.7 and 8.1]. Q.E.D.

DEFINTION. Let $H(A)$ denote the group $Gl(A) \ltimes St(A)$, the semidirect product of $Gl(A)$ and $St(A)$ with respect to the usual action of $Gl(A)$ on $St(A)$.

The existence of an action of $Gl(A)$ on $St(A)$ follows from the fact that $St(A)$ is the universal central extension of $E(A)$, which is a normal subgroup of $Gl(A)$. Let the map $St(A) \to Gl(A)$ be denoted by $x \to \bar{x}$. We choose the action so that $\overline{x^f} = \bar{x}^f = f^{-1}\bar{x}f \in Gl(A)$.

We use letters x, y, z to denote elements of $St(A)$, and letters f, g, h to denote elements of $Gl(A)$.

Let j, k, p denote the indicated maps in the short exact sequence

$$0 \to St(A) \xrightarrow{j} H(A) \underset{p}{\overset{k}{\leftrightarrows}} Gl(A) \to 0.$$

The multiplication in $H(A)$ is defined so that $j(x)\, k(f) = k(f)\, j(x^f)$.

LEMMA 4.4. *Suppose $x \in \operatorname{im}(St_{2n}(A) \to St(A))$, $f \in Gl_n(A)$, $m \geqslant 3$. Then $x^f = h^n_{1m}(f)^{-1}xh^n_{1m}(f)$.*

Proof. We begin by verifying that the expression $h^n_{1m}(f)^{-1}xh^n_{1m}(f)$ is independent of m. If x were of the form $x^n_{12}(g)$ or $x^n_{21}(g)$ then this would follow from [4, Cor. 9.4], which asserts that $h_{1m}(f)^{-1}x_{12}(g)\, h_{1m}(f) = x_{12}(f^{-1}g)$ and $h_{1m}(f)^{-1}x_{21}(g)\, h_{1m}(f) = x_{21}(gf)$. It is easy to see that elements of the form $x^n_{12}(g)$ and $x^n_{21}(g)$ generate $\operatorname{im}(St_{2n}(A) \to St(A))$, so the independence follows in general. In fact, the expression is also independent of n, because $h^n_{1,m+1}(f) = h^m_{1,n+1}(f)$, as before.

We may define an automorphism of $St(A)$ in this way which is compatible with the action of f on $E(A) \subset Gl(A)$. Thus, by universality of the central extension $St(A) \to E(A)$, this automorphism agrees with $x \to x^f$. Q.E.D.

LEMMA 4.5. *The map* φ_n: $St(M_n(A)) \to St(A)$ *is an isomorphism.**

Proof. We know that the square:

$$\begin{array}{ccc} St(M_n(A)) & \longrightarrow & Gl(M_nA) \\ \downarrow{\scriptstyle \varphi_n} & & \wr\downarrow \\ St(A) & \longrightarrow & Gl(A) \end{array}$$

commutes, and that the horizontal arrows express $St(M_n(A)$ (resp. $St(A)$) as the universal central extension of the commutator subgroup of the group $Gl(M_nA) = Gl(A)$. Q.E.D.

The following lemma is an easy exercise.

LEMMA 4.6. *Given* $m \neq 1$ *and* $f \in Gl_n(A)$, $h^n_{1m}(f^{-1})$ *differs from* $h^n_{1m}(f)^{-1}$ *by an element of* K_2A. *Moreover*, $h^n_{1m}(1) = 1$.

LEMMA 4.7. *We may define an action of* $Gl_A \times Gl_A$ *on* $H(A)$ *by setting*;

$$\begin{aligned} j(x)^{(1,f)} &= j(x^f) \\ k(g)^{(1,f)} &= j(f^{-1} \star g)\, k(g) \\ j(x)^{(f,f)} &= j(x^f) \\ k(g)^{(f,f)} &= k(g^f) \end{aligned}$$

Proof. We must verify:

(1a) $j(xy)^{(1,f)} = j(x)^{(1,f)} j(y)^{(1,f)}$

(1b) $k(gh)^{(1,f)} = k(g)^{(1,f)} k(h)^{(1,f)}$

(1c) $(j(x)k(g))^{(1,f)} = (k(g)j(x^g))^{(1,f)}$

(2a) $j(xy)^{(f,f)} = j(x)^{(f,f)} j(y)^{(f,f)}$

(2b) $k(gh)^{(f,f)} = k(g)^{(f,f)} k(h)^{(f,f)}$

(2c) $(j(x)k(g))^{(f,f)} = (k(g)j(x^g))^{(f,f)}$

(3a) $(j(x)^{(f,f)})^{(g,g)} = j(x)^{(fg,fg)}$

(3b) $(k(h)^{(f,f)})^{(g,g)} = k(h)^{(fg,fg)}$

(4a) $(j(x)^{(1,f)})^{(1,g)} = j(x)^{(1,fg)}$

(4b) $(k(h)^{(1,f)})^{(1,g)} = k(h)^{(1,fg)}$

* This result is well known; see, for instance, S. Klasa, On Steinberg Groups, pp. 131–138 in the Proceedings of the Conference on Orders, Group Rings and related topics, Lecture Notes in Math. #353, Springer-Verlag, Berlin, 1973.

(5a) $(j(x)^{(f,f)})^{(1,g^f)} = (j(x)^{(1,g)})^{(f,f)}$

(5b) $(k(h)^{(f,f)})^{(1,g^f)} = (k(h)^{(1,g)})^{(f,f)}$

(6a) $j(x)^{(1,f)} = j(x)$ if $f = 1$

(6b) $k(g)^{(1,f)} = k(g)$ if $f = 1$

(7a) $j(x)^{(f,f)} = j(x)$ if $f = 1$

(7b) $k(g)^{(f,f)} = k(g)$ if $f = 1$.

These properties are equivalent to the following identities:

(1b′) $(f^{-1} \star g)(f^{-1} \star h)^{g^{-1}} = f^{-1} \star (gh)$

(1c′) $(f^{-1} \star g)x^{gfg^{-1}} = x^f(f^{-1} \star g)$

(4b′) $(f^{-1} \star h)^g(g^{-1} \star h) = (fg)^{-1} \star h$

(5b′) $(g^{-1})^f \star h^f = (g^{-1} \star h)^f$

(6b′) $1 \star g = 1$,

which, in turn, follow from the usual commutator identities:

(i) $[x^{-1}, y][x^{-1}, z]^{y^{-1}} = [x^{-1}, yz]$

(ii) $[x^{-1}, y]z^{yxy^{-1}} = z^x[x^{-1}, y]$

(iii) $[x^{-1}, z]^y \cdot [y^{-1}, z] = [(xy)^{-1}, z]$

(iv) $[(y^{-1})^x, z^x] = [y^{-1}, z]^x$

(v) $[1, y] = 1$

together with the previous Lemmas 4.1–4.6. Q.E.D.

The following lemma is easy to prove:

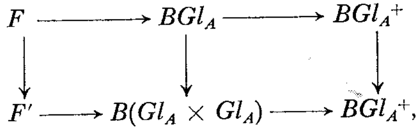
LEMMA 4.8. *Let* $H(A) \to Gl(A) \times Gl(A)$ *be the map defined by* $j(x) \to (1, \bar{x})$ *and* $k(f) \to (f, f)$. *This map is a group homomorphism which is compatible with the action of* $Gl(A) \times Gl(A)$.

5. COMPUTATION OF $G(A)$

In this section we identify $G(A)$ (defined in section 3) with $H(A) = Gl(A) \ltimes St(A)$, and show that the action of $Gl(A) \times Gl(A)$ on $G(A)$ agrees with the action on $H(A)$ defined in Section 4.

We begin by defining a map j': $St(A) \to G(A)$, analogous to the map j: $St(A) \to$

$H(A)$ of Section 4. Consider the map $Gl(A) \to Gl(A) \times Gl(A)$ defined by $f \to (1, f)$. There is a map induced on the homotopy fibers:



which induces the map $j' : \pi_1 F = St_A \to \pi_1 F' = G(A)$.

Let $p' : G(A) \to Gl_A$ be the composite $G(A) \to Gl(A) \times Gl(A) \to Gl(A)$, where the second map is projection on the first factor.

Consider the diagonal map $Gl(A) \to Gl(A) \times Gl(A)$. The composite $B(Gl(A)) \to B(Gl(A) \times Gl(A)) \to B\,Gl_A(A)^+$ is null-homotopic so we get a map $B(Gl(A)) \to F'$, which induces a map $Gl(A) \to \pi_1 F' = G(A)$ which we call k'.

LEMMA 5.1. *The diagram in Figure 6 has exact rows and columns, and* $p'k' = 1$. *In particular,* $G(A) = St(A) \rtimes Gl(A)$.

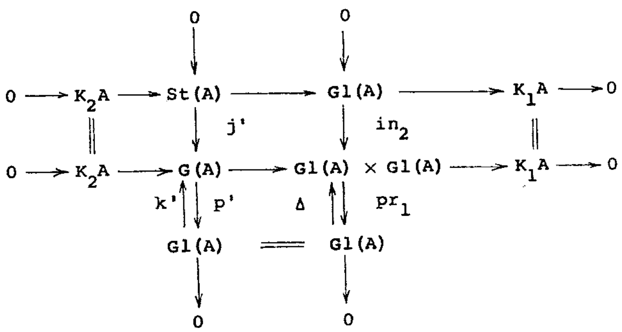


FIGURE 6

Proof. The rows are exact because they are part of long exact sequences associated to fibrations. The column containing $Gl(A) \times Gl(A)$ is certainly exact. A diagram chase shows that j' is injective. The identity $p'j' = 1$ is clear, as is $p'k' = 1$. Another diagram chase shows that $\ker p' = \operatorname{im} j'$. Q.E.D.

LEMMA 5.2. *Given* $f, g \in Gl_n A$, $j'(c(f * g)) = k'(g)^{-1}(k'(g))^{(1,f)}$.

Proof. This follows from Lemma 3.1. Q.E.D.

The following lemma shows that the action of $Gl(A)$ on $St(A)$ resulting from Lemma 5.1 is the usual one.

LEMMA 5.3. *For* $x \in St(A)$ *and* $f \in Gl(A)$, $j'(x)\,k'(f) = k'(f)\,j'(x^f)$.

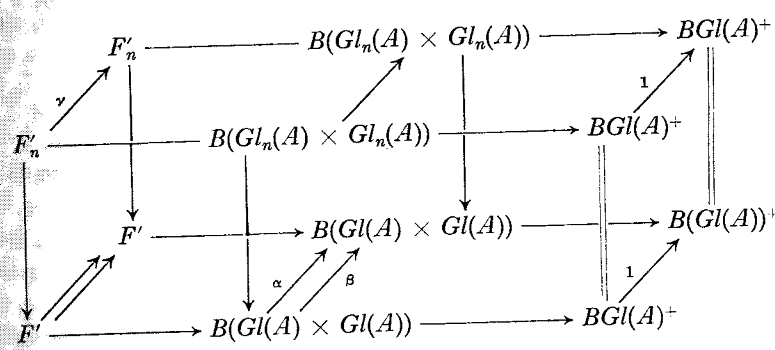
Proof. Consider the map $St(A) \to St(A)$ defined by $x \mapsto j'^{-1}(k'(f)^{-1} j'(x) k'(f))$. This map lies over the map $g \to g^f$ on $Gl(A)$, so by universality of the central extension $St(A) \to E(A)$, it must be the map $x \to x^f$. Q.E.D.

LEMMA 5.4. *Given* $g \in Gl(A)$, *conjugation by* g *induces a homotopy equivalence* $B(Gl(A))^+ \to B(Gl(A))^+$ *which is homotopic to the identity.*

Proof. This is a consequence of the fact that $B\,Gl(A)^+$, being an H-space, is simple, and thus its fundamental group acts trivially on it in the homotopy category. Q.E.D.

LEMMA 5.5. *Given* $f, g \in Gl_n(A)$, $k'(f)^{(1,g)} = j'(h^n_{1m}(g)^{-1})\,k'(f)\,j'(h^n_{1m}(g))$.

Proof. For convenience, suppose $m = 2$. We know that the right-hand side of the proposed equation is $k'(f)^{(1, g \oplus g^{-1})}$. Let F'_n denote the homotopy fiber of the map $B(Gl_n(A) \times Gl_n(A)) \to B(Gl(A))^+$. Now $g \oplus g^{-1}$ and g act the same way on $Gl_n(A)$, so we get the diagram, commutative up to homotopy:



Here α and β represent conjugation by $(1, g)$ and $(1, g \oplus g^{-1})$. By Lemma 5.4 the maps on the right, parallel to α and β, are the identity. The result follows from the fact that $k'(f)$ comes from $\pi_1 F'_n$, and both $k'(f)^{(1,g)}$ and $k'(f)^{(1, g \oplus g^{-1})}$ may be computed using the single map γ. Q.E.D.

LEMMA 5.6. *The isomorphism* $H(A) \cong G(A)$ *defined by sending* $j(x) \to j'(x)$ *and* $k(f) \to k'(f)$ *is compatible with the action of* $Gl(A) \times Gl(A)$ *on* $H(A)$ *and* $G(A)$.

Proof. We must show:

(a) $j'(x)^{(1,f)} = j'(x^f)$

(b) $k'(g)^{(1,f)} = j'(f^{-1} \star g)\,k'(g)$

(c) $j'(x)^{(f,f)} = j'(x^f)$

(d) $k'(g)^{(f,f)} = k'(g^f)$.

In the proof of Lemma 5.3 we obtained (c), and (a) is immediate from an examination of Figure 6. To obtain (d) is easy: $k'(g^f) = k'(g)^{k'(f)} = k'(g)^{(f,f)}$. To obtain (c) we use Lemmas 5.5 and 4.4:

$$\begin{aligned} k'(g)^{(1,f)} &= j\,(h^n_{13}(f))^{-1}\, k'(g) j'(h^n_{13}(f)) \\ &= j'(h^n_{13}(f))^{-1} j'(h^n_{13}(f))^{k'(g)^{-1}}\, k'(g) \\ &= j'(h^n_{13}(f))^{-1} j'(h^n_{13}(f))^{(g,g)^{-1}}\, k'(g) \\ &= j'(h^n_{13}(f)^{-1} h^n_{13}(f)^{g^{-1}})\, k'(g) \\ &= j'([h^n_{13}(f^{-1}),\, h^n_{13}(g)])\, k'(g) \\ &= j'(f^{-1} \star g)\, k'(g). \end{aligned}$$

Q.E.D.

LEMMA 5.7. *For $f, g \in Gl_n(A)$ which commute, $c(f * g) = f^{-1} \star g$.*

Proof. From 5.2 we have $j'(c(f * g)) = k'(g)^{-1}(k'(g))^{(1,f)}$, which, by 5.5 is just $j'(f^{-1} \star g)^{(g,g)} = j'(f^{-1} \star g)$.

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Golodideale der Gestalt $\mathfrak{a} \cap \mathfrak{b}$

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In seiner Arbeit [7] hat G. Levin den Begriff des Golodhomomorphismus eingeführt, mit dessen Hilfe man in vielen Fällen Ringwechselsätze für Poincaré–Reihen beweisen kann.

Beispiele von Golodhomomorphismen sind in [7], [8] und [1] angegeben.

Es soll hier untersucht werden, unter welchen Umständen für zwei Ideale $\mathfrak{a}, \mathfrak{b}$ eines lokalen Rings R der kanonische Epimorphismus $R \to R/\mathfrak{a} \cap \mathfrak{b}$ ein Golodhomomorphismus ist.

Alle betrachteten Ringe sollen noethersch und lokal sein. Unter einer Algebraauflösung X des Restklassenkörpers k eines lokalen Rings (R, m, k) wollen wir eine graduierte, differentielle Algebra verstehen, die als Komplex eine minimale freie Auflösung des Restklassenkörpers k darstellt. Jeder lokale Ring besitzt nach [4] eine solche Algebraauflösung des Restklassenkörpers. Wir wollen hier kurz die Definition eines Golodhomomorphismus zitieren.

Es sei $\varphi: R \to S$ ein lokaler Homomorphismus, der einen Isomorphismus der Restklassenkörper k induziert und X eine Algebraauflösung von k.

DEFINITION 1 (G. Levin, [8], 1.4; 1.3). φ heißt Golodhomomorphismus, wenn $X \otimes_R S$ triviale Masseyoperationen besitzt, d.h. für jede Folge von homogenen Elementen $v_1, \ldots, v_n \in \tilde{H}(X \otimes_R S)$ gibt es ein homogenes Element $\gamma(v_1, \ldots, v_n) \in m(X \otimes_R S)$, so daß folgende Regeln gelten:

(a) Für alle $v \in \tilde{H}(X \otimes S)$ ist $\gamma(v)$ ein Zykel, dessen Homologieklasse mit v übereinstimmt.

(b) Für alle $v_1, \ldots, v_n \in \tilde{H}(X \otimes_R S)$ ist

$$d\gamma(v_1, \ldots, v_n) = \sum_{k=1}^{n-1} \bar{\gamma}(v_1, \ldots, v_k)\, \gamma(v_{k+1}, \ldots, v_n).$$

Dabei sei für ein homogenes Element w einer graduierten Algebra $\bar{w} = (-1)^{1+\deg(w)} w$ gesetzt.