normalizes W_1 and W_2. However, $G_0 = G_1G_2 = \langle N_G(W_1)^{(\infty)}, N_G(W_2)^{(\infty)}\rangle$, so $G_0 = G_0^g$. This contradiction completes the proof of the main theorem.

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The *K*-Theory of Endomorphisms

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In this paper we prove an analog of the Fundamental Theorem [2, p. 663; 3, p. 236], which identifies the K-theory of the category of nilpotent endomorphisms of projective A-modules. Here we identify the K-theory of the category of arbitrary endomorphisms of projective A-modules, and recover a result of Almkvist for K_0.

If A is a ring with 1, let $\mathbf{P}_A$ denote the exact category of finitely generated projective left A-modules. Let $\mathbf{End}_A$ denote the exact category whose objects are all pairs (P, f), where P is an object of $\mathbf{P}_A$ and f is an endomorphism of P. An arrow from (P, f) to (Q, g) is an A-homomorphism $P \to Q$ such that

$$\begin{array}{ccc} P & \longrightarrow & Q \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ P & \longrightarrow & Q \end{array}$$

commutes. An exact sequence in $\mathbf{End}_A$ is one which is exact as a sequence of A-modules, forgetting the endomorphisms. The map $K_i\,\mathbf{End}_A \to K_iA$, induced by the forgetful functor $(P, f) \mapsto P$, is a split surjection; we let $\mathrm{End}_i\, A$ denote its kernel.

If A is commutative, we can form the ring $R = (1 + TA[T])^{-1}A[T]$; it is an augmented A-algebra, T generates the augmentation ideal (i.e., $A = R/TR$), and T is in the Jacobson radical of R. The map $K_iA \to K_iR$ is a split injection; we let EK_iA denote its cokernel. We shall prove the following theorem.

THEOREM 1. $EK_iA = \mathrm{End}_{i-1}A$.

The Fundamental Theorem is the statement $NK_iA = \mathrm{Nil}_{i-1}A$, where $NK_iA = \mathrm{ckr}(K_iA \to K_iA[T])$, and where Nil_iA is obtained from the category of nilpotent endomorphisms in the same way $\mathrm{End}_i\, A$ is obtained from the category of endomorphisms.

We can formulate a theorem which encompasses both Theorem 1 and the Fundamental Theorem; it is a statement about a ring A which is not necessarily

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commutative. Let $S \subseteq A[U]$ be a multiplicative set of monic central polynomials with U in S. Given $g(U) = U^n + a_{n-1}U^{n-1} + \cdots + a_0$, let $\tilde{g}(T) = g(1/T) \cdot T^n = 1 + a_{n-1}T + \cdots + a_0 T^n$. Let $\tilde{S} = \{\tilde{g} \mid g \in S\}$; it is a multiplicative set of central non-zero-divisors in $A[T]$.

Let $\mathbf{End}_A{}^S$ denote the exact category whose objects are all pairs (P, f), where P is an object of $\mathbf{P}_A$, and f is an endomorphism of P such that $g(f) = 0$ for some g in S. Let $\mathrm{End}_i{}^S A = \ker(K_i \mathbf{End}_A{}^S \to K_i A)$. Let R be the ring $\tilde{S}^{-1}A[T]$, and $EK_i{}^S = \mathrm{ckr}\,(K_i A \to K_i R)$.

THEOREM 2. $EK_i{}^S A = \mathrm{End}_{i-1}^S A$.

When A is commutative, and S is the set of all monic polynomials, we recover Theorem 1. The point is that the characteristic polynomial of an endomorphism is monic locally on Spec(A), and thus divides a monic polynomial. The Cayley–Hamilton theorem provides the vanishing [2, p. 631]. On the other hand, when S is the multiplicative set of powers of U, we recover the Fundamental Theorem.

Proof of Theorem 2. The proof is modeled after the proof of the Fundamental Theorem in [3]. We may assume that each $g(U)$ is S is divisible by U. Consider the diagram of rings:

$$\begin{array}{ccc} & & A[T] \\ & & \downarrow \\ A[U] & \longrightarrow & A[T, U]/TU - 1 = B \end{array}$$

Let $\mathbf{M}$ denote the exact category of triples (M, N, j) where M is an $A[T]$-module, N is an $A[U]$-module, and $j\colon M \otimes_{A[T]} B \simeq N \otimes_{A[U]} B$ is an isomorphism of B-modules. This category is better known as the category of quasi-coherent sheaves on the projective line $X = \mathbf{P}_A{}^1$.

Now given g in S, the pair $(\tilde{g}, g)$ determines a divisor Z in X because $\tilde{g}/g = T^n$ is a unit in B. The localization theorem for projective modules [3] yields a long exact sequence:

$$\cdots \to K_i X \to K_i A[T, 1/\tilde{g}] \to K_{i-1}\mathbf{H}(X, Z) \to \cdots$$

which ends at $K_0 A[T, 1/\tilde{g}]$. Here $K_i X$ denotes $K_i \mathbf{P}_X$, where $\mathbf{P}_X \subset \mathbf{M}$ is the category of vector bundles on X; (M, N, j) in $\mathbf{M}$ is a *vector bundle* if M (resp. N) is a finitely generated projective $A[T]$ (resp. $A[U]$)-module. In addition, $\mathbf{H}(X, Z)$ denotes the category of quasi-coherent sheaves on X which have a short resolution by vector bundles on X, and which are zero on $X - Z$; (M, N, j) is zero on $X - Z$ iff $M \otimes_{A[T]} A[T, 1/\tilde{g}] = 0$ and $N \otimes_{A[U]} A[U, 1/g] = 0$. From $M \otimes_{A[T]} A[T, 1/\tilde{g}] = 0$ and the fact that $\tilde{g}$ has constant term 1, it follows that $M = M \otimes_{A[T]} B$. It is easy to see that

$$N \mapsto (N \otimes_{A[U]} B, N, 1)$$

and

$$(M, N, j) \mapsto N$$

are exact functors which exhibit an equivalence of $\mathbf{H}(X, Z)$ with $\mathbf{H}(A[U], g)$, where the latter is the category of finitely presented g^n-torsion $A[U]$-modules of projective dimension 1. This equivalence is a standard sort of excision based on the inclusion of the closed set Z in the open set Spec($A[U]$).

If we pass to the limit over S we obtain the exact sequence:

$$\cdots \to K_i X \to K_i R \to K_{i-1}\mathbf{H}(A[U], S) \to \cdots$$

where $\mathbf{H}(A[U], S) = \bigcup_g \mathbf{H}(A[U], g)$.

We now use an argument of Quillen's from [3] to identify $\mathbf{H}(A[U], S)$ with $\mathbf{End}_A{}^S$. Define a functor

$$b\colon \mathbf{End}_A{}^S \to \mathbf{H}(A[U], S)$$

by

$$(P, f) \mapsto P_f,$$

where P_f is just P as A-module, with U acting on P as f:

$$U \cdot x = f(x) \qquad \text{for} \quad x \in P.$$

The characteristic exact sequence [2, p. 630]

$$0 \longrightarrow P[U] \xrightarrow{U-f} P[U] \longrightarrow P_f \longrightarrow 0$$

of $A[U]$-modules shows that P_f has projective dimension 1. To see that b is an equivalence, it suffices to show that every N in $\mathbf{H}(A[U], S)$ is a finitely generated projective A-module. Let $g(U)$ be a monic polynomial in S that annihilates N, and let

$$0 \to B \to C \to N \to 0$$

be a short resolution of N with B, C in $\mathbf{P}_{A[U]}$.

The surjection $C/gC \twoheadrightarrow N$ shows that N is a finite A-module, because $A[U]/g$ is. The exact sequence

$$0 \to N \to B/gB \to B/gC \to 0$$

shows that N is a flat A-module, for B/gB is flat, and B/gC has Tor-dimension 1. Since N is finite and flat, it is projective. This concludes the proof that b is an equivalence.

Now we use the calculation $K_iX = K_iA \oplus K_iA$ [6]. Given P in $\mathbf{P}_A$ let $P_X(n) = (P[T], P[U], T^n)$ in $\mathbf{P}_X$, and let $P_X = P_X(0)$. Let $h_n: \mathbf{P}_A \to \mathbf{P}_X$ be the exact functor $P \mapsto P_X(n)$. The statement of Quillen's theorem is that

$$\begin{pmatrix} h_0 \\ h_{-1} \end{pmatrix}: K_iA \oplus K_iA \longrightarrow K_iX$$

is an isomorphism. It follows that

$$\begin{pmatrix} h_0 - h_{-1} \\ h_0 \end{pmatrix}: K_iA \oplus K_iA \longrightarrow K_iX$$

is also an isomorphism.

The composite

$$K_iA \xrightarrow[h_0]{} K_iX \longrightarrow K_iR$$

is the natural map, and is thus a split injection.

The characteristic sequence for an object (P, f) in $\mathbf{End}_A$ extends to an exact sequence on X:

$$0 \longrightarrow P_X(-1) \xrightarrow{v} P_X \longrightarrow P_f \longrightarrow 0.$$

Here $v = (T - f, 1 - Uf)$. The exactness theorem [6, Sect. 8, Th. 3.1] shows, then, that the square

$$\begin{array}{ccc} K_i\,\mathbf{End}_A{}^S & \longrightarrow & K_i(\mathbf{H}(A[U], S)) \\ \downarrow & & \downarrow \\ K_iA & \xrightarrow[h_0 - h_{-1}]{} & K_iX \end{array}$$

commutes. The left vertical map in the diagram is the natural split surjection.

It is now clear that the long exact sequence splits into shorter ones:

$$0 \longrightarrow K_iA \underset{\longrightarrow}{\overset{\longleftarrow}{}} K_iR \longrightarrow K_{i-1}\,\mathbf{End}_A{}^S \underset{\longrightarrow}{\overset{\longleftarrow}{}} K_{i-1}A \longrightarrow 0,$$

yielding the theorem. Q.E.D.

COROLLARY 3 [1, p. 377]. *The map*

$$ch: \mathrm{End}_0 A \to 1 + TR,$$

given by

$$(P, f) \mapsto \det(1 - Tf),$$

is an isomorphism, where A is a commutative ring.

Proof. We take S to be the set of all monic polynomials in $A[U]$. Since T is in the Jacobson radical of R, $EK_1A = 1 + TR$. It is enough to show that the composite

$$1 + TR \xrightarrow{\sim} \mathrm{End}_0\, A \xrightarrow{ch} 1 + TR$$

is the identity, where the first map is the isomorphism induced by the boundary map

$$K_1R \to K_0\,\mathbf{End}_A$$

from the proof of Theorem 2. Notice that it is enough to check this identity for an arbitrary $\tilde{g} \in \tilde{S} \subset 1 + TR$.

If $Z \subset X$ is the divisor defined by $(\tilde{g}, g)$, its structure sheaf O_Z is $(A[T]/\tilde{g}, A[U]/g, 1) \in \mathbf{M}$. Let nH denote the divisor defined by $(1, U^n)$; H is the hyperplane at infinity. There are two exact sequences in $\mathbf{M}$:

$$0 \longrightarrow A_X(-n) \xrightarrow{v} A_X \longrightarrow O_Z \longrightarrow 0,$$
$$0 \longrightarrow A_X(-n) \xrightarrow{w} A_X \longrightarrow O_{nH} \longrightarrow 0.$$

We must now refer to the proof of the localization theorem for projective modules in [3] in order to compute the boundary map explicitly. We will show that the element $\tilde{g}$ of K_1R maps to the class $[O_Z] - [O_{nH}]$ in $K_0\mathbf{H}(A[U], S) = K_0\,\mathbf{End}_A$.

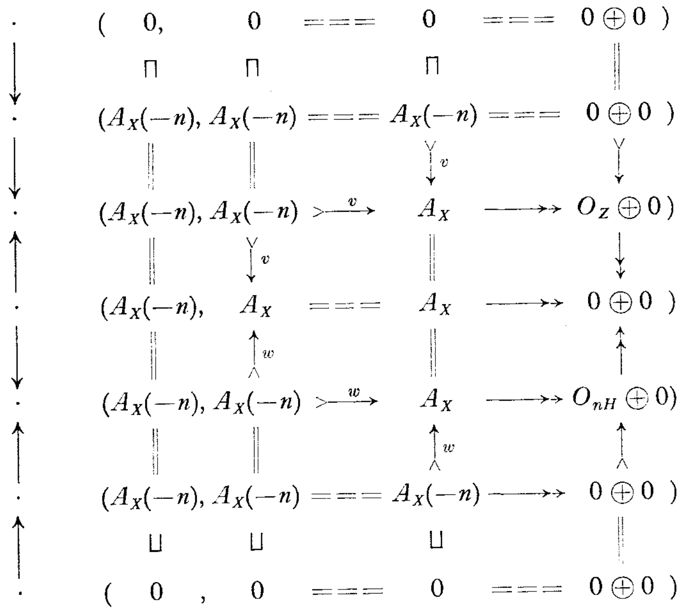

FIGURE 1

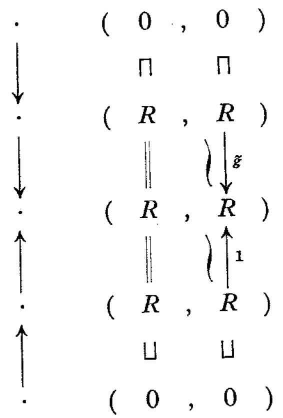

FIGURE 2

Consider the diagram in Fig. 1. Each row represents an object in the category G [3, p. 230], the data between adjacent rows determine arrows; the symbol represents an injection with splitting given, and all such injections associated with the same arrow of G have the same cokernel. The whole diagram is a loop and represents an element of $\pi_1 G$. The restriction of these data to Spec R yields the loop shown in Fig. 2, which represents the class $\tilde{g}(T)/1 \in K_1 R = \pi_1 S^{-1}S(\mathbf{P}_R)$ (different S, see [3, p. 224]). Forgetting all of the loop in Fig. 1 except for the left-hand summand in the right column yields:

$$0 \rightarrowtail O_Z \twoheadrightarrow 0 \twoheadleftarrow O_{nH} \leftarrowtail 0,$$

a loop in $Q\mathbf{H}(A[U], S)$ which represents the class

$$\left[A^n, \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{pmatrix}\right] - \left[A^n, \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 1 & 0 \\ & & & 1 & 0 \end{pmatrix}\right]$$

in $\mathrm{End}_0 A$, where $g(U) = U^n + a_{n-1}U^{n-1} + \cdots + a_0$. To compute ch applied to this difference is easy because these matrices are in rational canonical form; if we do so, we obtain the answer $\tilde{g}(T)/1$. Q.E.D.

This establishes the identity.

COROLLARY 4. *If A is commutative, then* $\mathrm{End}_1 A$ *has the following generators and relations as Abelian group*:

generators: $\langle r, s\rangle$, $\quad r, s \in R, \quad r \in TR, \quad$ *or* $\quad s \in TR.$

relations:
$$\langle r, s\rangle\langle -s, -r\rangle = 1,$$
$$\langle r, s\rangle\langle r, t\rangle = \langle r, s + t + rst\rangle,$$
$$\langle r, st\rangle\langle s, tr\rangle\langle t, rs\rangle = 1.$$

Proof. This is precisely the presentation given in [4] for $\ker(K_2 R \to K_2 A)$. Q.E.D.

What happens if we consider automorphisms instead of endomorphisms? Let $\mathbf{Aut}_A$ denote the full sub-exact-category of $\mathbf{End}_A$ consisting of all objects (P, f) for which f is an isomorphism. If A is a commutative ring, then we know that f is an isomorphism iff its determinant is a unit. Thus, if S is the set of monic polynomials in $A[U]$ whose constant terms are units in A, then $\mathbf{Aut}_A = \mathbf{End}_A{}^S$. Now S does not contain $g(U) = U$, but it does contain $g(U) = U - 1$, so a change of coordinates yields:

COROLLARY 5. *There is an exact sequence with splittings*:

$$0 \longrightarrow K_i A \overset{\longleftarrow}{\longrightarrow} K_i R_1 \longrightarrow K_{i-1}\,\mathbf{Aut}_A \overset{\longleftarrow}{\longrightarrow} K_{i-1} A \longrightarrow 0,$$

where Spec R_1 *is the intersection of the complements in* $X = \mathbf{P}_A{}^1$ *of the divisors* Z *defined by* $g \in S$.

See Fig. 3 for a picture of this situation.

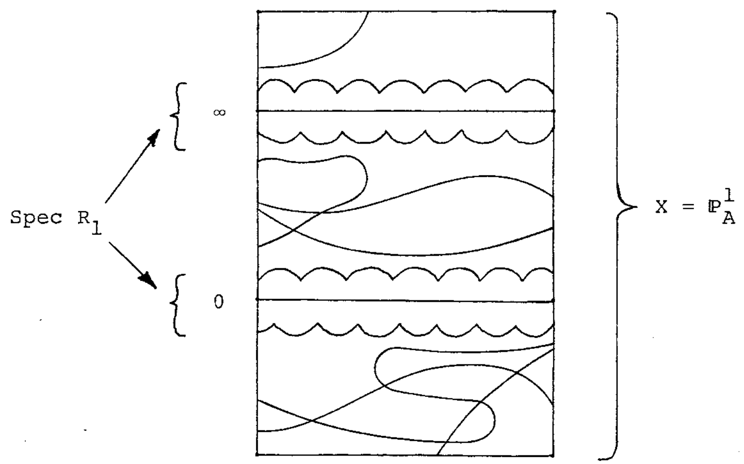

FIGURE 3

It is easy to see that R_1 comes equipped with two augmentations:

$$s_0, s_\infty : R_1 \rightrightarrows A$$

corresponding to the sections at 0 and ∞ of $X \to \operatorname{Spec} A$. Let $\operatorname{Aut}_i A$ denote the kernel of the split surjection $K_i\, \mathbf{Aut}_A \to K_i A$. The difference $s_0 - s_\infty$ defines a map

$$\operatorname{Aut}_{i-1} A \to K_i A. \tag{*}$$

When $i = 1$, this is the canonical surjection [2, p. 348]. To see this, one considers the class in $K_1 R_1$ determined by the automorphism $(U - f)/(U - 1)$ of $P \otimes_A R_1$, where (P, f) is an object of $\mathbf{Aut}_A$. A computation of the boundary map (similar to that in Corollary 3) yields $[P, f] - [P, 1]$ in $\operatorname{Aut}_0 A$, and an application of $s_0 - s_\infty$ yields the class of f in $K_1 A$.

When $i = 2$ this map is interesting. In a future paper we intend to show that Milnor's pairing $f \star g$ [5] in $K_2 A$ lifts in a natural way to $\operatorname{Aut}_1 A$. In the same paper we will see how to define the map $\operatorname{Aut}_{i-1} A \to K_i A$ when A is not necessarily commutative.

COROLLARY 6. *Let $A(U) = S^{-1}A[U]$, where S is the multiplicative set of all monic polynomials in $A[U]$. Then*

$$K_i A(U) = K_i A \oplus K_{i-1} A \oplus NK_i A \oplus EK_i A.$$

The proof of this corollary is analogous to the proof in [3] of the other half of the Fundamental Theorem: $K_i A[T, T^{-1}] = K_i A \oplus K_{i-1} A \oplus NK_i A \oplus NK_i A$. The point is that $A(U) = R[1/T]$; this observation yields a Mayer–Vietoris sequence that splits up.

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On the Boundedness and the Unboundedness of the Number of Generators of Ideals and Multiplicity

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For an unmixed ideal I in a regular formal power series ring $R = k[[x_1, ..., x_n]]$ with the Krull dim of $R/I = 1$ or dim $R/I = 2$ or dim $R/I = 3$ and R/I integrally closed in its field of quotients, one can give an upper bound for the number of generators for I over R in terms of the multiplicity of R/I and the embedding dimension of R/I. For dim $R/I = 4$ an example is given to show this is not possible, even for prime ideals.

In [3], I incorrectly claimed a proof of the following statement. Let $\mathcal{O}(V, p) = {}_n\mathcal{O}_p/I(V, p)$ be the ring of germs at p of holomorphic functions on a complex analytic subvariety V of $\mathbb{C}^n$ of pure dimension r and multiplicity μ. Let $\#I$ denote the minimal number of generators of the ideal $I(V, p)$ as a module over ${}_n\mathcal{O}_p$. Then $\#I \leqslant (\mu + 1)^n$. The proof given there for the cases $r = 1$ or 2 is correct, but the argument on p. 289 for $r \geqslant 3$ contains some rather large gaps and is in fact wrong. (Hochster has given a counterexample.)

In this paper we show that for unmixed ideals I in a regular formal power series ring R with residue field of characteristic zero it is possible to give an upper bound for the number of generators of I over R in terms of the multiplicity of R/I and the embedding codimension of R/I provided either Krull dim $R/I = 1$, or Krull dim $R/I = 2$, or Krull dim $R/I = 3$ and R/I is normal, or depth of R/I = dim R/I, or depth of R/I = dim $R/I - 1$. The proofs presented here arose in discussions with Professor William Heinzer and Professor Melvin Hochster. These generalize the correct portion of [3] and have the advantage of being easier to understand. In Section 2 we give Hochster's counterexample which is the local ring of a 4-dimensional nonnormal variety in affine 7 space. The author would like to thank Professor Hochster and Professor Heinzer for their permission to reproduce these arguments here and Professors Otto Forster and Raghavan Narasimhan for their skepticism concerning the validity of [3]. Sally has also obtained results similar to those in Section 1 for the Cohen–Macaulay case.

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