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EXACT SEQUENCES OF WITT GROUPS#

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An exact sequence of Witt groups, motivated by exact sequences obtained by Lewis and by Parimala, Sridharan and Suresh, is constructed. The behavior of the maps involved in these sequences with respect to isotropy is completely determined in the case of division algebras. In particular, the kernels of the maps involved in the previous sequences are explicitly given, leading to a new proof of their exactness. Similar exact sequences of equivariant Witt groups are constructed. As an application, relations between the cardinality of certain Witt groups are obtained.

Key Words: Central simple algebra with involution; Equivariant Witt group; Exact sequence of Witt groups; Hermitian form; Isotropy; Morita equivalence; Witt group.

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1. INTRODUCTION

Base change is an important tool in the algebraic theory of quadratic forms and of hermitian forms over division algebras. For a field extension L/K (of characteristic different from 2), one can consider base change from K to L; if moreover the extension has finite degree, then one also has the Scharlau transfer. The situation is especially well understood when L/K is of odd degree or a quadratic extension: see the book of Scharlau (1985) for these basic notions and results.

The Witt group (and Witt ring for quadratic forms) gives a very useful way to study quadratic and hermitian forms. The above results can be expressed very efficiently in this framework. One of the basic results in the theory of quadratic forms is a theorem of Pfister that determines the kernel of the restriction map $r^*_{L/K}: W(K) \to W(L)$ for a quadratic extension L/K. More precisely, this kernel is the ideal generated by the form $\langle 1, -\delta \rangle$, where $L = K(\sqrt{\delta})$. One can express this result by the exactness of the sequence:

$$W(K) \xrightarrow{t} W(K) \xrightarrow{r^*_{L/K}} W(L), \tag{1}$$

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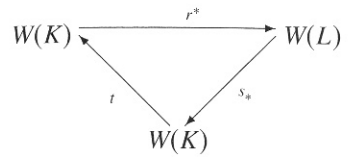
where t is multiplication by the two-dimensional form $\langle 1, -\delta\rangle$. By a result of Elman-Lam, the Scharlau transfer map $s_* : W(L) \longrightarrow W(K)$, can be used to embed (1) in the following exact triangle (cf. Scharlau, 1985, Ch. 2, 5.10):



(2)

For a quadratic extension L/K (resp. a quaternion division algebra $(a, b)_K = D$) with nontrivial automorphism $-$ (resp. with canonical involution $-$), one can consider the trace map $W(L, -) \to W(K)$ (resp. $W(D, -) \to W(K)$). By a result of Jacobson, these maps are injective (cf. Scharlau, 1985, Ch. 10, 1.1, 1.2, 1.7 and Jacobson, 1940).

Milnor and Husemoller (1973, App. 2), construct the following exact sequence of $W(K)$-modules:

$$0 \longrightarrow W(L, -) \longrightarrow W(K) \longrightarrow W(L), \tag{3}$$

where $-$ is the nontrivial automorphism of the quadratic extension L/K. The results concerning hermitian forms over quaternion division algebras are given in Lewis (1979, 1982a). He found the exact sequence

$$0 \longrightarrow W(D, -) \longrightarrow W(L, -) \longrightarrow W^{-1}(D, -) \longrightarrow W(L), \tag{4}$$

where $L = K(\sqrt{a}) \subset D$ is stable under $-$. In fact, in this sequence, D can also be split (cf. Scharlau, 1985, Ch. 10, 3.2). Lewis (1982b) uses this sequence to produce a non-cohomological version of the Bartels invariant, and in Lewis (1985), he constructed an exact octagon of Witt groups of Clifford algebras (of quadratic forms) in such a way that (4) and (6) below are special cases of it.

Parimala et al. (1995) obtained the following crucial exact sequence of Witt groups which is used by Bayer-Fluckiger and Parimala to prove Serre's conjecture II for classical groups:

$$W^{\varepsilon}(A, \sigma) \xrightarrow{\pi_1^{\varepsilon}} W^{\varepsilon}(\widetilde{A}, \sigma_1) \xrightarrow{\rho_1^{\varepsilon}} W^{-\varepsilon}(A, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^{\varepsilon}(\widetilde{A}, \sigma_2). \tag{5}$$

In (5), A is a central simple algebra over a field K with an involution σ of any kind, ε is an element of K with $\sigma(\varepsilon)\varepsilon = 1$, $\widetilde{A}$ denotes the centralizer of a skew-symmetric element $\lambda \in A^*$, $L = K(\lambda)$ is a quadratic extension of K, σ_1 is the restriction of σ to $\widetilde{A}$, and σ_2 is a certain involution over $\widetilde{A}$ that fixes L elementwise. The maps π_1^{ε} and $\pi_2^{-\varepsilon}$ above are transfers, and ρ_1^{ε} is a restriction map (for more details, see Section 2). Note that (4) is a particular case of (5).

Lewis (1982a), has found a longer exact sequence,

$$\begin{aligned} 0 &\to W(D, -) \to W(L, -) \to W^{-1}(D, -) \to W(L) \to W^{-1}(D, -) \\ &\to W(L, -) \to W(D, -) \to 0 \end{aligned} \tag{6}$$

This exact sequence motivated us to define further maps between Witt groups in order to continue (5) on the right. We obtain

Theorem 1.1. *There is an exact sequence of Witt groups (in fact of $W(K, \sigma|_K)$-modules):*

$$W^{-\varepsilon}(A, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^{\varepsilon}(\widetilde{A}, \sigma_2) \xrightarrow{\rho_2^{\varepsilon}} W^{-\varepsilon}(A, \sigma) \xrightarrow{\pi_3^{-\varepsilon}} W^{\varepsilon}(\widetilde{A}, \sigma_1), \tag{7}$$

where $\pi_3^{-\varepsilon}$ (resp. ρ_2^{ε}) is a transfer (resp. restriction) map defined in Section 2.

Of course, the exactness of (7) can be obtained as Parimala et al. (1995) did. Our proof is based on the one given by Lewis (1983) in the case of quaternion division algebras. The proof of 1.1 can be found in Section 3.

In spite of the exactness of (5), the kernel of each map is not explicitly given in the literature. In Section 4, we give an explicit description of the kernel of each map appearing in (5) and (7) in the case where A is a division algebra. More precisely, Theorem 4.4 describes the behavior of these maps with respect to isotropy when A is a division algebra. In particular, when we combine 4.4 with the explicit description of each image given by Bayer-Fluckiger and Parimala (1995), we obtain an alternative proof of the exactness of (5).

Lewis (1983), constructs an exact octagon of Witt groups of forms invariant under the action of a finite group G (Witt groups of equivariant forms) for quaternion division algebras. In Section 5, we show likewise that the sequences (5) and (7) are exact if we replace Witt groups by equivariant Witt groups. More precisely,

Theorem 1.2. *We suppose that A satisfies the same hypotheses as in Theorem 1.1. We have the two following exact sequences of $W(K, \sigma|_K)$-modules:*

$$W^{\varepsilon}(G, A, \sigma) \xrightarrow{\pi_1^{\varepsilon}} W^{\varepsilon}(G, \widetilde{A}, \sigma_1) \xrightarrow{\rho_1^{\varepsilon}} W^{-\varepsilon}(G, A, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^{\varepsilon}(G, \widetilde{A}, \sigma_2) \tag{8}$$

$$W^{-\varepsilon}(G, A, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^{\varepsilon}(G, \widetilde{A}, \sigma_2) \xrightarrow{\rho_2^{\varepsilon}} W^{-\varepsilon}(G, A, \sigma) \xrightarrow{\pi_3^{-\varepsilon}} W^{\varepsilon}(G, \widetilde{A}, \sigma_1). \tag{9}$$

In Section 6, by using (5) and (7) (resp. Theorem 1.2) we show how to construct an exact octagon of Witt groups (resp. equivariant Witt groups): see 6.1 and 6.2. In the literature one can find several octagons of Witt groups; for example, see Lewis (1983, 1985). Andrew Ranicki pointed out to us that the exact octagon of 6.1 can be viewed as a special case of an exact octagon of L-groups; in fact, in Ranicki (1987) he found a braid for any "ρ-twisted quadratic extension" of rings. When these rings are semisimple, they lead to an exact octagon. For more details see Ranicki (1987, 1992, p. 242) .

The octagons we obtain are useful when we try to estimate cardinalities of the underlying Witt groups. In fact, in Section 7, by using 6.1, we obtain the result

Corollary 1.3. *Let A be a K-central simple algebra with an involution σ of the first kind. Then we have $|W^{\varepsilon}(A, \sigma)||W^{-\varepsilon}(A, \sigma)| = |W(K)|$. In particular, $W(K)$ is finite if and only if $W^{\varepsilon}(A, \sigma)$ and $W^{-\varepsilon}(A, \sigma)$ are finite.*

This result was well known for quaternion algebras: see Lewis (1982a).

Although certain of our results can already be found in the literature, be it in a hidden or implicit form, be it as consequences of more general principles, we believe that our very explicit and at times computational approach has its own merits and may prove useful when working in particular situations, say an explicitly given field.

We use notions like Morita equivalence, isotropy of hermitian forms and Witt decomposition over a central simple algebra several times in this paper. For the convenience of the reader, we recall in Appendix A some facts about these notions. Of course, all of these results are well known, and one can find them in the literature: see McEvett (1969) for the notions of isotropy and Witt decomposition and the book of Knus (1991, I.9), a paper by Fröhlich and McEvett (1969), and a paper by Dejaiffe (1998) for the notion of Morita equivalence. Again, our approach to these notions is, however, somewhat different and may be of interest in its own right.

2. NOTATION AND DEFINITION OF THE MAPS

Let K be a field of characteristic different from 2. All modules in this paper are supposed to be right modules finitely generated, and all the ε-hermitian forms are supposed to be nondegenerate. Let A be a central simple algebra over K with an involution σ (of any kind). For $\varepsilon \in K$ with $\varepsilon\sigma(\varepsilon) = 1$, an ε-hermitian form (V, h) over (A, σ) consists of a right A-module V and a biadditive map $h : V \times V \to A$ such that $h(xa, yb) = \sigma(a)h(x, y)b$ and $h(y, x) = \varepsilon\sigma(h(x, y))$ for all $x, y \in V$ and for all $a, b \in A$. Let $S^\varepsilon(A, \sigma)$ denote the semigroup of isometry classes of ε-hermitian forms over (A, σ), and let $W^\varepsilon(A, \sigma)$ be the Witt group of (A, σ) (i.e., the quotient of the Grothendieck group corresponding to $S^\varepsilon(A, \sigma)$ by the subgroup generated by metabolic forms, an ε-hermitian form (V, h) being metabolic if there exists an A-submodule W of V such that $W = W^\perp$ for h). If σ is of the first kind, $\varepsilon = \pm 1$. If σ is of the second kind, $W^\varepsilon(A, \sigma)$ does not depend on ε, so one can always suppose that $\varepsilon = \pm 1$, whereas all the results of this paper can be adapted for an arbitrary $\varepsilon \in K$ with $\varepsilon\sigma(\varepsilon) = 1$.

Remark 2.1. As A is simple, there is of course no difference between the notions of metabolic and hyperbolic hermitian forms. We use any of these two notions subsequently (except in Section 5).

First, we define the maps involved in the different sequences of this paper. As in Bayer-Fluckiger and Parimala (1995) and Parimala et al. (1995), we suppose that there exist $\lambda, \mu \in A^*$ such that $\sigma(\lambda) = -\lambda$, $\sigma(\mu) = -\mu$, $\mu\lambda = -\lambda\mu$, and such that $L = K(\lambda)$ is a quadratic extension of K. We write $\widetilde{A}$ for the commutant of L in A: this is a central simple algebra over L. One can easily verify that $\mu\widetilde{A} = \widetilde{A}\mu$, $\mu^2 \in \widetilde{A}$, $\sigma(\widetilde{A}) = \widetilde{A}$, and $A = \widetilde{A} \oplus \mu\widetilde{A}$. We define two involutions on $\widetilde{A}$ in the following way: let $\sigma_1 = \sigma|_{\widetilde{A}}$, and let $\sigma_2 = \mathrm{Int}(\mu^{-1}) \circ \sigma_1$ (where $\mathrm{Int}(\mu^{-1})(x) = \mu^{-1}x\mu$).

◇ ***Definition of π_1^ε and π_2^ε.*** We have two L-linear projections: $\pi_1 : A \to \widetilde{A} : a_1 + \mu a_2 \mapsto a_1$ and $\pi_2 : A \to \widetilde{A} : a_1 + \mu a_2 \mapsto a_2$. If $h : V \times V \to A$ is an ε-hermitian space over (A, σ), we define (for $i = 1, 2$) $\pi_i^\varepsilon(h) : V \times V \to \widetilde{A}$ by $\pi_i^\varepsilon(h)(x, y) = \pi_i(h(x, y))$. One readily verifies that $\pi_1^\varepsilon(h)$ is an ε-hermitian space over $(\widetilde{A}, \sigma_1)$ and that $\pi_2^\varepsilon(h)$ is a $-\varepsilon$-hermitian space over $(\widetilde{A}, \sigma_2)$. In order to see that π_1^ε and π_2^ε

induce homomorphisms of Witt groups, we have to prove that these maps respect regularity, isometry classes, orthogonality, and hyperbolicity. All of these properties come from

$$h(x, y) = 0 \quad \forall y \iff \pi_1^\varepsilon(h)(x, y) = 0 \quad \forall y \iff \pi_2^\varepsilon(h)(x, y) = 0 \quad \forall y.$$

Hence π_1^ε and π_2^ε induce homomorphisms of semigroups of isometry classes of nondegenerate hermitian forms and homomorphisms of Witt groups (again denoted by π_1^ε and π_2^ε):

$$\pi_1^\varepsilon : S^\varepsilon(A, \sigma) \to S^\varepsilon(\widetilde{A}, \sigma_1); \qquad \pi_1^\varepsilon : W^\varepsilon(A, \sigma) \to W^\varepsilon(\widetilde{A}, \sigma_1)$$
$$\pi_2^\varepsilon : S^\varepsilon(A, \sigma) \to S^{-\varepsilon}(\widetilde{A}, \sigma_2); \qquad \pi_2^\varepsilon : W^\varepsilon(A, \sigma) \to W^{-\varepsilon}(\widetilde{A}, \sigma_2).$$

◇ ***Definition of ρ_1^ε.*** Let (V, f) be an ε-hermitian space over $(\widetilde{A}, \sigma_1)$. We associate to it $(V \otimes_{\widetilde{A}} A, \rho_1^\varepsilon(f))$, where $\rho_1^\varepsilon(f)(x \otimes \alpha, y \otimes \beta) = \sigma(\alpha)\lambda f(x, y)\beta$ for $x, y \in V$ and $\alpha, \beta \in A$. We can easily verify that ρ_1^ε is well-defined and that $(V \otimes_{\widetilde{A}} A, \rho_1^\varepsilon(f))$ is a $-\varepsilon$-hermitian space over (A, σ). Moreover ρ_1^ε induces the homomorphisms

$$\rho_1^\varepsilon : S^\varepsilon(\widetilde{A}, \sigma_1) \to S^{-\varepsilon}(A, \sigma); \qquad \rho_1^\varepsilon : W^\varepsilon(\widetilde{A}, \sigma_1) \to W^{-\varepsilon}(A, \sigma).$$

◇ ***Definition of ρ_2^ε.*** Let (V, f) be an ε-hermitian space over $(\widetilde{A}, \sigma_2)$. We associate to it $(V \otimes_{\widetilde{A}} A, \rho_2^\varepsilon(f))$, where $\rho_2^\varepsilon(f)(x \otimes \alpha, y \otimes \beta) = \sigma(\alpha)\lambda\mu f(x, y)\beta$ for $x, y \in V$ and $\alpha, \beta \in A$. ρ_2^ε induces the homomorphisms

$$\rho_2^\varepsilon : S^\varepsilon(\widetilde{A}, \sigma_2) \to S^{-\varepsilon}(A, \sigma); \qquad \rho_2^\varepsilon : W^\varepsilon(\widetilde{A}, \sigma_2) \to W^{-\varepsilon}(A, \sigma).$$

◇ ***Definition of π_3^ε.*** We define π_3^ε to be $\lambda\pi_1^\varepsilon$, so we obtain the homomorphisms

$$\pi_3^\varepsilon : S^\varepsilon(A, \sigma) \to S^{-\varepsilon}(\widetilde{A}, \sigma_1); \qquad \pi_3^\varepsilon : W^\varepsilon(A, \sigma) \to W^{-\varepsilon}(\widetilde{A}, \sigma_1).$$

◇ ***Definition of ρ_3^ε.*** We define: $\rho_3^\varepsilon(f) = \rho_1^{-\varepsilon}(\lambda^{-1}f)$, i.e., $\rho_3^\varepsilon(f)(x \otimes \alpha, y \otimes \beta) = \sigma(\alpha)f(x, y)\beta$ for $x, y \in V$ and $\alpha, \beta \in A$. By a straightforward verification we obtain

Table 1

Map		Definition
$S^\varepsilon(A, \sigma) \xrightarrow{\pi_1^\varepsilon} S^\varepsilon(\widetilde{A}, \sigma_1)$	$(V_A, h) \mapsto (V_{\widetilde{A}}, \pi_1^\varepsilon(h))$	$\pi_1^\varepsilon(h)(x, y) = \pi_1(h(x, y))$
$S^\varepsilon(A, \sigma) \xrightarrow{\pi_2^\varepsilon} S^{-\varepsilon}(\widetilde{A}, \sigma_2)$	$(V_A, h) \mapsto (V_{\widetilde{A}}, \pi_2^\varepsilon(h))$	$\pi_2^\varepsilon(h)(x, y) = \pi_2(h(x, y))$
$S^\varepsilon(A, \sigma) \xrightarrow{\pi_3^\varepsilon} S^{-\varepsilon}(\widetilde{A}, \sigma_1)$	$(V_A, h) \mapsto (V_{\widetilde{A}}, \pi_3^\varepsilon(h))$	$\pi_3^\varepsilon(h)(x, y) = \lambda\pi_1^\varepsilon(h(x, y))$
$S^\varepsilon(\widetilde{A}, \sigma_1) \xrightarrow{\rho_1^\varepsilon} S^{-\varepsilon}(A, \sigma)$	$(W, f) \mapsto (W \otimes_{\widetilde{A}} A, \rho_1^\varepsilon(f))$	$\rho_1^\varepsilon(f)(x \otimes \alpha, y \otimes \beta) = \sigma(\alpha)\lambda f(x, y)\beta$
$S^\varepsilon(\widetilde{A}, \sigma_2) \xrightarrow{\rho_2^\varepsilon} S^{-\varepsilon}(A, \sigma)$	$(W, f) \mapsto (W \otimes_{\widetilde{A}} A, \rho_2^\varepsilon(f))$	$\rho_2^\varepsilon(f)(x \otimes \alpha, y \otimes \beta) = \sigma(\alpha)\lambda\mu f(x, y)\beta$
$S^\varepsilon(\widetilde{A}, \sigma_1) \xrightarrow{\rho_3^\varepsilon} S^\varepsilon(A, \sigma)$	$(W, f) \mapsto (W \otimes_{\widetilde{A}} A, \rho_3^\varepsilon(f))$	$\rho_3^\varepsilon(f)(x \otimes \alpha, y \otimes \beta) = \sigma(\alpha)f(x, y)\beta$

the homomorphisms

$$\rho_3^\varepsilon : S^\varepsilon(\widetilde{A}, \sigma_1) \to S^\varepsilon(A, \sigma); \qquad \rho_3^\varepsilon : W^\varepsilon(\widetilde{A}, \sigma_1) \to W^\varepsilon(A, \sigma).$$

Remark 2.2. Note that, in these definitions, ε is arbitrary, so for example $\pi_1^{-\varepsilon}$ will be a homomorphism of Witt groups from $W^{-\varepsilon}(A, \sigma)$ to $W^{-\varepsilon}(\widetilde{A}, \sigma_1)$ and so on for the other maps.

A summary of these definitions can be found in Table 1.

3. PROOF OF THEOREM 1.1.

In this section we suppose that $(A, \widetilde{A}, \sigma, \sigma_1, \sigma_2)$ satisfies the same hypotheses as the ones mentioned in Section 2. We prove the result stated in 1.1:

Theorem 1.1. *There is an exact sequence of Witt groups (in fact of $W(K, \sigma|_K)$-modules),*

$$W^{-\varepsilon}(A, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^\varepsilon(\widetilde{A}, \sigma_2) \xrightarrow{\rho_2^\varepsilon} W^{-\varepsilon}(A, \sigma) \xrightarrow{\pi_3^{-\varepsilon}} W^\varepsilon(\widetilde{A}, \sigma_1).$$

Proof. First, we prove that this sequence is a complex.

$\rho_2^\varepsilon \circ \pi_2^{-\varepsilon} = 0$: Let (V, h) be a $(-\varepsilon)$-hermitian space over (A, σ), so $\pi_2^{-\varepsilon}(h)$ is an ε-hermitian form over $(\widetilde{A}, \sigma_2)$ and $\rho_2^\varepsilon \pi_2^{-\varepsilon}(h)$ will be a $(-\varepsilon)$-hermitian form over (A, σ). It is enough to find a self-orthogonal right A-submodule of $V \otimes_{\widetilde{A}} A$ with respect to $\rho_2^\varepsilon \pi_2^{-\varepsilon}(h)$. Let

$$W = \{x \cdot \mu \otimes 1 + x \otimes \mu \mid x \in V\}. \qquad (10)$$

Now W is readily seen to be a right A-submodule of $V_{\widetilde{A}} \otimes_{\widetilde{A}} A$, and an easy calculation shows that this space is a totally isotropic subspace of $\rho_2^\varepsilon \pi_2^{-\varepsilon}(h)$. By dimension count over K, we have $W = W^\perp$ (with respect to $\rho_2^\varepsilon \pi_2^{-\varepsilon}(h)$), and so $\rho_2^\varepsilon \pi_2^{-\varepsilon}(h)$ is hyperbolic.

$\pi_3^{-\varepsilon} \circ \rho_2^\varepsilon = 0$: Let (V, h) be an ε-hermitian space over $(\widetilde{A}, \sigma_2)$. Let

$$W' = \{x \otimes 1 \mid x \in V\} \subset V \otimes_{\widetilde{A}} A. \qquad (11)$$

Then W' is an $\widetilde{A}$-submodule of $V_A \otimes_{\widetilde{A}} A$, and it is a totally isotropic subspace for $\pi_3^{-\varepsilon} \rho_2^\varepsilon(h)$. By a dimension argument, one has $W' = W'^\perp$ and so $\pi_3^{-\varepsilon} \rho_2^\varepsilon(h)$ is hyperbolic.

Next we prove that the sequence is exact.

$\ker(\rho_2^\varepsilon) \subset \operatorname{im}(\pi_2^{-\varepsilon})$: Let (W, f) be an ε-hermitian form over $(\widetilde{A}, \sigma_2)$ such that $\rho_2^\varepsilon(f)$ is hyperbolic. We may assume that f is anisotropic thanks to A.7. There exists an A-submodule W_1 of $W \otimes_{\widetilde{A}} A$ such that $W_1^\perp = W_1$ (with respect to $\rho_2^\varepsilon(f)$). Let $W \otimes \mu = \{w \otimes \mu | w \in W\}$. Let $w_1 \in W_1 \cap (W \otimes \mu)$ ($w_1 = w \otimes \mu$ with $w_1 \in W_1$

and $w \in W$). As $\rho_2^\varepsilon(f)(w_1, w_1) = 0$, $f(w, w) = 0$, and so $w_1 = w \otimes \mu = 0$ since f is anisotropic. Moreover,

$$\dim_K W_1 = \frac{1}{2} \dim_K W \otimes_{\widetilde{A}} A = \dim_K W \otimes \mu,$$

so $W \otimes_{\widetilde{A}} A = W_1 \oplus (W \otimes \mu)$ as $\widetilde{A}$-modules. This implies that for all $w \in W$, there exists $w' \in W$ such that $w \otimes 1 + w' \otimes \mu \in W_1$. Since A is a free $\widetilde{A}$-module, w' is uniquely determined by w, and we write $J(w) := w'$. By definition of J, we have $J^2(w) = w\mu^{-2}$ and $J(wa) = J(w)\mu a \mu^{-1}$ for all $w \in W$ and $a \in \widetilde{A}$. As $W_1 = W_1^\perp$, $\rho_2^\varepsilon(f)(x \otimes 1 + J(x) \otimes \mu, y \otimes 1 + J(y) \otimes \mu) = 0$ for all $x, y \in W$, and we obtain the system

$$\begin{cases} f(x, y) + \mu f(J(x), J(y))\mu = 0 \\ f(x, J(y))\mu + \mu f(J(x), y) = 0 \end{cases} \qquad (12)$$

By means of J, we define an A-module structure on W by

$$w \cdot \mu = J(w)\mu^2$$

for all $w \in W$. We denote by W_J the A-module W equipped with this new action. Let h be the map defined by

$$h(x, y) = \mu f(x, J(y))\mu + \mu f(x, y) \qquad (13)$$

for all $x, y \in W_J$. By the definition of J and (12), we conclude that (W_J, h) is a $(-\varepsilon)$-hermitian space over (A, σ). Let us show that h is sesquilinear on the left with respect to σ: h is clearly biadditive, so it suffices to show this fact for μ and for elements of $\widetilde{A}$. We have

$$\begin{aligned} h(x \cdot \mu, y) &= h(J(x)\mu^2, y) \\ &= \mu f\big(J(x)\mu^2, J(y)\big)\mu + \mu f\big(J(x)\mu^2, y\big) \\ &= \mu^3 f\big(J(x), J(y)\big)\mu + \mu^3 f\big(J(x), y\big) \\ &= -\mu^2 f(x, y) - \mu^2 f(x, J(y))\mu \quad \text{(using (12))} \\ &= \sigma(\mu) h(x, y). \end{aligned}$$

Sesquilinearity on the left for elements of $\widetilde{A}$ and linearity on the right are done in the same way. Let us prove that h is $-\varepsilon$-hermitian with respect to σ:

$$\begin{aligned} h(y, x) &= \mu f(y, J(x))\mu + \mu f(y, x) \\ &= \varepsilon\sigma\big(f\big(J(x), y\big)\big)\mu^2 + \varepsilon\sigma\big(f(x, y)\big)\mu \\ &= -\varepsilon\sigma\big(\mu^{-1} f\big(x, J(y)\big)\mu\big)\mu^2 - \varepsilon\sigma\big(\mu f(x, y)\big) \quad \text{(using (12))} \\ &= -\varepsilon\sigma(h(x, y)). \end{aligned}$$

If h is degenerate, by (13), there exists a nonzero $x \in W$ such that $f(x, y) = 0$ for all $y \in W$. In particular, this implies that f is isotropic, which is a contradiction. Now (W_J, h) is the antecedent of (W, f) by $\pi_2^{-\varepsilon}$, i.e., $((W_J)_{\widetilde{A}}, \pi_2^{-\varepsilon}(h))$ is isometric to (W, f) (the isometry is given by the identity map). We conclude that $\ker(\rho_2^{\varepsilon}) \subset \operatorname{im}(\pi_2^{-\varepsilon})$.

$\ker(\pi_3^{-\varepsilon}) \subset \operatorname{im}(\rho_2^{\varepsilon})$: Let (V, h) be a $(-\varepsilon)$-hermitian space over (A, σ) such that $\pi_3^{-\varepsilon}(h)$ is hyperbolic. We can assume that h is anisotropic thanks to A.7. There exists an $\widetilde{A}$-submodule W of $V_{\widetilde{A}}$ such that $W^{\perp} = W$ (with respect to $\pi_3^{-\varepsilon}(h)$). This implies that $h(x, y) \in \mu\widetilde{A}$, for all $x, y \in W$. We define a map $f : W \times W \to \widetilde{A}$ where

$$f(x, y) = \mu^{-1}\lambda^{-1}h(x, y) \tag{14}$$

for $x, y \in W$. Since h is anisotropic and f is nondegenerate, we can easily see that (W, f) is an ε-hermitian form over $(\widetilde{A}, \sigma_2)$ and $(W \otimes_{\widetilde{A}} A, \rho_2^{\varepsilon}(f))$ is isometric to (V, h) via $\Phi(w \otimes a) = wa$ for all $w \in W$, $a \in A$. We conclude that $\ker \pi_3^{-\varepsilon} \subset \operatorname{im} \rho_2^{\varepsilon}$, thus completing the proof. □

4. THE BEHAVIOR OF THE MAPS FOR DIVISION ALGEBRAS

In this section (D, τ) is a division algebra with an involution τ of any kind. We assume that $(D, \widetilde{D}, \tau, \tau_1, \tau_2)$ satisfies the same hypotheses as the ones mentioned in Section 2 for $(A, \widetilde{A}, \sigma, \sigma_1, \sigma_2)$.

Proposition 4.1. (i) *If $h = \langle\delta\rangle$ is a one-dimensional ε-hermitian form over (D, τ) (with $\delta = d_1 + \mu d_2$, d_1, $d_2 \in \widetilde{D}$), then the matrix of $\pi_1^{\varepsilon}(h)$ over $(\widetilde{D}, \tau_1)$ with respect to the basis $\{1, \mu\}$ is*

$$\begin{pmatrix} d_1 & \mu d_2 \mu \\ -\mu^2 d_2 & -\mu d_1 \mu \end{pmatrix}.$$

(ii) *If P is the matrix of an ε-hermitian form f over $(\widetilde{D}, \tau_1)$ with respect to a basis B, then λP is the matrix of $\rho_1^{\varepsilon}(f)$ over (D, τ) with respect to the basis $B \otimes 1$.*

(iii) *If $h = \langle\delta\rangle$ is a one dimensional $-\varepsilon$-hermitian form over (D, τ) (with $\delta = d_1 + \mu d_2$, $d_1, d_2 \in \widetilde{D}$), then the matrix of $\pi_2^{-\varepsilon}(h)$ over $(\widetilde{D}, \tau_2)$ with respect to the basis $\{1, \mu\}$ is*

$$\begin{pmatrix} d_2 & \mu^{-1} d_1 \mu \\ -d_1 & -\mu d_2 \mu \end{pmatrix}.$$

(iv) *If P is the matrix of an ε-hermitian form f over $(\widetilde{D}, \tau_2)$ with respect to a basis B, then $\lambda\mu P$ is the matrix of $\rho_2^{\varepsilon}(f)$ over (D, τ) with respect to the basis $B \otimes 1$.*

(v) *For a $-\varepsilon$-hermitian form h over (D, τ), the matrix of $\pi_3^{-\varepsilon}(h)$ with respect to a basis B is λ times the matrix of $\pi_1^{\varepsilon}(h)$ with respect to the basis $B \cup \mu B$.*

(vi) *If P is the matrix of an ε-hermitian form f over $(\widetilde{D}, \tau_1)$ with respect to a basis B, then P is the matrix of $\rho_3^{\varepsilon}(f)$ over (D, τ) with respect to the basis $B \otimes 1$.*

Table 2

Map	Form	Conditions	Image
π_1^{ε}	$\langle\delta_1, \ldots, \delta_n\rangle$; $\delta_i = d_{2i-1} + \mu d_{2i}$, $d_{2i-1}, d_{2i} \in \widetilde{D}$	$\tau_1(d_{2i-1}) = \varepsilon d_{2i-1}$ $\tau_2(d_{2i}) = -\varepsilon d_{2i}$	$\bigoplus_{i=1}^{n} \begin{pmatrix} d_{2i-1} & \mu d_{2i}\mu \\ -\mu^2 d_{2i} & -\mu d_{2i-1}\mu \end{pmatrix}$
ρ_1^{ε}	$\langle\gamma_1, \ldots, \gamma_n\rangle$; $\gamma_i \in \widetilde{D}$	$\tau_1(\gamma_i) = \varepsilon\gamma_i$	$\langle\lambda\gamma_1, \ldots, \lambda\gamma_n\rangle$
$\pi_2^{-\varepsilon}$	$\langle\delta_1, \ldots, \delta_n\rangle$; $\delta_i = d_{2i-1} + \mu d_{2i}$, $d_{2i-1}, d_{2i} \in \widetilde{D}$	$\tau_1(d_{2i-1}) = -\varepsilon d_{2i-1}$ $\tau_2(d_{2i}) = \varepsilon d_{2i}$	$\bigoplus_{i=1}^{n} \begin{pmatrix} d_{2i} & \mu^{-1} d_{2i-1}\mu \\ -d_{2i-1} & -\mu d_{2i}\mu \end{pmatrix}$
ρ_2^{ε}	$\langle\gamma_1, \ldots, \gamma_n\rangle$; $\gamma_i \in \widetilde{D}$	$\tau_2(\gamma_i) = \varepsilon\gamma_i$	$\langle\lambda\mu\gamma_1, \ldots, \lambda\mu\gamma_n\rangle$
$\pi_3^{-\varepsilon}$	$\langle\delta_1, \ldots, \delta_n\rangle$; $\delta_i = d_{2i-1} + \mu d_{2i}$, $d_{2i-1}, d_{2i} \in \widetilde{D}$	$\tau_1(d_{2i-1}) = -\varepsilon d_{2i-1}$ $\tau_2(d_{2i}) = \varepsilon d_{2i}$	$\bigoplus_{i=1}^{n} \begin{pmatrix} \lambda d_{2i-1} & \lambda\mu d_{2i}\mu \\ -\lambda\mu^2 d_{2i} & -\lambda\mu d_{2i-1}\mu \end{pmatrix}$
ρ_3^{ε}	$\langle\gamma_1, \ldots, \gamma_n\rangle$; $\gamma_i \in \widetilde{D}$	$\tau_1(\gamma_i) = \varepsilon\gamma_i$	$\langle\gamma_1, \ldots, \gamma_n\rangle$

Proof. These are straightforward calculations. Almost all of these are mentioned in Bayer-Fluckiger and Parimala (1995, Sec. 3). □

The summary of the previous proposition can be found in Table 2.

Lemma 4.2. *If $\widetilde{D}$ is commutative, then D is a quaternion division algebra, say $D = (a, b)_K$, $\widetilde{D} = L = K(\lambda)$ with $\lambda^2 = a \in K$ and $\mu^2 = b \in K$.*

Proof. $\widetilde{D}$ is the commutant of L in D, so it is a central simple algebra with center L. As $\widetilde{D}$ is commutative, we have $\widetilde{D} = L$, and the lemma readily follows. □

Lemma 4.3. *If E is a noncommutative central simple algebra and $\star$ an involution on E, then for $\varepsilon = \pm 1$, the set of nonzero ε-hermitian elements with respect to $\star$ is nonempty.*

Proof. This is a consequence of Scharlau (1985, Ch. 8, 7.5). □

The following theorem determines completely the behavior of the maps π_1, ρ_1, π_2, ρ_2, π_3, and ρ_3 with respect to isotropy.

Theorem 4.4. (i) *Let $h \in S^{\varepsilon}(D, \tau)$. If $\widetilde{D}$ is commutative and $\varepsilon = 1$, then $\pi_1^{\varepsilon}(h) \in S^{\varepsilon}(\widetilde{D}, \tau_1)$ is isotropic if and only if h is isotropic. Otherwise, $\pi_1^{\varepsilon}(h)$ is isotropic if and only if there exists $c \in \widetilde{D}^*$ with $\tau_2(c) = -\varepsilon c$ such that h contains a subform isometric to $\langle\mu c\rangle$.*

(ii) *Suppose that $f \in S^{\varepsilon}(\widetilde{D}, \tau_1)$. If $\widetilde{D}$ is commutative, $\varepsilon = 1$ and $\dim(f) = 2$, then $\rho_1^{\varepsilon}(f) \in S^{-\varepsilon}(D, \tau)$ is isotropic if and only if f is isotropic or f is isometric to the two-dimensional anisotropic form $\langle c, -bc\rangle$, where $c \in K^*$ and $b = \mu^2$ (see 4.2). Otherwise $\rho_1^{\varepsilon}(f)$ is isotropic if and only if there exist $d_1, d_2 \in \widetilde{D}$ such that $\tau_1(d_1) = \varepsilon d_1$, $\tau_2(d_2) = -\varepsilon d_2$, and $f = f_1 \perp f_2$ for some nondegenerate*

$$f_1 \simeq \begin{pmatrix} d_1 & \mu d_2 \mu \\ -\mu^2 d_2 & -\mu d_1 \mu \end{pmatrix}.$$

(iii) *For* $h \in S^{-\varepsilon}(D,\tau)$, $\pi_2^{-\varepsilon}(h) \in S^{\varepsilon}(\widetilde{D},\tau_2)$ *is isotropic if and only if there exists* $c \in \widetilde{D}^*$ *with* $\tau_1(c) = -\varepsilon c$ *such that h contains a subform isometric to* $\langle c \rangle$.

(iv) *For* $f \in S^{\varepsilon}(\widetilde{D},\tau_2)$, $\rho_2^{\varepsilon}(f) \in S^{-\varepsilon}(D,\tau)$ *is isotropic if and only if there exist* $d_1, d_2 \in \widetilde{D}$ *such that* $\tau_1(d_1) = -\varepsilon d_1$, $\tau_2(d_2) = \varepsilon d_2$ *and* $f = f_1 \perp f_2$ *for some nondegenerate*

$$f_1 \simeq \begin{pmatrix} d_2 & \mu^{-1}d_1\mu \\ -d_1 & -\mu d_2 \mu \end{pmatrix}.$$

(v) *Let* $h \in S^{-\varepsilon}(D,\tau)$. *If* $\widetilde{D}$ *is commutative and* $\varepsilon = -1$, *then* $\pi_3^{-\varepsilon}(h) \in S^{\varepsilon}(\widetilde{D},\tau_1)$ *is isotropic if and only if h is isotropic. Otherwise,* $\pi_3^{-\varepsilon}(h)$ *is isotropic if and only if there exists* $c \in \widetilde{D}^*$ *with* $\tau_2(c) = \varepsilon c$ *such that h contains a subform isometric to* $\langle \mu c \rangle$.

(vi) *Suppose that* $f \in S^{\varepsilon}(\widetilde{D},\tau_1)$. *If* $\widetilde{D}$ *is commutative,* $\varepsilon = -1$, *and* $\dim(f) = 2$, *then* $\rho_3^{\varepsilon}(f) \in S^{\varepsilon}(D,\tau)$ *is isotropic if and only if f is isotropic or f is isometric to the two-dimensional anisotropic form* $\langle \lambda c, -\lambda bc \rangle$, *where* $c \in K^*$ *and* $b = \mu^2$ *(see 4.2). Otherwise* $\rho_3^{\varepsilon}(f)$ *is isotropic if and only if there exist* $d_1, d_2 \in \widetilde{D}$ *such that* $\tau_1(d_1) = -\varepsilon d_1$, $\tau_2(d_2) = \varepsilon d_2$, *and* $f = f_1 \perp f_2$ *for some nondegenerate*

$$f_1 \simeq \begin{pmatrix} \lambda d_1 & \lambda \mu d_2 \mu \\ -\lambda \mu^2 d_2 & -\lambda \mu d_1 \mu \end{pmatrix}.$$

Proof. Let $\mathbb{H}_\varepsilon$ denote an ε-hyperbolic plane.

(i) First, suppose that $\widetilde{D}$ is commutative and $\varepsilon = 1$. By 4.2, we know that D is a quaternion division algebra. The equivalence comes from the fact that, in this case, the trace form of h is isotropic if and only if h is isotropic.

Next, suppose that the previous case is excluded. If h is anisotropic and $\pi_1^{\varepsilon}(h)$ is isotropic, then we can find $x \in V - \{0\}$ such that $\pi_1^{\varepsilon}(h(x,x)) = 0$, i.e., $h(x,x) = \mu c$ with $c \in \widetilde{D}^*$. As $\tau(h(x,x)) = \varepsilon h(x,x)$, we conclude that $\tau_2(c) = -\varepsilon c$. It readily follows that h contains $\langle \mu c \rangle$.

Now consider the case where h is isotropic. We have $\mathbb{H}_\varepsilon \simeq \langle \mu c, -\mu c \rangle$ for all $c \in \widetilde{D}^*$ with $\tau_2(c) = -\varepsilon c$, provided such a c exists. If $\varepsilon = -1$ then we can take $c = 1$. If $\varepsilon = 1$ then we only have to show that there exists $c \in \widetilde{D}$ such that $\tau_2(c) = -c$. By assumption, $\widetilde{D}$ is noncommutative, and one can apply 4.3 to conclude the proof.

Conversely, suppose that h contains a subform isometric to $\langle \mu c \rangle$ as in the assertion. By applying 4.1(i) to the form $\langle \mu c \rangle$, we easily deduce that $\pi_1^{\varepsilon}(h)$ is isotropic.

(ii) First suppose that $\widetilde{D}$ is commutative, $\varepsilon = 1$, and $\dim(f) = 2$. We are in the situation of 4.2. If f is isotropic or $f \simeq \langle c, -bc \rangle$, it is obvious that $\rho_1^{\varepsilon}(f)$ is isotropic. Conversely suppose that for a two-dimensional form $f = \langle c, d \rangle$, $\rho_1^{\varepsilon}(f) = \langle \lambda c, \lambda d \rangle$ is isotropic. So there exists $q \in D^*$ such that

$$\tau(q)\lambda c q + \lambda d = 0. \tag{15}$$

Write $q = z_1 + \mu z_2$ with $z_1, z_2 \in \widetilde{D}$. By replacing q with $z_1 + \mu z_2$ in (15), by using the fact that $\{1, \mu\}$ is an L-basis of D, and by remarking that $\tau_2 = \mathrm{id}_L$, we obtain

$$\begin{cases} \tau_1(z_1)cz_1 + \tau_1(z_2)bcz_2 + d = 0 \\ z_1 z_2 = 0. \end{cases}$$

If $z_2 = 0$, then $\tau_1(z_1)cz_1 + d = 0$; this means that f is isotropic. If $z_1 = 0$, then $d = -\tau_1(z_2)bcz_2$, so $f \simeq \langle c, -\tau_1(z_2)bcz_2 \rangle \simeq \langle c, -bc \rangle$.

Next, suppose that the previous case is excluded. If f is anisotropic and $\rho_1^{\varepsilon}(f)$ is isotropic, let $z = x_1 \otimes 1 + y_1 \otimes \mu$ be a nonzero isotropic vector for $\rho_1^{\varepsilon}(f)$. By the definition of ρ_1^{ε}, we obtain

$$(\lambda f(x_1,x_1) - \mu\lambda f(y_1,y_1)\mu) + (\lambda f(x_1,y_1)\mu - \mu\lambda f(y_1,x_1)) = 0,$$

and so we obtain

$$\begin{cases} f(x_1,x_1) + \mu f(y_1,y_1)\mu = 0 \\ \mu f(y_1,x_1) + f(x_1,y_1)\mu = 0. \end{cases} \tag{16}$$

As f is anisotropic, thanks to this system, we can suppose that both x_1 and y_1 are nonzero. Moreover, x_1 and y_1 are linearly independent over $\widetilde{D}$. In fact, if x_1 and y_1 are linearly dependent, then $x_1 = y_1 d$ with $d \in \widetilde{D}^*$, and by replacing x_1 with $y_1 d$ in (16), we obtain

$$\begin{cases} \tau(d)f(y_1,y_1)d + \mu f(y_1,y_1)\mu = 0 \\ \mu f(y_1,y_1)d + \tau(d)f(y_1,y_1)\mu = 0. \end{cases} \tag{17}$$

From the second equation of (17), we obtain $\tau(d)f(y_1,y_1) = -\mu f(y_1,y_1)d\mu^{-1}$. By replacing $\tau(d)f(y_1,y_1)$ by $-\mu f(y_1,y_1)d\mu^{-1}$ in the first equation of (17), we obtain $\mu f(y_1,y_1)(-d\mu^{-1}d + \mu) = 0$. As the second factor is nonzero for all $d \in \widetilde{D}$, $f(y_1,y_1) = 0$, which is a contradiction with the anisotropy of f. Now y_1 and x_1 span a two-dimensional subspace W over $\widetilde{D}$, and if we denote $d_1 = f(y_1,y_1)$, $d_2 = \mu^{-1}f(y_1,x_1)\mu^{-1}$, the matrix M of $f|_W$ with respect to the basis $\{y_1, x_1\}$ is exactly the one given in the proposition. As $f_1 = f|_W$ is nondegenerate (since f is anisotropic), we can write $f = f_1 \perp f_2$, so f contains the given form.

Now consider the case where f is isotropic. If $\widetilde{D}$ is noncommutative, we take $d_1 = 0$, and we can find $d_2 \in \widetilde{D}$ such that $\tau_2(d_2) = -\varepsilon d_2$, and it is obvious that

$$\mathbb{H}_\varepsilon \simeq \begin{pmatrix} 0 & \mu d_2 \mu \\ -\mu^2 d_2 & 0 \end{pmatrix},$$

so f contains the given form. If $\widetilde{D}$ is commutative and $\varepsilon = -1$, we take $d_1 = 0$ and $d_2 = 1$, and $\mathbb{H}_{-1}$ is isometric to the matrix given in the proposition. If $\widetilde{D}$ is commutative, $\varepsilon = 1$, and $\dim(f) \geq 3$, then $f \simeq \mathbb{H}_1 \perp f_1$ with $\dim(f_1) \geq 1$. If $f_1 \simeq \langle a, \ldots \rangle$, then $f \simeq \langle \mu a \mu, -\mu a \mu, a, \ldots \rangle$, so, for $d_1 = a$, $d_2 = 0$, f contains the given form.

Conversely, with the same notations as in (ii), 4.1(ii) and a straightforward calculation show that $(\mu, 1)$ is an isotropic vector for $\rho_1^\varepsilon(f_1)$.

(iii) If h is anisotropic and $\pi_2^{-\varepsilon}(h)$ is isotropic, then we can find $x \in V - \{0\}$ such that $\pi_2^{-\varepsilon}(h(x,x)) = 0$, that is, $h(x,x) = c \in \widetilde{D}^*$. We conclude as in (i). If h is isotropic and $\varepsilon = 1$, then $h \simeq \langle \lambda, -\lambda \rangle \perp h_1$ and we can take $c = \lambda$. If h is isotropic and $\varepsilon = -1$, $h \simeq \mathbb{H}_1 \perp h_1$ and all we have to do is to find $c \in \widetilde{D}^*$ such that $\tau_1(c) = c$; we can take $c = 1$.

The converse is an easy consequence of 4.1(iii).

(iv) If f is anisotropic and $\rho_2^\varepsilon(f)$ is isotropic, let $z = x_1 \otimes 1 + y_1 \otimes \mu$ be an isotropic vector for $\rho_2^\varepsilon(f)$. With a straightforward computation we find the same system as in the proof of (ii). Proceeding in the same way, we can suppose that x_1 and y_1 are nonzero and span a two-dimensional subspace W over $\widetilde{D}$. If $d_1 = -f(x_1, y_1)$ and $d_2 = f(y_1, y_1)$, then the matrix of $f|_W$ with respect to the basis $\{y_1, x_1\}$ is exactly the one given in the proposition. Now consider the case where f is isotropic. If $\varepsilon = 1$, we take $d_2 = 0$ and $d_1 = \lambda$. If $\varepsilon = -1$, we take $d_2 = 0$ and $d_1 = 1$.

Conversely, $(\mu, 1)$ is an isotropic vector for $\rho_2^\varepsilon(f_1)$.

(v) If $\widetilde{D}$ is commutative and $\varepsilon = -1$, then the equivalence between $\pi_3^{-\varepsilon}(h)$ being isotropic and h being isotropic readily comes from (i). Next, we suppose that the previous case is excluded. If h is anisotropic and $\pi_3^{-\varepsilon}(h)$ is isotropic, we can conclude as in (i) and (iii). If h is isotropic, then $h \simeq \langle \mu c, -\mu c \rangle \perp h_1$ for all $c \in \widetilde{D}^*$ such that $\tau_2(c) = \varepsilon c$, and we only have to find such a c. If $\widetilde{D}$ is noncommutative, this is clear. If $\widetilde{D}$ is commutative, then $\tau_2 = \mathrm{id}_L$. As $\varepsilon = 1$ we can take $c = 1$. Conversely, we only have to apply 4.1(v).

(vi) First, suppose that $\widetilde{D}$ is commutative, $\varepsilon = -1$, and $\dim(f) = 2$. Then we are in the situation of 4.2. The proof goes as in (ii). We leave it to the reader.

Next suppose that the previous case is excluded. If f is anisotropic and $\rho_3^\varepsilon(f)$ is isotropic, let $z = x_1 \otimes 1 + y_1 \otimes \mu$ be an isotropic vector for $\rho_3^\varepsilon(f)$. We have the system

$$\begin{cases} f(x_1, x_1) - \mu f(y_1, y_1)\mu = 0 \\ f(x_1, y_1)\mu - \mu f(y_1, x_1) = 0 \end{cases}$$

Now let $d_1 = \lambda^{-1} f(y_1, y_1)$ and $d_2 = \mu^{-1}\lambda^{-1} f(y_1, x_1)\mu^{-1}$. Let W be the two-dimensional $\widetilde{D}$-subspace generated by x_1 and y_1 (the proof of the fact that x_1 and y_1 are linearly independent over $\widetilde{D}$ is similar to that of (ii)). The matrix of $f|_W$ with respect to the basis $\{y_1, x_1\}$ is exactly the one given in the proposition (the form $f|_W$ is nondegenerate because f is anisotropic).

Now consider the case where f is isotropic. If $\widetilde{D}$ is noncommutative, we take $d_1 = 0$ and $d_2 \in \widetilde{D}$ such that $\tau_2(d_2) = \varepsilon d_2$. If $\widetilde{D}$ is commutative and $\varepsilon = 1$, then $f \simeq \mathbb{H}_1 \perp f_1$ and we take $d_2 = 1$ and $d_1 = 0$. If $\widetilde{D}$ is commutative, $\varepsilon = -1$, and $\dim(f) \geq 3$, we conclude as in (ii). Conversely, $(\mu, 1)$ is an isotropic vector for $\rho_3^\varepsilon(f_1)$. □

In particular, if f and h are anisotropic, we obtain Table 3.

Table 3

Behavior with respect to hyperbolicity	Conditions
$\pi_1^\varepsilon(h)$ is hyperbolic $\Rightarrow h \simeq \langle \mu\gamma_1, \ldots, \mu\gamma_n \rangle$	$\tau_2(\gamma_i) = -\varepsilon\gamma_i$
$\rho_1^\varepsilon(f)$ is hyperbolic $\Rightarrow f \simeq \bigoplus_{i=1}^n \begin{pmatrix} d_{2i-1} & \mu d_{2i}\mu \\ -\mu^2 d_{2i} & -\mu d_{2i-1}\mu \end{pmatrix}$	$\tau_1(d_{2i-1}) = \varepsilon d_{2i-1}$ $\tau_2(d_{2i}) = -\varepsilon d_{2i}$
$\pi_2^{-\varepsilon}(h)$ is hyperbolic $\Rightarrow h \simeq \langle \gamma_1, \ldots, \gamma_n \rangle$	$\tau_1(\gamma_i) = -\varepsilon\gamma_i$
$\rho_2^\varepsilon(f)$ is hyperbolic $\Rightarrow f \simeq \bigoplus_{i=1}^n \begin{pmatrix} d_{2i} & \mu^{-1} d_{2i-1}\mu \\ -d_{2i-1} & -\mu d_{2i}\mu \end{pmatrix}$	$\tau_1(d_{2i-1}) = -\varepsilon d_{2i-1}$ $\tau_2(d_{2i}) = \varepsilon d_{2i}$
$\pi_3^{-\varepsilon}(h)$ is hyperbolic $\Rightarrow h \simeq \langle \mu\gamma_1, \ldots, \mu\gamma_n \rangle$	$\tau_2(\gamma_i) = \varepsilon\gamma_i$
$\rho_3^\varepsilon(f)$ is hyperbolic $\Rightarrow f \simeq \bigoplus_{i=1}^n \begin{pmatrix} \lambda d_{2i-1} & \lambda\mu d_{2i}\mu \\ -\lambda\mu^2 d_{2i} & -\lambda\mu d_{2i-1}\mu \end{pmatrix}$	$\tau_1(d_{2i-1}) = -\varepsilon d_{2i-1}$ $\tau_2(d_{2i}) = \varepsilon d_{2i}$

From 4.1 and 4.4, we obtain the following result.

Corollary 4.5. *The sequences* (5) *and* (7) *are exact when A is a division algebra.*

5. EXACT SEQUENCES OF WITT GROUPS OF EQUIVARIANT FORMS

Let K be a field of characteristic different from 2 and G be a finite group. Let A be a central simple algebra over K with an involution σ. We denote by $A[G]$ the group algebra of G over A. Let $\varepsilon = \pm 1$.

Definition 5.1. We say that an ε-hermitian space (M, h) over (A, σ) is a G-space if

- G acts on M on the right and for all $g \in G$, the map $M \to M : m \mapsto m{\cdot}g$ is A-linear on the right;
- We have $h(m{\cdot}g, n{\cdot}g) = h(m, n)$ for all $m, n \in M$ and for all $g \in G$.

In this case, h is called a G-form (or a G-equivariant form). It is obvious that M is a right $A[G]$-module.

Two ε-hermitian G-spaces (M, h) and (M', h') are said to be isometric if there exists an isomorphism $\Phi : M \to M'$ of right $A[G]$-modules such that

$$h'(\Phi(m), \Phi(n)) = h(m, n)$$

for all $m, n \in M$. If M is a right $A[G]$-module, $M^* = \mathrm{Hom}_A(M, A)$ has a natural structure of right $A[G]$-module: $(f \cdot g)(m) = f(m \cdot g^{-1})$ if $f \in M^*$, $g \in G$ and $m \in M$. A G-space (M, h) over (A, σ) is said to be nondegenerate if $M \to M^* : x \mapsto h(x, \cdot)$ is an isomorphism of A-modules. One can define the hyperbolic ε-hermitian G-space associated to such an M by $(M \oplus M^*, \mathbb{h}_M)$, where

$$\mathbb{h}_M(m \oplus f, m' \oplus f') = f(m') + \varepsilon\sigma(f'(m))$$

for all $m, m' \in M$ and $f, f' \in M^*$.

Remark 5.2. If char $K \nmid |G|$, then by Maschke's theorem the group algebra $A[G]$ is semisimple. Thanks to that, we can show that an ε-hermitian G-space (M, h) is hyperbolic if and only if it is metabolic (i.e., if there exists a right $A[G]$-submodule N of M such that $N = N^{\perp}$ for h): the proof can be adapted from Bayer-Fluckiger and Lenstra (1990, Corollary 1.4). This fact will not be used here.

Now one can construct a group (as for ε-hermitian forms) called the Witt group of ε-hermitian G-forms (also called the Witt group of equivariant forms), which will be denoted by $W^{\varepsilon}(G, A, \sigma)$ (i.e., the quotient of the Grothendieck group corresponding to isometry classes of nondegenerate ε-hermitian G-forms by the subgroup generated by metabolic forms). An element of this group is denoted by $[(M, h)]$ where (M, h) is an ε-hermitian G-space over (A, σ).

We can easily see that the maps involved in Section 2 induce group homomorphisms between the corresponding Witt groups of hermitian G-forms: if W is an $\widetilde{A}[G]$-module, then $W \otimes_{\widetilde{A}} A$ is an $A[G]$-module, where G acts on $W \otimes_{\widetilde{A}} A$ by $(w \otimes a) \cdot g = w \cdot g \otimes a$ for $w \in W$, $a \in A$, $g \in G$. The notion of anisotropy for G-forms that we will use is the following (as in Cibils, 1983, p. 29):

Definition 5.3. An ε-hermitian G-space (M, h) over (A, σ) is said to be anisotropic if for all $A[G]$-submodules N of M, we have $N \cap N^{\perp} = 0$ (for h).

Remark 5.4. Note that this notion of anisotropy coincides with the usual notion of anisotropy (see Section A) when (M, h) is an ε-hermitian space over a simple algebra with involution. But in the case of ε-hermitian G-forms, this notion of anisotropy is weaker than the usual one. For example, let $q : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be the quadratic form defined by $q(x, y) = x^2 + y^2$ and $G = \{1, \theta\}$, where θ is the reflection in the hyperplane orthogonal to $(1, 0)$. Then q is a quadratic G-form that is isotropic as a quadratic form but anisotropic as a G-form in the sense of 5.3.

Now one can prove a proposition analogous to Cibils (1983), Proposition 2:

Proposition 5.5. *If $[(M, h)] \in W^{\varepsilon}(G, A, \sigma)$, $[(M, h)] \neq 0$, then we can find an anisotropic ε-hermitian G-space (M', h') over (A, σ) such that $[(M, h)] = [(M', h')]$.*

Proof. The proof goes as in Cibils (1983), Proposition 2. □

Now we prove the result stated in 1.2.

Theorem. *We suppose that A satisfies the same hypotheses as in Theorem 1.1. We have the two following exact sequences of $W(K, \sigma|_K)$-modules:*

$$W^{\varepsilon}(G, A, \sigma) \xrightarrow{\pi_1^{\varepsilon}} W^{\varepsilon}(G, \widetilde{A}, \sigma_1) \xrightarrow{\rho_1^{\varepsilon}} W^{-\varepsilon}(G, A, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^{\varepsilon}(G, \widetilde{A}, \sigma_2)$$

$$W^{-\varepsilon}(G, A, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^{\varepsilon}(G, \widetilde{A}, \sigma_2) \xrightarrow{\rho_2^{\varepsilon}} W^{-\varepsilon}(G, A, \sigma) \xrightarrow{\pi_3^{-\varepsilon}} W^{\varepsilon}(G, \widetilde{A}, \sigma_1).$$

Proof. We only prove the exactness of the second sequence, the proof of the first one being similar. We only have to adapt the proof of 1.1. Let us keep the same notations. The fact that this sequence is a complex readily follows from the proof of 1.1, as we can easily verify that there exist right G-module structures on W and W' (see (10) and (11) for the definition of these spaces).

To show that this sequence is exact, we use Proposition 5.5 to exhibit an anisotropic representative (in the sense of Definition 5.3) of each form in the considered kernel.

Let us show that $\ker(\rho_2^{\varepsilon}) \subset \mathrm{im}(\pi_2^{-\varepsilon})$. We only point at the changes to the original proof. So let (W, f) be an anisotropic ε-hermitian G-space (in the sense of 5.3) lying in the kernel of ρ_2^{ε}. Thanks to 5.2, we know that there exists an $A[G]$-submodule W_1 of $W \otimes_{\widetilde{A}} A$ such that $W_1 = W_1^{\perp}$. To show that $W_1 \cap (W \otimes \mu) = 0$, we consider the following subspace: $V = \{w \in W | w \otimes \mu \in W_1\}$. It is an $\widetilde{A}$-submodule of W, and if $w \in W$, $g \in G$, we have $w \cdot g \otimes \mu = (w \otimes \mu) \cdot g \in W_1$, as W_1 is a right G-module. So V is an $\widetilde{A}[G]$-submodule of W. Now if v, $v' \in V$, then $\rho_2^{\varepsilon}(f)(v \otimes \mu, v' \otimes \mu) = 0$, and we have $f(v, v') = 0$. So $V \subset V^{\perp}$, and as (W, f) is anisotropic, we deduce that $V = 0$ and that $W_1 \cap (W \otimes \mu) = 0$. Now, $W \otimes_{\widetilde{A}} A = W_1 \oplus (W \otimes \mu)$ as $\widetilde{A}[G]$-modules, and this implies that for all $w \in W$, there exists $w' \in W$ such that $w \otimes 1 + w' \otimes \mu \in W_1$. One can define the map $J : W \to W$ as in the proof of 1.1 by $J(w) = w'$(the map J is well-defined because f is anisotropic). Thanks to the previous uniqueness, we have $J(w \cdot g) = J(w) \cdot g$ for all $w \in W$, $g \in G$. Thanks to that, we easily show that W_J is an $A[G]$-module. We define h as in (13). If h is degenerate, then there exists $x \neq 0$ in W_J such that $h(x, y) = 0$ for all $y \in W_J$. We deduce that $f(x, y) = 0$ for all $y \in W$, and this shows that $x \in W \cap W^{\perp}$, which is a contradiction to the anisotropy of (W, f). We can conclude as in 1.1.

Now let us show that $\ker(\pi_3^{-\varepsilon}) \subset \mathrm{im}(\rho_2^{\varepsilon})$. Let (V, h) be an anisotropic ε-hermitian G-space lying in the kernel of $\pi_3^{-\varepsilon}$. Then there exists an $\widetilde{A}[G]$-submodule W of $V_{\widetilde{A}}$ such that $W = W^{\perp}$. As in the proof of 1.1, we define a map f as in (14), and all we have to do is to show that the ε-hermitian form f is nondegenerate. If f is degenerate, let U be the right $A[G]$-module generated by W. Then U is an $A[G]$-submodule of V. Now there exists $x \in W$ such that $h(x, y) = 0$ for all $y \in W$, and this implies that $h(x, y) = 0$ for all $y \in U$; we have $x \in U \cap U^{\perp}$, which is a contradiction to the anisotropy of (V, h). □

Remark 5.6. If the group G is trivial, then (5) and (7) are special cases of 1.2.

6. EXACT OCTAGONS OF WITT GROUPS

Corollary 6.1. *With the same hypotheses as in Section 2, there is an exact octagon of Witt groups (in fact of $W(K, \sigma|_K)$-modules) as in Figure 1.*

Proof. The definition of the maps easily implies that the exactness at each point is equivalent to the exactness at the opposite point of the octagon. Now the result comes from the exact sequence of Parimala et al. (1995) and from Theorem 1.1. □

Corollary 6.2. *The octagon of* 6.1 *is also exact for G-forms.*

Proof. The proof is similar to 6.1 when we use Theorem 1.2. □

Remark 6.3. One can find a similar exact octagon of Witt groups of Clifford algebras of quadratic forms with their canonical involution in Lewis (1985).

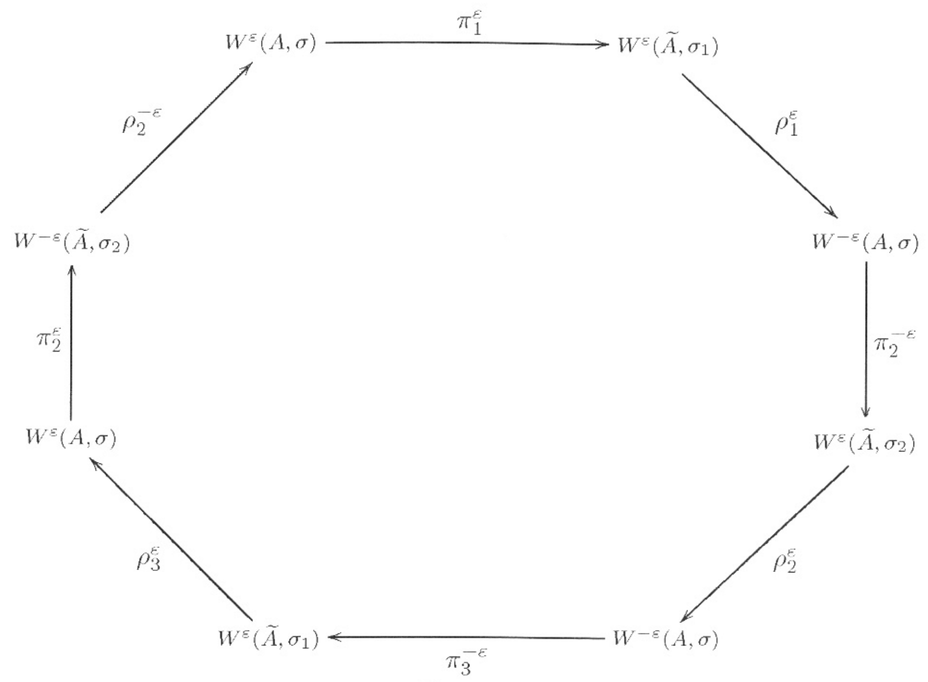


Figure 1

Remark 6.4. David Lewis suggested to us that, in 6.1, one can replace the bottom left entry of the octagon by $W^{-\varepsilon}(\widetilde{A},\sigma_1)$ because the multiplication by λ induces an isomorphism $W^{-\varepsilon}(\widetilde{A},\sigma_1)\simeq W^{\varepsilon}(\widetilde{A},\sigma_1)$. Thus we obtain "antipodal asymmetry." In this case, the exactness of 6.1 is equivalent to the exactness of the following square obtained by adding opposite entries of 6.1:

$$\begin{array}{ccc} W^{\varepsilon}(A,\sigma)\oplus W^{-\varepsilon}(A,\sigma) & \longrightarrow & W^{\varepsilon}(\widetilde{A},\sigma_1)\oplus W^{-\varepsilon}(\widetilde{A},\sigma_1) \\ \uparrow & & \downarrow \\ W^{\varepsilon}(\widetilde{A},\sigma_2)\oplus W^{-\varepsilon}(\widetilde{A},\sigma_2) & \longleftarrow & W^{-\varepsilon}(A,\sigma)\oplus W^{\varepsilon}(A,\sigma) \end{array} \tag{18}$$

Remark 6.5. If A is a split quaternion algebra and σ is symplectic, then the exact octagon of 6.1 becomes:

$$0\to W(L,-)\to W(K)\to W(L)\to W(K)\to W(L,-)\to 0,$$

where L/K is a quadratic extension with a nontrivial automorphism $-=\sigma|_L$. If A is a quaternion division algebra and σ is symplectic, then 6.1 becomes

$$0\to W(A,\sigma)\to W(L,\sigma|_L)\to W^{-1}(A,\sigma)\to W(L)\to W^{-1}(A,\sigma)$$
$$\to W(L,\sigma|_L)\to W(A,\sigma)\to 0,$$

where L is a maximal subfield of A that is stable under the involution. These sequences can be found in Lewis (1982a). These two sequences are particular cases of exact octagons found by Lewis (1983, 1985).

7. ORDER OF WITT GROUPS

Let L/K be a finite extension. One can ask for the relation between the orders of $W(K)$ and $W(L)$. If L/K is an extension of odd degree, then by the weak version of Springer's theorem there is a canonical injection $W(K)\hookrightarrow W(L)$, so the finiteness of $W(L)$ implies that of $W(K)$. If L/K is an extension of even degree, then this property fails. However, for a quadratic extension L/K, the finiteness of $W(K)$ implies that of $W(L)$; it is easy to see is that by the exact triangle of Elman–Lam (2) one has $|W(L)|\leq|W(K)|^2$. In Lewis (1982a) the defect of this inequality is calculated; in fact, the relation has been proved

$$|W(L)||W(L,-)|^2=|W(K)|^2, \tag{19}$$

where $-$ is the nontrivial automorphism of L/K. We have the same situation for a quaternion algebra Q with its symplectic involution $-$. In this case, the finiteness of $W(K)$ implies that of $W(Q,-)$ by the exact sequence of Jacobson. In fact by Lewis (1982a) we have

$$|W^{\varepsilon}(Q,-)||W^{-\varepsilon}(Q,-)|=|W(K)|. \tag{20}$$

More generally, as stated in 1.3, we have

Corollary. *Let A be a K-central simple algebra with an involution σ of the first kind. Then we have $|W^{\varepsilon}(A,\sigma)||W^{-\varepsilon}(A,\sigma)|=|W(K)|$. In particular, $W(K)$ is finite if and only if $W^{\varepsilon}(A,\sigma)$ and $W^{-\varepsilon}(A,\sigma)$ are finite.*

Proof. By Merkurjev's theorem, A is similar to a multiquaternion algebra, say $A\sim Q_1\otimes\cdots\otimes Q_n$. By Morita theory, we have $W^{\varepsilon}(A,\sigma)\simeq W^{\varepsilon'}(Q_1\otimes\cdots\otimes Q_n,\sigma_1\otimes\cdots\otimes\sigma_n)$, where $\varepsilon'=\varepsilon$ or $\varepsilon'=-\varepsilon$ and σ_i is the canonical involution of Q_i for $1\leq i\leq n$. So in order to prove the statement, we can suppose that

$$(A,\sigma)=(Q_1\otimes\cdots\otimes Q_n,\sigma_1\otimes\cdots\otimes\sigma_n).$$

We proceed by induction on n. If $n=0$, i.e., A is split, the statement becomes $|W^{\varepsilon}(K)||W^{-\varepsilon}(K)|=|W(K)|$, which is true because $\{W^{\varepsilon}(K),\ W^{-\varepsilon}(K)\}=\{W(K),0\}$. If $n=1$, the statement is a consequence of (20).

Suppose that $n>1$. Suppose that $Q_n=(a,b)_K$, where $(a,b)_K$ is the quaternion algebra generated by i and j with $i^2=a$, $j^2=b$ and $ij=-ji$, where $a,b\in K$. Take the split quaternion algebra $Q'_n=(a,1)_K$ generated by i' and j' with $i'^2=a$, $j'^2=1$ and $i'j'=-j'i'$. If we compare the exact octagon of 6.1 for $(A,\sigma)=(Q_1\otimes\cdots\otimes Q_{n-1}\otimes Q_n,\sigma_1\otimes\cdots\otimes\sigma_{n-1}\otimes\sigma_n)$ with $\lambda=1\otimes\cdots\otimes1\otimes i$ and $\mu=1\otimes\cdots\otimes1\otimes j$ and also for $(A',\sigma')=(Q_1\otimes\cdots\otimes Q_{n-1}\otimes Q'_n,\sigma_1\otimes\cdots\otimes\sigma_{n-1}\otimes\sigma'_n)$ (σ'_n is the canonical involution of Q'_n) with $\lambda'=1\otimes\cdots\otimes1\otimes i'$ and $\mu'=1\otimes\cdots\otimes1\otimes j'$

we deduce that $|W^\varepsilon(A,\sigma)||W^{-\varepsilon}(A,\sigma)| = |W^\varepsilon(A',\sigma')||W^{-\varepsilon}(A',\sigma')|$. By Morita theory, we have $W^\varepsilon(A',\sigma') \simeq W^{-\varepsilon}(A'',\sigma'')$ and $W^{-\varepsilon}(A',\sigma') \simeq W^\varepsilon(A'',\sigma'')$, where

$$(A'',\sigma'') = (Q_1 \otimes \cdots \otimes Q_{n-1}, \sigma_1 \otimes \cdots \otimes \sigma_{n-1}).$$

So by the induction hypothesis we obtain $|W^\varepsilon(A,\sigma)||W^{-\varepsilon}(A,\sigma)| = |W(K)|$. □

Remark 7.1. If A is a quaternion algebra over K, and σ is the canonical involution of A, then for $K = \mathbb{Q}_p$, both groups $W^{\pm 1}(A,\sigma)$ are finite. For $K = \mathbb{R}$, the group $W^{-1}(A,\sigma)$ is finite and $W^1(A,\sigma)$ is infinite.

Corollary 7.2. *Let A be a quaternion algebra over K with an involution σ of the second kind. Then $|W(A,\sigma)| = |W(K,\sigma|_K)|$.*

Proof. By a theorem of Albert (Scharlau, 1985, Ch. 8, 11.2) $(A,\sigma) = (A_0 \otimes_k K, \sigma_0 \otimes \sigma|_K)$, where k is the fixed field of σ in K, A_0 is a quaternion algebra over k, and σ_0 is its canonical involution. Suppose that $A_0 = (a,b)_k$, where $(a,b)_k$ is the quaternion algebra generated by i and j with $i^2 = a$, $j^2 = b$, and $ij = -ji$, where $a,\ b \in k$. Take the split quaternion algebra $A_0' = (a,1)_k$ generated by i' and j' with $i'^2 = a$, $j'^2 = 1$, and $i'j' = -j'i'$. Let σ_0' be the canonical involution of A_0'. If we compare the exact octagon of 6.1 for $(A_0 \otimes_k K, \sigma_0 \otimes \sigma|_K)$ with $\lambda = i \otimes 1$ and $\mu = j \otimes 1$ and for $(A',\sigma') := (A_0' \otimes_k K, \sigma_0' \otimes \sigma|_K)$ with $\lambda' = i' \otimes 1$ and $\mu' = j' \otimes 1$, we deduce that $|W^\varepsilon(A,\sigma)||W^{-\varepsilon}(A,\sigma)| = |W^\varepsilon(A',\sigma')||W^{-\varepsilon}(A',\sigma')|$. As $W^\varepsilon(A,\sigma) \simeq W^{-\varepsilon}(A,\sigma) \simeq W(A,\sigma)$ and $W^\varepsilon(A',\sigma') \simeq W^{-\varepsilon}(A',\sigma') \simeq W(A',\sigma')$, we deduce that $|W(A,\sigma)| = |W(A',\sigma')|$. By Morita theory, $W(A',\sigma') \simeq W(K,\sigma|_K)$ because A' is split. This implies the result. □

Corollary 7.3. *Let $A = Q_1 \otimes_K \cdots \otimes_K Q_n$ be a a multiquaternion algebra over K with the involution $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_n$, where σ_i is an involution of Q_i of the second kind for $i = 1,\ldots,n$. Then $|W(A,\sigma)| = |W(K,\sigma|_K)|$.*

Proof. The argument is similar to 1.3: we use an induction on n and the case $n = 1$ has already been proved in 7.2. □

The previous corollary gives the motivation to ask the following question for which we do not know the answer.

Question 7.4. Let A be a K-central simple algebra with an involution σ of the second kind. Is it true that $|W(A,\sigma)| = |W(K,\sigma|_K)|$?

Remark 7.5. Using 1.2 and applying the same type of arguments, we can show that, if A is a central simple algebra with an involution σ of the first kind, then $|W^\varepsilon(G,A,\sigma)||W^{-\varepsilon}(G,A,\sigma)| = |W(G,K)||W^{-1}(G,K)|$. In 7.3 one can replace the Witt groups by equivariant Witt groups.

APPENDIX

A. Some Basic Facts and Notions About Morita Theory

We recall some facts about isotropy, Witt decomposition of ε-hermitian forms over a central simple algebra with involution (A,σ), and Morita equivalence between two central simple algebras with involution.

Definition A.1. A nondegenerate ε-hermitian space (V,h) over (A,σ) is said to be isotropic if there exists $x \in V - \{0\}$ such that $h(x,x) = 0$.

The following approach is based upon Dejaiffe (1998). See also Fröhlich and McEvett (1969).

Definition A.2. Let (A,σ) and (B,τ) be two central simple algebras with involution over K such that σ and τ have the same restriction to K. Let $\delta = 1$ if σ and τ are of the second kind or of the first kind and of the same type, and $\delta = -1$ otherwise.

A δ-Morita equivalence $((A,\sigma),(B,\tau),M,N,f,g,\nu)$ between the algebra with involutions (A,σ) and (B,τ) is a tuple consisting of

- An (A,B)-bimodule M (i.e., a left A-module and a right B-module with compatible structures)
- A (B,A)-bimodule N
- Two nonzero bimodule homomorphisms $f : M \otimes_B N \to A$ and $g : N \otimes_A M \to B$, which are associative, i.e., $f(m \otimes n) \cdot m' = m \cdot g(n \otimes m')$ and $g(n \otimes m) \cdot n' = n \cdot f(m \otimes n')$ for all $m,\ m' \in M,\ n,\ n' \in N$
- A linear bijective map $\nu : M \to N$ that satisfies $\nu(amb) = \tau(b)\nu(m)\sigma(a)$ for all $a \in A,\ m \in M,\ b \in B$.

Remark A.3. Note that we do not suppose that σ and τ are of the same type as in Dejaiffe (1998): that is why we call this notion δ-Morita equivalence. This notion is a particular case of the Morita equivalence given in Fröhlich and McEvett (1969).

Remark A.4. In fact one can prove that f (resp. g) is a bimodule isomorphism between $M \otimes_B N$ and A (resp. $N \otimes_A M$ and B); see Dejaiffe (1998, Section 1.1).

Now we suppose that $B = D$ denotes the division algebra Brauer equivalent to A. By Albert's theorem, we know that there exists an involution τ over D such that τ is of the same kind as σ. By Dejaiffe (1998, Section 1.4) one can find $M,\ N,\ f,\ g,$ and ν as in A.2 and $\delta \in \{\pm 1\}$ such that $((A,\sigma),(D,\tau),M,N,f,g,\nu)$ is a δ-Morita equivalence. We can define semigroup homomorphisms:

$$F : S^\varepsilon(A,\sigma) \to S^{\delta\varepsilon}(D,\tau); \qquad (V,h) \mapsto (V \otimes_A M, b_0 h)$$

$$G : S^{\delta\varepsilon}(D,\tau) \to S^\varepsilon(A,\sigma); \qquad (W,\phi) \mapsto (W \otimes_D N, \delta b_0' \phi),$$

where:

$$(b_0h)(v\otimes m, v'\otimes m') = g(v(m)\otimes h(v,v')m') \qquad \forall v,\ v'\in V,\ m,\ m'\in M$$
$$(b_0'\phi)(w\otimes n, w'\otimes n') = f(v^{-1}(n)\otimes \phi(w,w')n') \qquad \forall w,\ w'\in W,\ n,\ n'\in N.$$

In fact, we can prove that F is a semigroup isomorphism and G is its inverse, and that they induce isomorphisms of Witt groups. The details of proofs can be found in Knus (1991, I.9, 3.5) or in Fröhlich and McEvett (1969). We have

Lemma A.5. (i) *M is a simple left A-module and N is a simple right A-module.*

(ii) *The maps F and G respect the rank of hermitian spaces (recall that the rank of a hermitian space (V, h) over (A, σ), where V is a right (resp. left) A-module, is defined to be the positive integer n such that $V \simeq T^n$, where T is a simple right (resp. left) A-module).*

Proof. (i) We prove it for N, the proof for M being similar. As A is a simple algebra, N is a semisimple right A-module, and we can write $N \simeq T^n$, where $n \in \mathbb{N} - \{0\}$ and T is a simple right A-module. But we know from A.4 that $N \otimes_A M \simeq D$ are (D, D)-bimodules, so

$$D \overset{g^{-1}}{\simeq} N \otimes_A M \simeq (T\otimes_A M)^n$$

are right D-vector spaces. A dimension argument shows that $n = 1$ and N is simple as a right A-module.

(ii) We prove it for F; the proof for G is similar. If (V, h) is a right A-module of rank n, then we have $V \simeq T^n$, where T is a simple right A-module; by (i), we can take $T = N$. As D is a division algebra, D is a simple right D-module and we have:

$$V\otimes_A M \simeq (N\otimes_A M)^n \overset{g}{\simeq} D^n.$$

So we deduce that the rank of $V \otimes_A M$ is n. □

Now we can prove (for fixed A, σ, and D as before)

Proposition A.6. *The following statements are equivalent:*

(i) *(V, h) is isotropic over (A, σ).*
(ii) *$F(V, h)$ is isotropic over (D, τ) for some δ-Morita equivalence.*
(iii) *$F(V, h)$ is isotropic over (D, τ) for every δ-Morita equivalence.*

Proof. This result can be found in McEvett (1969) in another context. Let $x \in V - \{0\}$ be such that $h(x, x) = 0$ and $((A,\sigma), (D,\tau), M, N, f, g, v)$ be a δ-Morita equivalence. We can easily see that there exists $m \in M$ such that $x \otimes m \neq 0$; in fact if $x \otimes m = 0$ for all $m \in M$, then we have $V_1 \otimes_A M = 0$, where $V_1 = xA \neq 0$. Now N is a simple right A-module by A.5, so we have $V_1 = \bigoplus_{i=1}^d N$, $d \geq 1$. Therefore we conclude that $0 = V_1 \otimes_A M \simeq \bigoplus_{i=1}^d (N\otimes_A M) \overset{g}{\simeq} D^d$, which is a contradiction.

Now $x \otimes m$ is clearly an isotropic vector for b_0h, so $F(V, h)$ is isotropic. If $F(V, h)$ is isotropic, and if $((A,\sigma), (D,\tau), M, N, f, g, v)$ is a δ-Morita equivalence, let $y \neq 0$ be an isotropic vector for b_0h. By the same argument as before, one can find $n \in N$ such that $y \otimes n \neq 0$. Using the definition of $F^{-1}(=G)$, we see that $y \otimes n$ is an isotropic vector for $\delta b_0'b_0h$ for all $n \in N$. But $(V\otimes_A M\otimes_D N, \delta b_0'b_0h)$ is isometric to (V, h), so we can conclude the existence of an $x \in V - \{0\}$ such that $h(x, x) = 0$. □

Using this proposition and the fact that the Witt decomposition exists over (D, τ), we conclude the existence of a Witt decomposition over (A, σ). Namely, if (V, h) is an ε-hermitian space over (A, σ), then $F(V, h) \simeq \phi_1 \perp \phi_2$, where ϕ_1 is hyperbolic and ϕ_2 anisotropic over (D, τ). We have

$$(V,h) \simeq (G\circ F)(V,h) \simeq G(\phi_1)\perp G(\phi_2).$$

By the previous proposition, $G(\phi_1)$ is hyperbolic, and we can show that $G(\phi_2)$ is anisotropic. By the same type of argument, we can show that this decomposition is unique up to isometry, because it is the case over (D, τ). Thus

Corollary A.7 (McEvett, 1969). (i) *There exists a Witt decomposition over (A, σ).*

(ii) *For all $[h] \in W^\varepsilon(A, \sigma)$, there exists an anisotropic form h_0 over (A, σ) such that $[h] = [h_0]$ in $W^\varepsilon(A, \sigma)$.*

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HIGHER TRACES ON GROUP RINGS#

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Motivated by the works of Cuntz and Quillen on cyclic homology and algebra extensions, we study higher traces on group rings.

Key Words: Bass' conjecture; Connes–Quillen homomorphism; Cyclic homology; Group rings; Higher traces.

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1. INTRODUCTION

Let k be a commutative ring with identity and let A be a k-algebra. A trace map $\tau : A \to M$, where M is a k-module, is a k-linear map satisfying $\tau(ab) = \tau(ba)$. If R is an extension of A by a two-sided ideal I, i.e., $R/I \simeq A$ as algebras, then we call the trace maps on R/I^n, $n \geq 1$, *higher traces on A* (relative to the extension $A \simeq R/I$); these are the *even higher traces* in the sense of Quillen (1989). In this paper we study higher traces on group rings. This investigation is motivated by the work of Quillen and of Cuntz and Quillen (1995a,b), where the authors study the relationship of higher traces on algebras with cyclic homology. Our interest in the study of higher traces on group rings stems particularly from Quillen's characterization (1989, Theorem 5.18) of cyclic homology groups of algebras.

For a group G, let $k[G]$ denote its group ring over k and $\Delta(G)(= \Delta_k(G))$ its augmentation ideal. Let H be a normal subgroup of G and $\Pi = G/H$. Then $k[G]$ is an algebra extension of $k[\Pi]$ with kernel the two-sided ideal $\Delta(H)k[G]$. To study higher traces on $k[\Pi]$, relative to this extension, it clearly suffices to investigate the k-module $k[G]/(\Delta^n(H)k[G] + k[G]^{[2]})$, where for a ring R, $R^{[2]} = [R, R]$ and for ideals I, J of R, $[I, J]$ denotes the additive subgroup of R generated by the elements of the type $xy - yx$ $(x \in I, y \in J)$. In Section 2 we describe the structure of the k-module $k[G]/(\Delta^n(H)k[G] + k[G]^{[2]})$ (resp. the related k-module $\Delta^{n+1}(H)k[G]/[\Delta^n(H)k[G], \Delta(H)k[G]])$ as a direct sum of k-modules indexed by the

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