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VOLUME AND BOUNDED COHOMOLOGY

by MICHAEL GROMOV

0. INTRODUCTION

0.1. Minimal Volume

Take a C^∞-manifold V and consider all complete Riemannian metrics g on V whose sectional curvatures are everywhere bounded in absolute value by one, $|K(g)| \leq 1$. We define the *minimal volume* of V as the lower bound of the total volumes of all such metrics:

$$\text{Min Vol}(V) = \inf_{|K(g)| \leq 1} \text{Vol}(V, g).$$

The minimal volumes of closed connected surfaces V are proportional to the Euler characteristics,

$$\text{Min Vol}(V) = 2\pi|\chi(V)|.$$

Indeed by Gauss-Bonnet one has, for $|K| = |K(g)| \leq 1$,

$$\text{Vol}(V) \geq \int_V |K\, dv| \geq \left|\int_V K\, dv\right| = 2\pi|\chi|,$$

with equality for K constant, 1 or -1. In particular, metrics g of constant curvature ± 1 are *extremal*: $\text{Vol}(V, g) = \text{Min Vol}(V)$, while the torus and the Klein bottle, who have $\chi = 0$, carry no extremal metrics since their minimal volumes are zero.

The Gauss-Bonnet formula also applies to complete non-compact surfaces V with $|K(V)| \leq 1$ and with $\text{Vol}(V) < \infty$. Again, metrics of constant curvature -1 are extremal and so for $\chi(V) < 0$ we get $\text{Min Vol}(V) = -2\pi\chi(V)$. In particular $\chi(V) = -\infty$ implies $\text{Min Vol}(V) = \infty$. Furthermore, the cylinder and the Möbius band have zero minimal volume, while for $V = \mathbf{R}^2$ we only obtain in Appendix 1 the following estimate

$$4\pi + 0.01 < \text{Min Vol}(\mathbf{R}^2) \leq (2 + 2\sqrt{2})\pi.$$

If $n = \dim V > 2$, then the Gauss-Bonnet formula yields the following inequality for closed manifolds V,

$$\operatorname{Min} \operatorname{Vol}(V) \geq \operatorname{const}_n |\chi(V)| \quad \text{for some } \operatorname{const}_n > 0.$$

There are similar inequalities for the Pontryagin numbers p of V,

$$\operatorname{Min} \operatorname{Vol}(V) \geq \operatorname{const}_n^1 |p(V)|.$$

In fact by the theorem of Chern-Weil there are certain polynomials $P(\Omega)$ in the curvature tensor Ω of V, such that

$$|p| = \left| \int_V P(\Omega) dv \right| \leq \sup ||P(\Omega)|| \operatorname{Vol}(V),$$

and $|K| \leq 1$ implies $\sup ||P(\Omega)|| \leq \operatorname{const}_n^1$.

Now, if V is an *open* complete manifold of dimension $n \geq 3$, then the theorem of Gauss-Bonnet does not apply, in general. In fact, there are manifolds V with zero minimal volume and with non-zero Euler characteristic. For example, *for every* $\varepsilon > 0$ *there is a complete metric* g_ε *on* $\mathbf{R}^3$ *with* $|K(g_\varepsilon)| \leq 1$ *and with* $\operatorname{Vol}(\mathbf{R}^3, g_\varepsilon) \leq \varepsilon$. (See Appendix 2.)

Pontryagin classes are more useful than the Euler characteristic for open manifolds.

Example. — *Let* V_0 *be a closed* $4m$*-dimensional manifold and let* $V = V_0 \times \mathbf{R}$. *If* V_0 *has a non-zero Pontryagin number, then* $\operatorname{Min} \operatorname{Vol}(V) = \infty$.

Proof. — We have a non-trivial integral characteristic class

$$p \in H^{4m}(V; \mathbf{R}) = H^{4m}(V_0; \mathbf{R}) = \mathbf{R},$$

and for any given metric g on V this p is represented by a $4m$-form $P(\Omega)$ on V which is a Chern-Weil polynomial in Ω of degree $2m$. Then for complete metrics g on V we consider concentric balls $B(R) \subset V$ of radii R around a fixed point $v_0 \in V$ and we observe that for sufficiently large R_0 the boundary spheres $S(R) = \partial B(R)$ for $R \geq R_0$ support a non-trivial class in $H_{4m}(V)$. Therefore

$$\int_{S(R)} ||P(\Omega)|| \, ds \geq 1, \quad \text{for } R \geq R_0,$$

and so

$$\int_{B(R)} ||P(\Omega)|| \, dv = \int_0^R dR \int_{S(R)} ||P(\Omega)|| \, ds \geq R - R_0.$$

Since $||P(\Omega)|| \leq \operatorname{const}'_m |K|^{2m}$ we also get

$$\int_{B(R)} |K|^{2m} dv \geq \operatorname{const} R \quad \text{for } R \geq 2R_0,$$

and in particular, for $|K| \leq 1$, we conclude

$$\liminf_{R \to \infty} R^{-1} \operatorname{Vol} B(R) \geq \operatorname{const} > 0.$$

This is even stronger than the required relation $\operatorname{Vol}(V) = \lim_{R \to \infty} \operatorname{Vol}(B(R)) = \infty$.

Remark. — If a manifold V' is homeomorphic to the infinite connected sum of copies of the manifold V above, then we have for $R \to \infty$,

$$\int_{S'(R)} ||P(\Omega')|| \to \infty, \quad \text{where } S' = \partial B' \subset V',$$

and so for $|K(V')| \leq 1$ we obtain

$$\lim_{R \to \infty} R^{-1} \mathrm{Vol}(B'(R)) = \infty.$$

At the end of Appendix 2, we shall, for *any given smooth manifold* V, and any positive function $\varepsilon(R)$ for which $\lim_{R \to \infty} \varepsilon(R) = 0$, construct a complete metric g on V, with $|K(g)| \leq 1$, such that

$$\lim_{R \to \infty} \varepsilon(R) R^{-1} \mathrm{Vol}\, B(R) = 0.$$

Now let us give five examples of manifolds V with $\mathrm{Min\,Vol}(V) = 0$.

(1) V is compact and admits a flat Riemannian metric.

(2) V admits a locally free S^1-action. In particular, $\mathrm{Min\,Vol}(V = V_0 \times S^1) = 0$, and also $\mathrm{Min\,Vol}(S^3) = 0$. The latter is the famous example of Berger (see [10]).

(3) V is a component of the boundary of a compact manifold W whose interior, Int W, either admits a complete locally symmetric metric of non-positive curvature and finite volume, or Int W admits the structure of a complex quasiprojective (for instance, affine) variety.

(4) V is the product of an arbitrary V_0 by a manifold in one of the above examples (1)-(3).

(5) V is odd dimensional and diffeomorphic to a finite or infinite connected sum of manifolds of example (4). For instance, connected sums of odd dimensional tori have zero minimal volume. Notice that such connected sums admit no non-trivial circle action.

The first example is obvious. For the rest see Appendix 2.

The main purpose of this paper is to provide new estimates from below for the minimal volume in terms of the *simplicial volume* defined in section (0.2). In particular, we exhibit in section (0.4) closed *odd dimensional* manifolds with non-vanishing minimal volume. One's interest in such estimates is motivated, in part, by the following theorem of J. Cheeger [9] which relates the " topological complexity " of a manifold to its geometric size:

Cheeger's finiteness theorem. — *For any given numbers* $D > 0$ *and* $\varepsilon > 0$ *there are at most finitely many diffeomorphism classes of closed Riemannian manifolds* V *of a fixed dimension* n *such that*

$$|K(V)| \leq 1,$$

$$\mathrm{Diameter}(V) \leq D,$$

$$\mathrm{Vol}(V) \geq \varepsilon.$$

0.2. Simplicial volume

Let X be any topological space. Denote by $C_* = C_*(X)$ the real chain complex of X: a chain $c \in C_*$ is a finite combination $\sum_i r_i \sigma_i$ of singular simplices σ_i in X with real coefficients r_i. We define the *simplicial ℓ^1-norm* in C_* by setting $||c|| = \sum_i |r_i|$. This norm gives rise to a pseudo-norm on the homology $H_* = H_*(X; \mathbf{R})$ as follows:

$$||\alpha|| = \inf_z ||z||,$$

where z runs over all singular cycles representing $\alpha \in H_*$.

For a closed manifold V we define its simplicial volume $||V||$ as the simplicial norm of its fundamental class. When V is not orientable we pass to the double covering $\tilde{V}$ and set $||V|| = \frac{1}{2}||\tilde{V}||$.

If V is open, then its fundamental class is represented by *locally finite* cycles $c = \sum_{i=1}^{\infty} r_i \sigma_i$, such that each compact subset of V intersects only *finitely many* (images of) simplices σ_i. Now, the ℓ^1-norm $||c|| = \sum_{i=1}^{\infty} |r_i|$ may be infinite and the corresponding simplicial volume $||V||$ also may be infinite.

Example. — For the real line one has $||R^1|| = \infty$.

The following functorial property of the simplicial volume is immediate from the definition:

Let $f: V \to V'$ be a proper map of degree d. Then $||V|| \geq d||V'||$. Furthermore, if f is a d-sheeted covering, then $||V|| = d||V'||$.

Corollary. — If a closed manifold V admits a self-mapping f of degree $d \geq 2$, then $||V|| = 0$.

For example, all spheres and tori have zero simplicial volume.

If V is an *open* manifold which admits a proper self-mapping of degree ≥ 2, then one can only claim that either $||V|| = 0$ or $||V|| = \infty$. In fact, both cases occur, but if V is homeomorphic to the interior of a compact manifold with boundary, $V = \operatorname{Int} \overline{V}$, then the case $||V|| = \infty$ is excluded. Moreover,

If the boundary $\partial\overline{V}$ of $\overline{V}$ admits a self mapping f with $|d| = |\deg f| \geq 2$, then $||\operatorname{Int} \overline{V}|| < \infty$.

Proof. — First we represent the fundamental class of $\overline{V}$ by a chain c with boundary b in $\partial\overline{V}$. Then we attach to $\overline{V}$ the cylinder $\partial\overline{V} \times [0, \infty)$, and observe that the resulting manifold V is exactly Int V. Now we extend the chain c to an ℓ^1-cycle of V as follows. Denote by b_1 the image $f_*(b)$ in $\partial\overline{V} \times 1 = \partial\overline{V}$ and let c_1 be a chain in $\partial\overline{V} \times [0, 1]$ with the boundary $\partial c_1 = d^{-1}b_1 - b$ for $d = \deg f$. Next we consider the iterates $f^{(k)}$ of f, for $k = 0, 1, \ldots$, and take the maps

$$f^k: \partial V \times [0, 1] \to \partial V \times [k, k+1] \subset V \times [0, \infty),$$

defined by f^k: $(v, t) \mapsto (f^{(k)}(v), t + k)$. Then the locally finite chain $\tilde{c} = c + \sum_{k=0}^{\infty} d^{-k} f_*^k(c_1)$ is a fundamental cycle of V of finite ℓ^1-norm,

$$||\tilde{c}|| = ||c|| + \sum_{k=0}^{\infty} d^{-k} ||c_1|| = ||c|| + \frac{|d|}{|d| - 1} ||c_1||.$$

Corollary. — *The simplicial volume of Euclidean spaces* $\mathbf{R}^n$ *is:* $||\mathbf{R}^n|| = 0$, *for* $n \geq 2$.

Indeed, the space $\mathbf{R}^n$ has self-mappings of any degree.

Now comes our first interesting example of a closed manifold V with $||V|| > 0$, which is a special case of a theorem of Thurston (see section (0.3)).

Example. — Let $||V||$ be a closed oriented surface with Euler characteristic $\chi < 0$ of constant negative curvature -1. The fundamental class is represented by a cycle $\sum_{i=1}^{q} r_i \sigma_i$, where $\sigma_1, \ldots,$ are singular 2-simplices. The total (algebraic) volume is

$$\sum_{i=1}^{q} r_i \mathrm{Vol}(\sigma_i) = -2\pi\chi.$$

Straighten all singular simplices involved, keeping the vertices fixed, by lifting to the universal covering, that is, the hyperbolic plane $\mathbf{H} = \tilde{V}$. Observe moreover that the absolute value of the volume of any straight triangle in $\mathbf{H}$ with positive angles is majorized by π (use the excess formula in hyperbolic geometry). So we find

$$2\pi |\chi| = \sum_{i=1}^{q} r_i \, (\mathrm{Vol\ straight}\ \sigma_i)$$

$$\leq \sum_{i=1}^{q} |r_i| . \pi.$$

Hence $$||V|| = \inf \sum_{i=1}^{q} |r_i| \geq 2 |\chi(V)|$$

$$||V|| \geq 2|\chi(V)| \quad \text{for } \chi < 0.$$

This is the precise bound. The simplicial volume is

$$||V|| = 2 |\chi(V)|$$

for closed surfaces V of constant negative curvature. In order to see elementarily that $||V|| \leq 2 |\chi|$ we consider the standard model of a fundamental domain in $\mathbf{H}$, a regular k-gon F_0, $k = 2 |\chi| + 4$. It can be covered by a cycle of $k - 2 = 2 |\chi| + 2$ straight triangles so that in any case $||V|| \leq 2 |\chi| + 2$. Then we apply this construction to d-sheeted coverings of V and thus we cover d times the fundamental class of V by $2d |\chi| + 2$ triangles. Hence we find $||V|| \leq 2 |\chi| + 2d^{-1}$ and by letting $d \to \infty$ we obtain

$$||V|| \leq 2 |\chi|.$$

Finally, we indicate two general useful properties of the simplicial volume.

(1) *If* V_1 *is a closed manifold and* V_2 *is arbitrary, then*

$$C\|V_1\|\,\|V_2\| \geq \|V_1 \times V_2\| \geq C^{-1}\|V_1\|\,\|V_2\| \qquad (*)$$

where $C > 0$ *is a constant which depends only on* $n = \dim(V_1 \times V_2)$.

Observe that the first inequality above is obvious and it also holds if both manifolds, V_1 and V_2 are open. The second inequality is more interesting since it gives an estimate *from below* for $\|V_1 \times V_2\|$. We prove this inequality in section (1.1) with the bounded cohomology technique. The requirement of V_1 to be closed is essential. Indeed for example $\|\mathbf{R}^1\| = \infty$, while $\|\mathbf{R}^2\| = 0$.

Notice that both inequalities claim nothing whatsoever for the case of $\|V_1\| = 0$ and $\|V_2\| = \infty$. However for all known examples of such manifolds V_1 and V_2 one has $\|V_1 \times V_2\| = 0$.

(2) *For* $n \geq 3$, *connected sums of n-dimensional manifolds satisfy*

$$\|V_1 \# V_2\| = \|V_1\| + \|V_2\| \qquad (**)$$

This is proven in section (3.5) where we also establish the following generalization.

Let V_0 *be a closed simply connected submanifold of* V *of codimension one. Then, for* $\dim V > 1$, *the simplicial volume of* V *does not change if the submanifold* V_0 *is deleted,* $\|V \setminus V_0\| = \|V\|$.

0.3. Inequalities of Milnor-Sullivan and Thurston

To take the simplicial volume seriously one needs additional examples of manifolds V for which $\|V\| \neq 0$. The following remarkable theorems provide a variety of such examples.

Theorem of Milnor-Sullivan ([39], [44]). — *If a closed manifold* V *supports an affine flat bundle of dimension* $n = \dim V$, *then* $\|V\| \geq |\chi|$ *for the Euler number* χ *of this bundle.*

Recently, Smillie refined Sullivan's argument and proved that $\|V\| \geq 2^n|\chi|$. This is presented in section (1.3), where we also study Pontryagin numbers of non-affine flat bundles.

The theorem of Milnor-Sullivan is only useful for *n even* since odd dimensional bundles have zero Euler numbers. Our next inequality works for all *n*.

Thurston's Theorem. — *Let* V *be a complete Riemannian manifold of finite volume,* $\mathrm{Vol}(V) < \infty$. *If the sectional curvature of* V *satisfies* $-\infty < -k \leq K(V) \leq -1$, *then* $\mathrm{Vol}(V) \leq \mathrm{const}_n \|V\|$.

Thurston's proof is presented in section (1.2). (See also [47] and [35].)

I wish to thank Denis Sullivan who introduced me to these results and to a circle of ideas important for this paper.

0.4. Complements to Thurston's inequality

Characteristic numbers of locally homogeneous manifolds V are proportional to Vol(V) by Hirzebruch's proportionality principle. In fact characteristic numbers can be obtained following Chern-Weil by integrating over V some universal polynomials in the curvature $\Omega = \Omega(V)$.

Unlike the characteristic numbers the simplicial volume of Riemannian manifolds V cannot be obtained by integration of local invariants of V, but the proportionality phenomenon remains valid (see [47] and also section (2.3)).

Proportionality theorem. — *If the universal coverings of two closed Riemannian manifolds* V *and* V′ *are isometric, then*

$$||V||/\mathrm{Vol}(V) = ||V'||/\mathrm{Vol}(V').$$

This theorem is clearly true for V and V′ with a common *finite* covering. In fact, the theorem is most useful if the universal covering $\tilde{V}$ of V is a symmetric space and then $||V||/\mathrm{Vol}(V) = \mathrm{const} = \mathrm{const}(\tilde{V})$. This constant is, probably, non-zero if $\tilde{V}$ has negative Ricci curvature. Indeed, const $\neq 0$ if $\tilde{V}$ has negative *sectional* curvature (rank 1 case) by Thurston's theorem. Also, as we shall prove in section (1.3), this constant is non-zero if some characteristic number p of V does not vanish. Furthermore, R. P. Savage recently proved (see [42]) that $\mathrm{const}(\tilde{V}) \neq 0$ for symmetric spaces $\tilde{V}$ whose isometry groups are special linear groups $\mathrm{Is}(\tilde{V}) = \mathrm{SL}_q(\mathbf{R})$ for $n = \dim V = \frac{1}{2}q(q+1) - 1$. These manifolds have $\mathrm{rank} = q - 1$ and all their characteristic numbers vanish for $q > 2$.

Example. — Let $\tilde{V}$ be a product of hyperbolic spaces ($K = -1$) of various dimensions. Then $||V|| = \mathrm{const}\ \mathrm{Vol}(V)$, and by Thurston's theorem combined with the inequality (*) of (0.2) we have $\mathrm{const} \geq \mathrm{const}_n > 0$, for $n = \dim V$. Furthermore, if all hyperbolic factors have even dimensions, then $||V|| = \mathrm{const}'\,|\chi(V)|$, where the last constant only depends on the dimensions of the factors.

Explicit constants are only known for negative curvature -1, $\tilde{V} = \mathbf{H}^n$. They are determined as follows. Take all n-dimensional simplices S in the hyperbolic space $\mathbf{H}^n$ of curvature -1 and denote by R_n the upper bound of their volumes,

$$R_n = \sup_{S \subset \mathbf{H}^n} \mathrm{Vol}(S).$$

(A simplex is, by definition, the convex hull of $n+1$ points in general position in $\mathbf{H}^n$.)

Then, we have the exact formula:

$$||V|| = R_n^{-1}\mathrm{Vol}(V).$$

This formula also holds for complete non-compact manifolds of finite volume (see [47] and section (1.2)) and the extremal value R_n is always assumed by the regular *ideal* simplex

in $\mathbf{H}^n$ with all vertices at infinity (Milnor [47] for $n = 3$, Haagerup and Munkholm [27] for $n \geq 4$). Furthermore, $R_2 = \pi$, $R_3 = \frac{3}{2}\sum_{i=1}^{\infty} i^{-2} \sin \frac{2\pi i}{3} \simeq 1.0149$, and asymptotically for $n \to \infty$ one has $R_n \simeq \frac{\sqrt{n}}{n!} e$. (See [27].)

Another useful relation for $\|V\|$ comes from the following.

The (Σb_i)-estimate. — Let V *be a complete connected real analytic manifold with bounded non-positive curvature,* $-k^2 \leq K(V) \leq 0$, *and let the Ricci tensor of* V *be negative definite at some point* $v \in V$. *Then*

$$\sum_{i=0}^{n} b_i(V) \leq \text{const}_n k^n \text{Vol}(V),$$

where $n = \dim V$ *and* b_i *are the Betti numbers with any given coefficients.* (See Appendix 3.)

With Thurston's theorem and (**) of (0.2) we come to the following

Corollary. — Let V *be a connected sum of manifolds of one of the following two types:*

(a) *Compact locally symmetric spaces with non-zero simplicial volume. For example those which have non-zero Euler characteristics.*

(b) *Complete manifolds of finite volume with sectional curvature pinched between two negative constants,*

$$-k_1 \leq K(V) \leq -k_2 < 0.$$

Then

$$\sum_{i=0}^{n} b_i(V) \leq \text{const}\, \|V\|,$$

where the constant depends only on $n = \dim V$ *and on the ratio* k_1/k_2.

It is unclear whether the constant can be chosen independently of this ratio.

0.5. Estimates from below for the minimal volume

For Riemannian manifolds V we denote by Ricci V the Ricci tensor, and we write $\text{Ricci} \geq -1$ if $\text{Ricci}(\tau, \tau) \geq -\langle \tau, \tau \rangle$ for all tangent vectors $\tau \in T(V)$. Observe that a bound from below for the sectional curvature, $K(V) \geq -k^2$, implies $\text{Ricci}\, V \geq -(n-1)k^2$, that is $\text{Ricci}(\tau, \tau) \geq -(n-1)k^2 \langle \tau, \tau \rangle$. We prove in section (2.5) the following

Main Inequality. — Let V *be a complete n-dimensional Riemannian manifold with* $\text{Ricci} \geq -1/(n-1)$. *Then* $\|V\| \leq \text{const}_n \text{Vol}(V)$ *for some constant in the interval,* $0 < \text{const}_n < n!$.

Corollaries. — (A) *The estimate for the Minimal Volume. — All differentiable manifolds* V *satisfy*

$$\|V\| \leq (n-1)^n\, n!\, \text{Min Vol}(V).$$

(B) *The* (Σb_i)*-estimate with Ricci curvature. — Let* V *be a complete Riemannian manifold with* $\mathrm{Ricci}\, V \geq -k^2$. *If* V *is homeomorphic to a compact hyperbolic manifold, or, more generally, o a connected sum of manifolds* (a) *and* (b) *of the last corollary of* (0.4), *then*

$$\sum_{i=0}^{n} b_i(V) \leq \mathrm{const}'_n\, k^n\, \mathrm{Vol}(V).$$

(C) *The Volume comparison theorem. — Let* V *and* V′ *be complete Riemannian manifolds of dimension n and let* $f: V \to V'$ *be a continuous proper map. Assume moreover*

$$\mathrm{Ricci}\, V \geq -1/(n-1),$$
$$-\infty < -k \leq K(V') \leq -1,$$

and $\mathrm{Vol}\, V' < \infty$. *Then*

$$|\deg(f)| \leq C_n\, \mathrm{Vol}(V)/\mathrm{Vol}(V') \qquad (*)$$

Unfortunately we do not know the explicit value of this constant C_n. The most optimistic conjecture would be $C_n = (n-1)^{-n}$. In fact, the proportionality theorem of (0.4) yields this conjecture when both V and V′ have constant negative curvature (notice that $\mathrm{Ricci}\, V = 1/(1-n)$ here corresponds to $K(V) = -(n-1)^{-2}$). One may conjecture further that the equality in (*) with this optimal hypothetic constant C_n may hold only if both manifolds V and V′ do have constant curvature, and then by *Thurston's rigidity theorem* ([47], [25]) the map f is homotopic to a locally isometric d-sheeted covering.

Now let V be homeomorphic to the interior of a (possibly non-compact) manifold W whose boundary ∂W is a disjoint union of *compact* manifolds, called $\partial_1 W$, $\partial_2 W$, ... We prove in section (2.5) the following estimate from below for the volumes of balls $B(R) \subset V$ around a fixed point v_0 in V. (Compare the Example of (0.1).)

The asymptotic inequality. — If V *is a complete manifold with* $\mathrm{Ricci}\, V \geq -1/(n-1)$ *then*

$$||\partial W|| = ||\partial_1 W|| + ||\partial_2 W|| + \ldots \leq \mathrm{const}_n \liminf_{R \to \infty} R^{-1}\, \mathrm{Vol}\, B(R),$$

for $0 < \mathrm{const}_n \leq (n-1)!$.

The estimate for the minimal volume will be generalized in section (2.5) to products $V = V_1 \times V_2$ where V_1 is a *closed* manifold with a nonzero Pontryagin number $p = p(V_1)$.

The product Inequality. — The product $V = V_1 \times V_2$ *satisfies*

$$p(V_1)||V_2|| \leq \mathrm{const}'_n\, \mathrm{Min}\, \mathrm{Vol}(V_1 \times V_2), \quad \textit{for } n = \dim V.$$

Example. — Take for V_1 the complex projective plane and let V_2 be a 3-dimensional manifold (compact or not) with $K(V_2) = -1$ and $\mathrm{Vol}(V_2) < \infty$. Then

$$\mathrm{Min}\, \mathrm{Vol}(V_1 \times V_2) \geq \mathrm{const}\, \mathrm{Vol}(V_2), \quad \text{for some const} > 0,$$

while $||V_1 \times V_2|| = 0$ since V_1 (and so $V_1 \times V_2$) has self-mappings of degree ≥ 2. Notice that all characteristic numbers of this $V_1 \times V_2$ are zero since $n = 7$. We have no example of a closed five dimensional manifold V with $||V|| = 0$ and $\operatorname{Min} \operatorname{Vol}(V) > 0$. On the other hand we shall show in Appendix 2 that for many 3-dimensional manifolds $\operatorname{Min} \operatorname{Vol}(V) \leq \text{const}' ||V||$.

We shall see in section (3.1) that the simplicial volumes of *closed* manifolds V are entirely determined by the fundamental groups $\Pi = \pi_1(V)$ and by the classifying maps $V \to K(\Pi, 1)$. For example, $||V|| = 0$ for all closed simply connected manifolds V. However we do not know whether the *minimal* volume is also zero for all closed *odd* dimensional simply connected manifolds V.

Another interesting problem concerns the sets of values of Min Vol V and $||V||$, when V runs over all manifolds of a given dimension n. The work of Thurston (see [47], [25]) may suggest that both sets are closed countable well ordered non-discrete subsets of the real line, but we do not even know whether the value zero is isolated, *i.e.* if $\operatorname{Min} \operatorname{Vol}(V) \leq \varepsilon$ for some $\varepsilon = \varepsilon_n > 0$ implies $\operatorname{Min} \operatorname{Vol}(V) = 0$. The same question is open for the simplicial volume. However, we shall see in section (3.4), that $\operatorname{Min} \operatorname{Vol}(V) \leq \varepsilon_n$, for some $\varepsilon_n > 0$, does imply $||V|| = 0$. Moreover, we have the following

Isolation theorem. — *Let* V *be a complete n-dimensional manifold with* $\operatorname{Ricci}(V) \geq -1$, *and let the unit ball in* V *around each point* $v \in V$ *satisfy* $\operatorname{Vol} B_v(1) \leq \varepsilon$ *for some sufficiently small positive* $\varepsilon = \varepsilon(n)$. *Then* $||V|| = 0$. *In particular, if* $\operatorname{Vol}(V) \leq \varepsilon(n)$, *then* $||V|| = 0$.

Corollary. — *If a manifold* V *with* $\operatorname{Ricci} V \geq -1$ *admits a proper map of positive degree onto a manifold* V′ *of negative sectional curvature, for which* $\operatorname{Vol}(V') < \infty$ *and* $-\infty < -k_1 \leq K(V') \leq -k_2 < 0$, *then for some point* $v \in V$ *the unit ball has* $\operatorname{Vol}(B_v(1)) \geq \varepsilon = \varepsilon(n) > 0$.

Remark. — If $V = V'$ and $V \to V'$ is the identity map then this corollary reduces to a theorem of Margulis (see [7]) and in this special case one can give an *effective* estimate for ε, for example, one can take $\varepsilon = \varepsilon_n = \exp(-\exp(\exp n^n))$. Notice, that our proof in section (3.4) depends on the polynomial growth theorem (see [23]) that gives no effective estimate. However, we prove in section (4.3) for manifolds with $|K| \leq 1$ the following more general result with the above *effective* value $\varepsilon = \varepsilon_n$.

The Injectivity Radius estimate. — *If* $|K(V)| \leq 1$ *then the simplicial volume* $||V||$ *is bounded from above by* $\text{const}_n \operatorname{Vol}(U_\varepsilon)$, *where* $U_\varepsilon \subset V$ *denotes the set of those points* $v \in V$, *for which the injectivity radius of* V *satisfies* $\operatorname{Inj} \operatorname{Rad}_v(V) \geq \varepsilon = \varepsilon_n$. *In particular, if for all* $v \in V$ *one has* $\operatorname{Inj} \operatorname{Rad}_v(V) \leq \varepsilon_n$, *then* $||V|| = 0$.

In the general case of $\operatorname{Ricci} \geq -k > -\infty$ we prove in section (4.2) the following weaker result.

The Geometric Finiteness Theorem. — Let V *be a complete manifold with*

$$\text{Ricci } V \geq -k > -\infty$$

and let the unit balls $B_v(1) \subset V$ *satisfy* $\text{Vol}(B_v(1)) \to 0$ *for* $v \to \infty$. *Then the simplicial volume is finite,* $||V|| < \infty$.

Corollary. — The above manifold V *admits no proper map of positive degree onto a connected sum of infinitely many hyperbolic manifolds of finite volumes.*

Plan of the paper. — We start section 1 with the translation of our problems into the language of *bounded cohomology*. Then we prove in this language the theorems of Milnor-Sullivan and Thurston. Next, we construct in section 2 *geometric smoothing operators* on bounded *Borel cochains* and prove the Main Inequality for *compact* manifolds. In fact, we prove this inequality for a modified notion of simplicial volume. The equivalence of the two simplicial volumes is established in section 3, by means of *algebraic averaging operators* on bounded cochains of *simplicial multicomplexes*. We also prove with these operators the isolation theorem for *compact* manifolds and the identity $||V_1 \# V_2|| = ||V_1|| + ||V_2||$ as well. Finally, in section 4 we return to ℓ^1*-chains* on *open* manifolds. We start with the *algebraic diffusion* of chains on these manifolds and then we combine the algebraic diffusion with the geometric smoothing operators. Thus we prove in section (4.3) the Main Inequality for *open* manifolds.

In fact, we establish in (D) of (4.3) a sharper result, called *Main Technical Theorem* which relates the simplicial norm on homology to the geometric *mass* of cycles.

Next, in section (4.4) we refine the simplicial volume for complete non-compact manifolds by also taking into account the geometric " size" of singular simplices of fundamental cycles. We prove with this refinement the existence of *extremal* manifolds V with $|K(V)| \leq 1$ whose volumes Vol(V) equal their minimal volumes. In fact we only prove a slightly weaker result for a modified notion of minimal volume, rather than for the original one. In the final section (4.5) we study " volumes " of maps between manifolds who themselves may have infinite volumes. In particular, we generalize the Volume Comparison Theorem to manifolds of infinite volumes. We also discuss there the Euler characteristic and the signature of complete non-compact manifolds. In the Appendices (1), (2) and (3), we briefly discuss surfaces, manifolds V with $\text{Min Vol}(V) = 0$ and manifolds with $K(V) \leq 0$ respectively.

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1. BOUNDED COHOMOLOGY

A general question, arising when a chain complex C_* is equipped with a norm, is the problem of finding a complete set of its chain homotopy invariants in the category of *topological* vector spaces and *bounded* operators. (If the topology in C_* is ignored, then the homology groups of C_* give us all these invariants.) The problem becomes simpler when we complete C_*. For the simplicial norm of section (0.2) the completion leads to the complex $\overline{C}_*$ of ℓ^1-chains $c = \sum_{i=1}^{\infty} r_i \sigma_i$ with $||c|| = \sum_{i=1}^{\infty} |r_i| < \infty$, like those we constructed in section (0.2).

It is more convenient, in some respects, to work with the dual complex $\hat{C}^* = \mathrm{Hom}(\overline{C}_*, \mathbf{R})$ whose elements admit the following independent description.

1.1. Bounded cochains

Denote by Σ the set of all singular simplices $\sigma : \Delta \to X$ and recall that real *cochains* $c \in C^* = C^*(X)$ are, by definition, certain functions $c : \Sigma \to \mathbf{R}$. We define the ℓ^∞-norm of c by setting

$$||c||_\infty = \sup_{\sigma \in \Sigma} |c(\sigma)|,$$

and we call a cochain c *bounded* if this norm $||c||_\infty$ is finite.

Counterexample. — Take a closed n-dimensional manifold V with the oriented volume form ω on V. Then the " standard " singular cocycle c which assigns to each $\sigma : \Delta^n \to V$ the integral $c(\sigma) = \int_{\Delta^n} \sigma^*(\omega)$ is not bounded.

Indeed, if the map σ " wraps " Δ^n around V many times, then $c(\sigma)$ becomes arbitrarily large.

For cohomology classes $\beta \in H^*(C_*; \mathbf{R})$ we set $||\beta|| = ||\beta||_\infty = \inf_y ||y||_\infty$ where y runs over all cocycles representing β. Of course, this " norm " can take infinite values, as will become clear below. We say that a cohomology class β is *bounded* if its " norm " $||\beta||_\infty$ is finite.

Cohomological definition of the simplicial volume

The C^∞-norm in $C^* = \mathrm{Hom}(C_*, \mathbf{R})$ is the dual of the ℓ^1-norm in C_*. We apply the Hahn-Banach theorem (compare [45]) and conclude that the norms $||\ ||_1$ in $H_*(X; \mathbf{R})$ and $||\ ||_\infty$ in $H^*(X; \mathbf{R})$ are also dual.

Corollary. — *Let* V *be a closed oriented n-dimensional manifold with dual fundamental classes* $\alpha \in H_n(V, \mathbf{R})$ *and* $\beta \in H^n(V; \mathbf{R})$. *Then*

$$\beta(\alpha) = 1 = ||\beta||_\infty ||\alpha||_1 = ||\beta||_\infty ||V||.$$

Therefore $||V||^{-1} = ||\beta||_\infty$; *in particular* β *is bounded if and only if the simplicial volume* $||V||$ *does not vanish.*

Now, if V is an open manifold then the fundamental cohomology class β of V is represented by cocycles $c : \Sigma \to \mathbf{R}$ with *compact support.* That is for some compact subset $V_0 \subset V$ which depends on c, the cochain $c(\sigma)$ vanishes at those singular simplices $\sigma : \Delta \to V$, whose images do not intersect V_0.

With these cochains we have again our ℓ^∞-norm, $||\beta||_\infty$, and we clearly have the inequality $||V||\,||\beta||_\infty \leq 1$. However, the equality $||V||^{-1} = ||\beta||_\infty$ does not hold in general.

Example. — Let V be the interior of a compact manifold with boundary, $V = \operatorname{Int} \overline{V}$. Then, with the ℓ^1-norm in the relative chain complex

$$C_*(\overline{V}, \partial\overline{V}) = C_*(\overline{V})/C_*(\partial\overline{V}),$$

one has the norm of the fundamental class $\bar{\alpha} \in H_*(\overline{V}, \partial\overline{V})$ and then one puts $||\overline{V}, \partial\overline{V}|| = ||\bar{\alpha}||_1$. Now *this* norm $||\ ||_1$ and the corresponding simplicial volume are dual to the ℓ^∞-norm on the cohomology with compact supports in $V = \operatorname{Int} \overline{V}$, and so

$$||\overline{V}, \partial\overline{V}||^{-1} = ||\beta||_\infty.$$

In particular, this simplicial volume $||\overline{V}, \partial\overline{V}|| = ||\beta||_\infty^{-1}$ is always *finite.* However if the simplicial volume $||\operatorname{Int} \overline{V}||$ is finite, then clearly the simplicial volume $||\partial\overline{V}||$ vanishes, and so one only can claim the inequality

$$||\operatorname{Int} \overline{V}|| \geq ||\overline{V}, \partial\overline{V}||.$$

In order to give a cohomological definition of $||V||$ we consider *locally finite* subsets of the set of singular simplices, $\Phi \subset \Sigma$, *i.e.* subsets such that every compact subset of V intersects only finitely many (images of) simplices in Φ. Then for cochains with compact supports in V we define seminorms $||C||_\Phi$ by putting

$$||C||_\Phi = \sup_{\sigma \in \Phi} |c(\sigma)|,$$

and next we have the corresponding seminorms on the cohomology with compact supports. Finally we take the upper bound of these seminorms over all locally finite sets of singular simplices,

$$||\beta||^\infty = \sup_{\Phi \subset \Sigma} ||\beta||_\Phi.$$

Now, the theorem of Hahn-Banach does apply and for the fundamental class β of V we have $||\beta||^\infty = ||V||^{-1}$.

As the first application we get the second inequality (*) of (0.2),

$$||V_1 \times V_2|| \geq C^{-1} ||V_1|| \, ||V_2||$$

by referring to the dual inequality for the cup-product,

$$||\beta_1 \cup \beta_2||^\infty \leq C ||\beta_1||^\infty \, ||\beta_2||^\infty,$$

which follows from the cup-product formula for singular cochains. Notice that if none of the manifolds V_1 and V_2 is closed, then the product of cochains with compact supports, c_1 in V_1 and c_2 in V_2, may have non-compact support in $V_1 \times V_2$, and so the failure of the product inequality is not surprising.

1.2 Thurston's theorem

We prove three slightly different forms of Thurston's theorem of section (0.3) and we start with the most transparent original version. We consider maps f of simplicial m-dimensional polyhedra P into complete Riemannian manifolds V. For a *Lipschitz* map f we denote by $\mathrm{mass}_m f$ the volume of f counted with geometric multiplicity. This mass equals the total volume of the singular Riemannian metric in P induced by the map $f: P \to V$. We denote by $[P]_m$ the number of m-dimensional simplices in P.

(A) *Homotopy theorem.* — *If the sectional curvature* $K = K(V)$ *satisfies* $K \leq -1$, *and if* $m = \dim P \geq 2$, *then every continuous map* $f_0 : P \to V$ *is homotopic to a Lipschitz map* f *such that* $\mathrm{mass}_m f \leq \mathrm{const}_m [P_m]$, *for some constant* $\mathrm{const}_m \leq \pi/(m-1)!$. *Furthermore, for manifolds* V *of constant curvature* -1 *one has* $\mathrm{const}_m = R_m \simeq e\sqrt{m}/m!$, *where* R_m *is the volume of the regular ideal simplex in the m-dimensional hyperbolic space* (see (0.3)).

Corollary. — *If* P *and* V *are moreover closed manifolds of the same dimension* $m \geq 2$ *then there is an upper bound for the degree of maps* $f_0 : P \to V$. *In particular, there is no self-mapping* $f_0 : V \to V$ *of degree* ≥ 2.

Indeed, $\deg f_0 = \deg f \leq \mathrm{const}\ \mathrm{mass}_m f$, where $\mathrm{const} = \mathrm{const}(P, V)$ depends on the manifolds P and V.

Proof of the theorem. — We construct the required map f by first " straightening " the map f_0 on all simplices Δ of P and then by estimating volumes of straight simplices in V.

First, take an ordered set of $m+1$ points $v_0, \ldots, v_m$ in the universal covering $\tilde{V}$ of V and span these by a *straight singular m-simplex* $\bar{\sigma}_m : \Delta^m \to \tilde{V}$ by induction: the ℓ-dimensional simplex $\bar{\sigma}_\ell : \Delta^\ell \to \tilde{V}$, $\ell = 1, \ldots, m$, with ordered vertices $v_0, \ldots, v_\ell$, is defined as the geodesic cone from v_ℓ over the $(\ell-1)$-dimensional straight face spanned by the first ℓ vertices $v_0, \ldots, v_{\ell-1}$. Next for an arbitrary simplex $\sigma : \Delta \to V$ we take a lift $\tilde{\sigma}$ to $\tilde{V}$, then we replace this $\tilde{\sigma}$ by the straight simplex $\bar{\sigma} : \Delta \to \tilde{V}$ spanned by the vertices of $\tilde{\sigma}$ and finally we project $\bar{\sigma}$ back to V. The resulting simplex $\sigma' : \Delta \to V$ is called the *straightening of* σ and denoted straight σ.

Now, for a map $f_0 : P \to V$, we order the vertices of P, and then for each simplex $\Delta \subset P$ we straighten the singular simplex $f_0 \mid \Delta : \Delta \to V$. Since these straightened simplices agree on their common faces, we obtain a map $f : P \to V$, which now is straight on all $\Delta \subset P$, and which is homotopic to f_0. Observe, that the only property of V we used here is the uniqueness of the geodesic segments joining pairs of points in $\widetilde{V}$.

What is left to prove is the following

Estimate of the volume of a straight simplex. — *If* $K(V) \leq -1$ *then straight simplices* $\sigma : \Delta^m \to \widetilde{V}$ *satisfy for* $m \geq 2$, $\mathrm{mass}_m \sigma \leq \pi/(m-1)!$.

Proof. — For $K = -1$ the sharper result, $\mathrm{mass}_m \sigma \leq R_m$, is established in [27].

This sharp inequality also holds in the general case for geodesic *triangles* $\Delta^2 \subset \widetilde{V}$. Indeed, the relative curvature of these triangles, that is their second quadratic form, is non-positive and so the Gauss curvature of the induced metric in Δ^2 satisfies $G(s) \leq -1$ for all $s \in \Delta^2$. Then, by Gauss-Bonnet, $\int_{\Delta^2} G(s)ds \geq -\pi$ and so

$$\mathrm{mass}\, \Delta^2 = \mathrm{area}\, \Delta^2 \leq \int_{\Delta^2} |G(s)|\, ds \leq \pi.$$

Now, for $K(\widetilde{V}) \leq -1$, volumes of geodesic cones over ℓ-dimensional submanifolds in $\widetilde{V}$ satisfy

$$\mathrm{Vol}(\mathrm{Cone}) \leq \ell^{-1}\, \mathrm{Vol}(\mathrm{Base})$$

(see [5], [10]) and thus the proof is finished. (Compare [35].)

Question. — Can one take $\mathrm{const}_m = R_m$?

Example. — Let P be a closed connected surface. Then, applying the second example of (0.2), we get for straight maps $f : P \to V$

$$\mathrm{mass}_2 f \leq \pi(2\,|\chi(P)| + 2).$$

In fact, one can even "triangulate" P into exactly $2\,|\chi(P)|$ *ideal* triangles (see [47]) and thus, when V is closed, obtain maps $f : P \to V$ with $\mathrm{mass}_2 f \leq 2\pi\,|\chi(P)|$.

One can also proceed in a more traditional way by deforming f_0 to a minimal or to a harmonic map $f : P \to V$, and then observe that the induced Gauss curvature in P is everywhere ≤ -1. This again yields, via Gauss-Bonnet, the sharp inequality $\mathrm{mass}_2 f \leq 2\pi\,|\chi(P)|$.

It is unclear how to obtain sharp inequalities for minimal maps of general polyhedra (or manifolds) P, but the next version of Thurston's theorem provides such an estimate for homology.

First for a singular m-dimensional chain $c = \sum_i r_i \sigma_i$, for $\sigma_i : \Delta^m \to V$, we put

$$\mathrm{mass}\, c = \sum_i |r_i|\, \mathrm{mass}_m \sigma_i,$$

and then for a homology class $\alpha \in H_m(V; \mathbf{R})$ we define $\mathrm{mass}(\alpha)$ as the lower bound of masses of cycles which represent α.

(B) *Homology theorem* (see [47], [35]). — *If* $K(V) \leq -1$ *and if* $m \geq 2$, *then all* $\alpha \in H_m(V; \mathbf{R})$ *satisfy*

$$\text{mass } \alpha \leq \text{const}_m ||\alpha||_1,$$

where $||\ ||_1 = ||\ ||$ *is the simplicial norm of section* (0.2) *and "* const_m *" is the same as in the homotopy theorem* (A).

Proof. — First we represent α by a cycle $c = \sum_i r_i \sigma_i$ whose norm $||c|| = \sum_i |r_i|$ is close to $||\alpha||_1$ and then we straighten all its singular simplices σ_i. Straightening commutes with the boundary operator on chains, and so the straightened chain

$$c' = \sum_i r_i \sigma_i', \quad \text{for } \sigma_i' = \text{straight } \sigma_i,$$

is, in fact, a cycle homologous to c. Therefore

$$\begin{aligned} \text{mass } \alpha \leq \text{mass } c' &= \sum_i |r_i| \, \text{mass}_m \, \sigma_i' \\ &\leq \text{const}_m \sum_i |r_i| = \text{const}_m ||c||, \quad \text{q.e.d.} \end{aligned}$$

Remark. — The homology theorem, when applied to the fundamental homology class of a closed n-dimensional manifold V, yields Thurston's theorem of (0.3),

$$\text{Vol}(V) \leq \text{const}_n ||V||.$$

Finally we come to the dual, cohomological version of (B), which relates the norm $||\ ||_\infty$ of section (1.1) to the *comass* norm on cohomology. Recall that the *comass* of a differential m-form ω on V is defined as the upper bound of the values of ω at the orthonormal m-frames in V. Then for a class $\beta \in H^m(V; \mathbf{R})$ we define

$$\text{comass } \beta = \inf_\omega \text{comass } \omega,$$

where ω runs over all closed m-forms representing β.

(C) *Cohomology theorem.* — *If* $K(V) \leq -1$ *and* $m \geq 2$ *then one has for all* $\beta \in H^m(V; \mathbf{R})$,

$$||\beta||_\infty \leq \text{const}_m \text{comass } \beta.$$

Proof. — First, represent β by a form ω with comass ω close to comass(β). Then construct a singular cocycle (!) c representing β by putting, for every singular simplex $\sigma : \Delta^m \to V$,

$$c(\sigma) = \int_{\sigma'} \omega,$$

where $\sigma' = \text{straight } \sigma$. Since $|c(\sigma)| \leq (\text{mass } \sigma')(\text{comass } \omega)$, we get

$$|c(\sigma)| \leq \text{const}_m \text{comass } \omega$$

for all σ, q.e.d.

Remark. — One can show with the theorem of Hahn-Banach, that the theorems (B) and (C) are, in fact, equivalent.

The relative case. — Consider a locally convex subset $V_0 \subset V$, for example a totally geodesic submanifold $V_0 \subset V$. Then for each singular simplex $\Delta \to V_0$ its straightening is also contained in V_0 and so the theorems (A), (B) and (C) hold in their respective relative forms.

Now we consider *proper* geodesics $\gamma : \mathbf{R} \to V$ that is both ends ($t \to \pm\infty$ for $t \in \mathbf{R}$) go to infinity in V. We say that V is *concave relative to infinity* (one might say that the infinity of V is convex) if no proper geodesic γ extends to a *proper* map $f: \mathbf{R} \times [0, \infty) \to V$ with $f \mid \mathbf{R} \times 0 = \gamma$ (identifying $\mathbf{R} \times 0$ with $\mathbf{R}$).

Examples. — If a manifold V_0 is closed, then the cylinder $V_0 \times \mathbf{R}$ is concave relative to infinity. The space $\mathbf{R}^n$ for $n \geq 2$ is not concave relative to infinity.

If V is concave relative to infinity then every cocycle b in V with *compact support* " straightens " to a cocycle b' which also has *compact support.* Furthermore, $b - b' = \delta c$, for some cochain c with *compact support.* Thus we obtain, for the fundamental cohomology class β of V, our old estimate

$$\|V\| = (\|\beta\|^\infty)^{-1} \geq (\|\beta\|_\infty)^{-1} \geq \text{const}_n^{-1} \text{Vol}(V).$$

Example. — If $-k \leq K(V) \leq -1$ and if $\text{Vol}(V) < \infty$, then V is concave relative to infinity (see Appendix 3) and so

$$\text{Vol}(V) \leq \text{const}_n \|V\|.$$

Thus the proof of Thurston's theorem of (0.3) is complete.

1.3. The theorem of Milnor-Sullivan

We start with the original geometric version of the theorem (see [39], [44]).

Let P *be a simplicial polyhedron and let* $Z \to P$ *be a flat n-dimensional vector bundle over* P. *Then the Euler class* $\chi(Z) \in H^n(P; \mathbf{Z})$ *can be represented by a simplicial cocycle whose value at each simplex of* P *is* 0, 1 *or* -1.

Proof. — Take a section $f: P \to Z$ which does not vanish on the $(n-1)$-skeleton of P. The cocycle, which assigns to each oriented n-dimensional simplex $\Delta \subset P$ the algebraic number of zeros of f in Δ, is cohomologous to $\chi(Z)$. (This is the definition of χ.) Since the bundle Z is flat we can choose the section $P \to Z$ piecewise linear, such that its restriction to each simplex Δ in P is the graph of an affine map $\Delta \to \mathbf{R}^n$. Now, a generic piecewise linear section has at most one simple zero inside each n-dimensional simplex. The assertion is proved.

The following argument due to Smillie, provides a rational cocycle whose value at each n-dimensional simplex equals $\pm 2^{-n}$ for n even and 0 for n odd.

A piecewise linear section $f: P \to Z$ is determined over each n-dimensional simplex $\Delta \subset P$ by its values $f_0, \ldots, f_i, \ldots, f_n$ at the vertices of Δ. If we multiply some of f_i by -1, we get a new linear section over Δ. There are 2^{n+1} such sections over Δ, but exactly two (opposite ones) of them have a zero. To see this, study $n+1$ points

$$\varepsilon_0 f_0, \ldots, \varepsilon_n f_n \text{ in } \mathbf{R}^n \quad \text{for } \varepsilon_i = \pm 1,$$

and examine whether their convex hull contains o in the interior. If we assign to each Δ the algebraic average number of zeros while averaging over P, we get the required cocycle representing $\chi(P)$.

Remarks. — *(a)* If P is a closed oriented manifold with fundamental class [P] then we can speak of the Euler numbers, $\chi = \langle \chi(Z), [P] \rangle$. Any lower bound for the number of n-simplices of a triangulation of P or of a multicomplex on P (see (3.2)) gives us an upper bound for $|\chi|$. One often obtains better estimates by using triangulations of finite coverings of P. For *surfaces* P, applying the second example of (0.2), one gets

$$|\chi| \leq \tfrac{1}{2} |\chi(P)| \qquad (*)$$

This estimate is sharp. Indeed, by taking a metric of constant curvature in P, we get $P = \mathbf{H}^2/\Pi$ where $\mathbf{H}^2$ is the hyperbolic plane and $\Pi \simeq \pi_1(P)$ is a subgroup in the orientation preserving isometry group $\mathrm{Is}(\mathbf{H}^2) = \mathrm{PSL}_2(\mathbf{R})$. The quotient $\mathrm{PSL}_2(\mathbf{R})/\Pi$ is canonically isomorphic to the total space of the unit tangent bundle $S \to P$, and since the Euler number of this bundle is even ($= \chi(P)$), there is a double covering $\tilde{S} \to S$ such that the pullbacks $\tilde{S}_p \subset \tilde{S}$ of the fibers $S_p = S^1 \subset S$, $p \in P$, are connected. Thus we get another circle bundle

$$\tilde{S} \to P \quad \text{(a "square root" of S)}$$

and $\tilde{S} = \mathrm{SL}_2(\mathbf{R})/\Pi$ for some lift

$$g: \Pi \to \mathrm{SL}_2(\mathbf{R}) \to \mathrm{PSL}_2(\mathbf{R}) = \mathrm{SL}_2(\mathbf{R})/(\pm 1).$$

The vector bundle $\tilde{T}$, associated to $\tilde{S}$, admits a flat structure, since it is also associated to the principle fibration $\mathbf{H}^2 \to P = \mathbf{H}^2/\Pi$ via the linear representation g. Now, $\tilde{T}$ is "the square root" of the tangent bundle T(V) and so

$$\chi(\tilde{T}) = \tfrac{1}{2} \chi(T(V)).$$

(b) There is another way (pointed out by Lusztig) to estimate Euler numbers of n-dimensional flat bundles over a *fixed* closed n-dimensional manifold P. In fact, flat bundles over P correspond to linear representations $\pi_1(P) \to \mathrm{GL}_n(\mathbf{R})$. The set of these representations is a real algebraic variety and so it has *finitely many* connected components. The bundles corresponding to the points of any given component are topologically equivalent, and in particular, their Euler numbers are equal. Observe furthermore, that

the number of different values of $\chi(Z)$ for all flat bundles $Z \to P$ is estimated from above by the fundamental group alone, while the *maximal value* of $|\chi|$ may also depend on the manifold P itself.

Example. — Let $\dim P \geq 4$ and let us delete from P an open regular neighborhood of the 1-skeleton of some triangulation of P. The boundary B of the resulting manifold P' admits an orientation reversing diffeomorphism D such that the induced homomorphism $D_* : \pi_1(B) \to \pi_1(B)$ is the identity. Next we take two copies of P' and glue their boundaries with D. We obtain in this way a closed manifold P'' which admits a map $f : P'' \to P$ of degree 2 and such that the induced homomorphism $f_* : \pi_1(P'') \to \pi_1(P)$ is an isomorphism. Now, for any flat bundle $Z \to P$ with Euler number χ, we observe that the induced bundle $f^*(Z)$ over P'' has Euler number 2χ, and by repeating this process we obtain flat bundles with arbitrary large χ without changing the fundamental group of the underlying manifolds.

The theorem of Milnor-Sullivan-Smillie can be generalized in the following abstract form.

Let Z be an n-dimensional flat bundle over an arbitrary topological space X. *Then the Euler class* $\chi = \chi(Z) \in H^n(X; \mathbf{R})$ *satisfies*

$$\|\chi\|_\infty \leq 2^{-n}.$$

(As before $\chi = 0$ *for n odd.) In particular, one gets the inequality* $\|V\| \geq 2^n |\chi|$ *of section* (0.3).

We shall show in section (3.2) how the geometric version of the theorem, when applied to the geometric realization of a particular semi-simplicial set of singular simplices in X, the " multicomplex K ", yields the abstract theorem.

Our version of the theorem of Milnor-Sullivan-Smillie implies in particular boundedness of the Euler class for affine flat bundles.

A generalization. — Let G be an algebraic subgroup in the linear group $GL_m = GL_m(\mathbf{R})$. Take the classifying space BG and consider also the classifying space BG^δ for G with the discrete topology.

Consider the natural homomorphism in cohomology, $H^*(BG; \mathbf{R}) \to H^*(BG^\delta; \mathbf{R})$, and call an $\alpha \in H^*(BG^\delta, \mathbf{R})$ a *characteristic class* if it is contained in the image of this homomorphism.

Theorem. — *Each characteristic class in* $H^*(BG^\delta; \mathbf{R})$ *is bounded.*

This theorem follows, as in the case of the Euler class (see (3.2)), from the following

Geometric version. — *Let Z be a flat m-dimensional* G*-bundle (*i.e. *a vector bundle with a flat* G*-structure) over a simplicial polyhedron* P. *Fix a class* $\beta \in H^n(BG; \mathbf{R})$ *and consider the characteristic class* $\beta^* \in H^n(P; \mathbf{R})$ *induced by the classifying map* $P \to BG$. *Then* β^* *can be represented by a simplicial cocycle which is bounded in absolute value at each n-dimensional simplex in* P *by a constant depending only on* β.

Proof. — We assume P of finite dimension $\leq n$ and construct the following finite dimensional " approximation " B_N of the classifying space BG for large N. It will suffice to take $N \geq n + m + 2$. Take first the total space K of the canonical principa GL_m-bundle over the Grassmanian $Gr_m(\mathbf{R}^N)$ of m-dimensional subspaces in $\mathbf{R}^N$. The group $G \subset GL_m$ acts on K and the space $B_N = K/G$ will be our approximation to BG. The space B_N now carries the natural structure of a real quasi-projective manifold.

Algebro-Geometric lemma. — Let B *be a real quasi-projective manifold. Then for each s there is a Zariski closed set* $C \subset B$ *of dimension s such that the inclusion homomorphism* $\tilde{H}_s(C) \to \tilde{H}_s(B)$ *is surjective, where* $\tilde{H}_*$ *denotes homology with non-compact supports.*

Proof of the lemma. — According to Hironaka [30] one can realize B by a Zariski open dense subset in a non-singular projective manifold A. Furthermore, A has a triangulation with the following properties (see [31]):

(a) each simplex of this triangulation is a semi-algebraic set in A;

(b) the complement $A \backslash B$ is a closed subcomplex of this triangulation. Denote by $\bar{C}$ the Zariski closure of the s-skeleton of our triangulation and take for the required C the intersection $\bar{C} \cap B$.

Proof of the theorem. — A classifying map $P \to B_N$ for any flat G-bundle Z over P now comes from the following construction. Take the trivial bundle $T = P \times \mathbf{R}^N \to P$ and an injective homomorphism $Z \to T$. Such a homomorphism assigns to each fiber Z_p, $p \in P$, an m-dimensional subspace $Y_p \subset \mathbf{R}^N$ and an isomorphism $Z_p \to Y_p$. A subspace Y_p with the G-structure induced from Z_p is interpreted as a point in B_N.

Since Z carries a flat structure we have a notion of piecewise linear homomorphisms $Z \to T$ whose corresponding classifying maps $P \to B_N$ are piecewise algebraic. Moreover, these maps are algebraic of degree d on each simplex in P where *d depends only on n* and of course on the group $G \subset GL_m$.

Now, the class $\tilde{\beta} \in \tilde{H}_s(B_N)$, $s = \dim(B_N) - n$, Poincaré dual to $\beta \in H^n(B_N)$, can be realized, according to the lemma, by a cycle $\tilde{b} = \sum_{i=1}^{k} r_i \tilde{\Delta}_i$ which is built of k s-dimensional semi-algebraic simplices $\tilde{\Delta}_i$ in C.

For a generic piecewise algebraic map $P \to B_N$ the image of each simplex $\Delta \subset P$ intersects each $\tilde{\Delta}_i \subset B_N$ transversally and only at interior points, whose number is at most $d.\deg(C)$. Therefore, the real intersection number ν_Δ of Δ with $\tilde{b}$ is at most

$$d \cdot \deg(C) \cdot \sum_{i=1}^{k} |r_i| \leq \text{const}_\beta.$$

Finally we observe that the cocycle $\Delta \to \nu_\Delta$ is cohomologous to $\beta^* \in H^n(P; \mathbf{R})$ thus concluding the proof of the theorem.

Remark. — In many interesting cases one can realize $\tilde{\beta}$ by a combination of Schubert cycles and then one gets more precise estimates for $||\beta^*||_\infty$ like those we obtained for the Euler class.

2. ESTIMATES FROM ABOVE FOR THE SIMPLICIAL VOLUME

2.1. A preliminary discussion

Consider a closed oriented Riemannian manifold of dimension n with sectional curvature bounded in absolute value by one, $|K(V)| \leq 1$. In order to prove that $||V|| \leq \text{const}_n \text{Vol}(V)$ (see (0.5)) we must "cover" the fundamental class of V by at most $\text{const}_n \text{Vol}(V)$ simplices. First, suppose that at all points $v \in V$ the *injectivity radius* of V satisfies $\text{Inj Rad}_v(V) \geq \varepsilon_0$ for some fixed number $\varepsilon_0 > 0$. Then the exponential map $\exp : T_v(V) \to V$ at every point $v \in V$ *embeds* the ε_0-ball $B(\varepsilon_0)$ in $T_v(V)$ around the origin into V. Furthermore, if ε_0 is small then the map $\exp_v : B(\varepsilon_0) \to V$ is almost isometric, that is its differential D has everywhere a norm close to one. In fact, even without the condition $\text{Inj Rad}(V) \geq \varepsilon_0$ one has $|\,||D|| - 1\,| < \varepsilon_0^2$ provided that $\varepsilon_0 \leq 1$ and $|K(V)| \leq 1$ (see [10]). Now if all ε_0-balls in a manifold V are roughly Euclidean, there is a triangulation with simplices of size about ε_0 and such that the total number of the simplices equals $\text{const}_n \varepsilon_0^{-n} \text{Vol}(V)$. In particular we obtain the following

Trivial Inequality. — *If* $|K(V)| \leq 1$ *and if* $\text{Inj Rad}_v(V) \geq \varepsilon_0$ *for all* $v \in V$, *then*

$$||V|| \leq \text{const}_n \varepsilon_0^{-n} \text{Vol}(V). \qquad (*)$$

This inequality gives no interesting estimates from below for Vol(V) and for Min Vol(V) since the condition $\text{Inj Rad}(V) \geq \varepsilon_0$ already implies $\text{Vol}(V) \geq C_n \varepsilon_0^n$. However, for manifolds of non-positive curvature one derives the following

Non-trivial Corollary. — *If* $0 \geq K(V) \geq -1$ *and if the fundamental group* $\pi_1(V)$ *is residually finite, then*

$$||V|| \leq \text{const}_n \text{Vol}(V). \qquad (**)$$

Proof. — Recall, that a group Π is called residually finite if the intersection of all subgroups in Π of finite index is the unit element e in Π. One can also express this condition in our context by saying that for every $\ell > 0$ there is a *finite* covering $\tilde{V} \to V$ such that every loop in $\tilde{V}$ of length $\leq \ell$ is contractible. The last condition implies, now for $K(V) \leq 0$, that $\text{Inj Rad}_v(\tilde{V}) \geq \frac{1}{2}\ell$ for all $v \in \tilde{V}$. Then for $\ell \geq 2$, the Trivial Inequality yields $||\tilde{V}|| \leq \text{const}_n \text{Vol}(\tilde{V})$, and as

$$||\tilde{V}||/||V|| = \text{Vol}(\tilde{V})/\text{Vol}(V) = d,$$

for d equal to the number of sheets of the covering $\tilde{V} \to V$, we also get $||V|| \leq \text{const}_n \text{Vol}(V)$.

Example. — *First take a manifold* V *with a metric* g_0 *of constant curvature* -1. *Then for an arbitrary metric* g *on* V *with* $0 \geq K(g) \geq -1$ *one has*

$$\mathrm{Vol}(V, g_0) \leq \mathrm{const}'_n \mathrm{Vol}(V, g). \qquad (***)$$

Indeed, manifolds of constant curvature have residually finite fundamental groups. In fact the same holds for all finitely generated subgroups of connected Lie groups, see [41]. Then (***) follows from Thurston's theorem of (0.3) and (**).

The arguments above can be extended to cover the general case of manifolds V with $|K(V)| \leq 1$. The idea is to represent the fundamental class of such a V by a cycle $c = \sum_i r_i \sigma_i$ with the following two properties:

(a) all simplices $\sigma_i : \Delta^n \to V$ are ε-small for $\varepsilon \simeq (20)^{-n^{10}}$. That is each σ_i admits a lift $\tilde{\sigma}_i$ to an exponential ε-ball,

$$\tilde{\sigma}_i : \Delta^n \to B(\varepsilon) \in T_{v_i}(V),$$

for some point $v_i \in V$, and $\exp_{v_i} \circ \tilde{\sigma}_i = \sigma_i$.

(b) $||c|| = \sum_i |r_i| \leq \mathrm{const}_n \mathrm{Vol}(V)$.

One can construct such a cycle c by first taking all " ε-cycles " c which satisfy *(a)* only, and then by minimizing the norm $||c||$ among " ε-cycles ". In fact, there is the following minimizing procedure which actually diminishes the norm of c, as long as this norm is too large compared to Vol(V). Namely, if $||c||$ is large then also a lift $\tilde{c}$ of c to the unit ball $B(1) \subset T_v(V)$ at some point $v \in V$ has a large norm $||\tilde{c}||$ compared to Vol(B(1)). Therefore, one can replace $\tilde{c}$ by a *smaller* standard " ε-chain " $\tilde{c}_0$ in B(1) which represents the relative fundamental class of the pair $(B(1), \partial B(1))$. Next, one can construct another chain $\tilde{c}_1$ which " interpolates " between $\tilde{c}$ and $\tilde{c}_0$, in the sense that $\tilde{c}_1$ equals $\tilde{c}$ in a small neighbourhood of $\partial B(1)$ and such that $\tilde{c}_1 = \tilde{c}_0$ far from the boundary $\partial B(1)$. This $\tilde{c}_1$ has smaller norm than $\tilde{c}$, and by projecting the difference $\tilde{c}_1 - \tilde{c}$ back to V and by adding this projection to the cycle c, we do diminish the norm of c.

A technical difficulty of this argument is the necessity to keep all singular simplices ε-small while constructing the interpolating chain $\tilde{c}_1$. This is the reason for the ridiculously small $\varepsilon \simeq (20)^{-n^{10}}$. In fact, one gets even worse estimates for " const_n " in the inequality $||V|| \leq \mathrm{const}_n \mathrm{Vol}(V)$, namely something like $\mathrm{const}_n \simeq (100)^{n^{10}}$.

We shall not pursue anymore this line of reasoning and turn to more efficient estimates for $||V||$.

2.2. Exact estimates for hyperbolic manifolds

Let V be a closed oriented manifold of constant curvature -1. Then there is the following very efficient way to " cover " the fundamental class of V by immersed

straight singular simplices σ_D with all edges of length D. Such a " covering " would yield, for $D \to \infty$, the inequality $||V|| \leq R_n^{-1} \mathrm{Vol}(V)$, and therefore (see (B) of (1.2)) the exact formula $||V|| = R_n^{-1} \mathrm{Vol}(V)$ of section (0.3).

We first generalize finite singular chains in V by admitting as a generalized chain any family F of singular simplices σ, parametrized by a manifold F with a finite measure, given by a form $\mu(\sigma)d\sigma$ and total measure the norm (see [47]) $||F|| = \int_F |\mu(\sigma)|\, d\sigma$ (instead of $\Sigma r_i \sigma_i$ with norm $\Sigma |r_i|$). Specifically we take the family F_D of all regular simplices σ_D with the orientations induced from V, that is each σ_D is covered by an embedded regular simplex $\tilde{\sigma}_D$ in the universal covering hyperbolic space H^n. The isometry group $\mathrm{Is}(\mathbf{H}^n)$ acts transitively on the simplices $\tilde{\sigma}_D$, and the Haar measure of $\mathrm{Is}(\mathbf{H}^n)$ induces a measure $\mu(\sigma)d\sigma$ on $F = F_D$. The generalized chain so defined is a cycle: under the boundary operator the contributions of two sides of any $(n-1)$-face $\tilde{\sigma}'$ cancel each other because these sides are symmetric under the orientation reversing (!) reflection of $\mathbf{H}^n$ in the hyperplane spanned by $\tilde{\sigma}'$. This chain F_D represents the fundamental class in case

$$\int \mu(\sigma)\, \mathrm{Vol}(\sigma) d\sigma = \mathrm{Vol}(V),$$

and then $$||F_D|| = \int \mu(\sigma) d\sigma = (\mathrm{Vol}\, \sigma_D)^{-1}\, \mathrm{Vol}\, V.$$

By definition $$||V|| \leq ||F_D||,$$

and since $\mathrm{Vol}(\sigma_D) \to R_n$ for $D \to \infty$, we get $||V|| \leq R_n^{-1} \mathrm{Vol}\, V$.

Now, we only must show that these generalized cycles F_D give the same value for $||V||$ as usual chains. First we take N points in V, each assigned with the weight Vol(V)/N. The resulting atomic measure in V is denoted by μ_N and, for $N \to \infty$, we require the sequence μ_N to converge, in the weak topology, to the Riemannian measure in V. Then we consider all straight simplices, whose edges now may have any length between D and $D + \delta$ for some $\delta > 0$, and whose vertices must be chosen among our N points. Next we consider the (formal) sum Σ of all these simplices and take the normalized chain

$$c = c(N, \delta, D) = \Sigma / ||\Sigma||.$$

Notice, that this $||\Sigma||$ equals the number of the simplices. Now, for fixed D and δ, one has

$$||\partial c|| \to 0 \quad \text{for } N \to \infty,$$

and then one can construct chains $c' = c'(N, \delta, D)$ such that $\partial c' = \partial c$ and $||c'|| \to 0$ for $N \to \infty$. Finally we take *cycles*

$$c' = c'(N, \delta, D) = c - c'$$

and observe that

$$\lim_{\delta \to 0} \lim_{N \to \infty} c'' = F_D / ||F_D||,$$

and so the limit of the cycles $||F_D||^{-1}c''$ for $N \to \infty$, $\delta \to 0$ and $D \to \infty$, is the fundamental class of V, while the norms of these cycles converge to $R_n^{-1}\,\mathrm{Vol}(V)$. (See [47] for more information.)

Nico Kuiper suggested the following direct elegant construction of *finite* singular cycles F'_D in V, for which $||F'_D|| \to R_n^{-1}\,\mathrm{Vol}(V)$ as $D \to \infty$. We fix a fundamental domain U in the hyperbolic space $\mathbf{H}^n = \tilde{V}$ for the Galois action of the group $\Pi = \pi_1(V)$ in the universal covering $\tilde{V}$ of V and we also fix a point $u \in U$. Then we consider straight simplices $\tilde{\sigma}$ in $\tilde{V}$ with vertices in the Π-orbit of u and we assign to each $\tilde{\sigma}$ with vertices $\gamma_0(u), \ldots, \gamma_n(u)$ a coefficient $\mu(\tilde{\sigma})$ equal to the (Haar) measure of those regular simplices $\tilde{\sigma}_D$ in $\tilde{V} = \mathbf{H}^n$ whose $n+1$ vertices lie in the translates $\gamma_0(U), \ldots, \gamma_n(U)$; precisely one vertex in each set $\gamma_i(U)$, $i = 0, \ldots, n$. As the family $\tilde{F}_D$ of all regular simplices $\tilde{\sigma}_D$ in $\tilde{V}$ is a (generalized) cycle, the Π-invariant chain $\tilde{F}'_D = \sum_{\tilde{\sigma}} \mu(\tilde{\sigma})\tilde{\sigma}$ also is a cycle. Furthermore, this cycle $\tilde{F}'_D$ projects to a finite straight singular cycle in V, called F'_D, which is homologous to F_D (*i.e.* F'_D is a fundamental cycle of V for a proper normalization of the Haar measure in the group $\mathrm{Is}(H^n)$), and also $||F'_D|| = ||F_D||$. Hence, $||F'_D|| \to R_n^{-1}\,\mathrm{Vol}(V)$ as $D \to \infty$.

Remark. — Generalized chains F_D can be defined, for $D \leq 1$, in all manifolds V with $K(V) \leq 1$ by taking equilateral geodesic simplices in balls $B(2) \subset T_v(V)$ with the metrics induced from V by the maps $\exp_v : B(2) \to V$. However, no canonical measure parametrizes these F_D into cycles, though

$$||\partial F_D||/||F_D|| \to 0 \quad \text{for } D \to 0.$$

It would be interesting to find a small perturbation of F_D to a cycle and thus obtain yet another proof of the inequality $||V|| \leq \mathrm{const}_n\,\mathrm{Vol}\,V$.

2.3. Straight invariant cochains

Consider a covering Y of manifold V with Galois group Π and first observe that singular cochains in V lift to those cochains in Y which are invariant under the action of Π. Next, we define a subcomplex, called $C^*(Y : \Pi) \subset C^*(Y)$ of real valued singular cochains c by imposing the following four conditions.

(1) Cochains $c \in C^*(Y : \Pi)$ are *straight*: the values $c(\sigma)$ for all $\sigma : \Delta^m \to Y$ only depend on the vertices $y_0, \ldots, y_i, \ldots, y_m \in Y$ of σ. The " straight " cochains are, in fact, functions in $m+1$ variables $y_0, \ldots, y_m \in Y$.

(2) The functions $c = c(y_0, \ldots, y_m)$ are antisymmetric, that is

$$c(y_0, \ldots, v, \ldots, y_j, \ldots, y_m) = -c(y_0, \ldots, y_j, \ldots, y_i, \ldots, y_m)$$

for all pairs of variables y_i and y_j.

(3) The functions c are *Borel* in the variables y_i, that is pullbacks of Borel subsets in $\mathbf{R}$ are Borel in the Cartesian products $\bigoplus_0^m Y$.

(4) The functions c are Π-invariant.

Remarks. — The condition (3) is purely technical. We could equally well require continuity (or even smoothness) of c, without changing the homology of $C^*(Y : \Pi)$.

The condition (2) will be quite important in the further calculations but again it does not affect the homology of our complex. Indeed, there is the natural (anti)-symmetrization operator which projects all functions to the antisymmetric ones, namely

$$c(y) \mapsto \frac{1}{(m+1)!} \sum_{\delta} [\delta] c(\delta(y)),$$

where δ runs over all permutations of the variables $y = (y_0, \ldots, y_m)$ and where $[\delta] = 1$ for even permutations δ and $[\delta] = -1$ for odd δ. This operator induces an isomorphism on the (real!) homology. In fact, it is a chain homotopy equivalence.

Without the condition (4) our complex would be acyclic (in fact, chain-contractible). Only the action of Π makes the story interesting. In fact, as we shall see in (3.3),

$$H_*(C^*(Y : \Pi)) \simeq H^*(\Pi; \mathbf{R}) \overset{\text{def}}{=} H^*(K(\Pi, 1); \mathbf{R}),$$

where $K(\Pi, 1)$ denotes the Eilenberg MacLane space.

Example. — In the proof of the homotopy theorem of section (1.2), we constructed a cocycle $c = \int_{\sigma'} \omega$, which lifts to the universal covering $Y = \widetilde{V} \to V$ with the properties (1), (3) and (4). In fact, this c is a *continuous* function in $y_0, \ldots, y_m$ and it even extends to the *ideal boundary* of $\widetilde{V}$. With the (anti)symmetrization operator, one can also make this cocycle c antisymmetric, without changing its cohomology class and keeping it bounded with the norm $\leq \text{const}_m$ comass ω.

New simplicial volume. — The ℓ^∞-norm on functions $c = c(y_0, \ldots, y_m)$ induces a norm in cohomology $H^*(Y : \Pi) \overset{\text{def}}{=} H_*(C^*(Y : \Pi))$. Furthermore, the inclusion $C^*(Y : \Pi) \to C^*(V)$ induces a homomorphism $H^*(Y : \Pi) \to H^*(V)$. Now let us take for Y the universal covering $\widetilde{V} \to V$ with $\Pi = \pi_1(V)$ and let us denote by F the homomorphism $H^*(\widetilde{V} : \Pi) \to H^*(V)$. For all $\beta \in H^*(V)$ we put $\|\beta\|_\infty^{\text{new}} = \inf_\alpha \|\alpha\|_\infty$, where α runs over the pullback $F^{-1}(\beta) \subset H^*(\widetilde{V} : \Pi)$. In particular, $\|\beta\|_\infty = \infty$ if β does not come from $H^*(\widetilde{V} : \Pi)$. Then, by duality, we define the following new norm for all $\gamma \in H_m(V)$:

$$\|\gamma\|^{\text{new}} = \sup_\beta (\|\beta\|_\infty^{\text{new}})^{-1} |\beta(\gamma)|$$

over all $\beta \in H^m(V)$. It is clear that $||\gamma||^{new} \leq ||\gamma||^{old}$ and in particular, for closed manifolds V, we have $||V||^{new} \leq ||V||^{old}$. In fact, with the remark above, the new norm $||\gamma||^{new}$ equals the old norm of *the image* of γ *under the natural homomorphism* $H_*(V) \to H_*(K(\Pi, 1))$, for $\Pi = \pi_1(V)$. In particular, the new norm (and thus the new simplicial volume) vanishes for simply connected manifolds. We shall prove in section (3.3) an equivalence theorem claiming the equality $||\ ||^{new} = ||\ ||^{old}$, but even without this theorem (which requires a bit of abstract machinery) one can use the new norm as efficiently as the old one. Indeed, the principal estimate from below for $||\ ||^{old}$, namely Thurston's theorem (C) of (1.2), provides, in fact, the same estimate for the new norm, because the (anti-symmetrized) cocycle $c = \int_{\sigma'} \omega$ is contained in the complex $C^*(Y : \Pi)$. Moreover, the functorial property for maps $f: V \to V'$,

$$||f(\gamma)|| \leq ||\gamma||,$$

also holds for the new norm and so our main geometric applications in section (0.4) will not suffer if the old norm there is understood in the new sense. In any case, since the new and the old norms are equal (see (3.3)) we do not bother to distinguish them anymore.

Finally, a word of caution: the new simplicial volume has not yet been defined for *open* manifolds V and so $||V||$ must be still understood for such V in the old sense. We return to open manifolds in section 4.

Now, let us observe that the new norm on cohomology $H^*(Y : \Pi)$ also makes sense for an arbitrary locally compact group of homeomorphisms (instead of Π) of any space Y.

Examples. — Let Π be a discrete group of isometries of a Riemannian manifold Y. By taking a subgroup of index two we always can make Π orientation preserving. Then the space Y/Π is a pseudo-manifold (in fact it is a rational homology manifold) and if Y/Π is compact of dimension n we have the n-dimensional fundamental class $\beta \in H^n(Y/\Pi; \mathbf{R})$. We represent this β by a singular cocycle $b \in C^*(Y/\Pi)$ and take its pull-back $\tilde{b}$ in the complex $\tilde{C}^*_\Pi(Y)$ of Π-invariant cochains in Y. Finally we consider all cocycles $a \in C^*(Y : \Pi) \subset \tilde{C}^*_\Pi(Y)$ which are cohomologous to $\tilde{b}$ (in $\tilde{C}^*_\Pi(Y)$) and put $||\beta||_\infty = \inf_a ||a||_\infty$. In particular, we may define in this way *the simplicial volume of an orbifold* V (see [47]) by taking the " universal covering " $Y = \tilde{V} \to V = \tilde{V}/\Gamma$ and by putting $||V|| = (||\beta||_\infty)^{-1}$. For instance, if $\tilde{V}$ has constant curvature -1 then $||V|| = R_n^{-1} \mathrm{Vol}(V)$, as in section (0.3).

Now, let G be the full group of isometries of the manifold Y and let $\mathscr{D}^*_G(Y)$ denote the de Rham complex of G-invariant forms on Y. By integrating forms over singular simplices in Y we get a homomorphism of $\mathscr{D}^*_G = \mathscr{D}^*_G(Y)$ into the complex $\tilde{C}^*_G(Y)$. Let $\tilde{\omega} \in \tilde{C}^*_G(Y)$ denote the image of a form $\omega \in \mathscr{D}^*_G$, and let us define $||\omega||_\infty$ as the infimum of ℓ^∞-norms of cocycles $\alpha \in C^*(Y : G)$ which are cohomologous (in $\tilde{C}^*_G(Y)$) to ω. Next we take a discrete subgroup $\Pi \subset G$ and for simplicity we assume that Π acts freely on Y. Then ω also defines a cohomology class in $H^*(Y/\Pi)$, called $[\omega]_\Pi$, and one has

$$||\ [\omega]_\Pi||_\infty \leq ||\omega||_\infty \qquad (*)$$

Assume furthermore that the subgroup Π is cocompact in G. Then every Π-invariant cocycle $\alpha \in C^*(Y : \Pi)$ can be averaged over G/Π to a G-invariant cocycle $\bar{\alpha} \in C^*(Y : G)$ with $||\bar{\alpha}||_\infty \leq ||\alpha||_\infty$, and thus the inequality (*) becomes an equality:

$$||[\omega]_\Pi||_\infty = ||\omega||_\infty \qquad (**)$$

In particular, $||[\omega]_\Pi||_\infty$ does not depend on Π as long as Π is cocompact in G.

If Y/Π is a compact manifold, then the equality (**), when applied to the volume form ω on Y, yields Thurston's proportionality theorem of section (0.3). Indeed, with this ω we have, by definition,

$$||V = Y/\Pi|| = \mathrm{Vol}(V)(||[\omega]_\Pi||_\infty)^{-1}.$$

(See [47] for a " dual " proof in the language of " smeared homology ".)

2.4. Smoothing of Borel cochains

Our main estimates from above for the simplicial volume are based on the following averaging (smoothing) construction, which is first explained in the geometric language of straight chains. Let, for example, V be an n-dimensional manifold of non-positive curvature which is triangulated into small straight simplices Δ. Then we continuously move the vertices of this triangulation, called $v_i \in V$, $i = 0, \ldots, k$, into new positions, $v_i' \in V$, and we observe that any such move $v_i \to v_i'$ uniquely extends to a map f of V into itself such that f sends every simplex Δ with some vertices v_i onto the straight simplex Δ' with vertices v_i'. The result of any move $v_i \to v_i' = f(v_i)$ can be obtained by going along a geodesic between v_i and v_i' and so the space F of maps $f : V \to V$ is parametrized by the Cartesian product of $k + 1$ copies of the universal covering $\tilde{V}$ of V.

Next we take the fundamental class $c = \Sigma\Delta \in C_n(V)$ of our triangulation and then we have a family of cycles $f_*(c) \in C_n(V)$ for all $f \in V$. Now, with some positive normalized (probability) measure μ on F we average this family to the generalized cycle $c_\mu = \int_F f_*(c)d\mu$. (Compare (2.2).) As all maps $f \in F$ are homotopic to the identity, the cycle c_μ is homologous to c, while its simplicial norm clearly satisfies $||c_\mu|| \leq ||c|| \int_F d\mu = ||c||$. In fact, the norm $||c_\mu||$ may even become *strictly* less than $||c||$. Indeed, a simplex Δ may be sent by two different maps f in F onto the same geometric simplex Δ' but with opposite orientations and these simplices algebraically cancel.

Example. — Let Δ be the oriented (!) unit interval in $\mathbf{R}^1$, $\Delta = [0, 1]$. Then the moving pairs of ends v_0' and v_1' are points in $\mathbf{R}^2$ and with the normalized Lebesgue measure μ in the square $\{v_0' \in [-\rho, \rho]$ and $v_1' \in [1 - \rho, 1 + \rho]\}$

$$c_\mu = \rho^{-2} \int_{-\rho}^{\rho} dv_0' \int_{1-\rho}^{1+\rho} dv_1' [v_0', v_1'].$$

This chain c_μ is understood as a measure on ordered (!) pairs of points $[v_0', v_1']$, and this measure *vanishes* at the pairs of points v_0' and v_1' in the interval $[1-\rho, \rho]$, since there are exactly two opposite moves of the interval $\Delta = [0, 1]$ onto $\Delta' = [v_0', v_1']$ with equal μ-probabilities. It follows, that indeed $||c_\mu|| < ||c = \Delta|| = 1$ for $\rho > \frac{1}{2}$, and furthermore, $||c_\mu|| \to 0$ as $\rho \to \infty$.

We shall return to the averaging (or diffusion) of chains in section (4.3), but now, we develop the more flexible language of smoothing of cochains in manifolds V. To do that we consider a covering $Y \to V$ with Galois group Π and we construct the following averaging (smoothing) in the subcomplex of *bounded* cochains in $C^*(Y:\Pi)$, called $\hat{C}^*(Y:\Pi)$. We denote by $\mathscr{M} = \mathscr{M}(Y)$ the Banach space of finite measures μ on Y with the norm $||\mu|| = \int_Y |\mu|$, and we denote by $\mathscr{M}^+ \subset \mathscr{M}$ the cone of positive measures. We observe that every bounded Borel m-cochain $c = c(y_0, \ldots, y_m)$ uniquely extends to an $(m+1)$-linear function on $\mathscr{M}$,

$$c(\mu_0, \ldots, \mu_m) = \underbrace{\int\int \ldots \int}_{m+1} c(y_0, \ldots, y_m)\mu_0(y_0) \ldots \mu_m(y_m).$$

This extended cochain is again denoted by c. We observe that

$$\sup_{||\mu_i|| \leq 1} |c(\mu_0, \ldots, \mu_i, \ldots, \mu_m)| = ||c||_\infty = \sup_{y_i \in Y} |c(y_0, \ldots, y_i, \ldots, y_m)|.$$

Let us call a smooth Π-invariant map $\mathscr{S} : Y \to \mathscr{M}^+$ a *smoothing operator.* We consider the induced cochains $\mathscr{S}^*(c)$ on Y for $c = c(\mu_0, \ldots, \mu_m)$ and we normalize them to $\mathscr{S} * c$ defined by

$$(\mathscr{S} * c)(y_0, \ldots, y_m) = \mathscr{S}^*(c)(y_0, \ldots, y_m) \Big/ \prod_{i=0}^{m} ||\mathscr{S}(y_i)||$$
$$= c(\mathscr{S}y_0, \ldots, \mathscr{S}y_m) \Big/ \prod_{i=0}^{m} ||\mathscr{S}(y_i)||.$$

Remark. — If we assign to each $y \in Y$ the Dirac δ-measure at y, then for this map $\delta : Y \to \mathscr{M}$, $\delta : y \mapsto \delta_y$, we take $\delta^*(c) = \delta * c = c$ for all $c \in C^*(Y:\Pi)$. However the map δ is not smooth and not even continuous relative to our norm in $\mathscr{M}$.

As the map $\mathscr{S} : Y \to \mathscr{M}^+$ is Π-invariant, the smoothing operator $c \to \mathscr{S} * c$ sends the complex $C^*(Y:\Pi)$ into itself. Furthermore this operator $C^*(Y:\Pi) \to C^*(Y:\Pi)$ is a chain homomorphism commuting with boundary operators, and due to our normalization this operator induces the identity homomorphism on the cohomology $H^*(Y:\Pi)$.

Now let V be a Riemannian manifold. Then for each $y \in Y$ we define the norm of the differential of $\mathscr{S}$,

$$||D_y\mathscr{S}|| = \sup_\tau ||(D_y\mathscr{S})(\tau)||,$$

where τ runs over the unit tangent sphere $S_y \subset T_y(Y)$. Then we put

$$[\mathscr{S}]_y = ||\mathscr{S}(y)||^{-1} ||D_y \mathscr{S}||$$

and

$$[\mathscr{S}] = \sup_{y \in Y} [\mathscr{S}_y].$$

Proposition. — *Let $\mathscr{S}$ be a smoothing with $[\mathscr{S}] < \infty$. Then any bounded cocycle $c \in C^m(Y : \Pi)$ is De Rham cohomologous to a closed Π-invariant form ω on Y for which*

$$\text{comass } \omega \leq m! \, ||c||_\infty ([\mathscr{S}])^m.$$

Proof. — First we take $\bar{c} = \mathscr{S} * c$ and then we define informally $\omega = \omega(t_1, \ldots, t_m)$ for tangent vectors $t_i \in T_y(Y)$, $y \in Y$, as the limit for $t \to 0$ of

$$m! \bar{c}(y, y + t_1, \ldots, y + t_m) \Big/ \prod_{i=1}^{m} ||t_i||.$$

We put the normalizing factor $m!$ to make the integral of ω over the infinitesimal simplex with vertices $y, y + t_1, \ldots, y + t_m$ equal to $\bar{c}(y, y + t_1, \ldots, y + t_m)$.

Now, to be precise, we first identify, by parallel translations in $\mathscr{M}$, the tangent spaces $T_\mu(\mathscr{M})$ for all $\mu \in \mathscr{M}$ with $T_0(\mathscr{M}) = \mathscr{M}$. Then we introduce the *differential* m-form $\tilde{c}$ on $\mathscr{M}$, whose value at the frame of tangent vectors $\mu_1, \ldots, \mu_m$ in $T_\mu(\mathscr{M}) = \mathscr{M}$ equals $m! \, c(\mu, \mu_1, \ldots, \mu_m)$ by definition. This form $\tilde{c}$ is uniquely characterized by the property that its integral over every linear simplex in $\mathscr{M}$ with vertices $\mu, \mu_1, \ldots, \mu_m$ in $\mathscr{M}$ equals $c(\mu, \mu_1, \ldots, \mu_m)$.

Observe, that comass $\tilde{c}$, that is the upper bound of the values of $\tilde{c}$ on the frames of unit vectors, does not exceed $m! \, ||c||$.

Now, with the map $\mathscr{S} : Y \to \mathscr{M}$ we take the induced form $\tilde{c}^* = \mathscr{S}^*(\tilde{c})$ and we define the required form ω by

$$\omega(t_1, \ldots, t_m) = \tilde{c}^*(t_1, \ldots, t_m) / ||\mathscr{S}(y)||^m$$

for $t_i \in T_y(Y)$, $y \in Y$.

Remark. — Since the form $\tilde{c}$ vanishes on the radial tangent field in $\mathscr{M}$ one could take first

$$\tilde{\mathscr{S}}(y) = ||\mathscr{S}(y)||^{-1} \mathscr{S}(y)$$

and then the form ω is induced by $\tilde{\mathscr{S}}$,

$$\omega = \tilde{\mathscr{S}}^*(\tilde{c}).$$

Let us slightly sharpen the Proposition for the important case of $m = n = \dim V$, by introducing a new quantity, $[\mathscr{S}]^* \leq [\mathscr{S}]$. Namely, we take the *average* of $||D_y \mathscr{S}(\tau)||$ over $\tau \in S_y \subset T_y(Y)$, called $||D_y \mathscr{S}||^*$, and then we put

$$[\mathscr{S}]^*_y = ||\mathscr{S}(y)||^{-1} ||D_y \mathscr{S}||^*$$

and

$$[\mathscr{S}]^* = \sup_{y \in Y} [\mathscr{S}]^*_y.$$

Example. — Let the smoothing $\mathscr{S}(y)$ be given by a function $\mathscr{S}(y,y')$, such that

$$\mathscr{S} * c(y') = \int_Y \mathscr{S}(y,y')c(y')dy',$$

for all functions $c(y')$, $y' \in Y$. Then

$$||\mathscr{S}(y)|| = \int_Y \mathscr{S}(y,y')dy',$$

and

$$||D_y\mathscr{S}|| \leq \int_Y ||\operatorname{grad}_y \mathscr{S}(y,y')||\, dy'.$$

Furthermore, for symmetric functions, $\mathscr{S}(y,y') = \mathscr{S}(y',y)$,

$$||D_y\mathscr{S}|| \leq \int_Y ||\operatorname{grad}_{y'} \mathscr{S}(y,y')||\, dy',$$

and

$$||D_y||^* = \int_Y dy' \int_{S_{y'}} |\langle \tau, \operatorname{grad}_{y'} \mathscr{S}(y,y')\rangle|\, d\tau,$$

where the interior integral of the scalar product is taken with the normalized measure in the sphere $S_{y'}$, and so

$$||D_y\mathscr{S}||^* = C_n \int_Y ||\operatorname{grad}_{y'} \mathscr{S}(y,y')||\, dy',$$

for $C_n = \Gamma(n/2)\Big/\left(\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)\right) < 1$, $n = \dim Y$.

The Integral Inequality. — *If* $m = n = \dim Y$, *then the form* ω *above satisfies*

$$\left|\int_U \omega\right| \leq n!\, ||c||_\infty \int_U ([\mathscr{S}]_y^*)^n\, dy \leq n!\, ||c||_\infty ([\mathscr{S}]^*)^n \operatorname{Vol}(Y/\Pi)$$

for all fundamental domains $U = U(\Pi) \subset Y$.

Proof. — Indeed, by the geometric arithmetic mean inequality,

$$\operatorname{comass}_y \mathscr{S}^*(\tilde{c}) \leq \operatorname{comass}_\mu \tilde{c}\,(||D_y\mathscr{S}||^*)^n,$$

for all n-forms $\tilde{c}$ on $\mathscr{M}$ and for all $\mu = \mathscr{S}(y)$.

Corollary.— Every smoothing $\mathscr{S}$ *in the universal covering* Y *of a closed Riemannian n-dimensional manifold* V *satisfies*

$$\int_U ([\mathscr{S}]_y^*)^n\, dy \geq ||V||/n! \qquad (*)$$

Remark. — If we introduce the quantity

$$[V]^* = \inf_{\mathscr{S}} \int_U ([\mathscr{S}]_y^*)^n\, dy,$$

for $\mathscr{S}$ running over all smoothing operators $\mathscr{S}$ in the universal covering $Y \to V$, then $(*)$ reads: $||V|| \leq n!\,[V]^*$. Notice that the " norm " $[V]^*$ is a conformal invariant of the Riemannian manifold V.

Let us sketch a geometric explanation of $(*)$. First observe that the cone $\mathscr{M}^+ \subset \mathscr{M}$ projects to a convex body P^+, call it a " simplex ", in the projective space

$P(\mathscr{M}) = [\mathscr{M}\setminus\{0\}]/\mathbf{R}^*$. Our comass in $\mathscr{M}$ induces a comass in P^+ and then we also have the induced comass in the quotient space P^+/Π as well as the dual mass on chains c in P^+/Π:

$$\text{mass } c = \sup_{\omega} \int_c \omega,$$

where ω runs over all forms of comass ≤ 1. Next, by assigning Dirac δ_y-measures to the points $y \in Y$ we get a canonical map $\delta : V = Y/\Pi \to P^+/\Pi$. Now, one can show that $\text{mass}(\delta_*[V]) = ||V||/n!$, where $\delta_*[V] \in H_n(P^+/\Pi)$ denotes the image of the fundamental class of V whose mass is defined as the lower bound of masses of the cycles in $C_n(P^+/\Pi)$ which are homologous to $\delta_*[V]$. On the other hand, the " norm " $[V]^*$ is the lower bound

$$[V]^* = \inf_{\mathscr{S}} \int_S ||(D\mathscr{S})(\tau)||^n \, d\tau.$$

Here $S = S(V)$ denotes the unit tangent bundle of V and $\mathscr{S}$ runs over the smooth maps $V \to P^+/\Pi$ homotopic to δ.

Observe that one could equally well use the space of measures on the group Π, rather than the space $\mathscr{M}$ of measures on Y. Another interesting space of measures lives on the Fürstenberg boundary of Π (see [14]) and it would be nice to find cycles of least mass in the corresponding space P^+/Π.

2.5. Growth functions and Ricci curvature

Let V be a complete Riemannian manifold and let $Y \to V$ be the universal covering of V. For each point $v \in V$ we take a point $y \in Y$ over v and then we consider balls $B_y(R) \subset Y$. Put

$$\ell_v(R) = \log \text{Vol } B_y(R), \quad \text{for } R \in [0, \infty),$$

and

$$\ell'_v(R) = \frac{d\ell_v(R)}{dR} = \text{Vol } \partial B_y(R) \,/\, \text{Vol } B_y(R).$$

Warning. — In some exceptionally "irregular" cases the topological boundary $\partial B_y(R)$ does not coincide with the sphere $S_y(R)$ and then the derivative ℓ' is not well defined. To avoid any ambiguity we can use the all-purpose definition

$$\ell'_v(R) \overset{\text{def}}{=} \limsup_{\substack{w \to v \\ \varepsilon \to 0}} \varepsilon^{-1}(\ell_w(R) - \ell_w(R - \varepsilon)).$$

Finally, let $\ell' = \ell'(V) = \inf_{R>0} \sup_{v \in V} \ell'_v(R)$.

Theorem. — *Every cohomology class* $\beta \in H^m(V)$ *satisfies*

$$\text{comass } \beta \leq m! \, (\ell')^m \, ||\beta||_\infty \qquad (*)$$

Therefore

$$||\gamma|| \leq m! \, (\ell')^m \, \text{mass } \gamma, \quad \textit{for any } \gamma \in H_m(V) \qquad (**)$$

In particular, for closed n-dimensional manifolds V,

$$||V|| \leq n!\,(\ell')^n \operatorname{Vol}(V).$$

Moreover, for $C_n = \Gamma(n/2)/\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right) < 1$,

$$||V|| \leq C_n n!\,(\ell')^n \operatorname{Vol}(V). \qquad (***)$$

Proof. — First we consider the smoothing $\mathscr{S} = \mathscr{S}_R$ which assigns to $y \in Y$ the Riemannian measure dy' in the ball $B_y(R) \subset Y$:

$$\mathscr{S}(y,y') = \begin{cases} 1 & \text{for } \operatorname{dist}(y,y') \leq R \\ 0 & \text{for } \operatorname{dist}(y,y') > R. \end{cases}$$

Then, in the " regular " case we have

$$||\mathscr{S}_R(y)|| = \operatorname{Vol} B_y(R),$$

$$||D_y\mathscr{S}_R||^* = C_n \operatorname{Vol} \partial B_y(R),$$

and $$||D_y\mathscr{S}_R|| \leq \operatorname{Vol} \partial B_y(R).$$

Therefore $$\inf_{R>0} [\mathscr{S}_R]^* = C_n\ell',$$

$$\inf_{R>0} [\mathscr{S}_R]^* \leq \ell',$$

and the results of the previous section apply.

To handle possible " irregularities " we slightly refine the definition of the smoothing $\mathscr{S}_R$. Namely we take a small positive function $\varphi : V \to (0, R) \subset \mathbf{R}$ and eventually we send this φ to zero in the fine C^1-topology. (Recall that fundamental neighbourhoods of zero in this topology $U = U_\varepsilon$, are sets of functions φ for which $|\varphi(v)| \leq \varepsilon(v)$ and $||\operatorname{grad} \varphi(v)|| \leq \varepsilon(v)$ for all positive functions $\varepsilon = \varepsilon(v)$ on V.) Then we define $\mathscr{S}_{R,\varphi}$ by the following averaging

$$\mathscr{S}_{R,\varphi}(y) = (\varphi(v))^{-1} \int_{R-\varphi(v)}^{R} \mathscr{S}_{R'}(y)\, dR',$$

for the points $y \in Y \to V$ over all points $v \in V$. As $\varphi \to 0$ we have

$$\limsup_{\varphi \to 0} [\mathscr{S}_{R,\varphi}]^* \leq C_n\ell',$$

and $$\limsup_{\varphi \to 0} [\mathscr{S}_{R,\varphi}] \leq \ell', \quad \text{q.e.d.}$$

Corollary (An estimate for $\operatorname{Ricci} \geq -k^2$*).* — *If* $\operatorname{Ricci} V \geq -1/(n-1)$ *then*

$$\operatorname{comass}(\beta) \leq m!\,||\beta||_\infty \qquad (+)$$

$$||\gamma|| \leq m!\,\operatorname{mass}(\gamma) \qquad (++)$$

$$||V|| \leq C_n n!\,\operatorname{Vol}(V). \qquad (+++)$$

Proof. — Bishop's inequality [5] for $\mathrm{Ricci} \geq -1/(n-1)$ implies $\limsup_{R\to\infty} \ell'(R) \leq 1$.

This corollary yields all estimates of section (0.5) for closed manifolds V with the exception of the isolation theorem. Moreover, we have the following generalization of the product inequality of (0.5).

If $|K(V)| \leq 1$, *then an arbitrary product* p *of Pontryagin classes of* V *satisfies*

$$(p \cup \beta)[V] \leq \mathrm{const}_n \, ||\beta||_\infty \, \mathrm{Vol}(V),$$

where β *is an arbitrary cohomology class.*

Indeed, this follows from $(p \cup \beta)[V] \leq \mathrm{const}'_n(\mathrm{comass}\, p)(\mathrm{comass}\, \beta)\, \mathrm{Vol}\, V$ with the estimate by Chern-Weil for $|K| \leq 1$,

$$\mathrm{comass}\, p \leq C'_n;$$

and for $\mathrm{Ricci} \geq 1 - n$,

$$\mathrm{comass}\, \beta \leq (n-1)^n n! \, ||\beta||_\infty,$$

by (+) above.

Now, we can prove a result for *open* manifolds, namely the asymptotic estimate of (0.5). To see this, observe that, for increasing R, the boundary of the ball $B(R) \subset V$ will eventually support the cycles homologous to $\partial_1 W$, $\partial_2 W$, and so on. Therefore (++) above yields

$$\liminf_{R\to\infty} \mathrm{Vol}(\partial B(R)) \geq (||\partial_1 W|| + ||\partial_2 W|| + \ldots)/(n-1)!,$$

while asymptotically for $R \to \infty$,

$$\mathrm{Vol}(B(R)) = \int_0^R \mathrm{Vol}(\partial B(R))\, dR \sim R\, \mathrm{Vol}(\partial B(R)).$$

Estimates with the entropy. — Let the manifold V be compact. Then the limit $\lim_{R\to\infty} R^{-1}\ell_v(R)$ exists. It is called *entropy* (Ent V), and does not depend on $v \in V$. One can modify the theorem by substituting this entropy for ℓ' in the inequalities (*)-(***). To see this, fix a number $\lambda > \mathrm{Ent}\, V$ and use the following smoothing $\mathscr{S} = \mathscr{S}_{R,\lambda}$:

$$\mathscr{S}(y,y') = \begin{cases} \exp(-\lambda\, \mathrm{dist}(y,y')) - \exp(-\lambda R), & \text{for } \mathrm{dist} \leq R, \\ 0, & \text{for } \mathrm{dist}(y,y') > R. \end{cases}$$

If $R \to \infty$, then $[\mathscr{S}_{R,\lambda}]^* \to C_n\lambda$ and also the norm $[\mathscr{S}]$ is asymptotically bounded by λ. Hence, the inequalities (*)-(***) hold with any $\lambda > \mathrm{Ent}\, V$ in place of ℓ' and so with $\lambda = \mathrm{Ent}\, V$ as well.

Recall that Ent V bounds from below *the topological entropy of the geodesic flow of* V (see [12], [24], [37]) and so the modified inequality (***) implies

$$(\mathrm{Top\ Ent}\, V)^n \geq ||V||/C_n n!\, \mathrm{Vol}(V).$$

This inequality for $n = 2$ with the *sharp* constant $1/2\pi < C_2 = 2/\pi$ is due to Katok [36].

Finally, if V has negative curvature, then $\ell'_v(R) \to \mathrm{Ent}\, V = \mathrm{Top\ Ent}\, V$ as $R \to \infty$ (see [38], [37]) and so the modified version of the theorem follows from the original one.

3. BOUNDED COHOMOLOGY OF SIMPLICIAL MULTICOMPLEXES

3.0. We denote by $\hat{C}^* = \hat{C}^*(X) \subset C^*(X)$ the complex of bounded real-valued singular cochains of the space X. A continuous map $f: X \to Y$ induces a homomorphism $\hat{f}: \hat{C}^*(Y) \to \hat{C}^*(X)$ which is bounded relative to the ℓ^∞-norms. Indeed f is even bounded by *one* as $||\hat{f}(c)||_\infty \leq ||c||_\infty$ for all $c \in C^*(Y)$.

We denote by $\hat{H}^*(X)$ the homology of the complex $\hat{C}^*(X)$. Observe that the space $\hat{H}^*(X)$ carries a natural pseudo-norm which is also denoted by $||\ ||_\infty$. (" Pseudo " means that for a non-zero $\alpha \in \hat{H}^*(X)$ one can have $||\alpha||_\infty = 0$. This might happen when the image of the coboundary operator $\delta: \hat{C}^*(X) \to \hat{C}^*(X)$ is not closed. I do not know whether it actually occurs.)

A homotopy between two maps $f, g: X \to Y$ provides a chain-homotopy h between $\hat{f}$ and $\hat{g}$. The standard construction (see [40] for instance) gives an h which is bounded in each dimension relative to the ℓ^∞-norm, and hence the homomorphisms $\hat{f}^*$ and $\hat{g}^*: \hat{H}^*(Y) \to \hat{H}^*(X)$ are equal. In particular, $\hat{H}^*(X)$ depends only on the homotopy type of X. We shall see below that $\hat{H}^*(X)$ depends only on the fundamental group $\pi_1(X)$.

Observe, that $\hat{H}^1(X)$ is always zero. This is clear, because each real-valued 1-cocycle z determines a homomorphism $\pi_1(X) \to \mathbf{R}$ (we assume X to be path-connected). When z is bounded this homomorphism is also bounded, and hence trivial. It follows that z is the coboundary of a 0-cocycle.

The results of Milnor-Sullivan and Thurston provide many examples of non-trivial groups $\hat{H}^*$. In fact, Thurston's theorem says that the homomorphisms $\hat{H}^m(X) \to H^m(X, \mathbf{R})$ are surjective for $m \geq 2$ if X is a closed manifold of negative curvature. These homomorphisms are in general not injective. We shall see below for example that the groups $\hat{H}^2$ and $\hat{H}^3$ of the infinite wedge of circles do not vanish.

Bounded cohomology first appeared in the group theoretic context. I learned this notion from Phillip Trauber who explained to me his (unpublished) version of the Theorem of Hirsch-Thurston [33]:

If a group Π *is amenable then the bounded cohomology vanishes*:

$$\hat{H}^i(\Pi) \overset{\text{def}}{=} \hat{H}^i(K(\Pi, 1)) = 0 \quad \text{for } i > 0.$$

Recall (see [19]) that a group Π is called *amenable* if its action on the space of bounded functions $\Pi \to \mathbf{R}$ has a left invariant mean (average), that is a Π-invariant projection A of norm one from the space $L^\infty(\Pi)$ onto the (one-dimensional) subspace of constant functions.

Examples (see [19]). — Abelian groups are amenable. Finite groups are amenable. If a normal subgroup $\Gamma \subset \Pi$ as well as the quotient group Π/Γ is amenable, then Π is itself amenable.

Unions of increasing families of amenable groups are amenable. In particular, if Π is locally solvable (*i.e.* all finitely generated subgroups in Π are solvable) then it is amenable.

It is unknown whether all (discrete) amenable groups are built out of Abelian and finite group by taking extensions and " increasing unions ". Also notice, that subgroups and factor groups of amenable groups are amenable.

The simplest examples of non-amenable groups are free non-abelian groups. All known *finitely presented* non-amenable groups contain free non-abelian subgroups.

If Π is the fundamental group of a closed Riemannian manifold V, then non-amenability can be expressed geometrically in terms of the universal covering $\widetilde{V} \to V$, by requiring all bounded domains $\Omega \subset \widetilde{V}$ to satisfy the inequality $\mathrm{Vol}(\Omega) \leq \mathrm{const}\, \mathrm{Vol}(\partial\Omega)$ for some positive " const " $= \mathrm{const}(V)$. According to Avez (see [2], [21]) this inequality holds, for example, if V is a non-flat manifold of non-positive curvature, $K(V) \leq 0$, and then the fundamental group $\Pi = \pi_1(V)$ is non amenable. It is unknown whether every such group Π contains a free non-abelian subgroup.

One can also express the amenability of $\pi_1(V)$ in terms of the smoothing operators $\mathscr{S}$ of section (2.4): the group $\Pi = \pi_1(V)$ is amenable if and only if for every $\varepsilon > 0$ there exists a Π-invariant smoothing $\mathscr{S} : \widetilde{V} \to \mathscr{M}(\widetilde{V})$ for which $[\mathscr{S}] \leq \varepsilon$. The existence of such operators $\mathscr{S}$ implies Trauber's vanishing theorem, but the original argument of Trauber is shorter and also yields the following more general fact.

Let $f : Y \to X$ be a regular covering with an amenable Galois group Π. Then the induced map $\hat{f}^* : \hat{H}^*(X) \to \hat{H}^*(Y)$ is injective and isometric relative to the norm $\| \ \|_\infty$.

Proof. — The Galois group Π acts on Y and thus on $\hat{C}^*(Y)$. Then the complex $\hat{C}^*(X)$ can be identified with the complex of Π-invariant cochains in $\hat{C}^*(Y)$. Fix an averaging A on Π and consider the corresponding averaging in $\hat{C}^*(Y)$, a Π-invariant projection $\hat{C}^*(Y) \to \hat{C}^*(X)$. This projection, call it $\hat{A}$, commutes with differentials and satisfies $\hat{A} \circ \hat{f} = \mathrm{Id}$, where $\hat{f} : C^*(X) \to C^*(Y)$ is the cochain homomorphism induced by f. Now, by the very definition of the averaging, we have $\|\hat{A}\| = 1$ and so $\|\hat{A}^*\| \leq 1$ for $\hat{A}^* : \hat{H}^*(Y) \to H^*(X)$. Then with the identity $\hat{A}^* \circ \hat{f}^* = (\hat{A} \circ \hat{f})^* = \mathrm{Id}$, we get $\|\hat{f}^*(\alpha)\|_\infty \geq \|\alpha\|_\infty$ for all $\alpha \in H^*(X)$ and since also $\|\hat{f}^*\| \leq \|\hat{f}\| = 1$, the map $\hat{f}^*$ is isometric as well as injective.

Examples of non-vanishing $\hat{H}^i$. — Let X be a closed surface of genus ≥ 2 and let $f : Y \to X$ be an infinite Abelian covering. Since X supports a metric of negative curvature, the fundamental class α of X is bounded, $\alpha \in \hat{H}^2(X)$, and since Abelian groups are amenable, the pullback $f^*(\alpha) \in \hat{H}^2(Y)$ is non-zero. Notice that Y is homotopy equivalent to an infinite wedge of circles.

Also the wedge of two circles has a non-trivial group $\hat{H}^2$. In fact, this group $\hat{H}^2(S^1 \vee S^1)$ is infinitely generated (see [6]). Here is a geometric construction of a non-trivial bounded 2-cocycle. Take a complete surface V of constant negative curvature such that $\mathrm{Vol}(V) = 2\pi < \infty$ and $\pi_1(V) = \mathbf{Z} * \mathbf{Z}$. Let ω be the volume form and let the cocycle $z \in C^2(V)$ assign to each simplex $\sigma : \Delta^2 \to V$ the integral of ω over straight(σ), as we did in (C) of (1.2). Then we have $V = \mathrm{Int}\,\overline{V}$ where the boundary $\partial\overline{V}$ has self mappings of all degrees and for the ℓ^1-cycle $\tilde{c}$ constructed in section (0.2) we observe that $z(\tilde{c}) \neq 0$. Therefore z is not the coboundary of a *bounded* cochain, and $\tilde{c}$ is not the boundary of an ℓ^1-chain.

Jörgensen [34] constructed a closed 3-dimensional manifold V of constant negative curvature which admits an infinite cyclic covering $\tilde{V} \to V$ such that $\tilde{V}$ is homotopy equivalent to a surface. The pull-back of the fundamental class of V gives a non-trivial element in $\hat{H}^3(\tilde{V})$ and thus a non-trivial element for the infinite wedge of circles.

3.1. Vanishing theorems for bounded cohomology

Trauber's theorem implies the vanishing of the bounded cohomology of $K(\Pi, 1)$ spaces with amenable groups Π. In particular, the simplicial volume of solv-manifolds is zero. By using another averaging procedure in simplicial models of arbitrary spaces X we prove in section (3.3) the following generalization:

The Mapping theorem. — *Let* $f : X_1 \to X_2$ *be a continuous map such that the induced homomorphism* $f_* : \pi_1(X_1) \to \pi_1(X_2)$ *is a surjection with an amenable kernel. Then the homomorphism* $\hat{f}^* : \hat{H}^*(X_2) \to \hat{H}^*(X_1)$ *is an isometric isomorphism.*

Corollaries. — (A) *If* f_* *is an isomorphism then* $\hat{f}^*$ *is also an isomorphism. In particular, if* $f : X \to K(\Pi, 1)$ *is the classifying map for* $\Pi = \pi_1(X)$, *then* $\hat{f}^*$ *is an isometric isomorphism.*

(B) *Let* X *be a closed oriented manifold with fundamental class* $[X]$ *and let* $f : X \to K(\Pi, 1)$, $\Pi = \pi_1(X)$, *denote the classifying map. Then* $\|X\| = \|f_*[X]\|$.

(C) *If the fundamental group of the manifold* X *above is amenable, for example if* $\pi_1(X) = 0$, *then* $\|X\| = 0$.

Observe that (A) $\Rightarrow$ (B) $\Rightarrow$ (C).

The assertion (C) will be generalized in section (3.3) as follows. First, a subset $Y \overset{i}{\subset} X$ is called " amenable " if for every path-connected component Y' of Y the image of the inclusion homomorphism $i_* : \pi_1(Y') \to \pi_1(X)$ is an amenable subgroup of $\pi_1(X)$.

The Vanishing theorem. — *If a manifold* X *can be covered by some open amenable subsets such that every point* $x \in X$ *is contained in no more than m subsets, then the homomorphism* $\hat{H}^i(X) \to H^i(X)$ *vanishes for* $i \geq m$.

Corollaries. — (1) *If a closed manifold* X *can be covered by* $m \leq \dim X$ *" amenable " open sets then* $||X|| = 0$.

(2) *If the above manifold* X *can be mapped into a manifold* Y *with* $\dim Y < \dim X$ *such that the pullback of every point in* Y *has an " amenable " neighbourhood in* X, *then* $||X|| = 0$.

Indeed, one can cover Y by $1 + \dim Y$ open subsets, such that each of them is a union of small disjoint subsets. Then the previous corollary applies to the pullback of this covering.

(3) (Koichi Yano [49]). *If* X *admits a nontrivial circle action then* $||X|| = 0$.

Indeed, (2) above applies to the quotient map $X \to X/S^1$.

Remark. — If the action is free, then the mapping theorem also yields $||X|| = 0$. Furthermore, if the action is locally free, then not only $||X|| = 0$ but also $\operatorname{Min} \operatorname{Vol} ||X|| = 0$ (see Appendix 2). Finally, if the action is not locally free, then the classifying map $X \to K(\Pi, 1)$ sends the fundamental class $[X]$ to zero (see Appendix 2) and, again by the mapping theorem, $||X|| = 0$.

3.2 Simplicial multicomplexes and the isometry $\hat{H}^* \to \hat{H}^*_s$

A *simplicial multicomplex* (for short a multicomplex) is defined as a set K divided into the union of closed affine simplices $\Delta_i \subset K$, $i \in I$, such that the intersection of any two simplices $\Delta_i \cap \Delta_j$ is a (simplicial) subcomplex in Δ_i as well as in Δ_j. The set K with the weakest topology which agrees with the decomposition $K = \bigcup_i \Delta_i$ is denoted by $|K|$. The union of all m-dimensional simplices in K is called the *m-skeleton* of K and denoted by $K^m \subset K$. A map between two multicomplexes, $K = \bigcup_i \Delta_i \to L = \bigcup_j \Delta_j$, is called *simplicial* if it maps each Δ_i linearly onto some Δ_j.

Examples. — *(a)* Every simplicial complex is also a multicomplex. The simplest multicomplex which is not a simplicial complex consists of two one dimensional simplices, $K = \Delta^1_1 \cup \Delta^1_2$, which intersect over their common boundary, $\partial\Delta^1_1 = \partial\Delta^1_2 = \Delta^1_1 \cap \Delta^1_2$. (See fig. 1.)

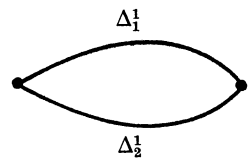

FIG. 1

Also observe that the first barycentric subdivision of any multicomplex is an ordinary simplicial complex.

(b) Every simplicial multicomplex $K = \bigcup_{m=0}^{\infty} K^m$ can be built inductively starting from the discrete set K^0, and then each K^m is obtained from K^{m-1} by attaching m-simplices Δ_ν^m by some simplicial embeddings of the boundaries $\partial\Delta_\nu^m \to K^{m-1}$.

(c) Our important example is the following. Take a topological space X and consider the set Σ of all those singular simplices $\sigma : \Delta^m \to X$, $m = 0, 1, \ldots$, which are injective on the vertices of Δ^m. Then take one copy of Δ^m for each σ, call it Δ_σ^m, and put $K(X) = \bigcup_{\sigma \in \Sigma} \Delta_\sigma^m$. This union has in a natural way the structure of a multicomplex such that the canonical map $S : |K(X)| \to X$ defined by the condition

$$S \mid \Delta_\sigma^m = \sigma : \Delta_\sigma^m = \Delta^m \to X, \quad \text{for all } \sigma \in \Sigma,$$

is continuous. Moreover, by a standard argument (see [40]), this map S is a weak homotopy equivalence.

A multicomplex K is called *complete* if every continuous map $f : \Delta^m \to K$ whose restriction to the boundary $f \mid \partial\Delta^m : \partial\Delta^m \to K$ is a simplicial embedding, is homotopic, relative to $\partial\Delta^m$, to a simplicial embedding $f' : \Delta^m \to K$.

Examples. — *(a)* The complex K(X) is complete.

(b) If K is complete and connected then for any finite set of vertices $\{k_0, \ldots, k_\ell\} \subset K^0$ there is an ℓ-dimensional simplex $\Delta^\ell \subset K$ with vertices $k_0, \ldots, k_\ell$.

(c) If a connected 1-dimensional complex is complete then it consists of a single 1-simplex.

(d) If a connected simplicial complex K is complete as a multicomplex then it equals the simplex spanned by the vertices of K.

The role of the completeness is explained by the following simple fact:

(e) Let K *be a complete multicomplex with at least* $n + 1$ *vertices in every connected component. Let f be a continuous map of an n-dimensional multicomplex* L *into* K. *Then there is a simplicial map f′ of the first barycentric subdivision of* L *into* K, *homotopic to f and injective on every simplex of the subdivision of* L.

Call K *large* if every component has infinitely many vertices. Then we state the following relative version of *(e)*:

Homotopy Lemma. — *Let* K *be large and complete. Then for any two homotopic simplicial maps* $f_0, f_1 : L \to K$, *both injective on each simplex in* L, *there exists a simplicial map f of a canonically subdivided cylinder* $L \times [0, 1]$ *into* K *such that* $f_{t=0} = f_0$ *and* $f_{t=1} = f_1$, *and such that f is injective on each simplex of the subdivision.*

The completeness property is reminiscent of Kan's property in the theory of semi-simplicial sets. Since multicomplexes have no degenerate simplices, they are less convenient than semi-simplicial sets from the algebraic point of view, but they are better adapted to our geometric applications. In any case the standard techniques of semi-simplicial

sets (see [40]) apply to multicomplexes with minor changes. Our exposition is essentially independent of [40] as we only need simple facts which can be easily proved directly. For example we consider simplicial m-cochains on K. They are, by definition, antisymmetric functions of oriented m-simplices of K. We take only *bounded* functions and get the *complex of bounded antisymmetric cochains*, called $\hat{C}_a^*(K)$. We denote by $\hat{H}_a^*(K)$ the homology of $\hat{C}_a^*(K)$ with the ℓ^∞-norm. If the complex K and the maps $f_0, f_1 : L \to K$ are as in the homotopy lemma, then the cylinder $f : L \times [0, 1] \to K$ provides a chain homotopy equivalence between the induced maps f_0^* and $f_1^* : \hat{C}_a^*(K) \to \hat{C}_a^*(L)$. This chain homotopy equivalence $\{\hat{C}_a^i(L) \to \hat{C}_a^{i-1}(K)\}_{i=0,1,\ldots}$ is bounded in every dimension i. Therefore the corresponding homomorphisms on homology, $\hat{f}_0^*$ and $\hat{f}_1^*$ from $H_a^*(L)$ to $H_a^*(K)$, are equal.

Next we identify $\hat{H}_a^*(K)$ with the bounded *singular* cohomology $\hat{H}^*(|K|)$. To do that we start with a natural homomorphism $h : \hat{C}^*(|K|) \to \hat{C}_a^*(K)$, defined as follows. For each $c \in \hat{C}^i(|K|)$ and for every oriented i-simplex $\Delta \subset K$ we consider all affine isomorphisms δ of the standard i-simplex Δ^i onto Δ. Then we define $c' = h(c)$ by the formula

$$c'(\Delta) = \frac{1}{(i+1)!} \sum_{\delta} [\delta] c(\delta),$$

where we sum over the affine maps $\delta : \Delta^i \to \Delta \subset K$ with $[\delta] = 1$ for the orientation preserving maps and with $[\delta] = -1$ for the others. The desired identification is given in the

Isometry Lemma. — For a large complete complex K *the induced homomorphism on the bounded cohomology,*

$$h^* : \hat{H}^*(|K|) \to \hat{H}_a^*(K),$$

is an isometric isomorphism.

Proof. — Fix an integer $N \geq 0$, take the standard simplex Δ^N, and let $\sigma_\mu^i : \Delta^i \to \Delta^N$ denote the isomorphisms of the standard simplex Δ^i onto the i-faces of Δ^N, for $i = 0, \ldots, N$ and $\mu = 1, \ldots, \frac{(N+1)!}{(N-i)!}$. We consider all those singular simplices $\Delta^i \to |K| \times \Delta^N$ which project to isomorphisms σ_μ^i. These singular simplices $\Delta^i \to |K| \times \Delta^N$ form in a natural way an N-dimensional multicomplex, called $\tilde{K}_N$. Notice that each i-dimensional simplex Δ in $\tilde{K}_N$ is *canonically* isomorphic to Δ^i and thus all $\Delta \subset \tilde{K}_N$ come with canonical orientations.

Now let us construct a chain homomorphism $A : \hat{C}^{i \leq N}(\tilde{K}_N) \to \hat{C}_a^{i \leq N}(|K|)$ by first considering for every singular simplex $\sigma : \Delta^i \to |K|$ the (oriented!) simplices $\Delta_\mu \subset \tilde{K}_N$, $\mu = 1, \ldots, M = \frac{(N+1)!}{(N-i)!}$, which lie in $|K| \times \Delta^N$ over σ.

Then for $c \in \hat{C}^i_a(\widetilde{K}_N)$ we define $c' = A(c)$ by the formula

$$c'(\sigma) = M^{-1} \sum_{\mu=1}^{M} c(\Delta_\mu).$$

Next we take the natural map $f: |\widetilde{K}^N| \to |K|$. With our assumptions on K we can replace it by a simplicial map $\widetilde{f}: \widetilde{K}_N \to K$ which is homotopic to f and injective on all simplices Δ in $\widetilde{K}_N$.

We have now two maps:

$$h: \hat{C}^*(|K|) \to \hat{C}^*_a(K)$$

and

$$g = A \circ \widetilde{f}^*: \hat{C}^{i \leq N}_a(K) \to \hat{C}^{i \leq N}(|K|),$$

both chain homomorphisms of ℓ^∞-norm ≤ 1. The homotopy lemma implies that the composed maps $h \circ g$ and $g \circ h$ are chain homotopic to the identity (in dimensions below N), and so the homomorphism h^* is an isomorphic isometry.

An application: the abstract version of the theorem of Milnor-Sullivan-Smillie

In order to estimate the norm of a characteristic class $\beta \in H^*(X)$, we may pass to the multicomplex $K = K(X)$. Then we only need the geometric version of the theorem, but now for the multicomplex K rather than for the simplicial complex P as in (1.3). The arguments of (1.3) immediately apply to all countable multicomplexes (we need " countable " to have a good notion of *generic* piecewise linear sections). The uncountable case however only requires the following simple lemma:

If, for every finite subcomplex K' *of* K, *a class* $\beta \in H^*_a(K)$ *can be represented by a cocycle* $c \in H_a(K)$ *which is bounded on* K' *by a given constant* b, *then* $||\beta||_\infty \leq b$.

3.3. Minimal multicomplexes and their automorphisms

A simplicial multicomplex K is called *minimal* (compare [40]) if each continuous map of a simplex Δ into K whose restriction to the boundary is a simplicial embedding is homotopic relative to the boundary $\partial\Delta$ to at most one simplicial embedding $\Delta \to K$.

(A) *Lemma.* — *Let a multicomplex* K *be large and complete. Then there is a subcomplex* $\widetilde{K}$ *in* K *which is complete, minimal and such that the inclusion* $\widetilde{K} \subset K$ *is a homotopy equivalence. Furthermore, one can take* $\widetilde{K}$ *with countably many vertices in every connected component. With this last property the complex* K *is uniquely determined, up to simplicial isomorphism, by the homotopy type of* K.

Proof. — Two i-dimensional simplices Δ_1 and $\Delta_2 \neq \Delta_1$ in K are called *homotopic* if they have a common geometric boundary, $\partial\Delta_1 = \partial\Delta_2 = \Delta_1 \cap \Delta_2$, and if the sphere $S^i = \Delta_1 \cup \Delta_2 \subset K$ is homotopic to zero.

Now, take countably many vertices in each component of K and denote this set of vertices by $\widetilde{K}^\circ \subset K^\circ \subset K$. Then define $\widetilde{K}$ inductively, starting from the zero skeleton $\widetilde{K}^\circ$, and then taking for $\widetilde{K}^i$ a maximal system of pairwise non-homotopic simplices in K with boundaries in $\widetilde{K}^{i-1}$.

To prove uniqueness, let $\widetilde{K}_1$ and $\widetilde{K}_2$ be two minimal complete complexes and let $f: \widetilde{K}_1 \to \widetilde{K}_2$ be a homotopy equivalence which is bijective on the zero skeletons, $\widetilde{K}_1^\circ \xrightarrow{\sim} \widetilde{K}_2^\circ$. Since $\widetilde{K}_2$ is complete there is a simplicial map $\widetilde{f}: \widetilde{K}_1 \to \widetilde{K}_2$ which is homotopic to f and equal to f on $\widetilde{K}_1^\circ$. Now, the uniqueness of K follows from the following

Sublemma. — If a simplicial map $f: \widetilde{K}_1 \to \widetilde{K}_2$ is a homotopy equivalence and if it is bijective on the zero skeletons, then $\widetilde{f}$ is bijective.

Proof. — Let us assume, by induction, that $\widetilde{f}$ is bijective on the $(i-1)$-skeletons and let us first show that $\widetilde{f}$ is injective on the i-skeleton of $\widetilde{K}_1$. Indeed, if two i-simplices, say Δ_1 and Δ_2 in $\widetilde{K}_1^i \subset \widetilde{K}_1$, are sent onto the same simplex in $\widetilde{K}_2^i \subset \widetilde{K}_2$, then they have a common boundary, $\partial\Delta_1 = \partial\Delta_2 = \Delta_1 \cap \Delta_2$. The map $\widetilde{f}$ of the sphere $S^i = \Delta_1 \cup \Delta_2 \subset \widetilde{K}_1$ into $\widetilde{K}_2$ is contractible. Since $\widetilde{f}$ is a homotopy equivalence, the sphere $S^i \subset \widetilde{K}_1$ is also contractible and so the simplices Δ_1 and Δ_2 are homotopic. Then Δ_1 and Δ_2 coincide by the minimality of $\widetilde{K}_1$.

To prove the surjectivity of $\widetilde{f}$, we take an i-dimensional simplex $\Delta \subset \widetilde{K}_2^i$ with boundary $S^{i-1} = \partial\Delta \subset \widetilde{K}_2^{i-1}$ and we consider $\widetilde{S}^{i-1} = \widetilde{f}^{-1}(S^{i-1}) \subset \widetilde{K}_1^{i-1}$. Since the sphere S^{i-1} is contractible in $\widetilde{K}_2$, the sphere $\widetilde{S}^{i-1} \subset \widetilde{K}_1$ is also contractible, and by the completeness of $\widetilde{K}_1$ we have a simplex $\Delta \subset \widetilde{K}_1^i$ with $\partial\widetilde{\Delta} = \widetilde{S}^{i-1}$. Then we consider the sphere $S^i = \widetilde{f}(\widetilde{\Delta}) \cup \Delta \subset \widetilde{K}_2$. Since f is a homotopy equivalence and as $\widetilde{K}_1$ is complete, there is a simplex $\Delta' \subset \widetilde{K}_1^i$ with boundary $\widetilde{S}^{i-1}$, such that the sphere $\widetilde{f}(\widetilde{\Delta} \cup \Delta') \subset \widetilde{K}_2$ is homotopic to S^i relative to $\widetilde{\Delta}$. Finally, by the minimality of $\widetilde{K}_2$ we conclude that $\widetilde{f}(\Delta') = \Delta$, q.e.d.

Corollary. — Any space X is weakly homotopy equivalent to a large complete minimal multicomplex.

Indeed, take $\widetilde{K}$ in the multicomplex K(X) defined in the previous section.

Now, denote by $\Gamma = \Gamma(K)$ the group of those simplicial automorphisms of a multicomplex K which are homotopic to the identity. Let $\Gamma_i \subset \Gamma$ be the subgroup which keeps the i-skeleton of K pointwise fixed. Observe that each Γ_i is a *normal* subgroup in Γ.

Take an $(i+1)$-dimensional simplex $\Delta_0 \subset K^{i+1}$ and denote by $\pi(\Delta_0)$ the set of all $(i+1)$-simplices $\Delta \subset K$ for which $\partial\Delta = \partial\Delta_0$.

(B) *Lemma. — If K is a complete minimal multicomplex then the group Γ_i is transitive on every set $\pi(\Delta_0)$.*

Proof. — Take any $(i+1)$-simplex $\Delta \subset \pi(\Delta_0)$ and a linear isomorphism $f_0 : \Delta_0 \to \Delta$ which keeps fixed the vertices of Δ_0. Then, while keeping fixed the i-skeleton K^i, we extend f_0 to a map $\tilde{f}: K^i \cup \Delta_0 \to K$ which sends $K^i \cup \Delta_0$ isomorphically onto $K^i \cup \Delta$. This map $\tilde{f}$ is homotopic to the inclusion $K^i \cup \Delta_0 \subset K$. In fact, for any multicomplex K there is a homotopy of $K^i \cup \Delta_0$ to $K^i \cup \Delta$ which keeps fixed K^i minus one i-face of Δ_0, and moves that i-face over $\Delta \cup \Delta_0$ back to its original position. Now, we extend $\tilde{f}$ to a continuous map $f: K \to K$ which is homotopic to the identity by an extension of the above homotopy. Since K is complete, this f can be chosen simplicial. Finally, as K is minimal, we apply the sublemma and conclude that f is an automorphism.

(C) *Lemma.* — *If* K *is minimal, then the quotient groups* Γ_1/Γ_i *are amenable for all* $i = 2, 3 \ldots$

Proof. — It suffices to show that the groups Γ_{i-1}/Γ_i are abelian groups for $i \geq 2$. Take one i-simplex, say $\Delta = \Delta_\alpha$ with $\alpha \in J$, in every orbit of Γ_{i-1}. The Γ_{i-1}-orbit of every Δ_α is contained in the set $\pi(\Delta_\alpha)$ and so for every $\gamma \in \Gamma_{i-1}$, we get a sphere $S^i_\gamma = \Delta \cup \gamma(\Delta) \subset K$ that represents an element of the homotopy group, $[S^i_\gamma] \in \pi_i(K, p_\alpha)$, for some base point $p_\alpha \in \Delta = \Delta_\alpha$. As all $\gamma \in \Gamma_{i-1}$ are homotopic to the identity, the maps $H_\alpha : \gamma \to [S^i_\gamma]$ are, in fact, homomorphisms $\Gamma_{i-1} \to \pi_i(K, p_\alpha)$. Denote by $H^{(i)} : \Gamma_{i-1} \to \Pi^{(i)} = \bigoplus_{\alpha \in J} \pi_i(K, p)$ the direct product of the homomorphisms H_α and observe, as K is minimal, that the kernel of the homomorphism $H^{(i)}$ equals Γ_i. Thus the group Γ_{i-1}/Γ_i is *embedded* into the *abelian* group $\Pi^{(i)}$ for $i \geq 2$. Therefore, the groups Γ_1/Γ_i are solvable and so amenable. (One can even show that they are nilpotent.)

(D) *Corollary.* — *If* K *is large then the homomorphism* $I^* : H^*(K/\Gamma_1) \to H^*(K)$ *induced by the quotient map* $K \to K/\Gamma_1$ *is an isometric isomorphism.*

Proof. — The chain complex $\hat{C}^*_a(K/\Gamma_1)$ is canonically isomorphic to the subcomplex of Γ_1-invariant cochains in $\hat{C}^*_a(K)$ and hence by averaging over the group Γ_1/Γ_m we get a chain homomorphism $A : \hat{C}^{i \leq m}(K) \to \hat{C}^{i \leq m}_a(K/\Gamma_1)$ such that $A \circ I = \mathrm{id}$ for $I : C^*_a(K/\Gamma_1) \to \hat{C}^*_a(K)$. Now, since K is large as well as complete and minimal, the transformation of $\hat{C}^*_a(K)$ induced by any $\gamma \in \Gamma$ is chain homotopic to the identity by the homotopy lemma. Therefore, we also have $I \circ A \sim \mathrm{id}$, and so the homomorphism A is a chain homotopy equivalence for all m, q.e.d.

(E) *Remark.* — *The quotient complex* K/Γ_1 *is a* $K(\Pi, 1)$*-complex for* $\Pi = \pi_1(K)$ *and the map* $K \to K/\Gamma_1$ *induces an isomorphism of fundamental groups.*

Proof of the mapping theorem of (3.1). — Since every space X is weakly homotopy equivalent to a large complete and minimal multicomplex, (D) and (E) above imply the corollary (A) of the mapping theorem, first for maps of the kind $f: X \to K(\Pi, 1)$, and then for all maps $f: X_1 \to X_2$ for which $f_* : \pi_1(X_1) \to \pi_1(X_2)$ is an isomorphism. The mapping theorem itself follows from the following considerations.

Take a large complete and minimal multicomplex K_1 of the $K(\Pi, 1)$ type, $\Pi = \pi_1(K_1, 1)$, choose a vertex $x \in K_1^\circ \subset K_1$ and let the fundamental group $\pi_1(K_1, x)$ act on K_1 as follows. For every edge e which joins x with another vertex x', and for each $\gamma \in \pi_1(K_1, x)$, we compose the path e with a loop which represents γ and then we take the (unique) edge in K_1, called $\gamma(e)$, which is homotopic to this composition. Thus the direct *sum* of the groups $\pi_1(K_1, x)$, over all $x \in K_1^\circ$, acts on the *one-skeleton* of K_1. As K_1 is a $K(\Pi, 1)$-complex this action uniquely extends to all of K_1.

Recall, that the direct *product* of groups G_j, over $j \in J$, consists by definition of all sequences $\{g_j\}$, $g_j \in G_j$, while in the direct sum the sequences only have finitely many non-identity entries. The important point is that direct sums of amenable groups are amenable, while direct products may be not amenable (see [19]).

Now, with a normal amenable subgroup $\Pi' \subset \Pi$, we have isomorphic normal subgroups $\Pi'_x \in \pi_1(K_1, x)$ for all $x \in K_1^\circ$, and the direct sum of these subgroups, called $\widetilde{\Pi}'$, acts on K_1. The quotient complex $K_1/\widetilde{\Pi}'$ has $K(\Pi/\Pi', 1)$ type and the quotient map $K_1 \to K_1/\widetilde{\Pi}'$ induces the quotient homomorphism of the fundamental groups $\Pi \to \Pi/\Pi'$. Since the group $\widetilde{\Pi}'$ is amenable, we conclude as before that this quotient map induces isometric isomorphisms of the bounded cohomology groups,

$$\hat{H}^*_a(K_1/\Pi') \xrightarrow{\sim} \hat{H}^*_a(K_1) \qquad \text{q.e.d.}$$

Proof of the vanishing theorem of (3.1). — Take a subset $X_0 \overset{I_0}{\subset} X$ and consider all paths continuous in t of (possibly discontinuous) maps $X_0 \to X$, called $I_t : X_0 \to X$, $t \in [0, 1]$, with the following three properties:

(1) $I_{t=0} = I_0 : X_0 \subset X$.

(2) For almost all points $x \in X$ (*i.e.* only with finitely many exceptions) $I_t(x) = I_0(x)$ for all $t \in [0, 1]$.

(3) The map $I_{t=1}$ sends X_0 bijectively onto itself.

Denote by $\pi(X, X_0)$ the group of homotopy classes of the paths I_t. Observe that, for a single point $x_0 \in X$, $\Pi(X, x_0) = \pi_1(X, x_0)$; in general, there is a natural homomorphism of the group $\Pi(X, X_0)$ to the group of permutations of X_0 with finite supports, and the kernel of this homomorphism is the direct sum of the fundamental groups, $\bigoplus_{x \in X_0} \pi_1(X, x)$.

Next, we take a complete minimal multicomplex K and a homotopy equivalence $h : K \to X$ which maps the zero-skeleton $K_0 \subset K$ onto X bijectively. Then the group $\Pi(X, X)$ acts, by compositions of paths I_t with edges of K sent by h to X, on the one-skeleton K^1 of K, and the actions of all $\gamma \in \Pi(X, X)$ extend (not uniquely) to automorphisms of K. Therefore, the group $\Pi(X, X)$ acts on the complex $\hat{C}^*_a(K/\Gamma_1) \subset \hat{C}^*_a(K)$ of bounded Γ_1-invariant cochains.

Then, for all subsets $U \subset X$ and $V \subset U$, the group $\Pi(U, V)$ also acts on K^1 via

the natural homomorphism $\Pi(U, V) \to \Pi(X, V) \subset \Pi(X, X)$ and if the subset $U \subset X$ is " amenable " (see (3.1)) then the image of this homomorphism is an amenable subgroup in $\Pi(X, V)$ and so the resulting action of $\Pi(U, V)$ on the complex $\hat{C}^*(K/\Gamma_1)$ is amenable.

Now, let $U_j \subset X$, $j \in J$, be some amenable open subsets and let $V_j \subset U_j$ be arbitrary mutually disjoint subsets. Then the direct sum of the groups $\Pi(U_j, V_j)$ over $j \in J$ also acts on the complex $\hat{C}^*_a(K/\Gamma_1)$ and with the averaging process we conclude to the following

Proposition. — Every bounded cohomology class $\beta \in \hat{H}^*(X)$ *can be represented by a cocycle* $b \in \hat{C}_a(K/\Gamma_1) \subset C^*_a(K)$ *which is also invariant under the action of the direct sum of the groups* $\Pi(U_j, V_j)$.

Now, we prove the vanishing theorem for the covering $\{U_j\}$ of X as follows. We take a sufficiently fine triangulation L of the manifold X and we divide the vertices of L into disjoint subsets called V_j, such that the stars of the vertices $v \in V_j$ are contained n the sets U_j. Then, we choose the multicomplex K and the map h such that L becomes the homeomorphic image of a subcomplex L′ in K, $h(L') = L$. As no $m + 1$ sets U_j intersect, every simplex in L of dimension $\geq m$ has an edge, say $e \subset \Delta$, which is contained in some set U_j and whose two vertices are in V_j. Now, there is a transformation by the group $\Pi(U_j, V_j)$ which permutes the corresponding two vertices of the simplex $\Delta' \subset L'$, $h(\Delta') = \Delta$, while keeping fixed the other vertices of Δ'. Moreover, there is such a transformation which also sends the one-skeleton of Δ' onto itself. Therefore, every (anti-symmetric!) cocycle b in K, which is invariant under the groups Γ_1 and $\Pi(U_j, V_j)$, vanishes at the simplex Δ'. According to the Proposition, the bounded cocycles can be made invariant under all groups $\Pi(U_j, V_j)$, and then they vanish at all simplices of the complex $L' \approx L$. Therefore, these cocycles are cohomologous to zero, q.e.d.

Proof of the identity $|| \; ||^{new} = || \; ||$ *of section* (2.3). — Let first X be a $K(\Pi, 1)$ space and let $Y \to X$ be the universal covering. Then the complex $\hat{C}^*(X)$ is canonically isomorphic to the complex of bounded Π-invariant singular cochains in Y. Next, we take the complex $\hat{C}^*(Y : \Pi)$ whose i-cochains are bounded functions $c = c(y_0, \ldots, y_i)$ and consider the natural embedding $I : \hat{C}^*(Y : \Pi) \to \hat{C}^*(X) \subset \hat{C}^*(Y)$. As the space Y is contractible, we can Π-equivariantly assign to each $(i + 1)$-tuple of points, $(y_0, \ldots, y_i)$, a singular simplex $\sigma : \Delta^i \to Y$ with vertices $y_0, \ldots, y_i$, such that the subsets of $\{y_0, \ldots, y_i\}$ go under this assignment to the corresponding faces of σ. Thus we get a chain homomorphism $S : \hat{C}^*(X) \to \hat{C}^*(Y : \Pi)$. Both homomorphisms, I and S have norms ≤ 1. Furthermore, $S \circ I = Id$ and also $I \circ S \sim Id$. Indeed, the standard construction usually applied to unbounded cochains (see [40] for instance) gives a chain homotopy equivalence $I \circ S \sim Id$ which is *bounded* in every dimension. Therefore, the homomorphism I induces isometric isomorphisms on bounded cohomology

$$I^* : H^*(Y : \Pi) \tilde{\to} H^*(X).$$

Next, we extend the definitions of $\hat{C}^*(Y : \Pi)$ and $\hat{H}^*(Y : \Pi)$ to any action of Π on an arbitrary set Y. In particular, if $Y = \Pi$, we get for $\hat{C}^*(\Pi : \Pi)$ the subcomplex of the bounded cochains in the usual complex of (homogeneous) real cochains on Π. More generally, for any set X we may consider the natural action of Π on the product $\Pi \times X$, and, for coverings $Y \to X$, we have the canonical isomorphism $\hat{C}^*(Y : \Pi) = \hat{C}^*((\Pi \times X) : \Pi)$. Maps between sets, $X_1 \to X_2$, induce chain homomorphisms,

$$\hat{C}^*((\Pi \times X_2) : \Pi) \to \hat{C}^*((\Pi \times X_1) : \Pi),$$

of norm ≤ 1, and again, by the standard argument, we conclude that these maps induce isometric isomorphisms on cohomology. As a result, we obtain an isometric isomorphism

$$\hat{H}^*(X) \approx \hat{H}^*(\Pi),$$

where X is a $K(\Pi, 1)$-space and $\hat{H}^*(\Pi)$ denotes the cohomology of the complex $\hat{C}^*(\Pi; \mathbf{R}) = \hat{C}^*(\Pi : \Pi)$ of bounded real cochains $c(\gamma_0, \ldots, \gamma_i)$, $\gamma_i \in \Pi$.

Finally, we denote by $\hat{C}_a^*(\Pi) \subset \hat{C}^*(\Pi; \mathbf{R})$ the subcomplex of antisymmetric cochains, and then with the (anti)symmetrization over the permutations of $\{\gamma_0, \ldots, \gamma_i\}$ we again obtain an isometry of cohomology groups,

$$\hat{H}_a^*(\Pi) = \hat{H}^*(\Pi).$$

Now, let X be any path-connected space with $\pi_1(X) = \Pi$ and let $Y \to X$ be the universal covering. We represent Π by an orbit of a point in Y and then we take an arbitrary Π-equivariant map $r : Y \to \Pi \subset Y$. With the discussion above, the mapping theorem for maps $X \to K(\Pi, 1)$ now provides an isometric isomorphism

$$r^* : \hat{H}_a^*(\Pi) \xrightarrow{\sim} \hat{H}^*(X).$$

With this isomorphism the proof of the inequality $\| \|_\infty^{\text{new}} = \| \|_\infty$ is immediate. Indeed, we take first the complex $\tilde{C}^*(Y : \Pi)$ of bounded Borel cochains, that is the intersection of the complex $C^*(Y : \Pi)$ of section (2.3) with $\hat{C}^*(Y : \Pi)$. Then we have a natural homomorphism $\tilde{F} : \tilde{H}^*(Y : \Pi) \to \hat{H}^*(X)$ (compare with F of (2.3)), and with a *Borel* map r we also have

$$\tilde{r} : H_a^*(\Pi) \to \tilde{H}^*(Y : \Pi),$$

such that $r^* = \tilde{F} \circ \tilde{r}$. As $\|\tilde{r}\|_\infty \leq 1$ and r is an isometry, the homomorphism $\tilde{F}$ is surjective and such that for any $\hat{\beta} \in \hat{H}^*(X)$ we have $\|\hat{\beta}\|_\infty = \inf \|\hat{\alpha}\|_\infty$, where $\hat{\alpha}$ runs over the pullback $\tilde{F}^{-1}(\hat{\beta}) \subset \tilde{H}^*(Y : \Pi)$. Now, by the definition of $\|\beta\|_\infty^{\text{new}}$ of (2.3), we get $\| \|_\infty^{\text{new}} \leq \| \|_\infty$ and since the opposite inequality is obvious, we get the required identity $\| \|^{\text{new}} = \| \|^{\text{old}}$.

3.4. Proof of the isolation theorem of section 0.4

For a metric ball B of radius R in a Riemannian manifold V, we denote by λB, $\lambda > 0$, the concentric ball of radius λR. We denote by $\widetilde{B}$ a ball of radius R in the universal covering $\widetilde{V} \to V$ which projects onto B.

(A) *Lemma.* — *For any given positive numbers* C *and* d *there is a constant* $\mu = \mu(C, d)$ *such that the inequalities*

$$\mathrm{Vol}(\lambda\widetilde{B})/\mathrm{Vol}(B) \leq C\lambda^d, \quad \textit{for all } \lambda \in [1, \mu]$$

imply the "amenability" (see (3.1)*) of the ball* $B \subset V$.

Proof. — Let $\widetilde{B} \subset \widetilde{V}$ be a ball over $B \subset V$ and consider all those deck transformations $\gamma \in \Pi = \pi_1(V)$ for which the intersection $\widetilde{B} \cap \gamma(\widetilde{B})$ is not empty. Denote these transformations by $\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m \in \Pi$ and observe that the subgroup $\Pi' \subset \Pi$ generated by γ_i, $i = 1, \ldots, m$, equals the image of $\pi_1(B)$ in $\pi_1(V) = \Pi$. We associate for every $\lambda' \geq 1$ the set $\Pi'(\lambda') \subset \Pi'$ of all those $\gamma \in \Pi$ which can be represented by words in $\gamma_1, \ldots, \gamma_m$ of length $\leq \lambda'$, namely

$$\gamma = \gamma_{i_1}^{p_1}, \ldots, \gamma_{i_k}^{p_k} \quad \text{for } |p_1| + \ldots + |p_k| \leq \lambda'.$$

For every $\gamma \in \Pi'(\lambda')$, the ball $\gamma(\widetilde{B})$ is contained in the ball λB for $\lambda = 2\lambda' + 1$; therefore, the number of elements in $\Pi'(\lambda')$ satisfies

$$\#\, \Pi'(\lambda') \leq \mathrm{Vol}(\lambda\widetilde{B})/\mathrm{Vol}(B).$$

Finally, if $\#\, \pi'(\lambda') \leq C\lambda^d = C(2\lambda' + 1)^d$ for all $\lambda' \in \left[1, \frac{1}{2}\mu - 1\right]$, then, for a sufficiently large $\mu = \mu(C, d)$, the group Π', being of initial polynomial growth, contains a nilpotent subgroup of finite index (see the end of [23]) and so Π' is amenable, q.e.d.

Let us call an open ball in V *extremal* if it is "amenable" and if all larger concentric balls are not "amenable".

(B) *Lemma.* — *For any given number* $\rho > 0$ *there exists a system of open balls* $B_1, \ldots, B_j, \ldots$, *in an arbitrary complete manifold* V *such that the following four properties hold*:

(1) *Each ball* B_j *has radius at most* ρ *and each concentric ball* $4B_j$ *is "amenable". Furthermore, if some ball* B_j *has a radius strictly less than* ρ, *then the concentric ball* $4B_j$ *is extremal.*

(2) *The balls* $\frac{1}{4}B_j \subset V$ *are mutually disjoint.*

(3) *The balls* $\frac{3}{4}B_j \subset V$ *cover the manifold* V.

(4) *If two balls* B_{j_1} *and* B_{j_2} *intersect, then their radii* ρ_1 *and* ρ_2 *satisfy*

$$2\rho_1 \geq \rho_2 \geq \frac{1}{2}\rho_1.$$

Proof. — Observe that around each point in V there is exactly one ball which satisfies (1). Furthermore, if two such balls, B of radius ρ and B′ of radius $\rho' \leq \rho$, intersect, then the ball $B'' = \frac{2\rho}{\rho'}B'$ of radius 2ρ is contained in the ball $4B$. As the ball $4B$ is " amenable ", so is the ball B″, and since the ball $4B'$ is extremal it contains B″. Therefore $\rho'' \geq \frac{1}{2}\rho$ and so the property (4) holds for our balls.

Now, let us take a *maximal* system of balls which satisfies the properties (1) and (2) and prove the property (3) for this system of balls B_j. Indeed, take an arbitrary point $v \in V$ and the ball B around v which satisfies (1). As the system (B_j), with $j = 1, \ldots,$ is maximal, the ball $\frac{1}{4}B$ intersects some ball $\frac{1}{4}B_j$ and then by (4) the concentric ball $\frac{3}{4}B_j$ contains v, q.e.d.

(C) *Lemma.* — *Let* V *be a complete n-dimensional manifold with* $\mathrm{Ricci}(V) \geq -1$ *and let* B *and* B′ *be some balls in* V *of radii* ρ *and* ρ' *respectively. If the balls* B *and* B′ *intersect and if their radii* ρ *and* ρ' *are less than one, then*

$$\mathrm{Vol}(B)/\mathrm{Vol}(B') \leq \mathrm{const}_n\, \rho^n(\rho')^{-n}.$$

In particular, $\mathrm{Vol}(B) \leq \mathrm{const}'_n\, \rho^n$.

Proof. — This is a special case of Bishop's inequality (see [5], (11.10), and [22]).

Corollary. — *Balls* B *of radius* $\rho \leq 1$ *contain no more than* $\mathrm{const}_n\, \rho^n(\rho')^{-n}$ *disjoint balls of radius* ρ' *and, therefore,* B *can be covered by a number not greater than* $\mathrm{const}'_n\, \rho^n \varepsilon^{-n}$ *of balls of radius* ε, *for any given positive* $\varepsilon < \rho$.

Now, let $B_j \subset V$ of radii ρ_j, $j = 1, \ldots,$ be balls which satisfy the properties (2), (3) and (4) of lemma (B) and let us take the following functions $\varphi_j : V \to \mathbf{R}$ with supports in B_j, $j = 1, 2, \ldots$ The function φ_j is zero outside B_j, it is equal to one on $\frac{3}{4}B_j$ and for $v \in B_j \setminus \frac{3}{4}B_j$ we define it by $\varphi_j(v) = 4\rho^{-1}\left(1 - \mathrm{dist}\left(v, \frac{3}{4}B_j\right)\right)$. Observe that the functions φ_j are Lipschitz with $\|\mathrm{grad}\, \varphi_j\| \leq 4\rho^{-1}$. Then, we assume the covering $\{B_j\}$ to be locally finite and we take the functions $f_j = (\sum_j \varphi_j)^{-1}\varphi_j$ which send V into the unit simplex $\bar{\Delta} = \{x_1, \ldots) : \Sigma_j x_j = 1,\ x_j \geq 0\ \forall j\}$ in the Euclidean space with coordinates $x_1, \ldots, x_j, \ldots$ The map $f = (f_1, \ldots, f_j, \ldots)$ sends V into the *nerve* of the covering $\{B_j\}$, which is realized as a subcomplex P in $\bar{\Delta}$. The dimension of P at any point $v \in V$ is equal to the multiplicity of the covering B_j. If a point $v \in V$ is covered by at most m balls B_j, then the norm of the differential of f satisfies $\|D_v f\| \leq \mathrm{const}_m\, \rho^{-1}$ where ρ denotes the minimum of the radii of the m balls B_j.

Now, let $\mathrm{Ricci}(V) \geq -1$ and let all balls B_j have radii ≤ 1. Then by lemma (C),

there are at most $N = N(n)$ balls B_j which intersect any given fixed ball B_{j_0}, and in particular $\dim(P) \leq N$. Furthermore, if a point $v \in V$ is contained in some ball B_j then

$$||D_v f|| \leq 2\,\mathrm{const}_N\, \rho_j^{-1} \leq \mathrm{const}'_n\, \rho_j^{-1}$$

where ρ_j denotes the radius of B_j.

(D) *Lemma.* — *There exists a map g of V into the n-skeleton $P^n \subset P$ with the following two properties*:

(a) *The norm of the differential Dg is bounded on every ball B_j by $\mathrm{const}\,\rho_j^{-1}$ for some $\mathrm{const} = \mathrm{const}''_n$.*

(b) *The pullback under g of the star of every vertex in P is contained in the union of some balls B_j which intersect a certain fixed ball B_{j_0}.*

Proof. — The map $f: V \to P$ satisfies (a) and (b). Let us first construct $f_1 : V \to P^{N-1}$, then $f_2 : V \to P^{N-2}$, and so on, until we eventually get $g = f_{N-n}$. We obtain the map f_1 by choosing a point x inside each N-dimensional simplex $\Delta = \Delta^N \subset P$, $x = x(\Delta) \in \mathrm{Int}\,\Delta$, but not in the image $f(V) \subset P$, and then by projecting the intersections $f(V) \cap \Delta$ radially from x to the boundary of Δ. In order to guarantee the property (a), we choose the points $x \in \mathrm{Int}\,\Delta$ with $\mathrm{dist}\,(x, f(V)) \geq \varepsilon = \varepsilon_n > 0$. This is possible since the numbers

$$\varepsilon = \varepsilon(\Delta) = \sup_{x \in \Delta} \mathrm{dist}(x, f(V))$$

are bounded from below in terms of N. Indeed, as $\dim \Delta = N$ there are at least $k \geq \mathrm{const}^{(1)} \varepsilon^{-N}$ disjoint ε-balls in Δ with centers in $f(V) \cap \Delta$, for some $\mathrm{const}^{(1)} = \mathrm{const}^{(1)}(N)$. By the property (b) of the map f the pullbacks of these balls to V, called $\widetilde{B}_1, \ldots, \widetilde{B}_i, \ldots, \widetilde{B}_k \subset V$, are contained in the union of some balls $B_{j_1}, \ldots, B_{j_N}$ with non-empty intersection, and by the property (a) all sets $\widetilde{B}_i$ contain balls of radii $\geq \mathrm{const}^{(2)} \varepsilon \rho_1^{-1}$, where ρ_1 is the radius of B_{j_1} and again $\mathrm{const}^{(2)} = \mathrm{const}^{(2)}(N)$. Now, by lemma (C) we have $k \geq \mathrm{const}^{(3)} \varepsilon^{-n}$ and thus $\varepsilon \geq (\mathrm{const}^{(1)}/\mathrm{const}^{(3)})^{1/N-n} \geq \varepsilon(N)$, q.e.d.

Corollary. — *If all balls B_j satisfy*

$$\mathrm{Vol}(B_j) \leq \delta(n) \rho_j^n$$

for some sufficiently small $\delta(n) > 0$, then, there is a map $g' : V \to P^{n-1} \subset P^n$ which satisfies property (b) *of lemma* (D).

Proof. — Indeed, with the small $\mathrm{Vol}(B_j)$ we conclude according to (a), that no n-simplex in P^n is covered by the image $g(V)$ and so this image can be pushed further to the $(n-1)$-skeleton P^{n-1}.

Now, we can prove the Isolation theorem. First, choose a sufficiently large constant $C = C(n)$ in lemma (A) and take $d = n$. Next, take ρ of lemma (B) equal

to $\frac{1}{4}\mu^{-1}$. Thus, $\mathrm{Vol}(B_j) \leq \delta(n)\rho_j^n$ with $\delta(n)$ small if C is large. For $\rho_j = \rho = \rho(n)$ this follows from the assumption $\mathrm{Vol}(B(1)) \leq \varepsilon(n)$. Then, if the ball $4B_j$ is extremal, so that larger concentric balls are not " amenable ", we get, according to lemma (A),

$$\mathrm{Vol}(B_j) \leq \mathrm{Vol}(4B_j) \leq C^{-1}\lambda^{-n}\,\mathrm{Vol}(4\lambda\widetilde{B}_j) \quad \text{for some } \lambda \leq \mu.$$

Now, by lemma (C), we have

$$\mathrm{Vol}(4\lambda B_j) \leq \mathrm{const}'_n(4\lambda\rho)^n$$

and so $$\mathrm{Vol}(B_j) \leq 4^n C^{-1}\,\mathrm{const}'_n.$$

Finally, using property (b) of lemma (D), we obtain that the pullback under g' of every point $x \in K^{n-1}$ is contained in some ball $3B_j$ that is an " amenable " subset of V, and so the corollary (2) of the vanishing theorem applies, provided V is a *closed* manifold. The open case will be treated later, in section (4.2).

Final Remarks. — Some complications we met above would disappear if we could prove that balls of radius $\leq \varepsilon = \varepsilon(n)$ in V with $\mathrm{Ricci}\, V \geq -1$ are " amenable ". As the matter stands now, we only have lemma (A) and so we must be careful in choosing the covering $\{B_j\}$. However, if $|K(V)| \leq 1$, then, by Margulis' lemma, small balls are amenable and the proof of this section can be simplified for manifolds V with $|K(V)| \leq 1$. In fact, in (C) of section (4.3), we shall prove the sharper Injectivity Radius estimate of (0.4).

3.5. Proof of the identity $\|V_1 \# V_2\| = \|V_1\| + \|V_2\|$ and some generalizations

As the manifolds V_1 and V_2 are assumed of dimension ≥ 3 (see (0.2)), the (pinching) map $V_1 \# V_2 \to V_1 \vee V_2$ yields an isomorphism of the fundamental groups. Suppose the manifolds closed (the open case will be studied in section (4.2)). Then, by the mapping theorem (see (3.1)), we can replace the connected sum $V_1 \# V_2$ by the wedge $V_1 \vee V_2$. The homology of this wedge is

$$H_*(V_1 \vee V_2) = H_*(V_1) \oplus H_*(V_2),$$

and we must show that $\|[V_1] + [V_2]\| = \|V_1\| + \|V_2\|$ for the fundamental classes $[V_1]$ and $[V_2]$. Observe that the inequality

$$\|[V_1] + [V_2]\| \leq \|V_1\| + \|V_2\|$$

is immediate from the definition of the norm. To prove the opposite inequality we first pass to cohomology and then we shall construct a homomorphism $Q: \hat{C}^*(V_1) \oplus \hat{C}^*(V_2) \to \hat{C}^*(V_1 \vee V_2)$ such that $P \circ Q = \mathrm{Id}$ for the obvious homomorphism

$$P : \hat{C}^*(V_1 \vee V_2) \to \hat{C}^*(V_1) \oplus \hat{C}^*(V_2),$$

and such that $\|Q\|_\infty \leq 1$, with the convention $\|c_1 \oplus c_2\|_\infty = \max(\|c_1\|_\infty, \|c_2\|_\infty)$. Given such a homomorphism Q, we have, for every cohomology class $h = (h_1, h_2) \in H^*(V_1 \vee V_2)$,

the inequality $||h||_\infty \leq \max(||h_1||_\infty, ||h_2||_\infty)$, and by duality we get the required relation in homology.

We start the construction of Q with the following combinatorial considerations. Call a simplicial complex K *tree-like* if it is simply connected and if it can be decomposed into the union of closed simplices, called Δ_j, $j \in J$, such that any two of them have at most one common vertex. A piecewise linear path in K made out of some edges in K is called *straight* if it has at most one edge in every simplex Δ_j. Such a path is topologically a closed interval divided into edges, and any two vertices in the (tree-like!) complex K can be joined by a unique straight path.

Now, take $n + 1$ vertices in K, say $x_0, \ldots, x_n$, and join any two of them by the straight path called $[x_k, x_\ell]$, for $k, \ell = 0, \ldots, n$ and $k \neq \ell$. Let us intersect each $[x_k, x_\ell]$ with a fixed simplex Δ_j. Observe, that if two paths $[x_{k_0}, x_{\ell_0}]$ and $[x_{k_1}, x_{\ell_1}]$ intersect Δ_j along the edges $[y_{k_0}, y_{\ell_0}]$ and $[y_{k_1}, y_{\ell_1}]$, for some vertices $y_{k_0}, y_{\ell_0}, y_{k_1}, y_{\ell_1}$ of Δ_j, then the path $[x_{k_0}, x_{k_1}]$ intersects Δ_j in $[y_{k_0}, y_{k_1}]$. Therefore the union of all $[x_k, x_\ell]$ intersects Δ_j along the one-skeleton of some m-dimensional face of Δ_j for $m \leq n$. Furthermore, if $m = n$, then the union of all paths $[x_k, x_\ell]$ consists exactly of this one-skeleton, spanned by some vertices $y_0, \ldots, y_n$ in Δ_j, plus the paths $[x_k, y_k]$, $k = 0, \ldots, n$. Moreover, there is at most one n-simplex Δ_j for which $m = n$. If there is none, we say that the "simplex" $(x_0, \ldots, x_n)$ is *degenerate*, and our construction assigns to each non-degenerate "simplex", an actual n-dimensional simplex in K, namely

$$(x_0, \ldots, x_n) \mapsto (y_0, \ldots, y_n).$$

Now, we return to V_1 and V_2 with the fundamental groups Π_1 and Π_2. We may replace the complexes $\hat{C}^*(V_1)$ and $\hat{C}^*(V_2)$ by $\hat{C}^*_a(\Pi_1)$ and $\hat{C}^*_a(\Pi_2)$, as shown in the proof of the identity $||\ ||^{\text{new}} = ||\ ||^{\text{old}}$ in section (3.3).

We consider the (infinite dimensional) simplices $\Delta(\Pi_1)$ and $\Delta(\Pi_2)$ spanned by the groups Π_1 and Π_2 as sets of vertices, and then we interpret our complexes as the complexes of Π_1- and Π_2-invariant simplicial cochains in $\Delta(\Pi_1)$ and $\Delta(\Pi_2)$. The group $\Pi = \Pi_1 * \Pi_2 = \pi_1(V_1 \vee V_2)$ acts on a tree-like complex K built of infinitely many copies of $\Delta(\Pi_1)$ and $\Delta(\Pi_2)$, as seen by looking at the universal covering of the wedge $V_1 \vee V_2$, a tree-like union of copies of the universal coverings of V_1 and V_2. The zero-skeleton of K is identified with the zero-skeleton of the simplex $\Delta(\Pi)$, and so every n-face of $\Delta(\Pi)$ becomes an n-dimensional "simplex" $\Delta = (x_0, \ldots, x_n)$. Now, every pair of cochains, $(c_1, c_2) \in \hat{C}_a(\Pi_1) \oplus \hat{C}_a(\Pi_2)$, uniquely defines a simplicial Π-invariant cochain c on K. Then, with our correspondence $\Delta \to \Delta' = (y_0, \ldots, y_n)$, we define $c' = Q(c)$ by putting $c'(\Delta) = c(\Delta')$ in case Δ is a non-degenerate "simplex", and $c'(\Delta) = 0$ otherwise, q.e.d.

A generalization: amalgamated products

Let a space V be divided into a union of two closed oriented manifolds, $V = V_1 \cup V_2$, where it is assumed for simplicity that V_1, V_2 and the intersection $V_0 = V_1 \cap V_2$ are connected, and that V_0 is a subpolyhedron in V_1 as well as in V_2. We set $\Pi_0 = \pi_1(V_0)$ and we assume the inclusion homomorphisms $\Pi_0 \to \Pi_1 = \pi_1(V_1)$ and $\Pi_0 \to \Pi_2 = \pi_1(V_2)$ injective. Then the group $\Pi = \pi_1(V)$ is called the *amalgamated* product of Π_1 and Π_2. The fundamental homology classes of V_1 and V_2 define a class in V called

$$[V_1] + [V_2] \in H_*(V).$$

Theorem. — *If the group* Π_0 *is amenable then* $\|[V_1] + [V_2]\| = \|V_1\| + \|V_2\|$.

Corollary. — *Let* $\dim V_1 = \dim V_2$ *and take away from* V_1 *and* V_2 *open regular neighbourhoods of* V_0, *assumed of codimension* ≥ 3. *Glue the resulting manifolds by some diffeomorphism of the boundaries (assuming such a diffeomorphism exists). Then the resulting manifold* V' *satisfies*

$$\|V'\| = \|V_1\| + \|V_2\|.$$

Indeed, as in the case of the connected sum when V_0 is only one point, we can apply the mapping theorem since

$$\pi_1(V') = \Pi = \pi_1(V).$$

Proof of the theorem. — Look at the universal covering $\tilde{V}_1 \to V_1$ and observe that $V_0 \subset V_1$ lifts to infinitely many disjoint copies of the universal covering of V_0. These copies correspond to the cosets in Π_1/Π_0. In terms of the simplex $\Delta(\Pi_1)$ we have some faces $\Delta_i(\Pi_0) \subset \Delta(\Pi_1)$, for some set of indices, $i \in I = \Pi_1/\Pi_0$. The same applies to $\Delta(\Pi_2)$, where we have simplices $\Delta_j(\Pi_0)$ for $j \in J = \Pi_2/\Pi_0$. We then form in a natural way a " tree-like " union of infinitely many copies of $\Delta(\Pi_1)$ and $\Delta(\Pi_2)$, such that any two copies may intersect along at most one copy of the simplex $\Delta(\Pi_0)$. The group Π acts on the resulting " tree-like " complex K', whose zero-skeleton is identified with the zero-skeleton of the simplex $\Delta(\Pi)$.

Now, as the group Π_0 is amenable, we can average the bounded cocycles over the group $\Pi(V_0, V_0)$ (see the proof of the vanishing theorem in section (3.3)) and interpret the averaged cochains in $\hat{C}^*(\Pi_1)$ as Π_1-invariant cochains on the quotient Π_1/Π_0.

In other words, we may work with the cochain complex of bounded Π_1-invariant cochains on the simplex $\Delta(\Pi_1/\Pi_0)$, which may also be obtained by pinching to points the copies $\Delta_i(\Pi_0) \subset \Delta(\Pi_1)$. Finally, by pinching to points all copies of $\Delta(\Pi_0)$ in K' we get a tree-like complex K, which is built of some copies of $\Delta(\Pi_1/\Pi_0)$ and $\Delta(\Pi_2/\Pi_0)$ by glueing these simplices at some vertices. Furthermore, the zero-skeleton of this K is identical to the zero-skeleton of $\Delta(\Pi/\Pi_0)$, and the argument we used above for free products applies.

4. SIMPLICIAL VOLUMES OF COMPLETE MANIFOLDS

If V is an open manifold, then, as was shown in section 1, there are several non-equivalent notions of simplicial volume. For complete Riemannian manifolds one can take into account the geometry of V at infinity as well as the topology of V, and then more simplicial volumes emerge.

4.1. Relative bounded cohomology

For a pair of topological spaces X and $X' \subset X$, one has a natural ℓ^1-norm in the relative chain complex $C_*(X, X') = C_*(X)/C_*(X')$. One also has the dual ℓ^∞-norm on relative cochains in $C^*(X, X')$ induced from the ℓ^∞-norm in $C^*(X)$ by the inclusion $C^*(X, X') \subset C^*(X)$. The bounded cohomology groups $\hat{H}^*(X, X')$ with the pseudo-norm $\| \|_\infty$, dual to the ℓ^1-norm $\| \|$ on the relative homology $H_*(X, X')$, then follow as in section (3.0). Furthermore, for every $\theta \geq 0$, one defines a norm $\| \|(\theta)$ on $C_*(X)$ by putting

$$\|c\|(\theta) = \|c\| + \theta \|\partial c\|.$$

Then, using the quotient homomorphism $q : C_*(X) \to C_*(X, X')$, one defines the norm $\|c'\|(\theta)$ of $c' \in C_*(X, X')$ by taking all $c \in q^{-1}(c') \subset C_*(X)$ and setting

$$\|c'\|(\theta) = \inf_c \|c\|(\theta).$$

The dual norms on relative cochains are denoted by $\| \|_\infty(\theta)$. Observe that all norms $\| \|(\theta)$ are equivalent, but not equal, to the usual ℓ^1-norm $\| \| = \| \|(0)$, and the dual norms $\| \|_\infty(\theta)$ are all equivalent to $\| \|_\infty$. Now with these norms on relative chains and cochains we have the corresponding norms $\| \|(\theta)$ on $H_*(X, X')$ and the dual norms $\| \|_\infty(\theta)$ on the bounded cohomology $\hat{H}^*(X, X')$. Finally we define these norms for all θ in the closed interval $[0, \infty]$ by passing to the limits. Observe that the limit " norm " $\| \|(\infty)$ may be nonequivalent to $\| \|(\theta)$ for $\theta < \infty$. For example, $\|h\|(\infty) = \infty$ for those $h \in H_*(X, X')$ for which the boundary $\partial h \in H_*(X')$ has non-zero ℓ^1-norm, $\|\partial h\| > 0$.

Remark. — Our norm $\| \|(\infty)$ is equal to Thurston's norm $\| \|_0$ defined in [47]. Warning: Thurston's approximations $\| \|_\varepsilon$ in [47] are different from our $\| \|(\theta^{-1})$.

In [47], Thurston proves the equality $\| \| = \| \|(\infty)$ for the fundamental classes of compact 3-manifolds whose boundaries are tori. This result generalizes to the following

Equivalence theorem. — *If the fundamental groups of all path-connected components of the space* X' *are amenable, then the norms* $\| \|_\infty(\theta)$ *on the groups* $\hat{H}^i(X, X')$, *for* $i \geq 2$, *are equal for all* $\theta \in [0, \infty]$, *and so are the dual norms* $\| \|(\theta)$ *on the homology groups* $H_i(X, X')$ *for* $i \geq 2$.

Proof. — With the averaging technique of section 3 one first gets the

Relative mapping theorem. — *Let* f *be a map of pairs of spaces,* $f: (X, X') \to (Y, Y')$ *which is bijective on the sets of path-connected components,* $\pi_0(X) \xrightarrow{\sim} \pi_0(Y)$ *and* $\pi_0(X') \xrightarrow{\sim} \pi_0(Y')$. *Furthermore, let* g *be surjective on the fundamental groups of all these components and let the kernels of these surjective homomorphisms be amenable groups for all components of* X *and of* X'. *Then the induced homomorphism*

$$f^*: \hat{H}^*(Y, Y') \to \hat{H}^*(X, X')$$

is an isometric isomorphism for the norms $\| \|_\infty(\theta)$ *for all* $\theta \in [0, \infty]$. *In particular, if the fundamental groups of all components of* X *and* X' *are amenable, then, for* $i \geq 2$, $\hat{H}^i(X, X') = 0$ *and the norms* $\| \|(\theta)$ *on* $H_i(X, X')$ *vanish for all* $\theta \in [0, \infty]$.

Remark. — The group $\hat{H}^1(X, X')$ vanishes if and only if the inclusion map $\pi_0(X') \to \pi_0(X)$ is injective.

Now, by averaging over the *amenable* group $\Pi(X, X')$, as in section (3.3), one makes all bounded cochains vanish at all those singular simplices $\Delta \to X$ which have an edge contained in X'. In particular, the values of such cocycles on relative cycles c of dimension ≥ 2 do not depend any more on their boundaries, $\partial c \subset X'$, and thus the equivalence theorem is established.

4.2. Diffusion of chains in open manifolds

For a sequence of subsets of an open manifold, $U_j \subset V$, we write $U_j \to \infty$ if only finitely many of these sets U_j intersect any given compact subset of V. A subset $U \subset V$ is called *large* if the complement $V \setminus U$ is relatively compact. Then we have a canonical homomorphism of the homology defined with *locally finite* singular chains (see (0.2)), $\overline{H}_*(V)$, into the relative homology $H_*(V, U)$. For a sequence of large sets $U_j \subset V$ we denote by $h_j \in H_*(V, U_j)$ the values of these homomorphisms on elements $h \in \overline{H}_*(V)$. Then, for $U_j \to \infty$ and $\theta \in [0, \infty]$, the limit $\lim_{j \to \infty} \|h_j\|(\theta)$ depends only on $h \in \overline{H}_*(V)$; we denote it by $\|h\|(\theta)$. Furthermore, we also have the ℓ^1-norm called $\| \|$ on $\overline{H}$, which satisfies

$$\|h\| \geq \|h\|(\theta) \quad \text{for all } \theta \in [0, \infty].$$

Notice that

$$\| \|(\theta_1) \geq \| \|(\theta_0) \quad \text{for } \theta_1 \geq \theta_0.$$

Observe that for interiors of compact manifolds, $V = \operatorname{Int} \overline{V}$, the canonical isomorphism $\overline{H}_*(V) \xrightarrow{\sim} H_*(\overline{V}, \partial\overline{V})$ is *isometric* in all norms $\| \ \|(\theta)$ and in particular, the simplicial volume $\|\overline{V}, \partial\overline{V}\|$ of section (1.1) equals $\| [V] \|(o)$ for the fundamental homology class $[V]$ of V.

The proofs of the following three theorems are given at the end of this section.

First, let us call the manifold V " *amenable* " *at infinity* if every large set $U \subset V$ contains another large set, $U' \subset U$, such that U' is an " amenable " subset of U (see (3.1)).

(1) *Equivalence theorem.* — *If* V *is* " *amenable* " *at infinity then the norms on* $\overline{H}_i(V)$, *for* $i \geq 2$, *satisfy* $\| \ \|(\theta) = \| \ \|$ *for all* $\theta \in [o, \infty]$.

Next we consider closed subsets V' in V and, for $h \in \overline{H}_*(V)$, we denote by $h' \in \overline{H}_*(V \setminus V')$ the value at h of the canonical homomorphism $\overline{H}_*(V) \to \overline{H}_*(V \setminus V')$. Then we say that a sequence of subsets $V_j' \subset V$ is " amenable " at infinity if there is a sequence of large open sets $U_j \subset V$ which goes to infinity, $U_j \to \infty$, such that $V_j' \subset U_j$ and such that for all sufficiently large j each V_j' is an " amenable " subset of U_j.

(2) *Cutting-of theorem.* — *Let* $V' \subset V$ *be a union of disjoint compact submanifolds (with boundaries),* $V' = \bigcup_j V_j'$, $j = 1, \ldots$ *If the set* V' *is* " *amenable* " *in* V *and if the sequence* V_j' *is* " *amenable* " *at infinity, then for all* $h \in \overline{H}_i(V)$, $i \geq 2$, *one has the inequalities*

$$\|h'\| \geq \|h\| \quad \text{and} \quad \|h'\|(\theta) \geq \|h\|(\theta),$$

for all $\theta \in [o, \infty]$. *Furthermore, if also all* V_j' *are closed connected orientable manifolds of codimension one in* V *and if the inclusion homomorphisms* $\pi_1(V_j') \to \pi_1(V)$ *are injective for all* $j = 1, \ldots,$ *then*

$$\|h'\| = \|h\| \quad \text{and} \quad \|h'\|(\theta) = \|h\|(\theta),$$

for all $\theta \in [o, \infty]$. *For example, for* $\dim V \geq 2$, *the simplicial volume does not change if* V' *is deleted from* V.

Corollary (Thurston [47], Soma [43]). — *Simplicial volumes of 3-manifolds do not change if the manifolds are cut along spheres and incompressible tori.*

Finally, we consider a cover of V by relatively compact open subsets, $U_j \subset V$, such that $U_j \to \infty$ (*i.e.* the cover is locally finite).

(3) *Vanishing-Finiteness theorem.* — *If the sequence* U_j *is* " *amenable* " *at infinity and if there is a large set every point of which is contained in at most* m *subsets* U_j, *for some* $m = 1, \ldots,$ *then the norm* $\| \ \|$ *is finite on the group* $\overline{H}_i(V)$ *for* $i \geq m$. *Furthermore, if also the subsets* U_j *in* V *are* " *amenable* " *for all* $j = 1, \ldots,$ *and if every point of* V *is contained in at most* m *subsets* U_j, *then* $\|h\| = o$ *for all* $h \in H_i(V)$ *and for* $i \geq m$.

Corollaries. — (A) *If* $m = \dim V$, *then under the first set of assumptions one has* $\|V\| < \infty$, *and in the second case* $\|V\| = o$.

(B) *Proof of the Isolation and Geometric finiteness theorems of section* (0.4). — With the covering technique of section (3.4), one satisfies the requirements of (A) above.

(C) Let P be a piecewise smooth subpolyhedron in V and let us consider small round spheres $S_v^{n-1} \subset V$, $n = \dim V$, around all points $v \in P$. We say that P is locally " coamenable " if the fundamental groups of the differences,

$$S_v^{n-1} \backslash P = S_v^{n-1} \backslash (P \cap S_v^{n-1}) \subset S_v^{n-1},$$

are amenable for all $v \in P$.

Examples. — If $\operatorname{codim} P \geq 3$ then P is " coamenable ".

A one-dimensional polyhedron P in $\mathbf{R}^3$ is locally " coamenable " if and only if P is a manifold.

If P is the image of a properly immersed manifold $V_0 \to V$ with normal crossings, then P is locally " coamenable ".

Let P *be a locally " coamenable " polyhedron in* V *of dimension* $\dim P \leq m - 2$ *for* $m \leq n = \dim V$. *Then the homology of a small regular neighbourhood* U *of* P *has zero* ℓ^1*-norm in the dimensions* $\geq m$, *that is* $\|h\| = 0$ *for all* $h \in \overline{H}_{i \geq m}(P)$. *Furthermore, if the manifold* V *is compact, then the* ℓ^1*-norms are finite on the homology groups* $\overline{H}_{i \geq m}(V \backslash P)$.

Indeed, for small round balls $B_v \subset V$ with center $v \in P$ the complements $B_v \backslash P$ have amenable fundamental groups. Thus we have an " amenable " covering of $U \backslash P$ by small subsets with multiplicity $m - 1$, and then, there are finer coverings of U and of $V \backslash P$ which satisfy the assumptions of the vanishing and of the finiteness theorems respectively.

Examples. — *(a)* Let V be the Cartesian product of three open manifolds. Then V can be realized by a regular neighbourhood of some $P \subset V$ with $\operatorname{codim} P \leq 3$, and so $\|V\| = 0$. In particular the proportionality theorem (see (0.3)) fails to hold for products of open hyperbolic manifolds of finite volume. (We shall rescue this theorem in section (4.5) by introducing yet another simplicial volume.)

(b) *Algebro-geometric finiteness theorem.* — *Let* V *be a non-singular complex quasi-projective algebraic variety. Then the simplicial volume of* V *is finite. In particular, the interior of a compact manifold,* $V = \operatorname{Int} \overline{V}$, *admits no quasi-projective structure if* $\|\partial \overline{V}\| \neq 0$.

Proof. — By Hironaka's theorem [30], V can be realized as the complement of a subvariety with normal crossings in a non-singular variety.

Remarks. — For *singular* varieties V one also has the fundamental class, $[V] \in \overline{H}_*(V)$, and non-singular resolutions $\widetilde{V} \to V$ are maps of degree one. Therefore $\|[V]\| \leq \|\widetilde{V}\| < \infty$.

Birational maps between non-singular projective varieties have degree one and they induce isomorphisms of the fundamental groups. It follows that the simplicial volume of such varieties is a birational invariant.

Proof of the theorems (1), (2) *and* (3). — The norms $\| \|(\theta)$ on the group $\overline{H}_*(V)$ are limits of the corresponding norms on relative homology groups, and with the dual norms on the bounded cohomology one can use the averaging techniques of section 3. However, for the most interesting ℓ^1-norm $\| \|$, there is no apparent bounded cohomology theory and so we must apply the averaging operators to ℓ^1-chains rather than to bounded cochains. Unfortunately, there are no such operators for *infinite* amenable groups. Indeed, the only invariant ℓ^1-function on an infinite group is zero.

This problem can be solved by adding to the ℓ^1-functions on an infinite group Γ *all* linear functionals on the space of bounded functions $\Gamma \to \mathbf{R}$. These functionals are, in fact, measures on the Čech-Stone compactification of Γ. In particular, positive Γ-invariant measures of total mass one are exactly our old averaging operators on Γ.

There is an elementary alternative to the Čech-Stone compactification. Namely one can replace averaging operators by *locally finite diffusion operators* as defined below.

First, we consider a probability measure μ on a group Γ, that is, a non-negative real-valued function μ on Γ such that $\|\mu\|_{\ell^1} = \sum_{\Gamma} |\mu(\gamma)| = 1$. For $\varphi \in \Gamma$ we define the " derivative " $\mu' = D_\varphi \mu$ by setting $\mu'(\gamma) = \mu(\gamma\varphi) - \mu(\gamma)$, and for subsets Φ in Γ we put $\|D_\Phi \mu\| = \sup_{\varphi \in \Phi} \|D_\varphi \mu\|_{\ell^1}$.

Next we let Γ act on a set X, we then have the (diffusion) operators, $f \mapsto \mu * f$, on ℓ^1-functions $f(x)$:

$$(\mu * f)(x) = \sum_{\Gamma} \mu(\gamma) f(\gamma^{-1}x).$$

Lemma. — *Let* Γ *be transitive on* X *and let a subset* $\Phi \subset \Gamma$ *be transitive on the support of a function* $f(x)$, *that is, for some point* $x_0 \in X$, *the* Φ*-orbit* $\Phi x_0 = \bigcup_{\Phi} \varphi x_0$ *contains the support of* f. *Then*

$$\|\mu * f\|_{\ell^1} \leq \left| \sum_{x \in X} f(x) \right| + \|D_\Phi \mu\| \, \|f\|_{\ell^1}.$$

Proof. — Let $q : \Gamma \to X$ be the quotient map, $q(\gamma) = \gamma x_0$, and let q_* denote the push-forward operator on functions $g = g(\gamma)$:

$$(q_* g)(x) = \sum_{\gamma \in q^{-1}(x)} g(\gamma), \quad \text{for } x \in X.$$

Observe that q_* commutes with the diffusions $\mu *$,

$$q_*(\mu * g) = \mu * (q_* g),$$

where $(\mu * g)(\gamma) = \sum_{\lambda \in \Gamma} \mu(\lambda) g(\gamma^{-1}\lambda)$. Now, for the function $f = f(x)$, there is a function $g = g(\gamma)$ with support in Φ, such that $q_* g = f$, and so $||\mu * f||_{l^1} = ||q_*\mu * g||_{l^1} \leq ||\mu * g||_{l^1}$. Next,

$$||\mu * g||_{l^1} = \sum_{\gamma \in \Gamma} |\sum_{\lambda \in \Gamma} \mu(\lambda) g(\gamma^{-1}\lambda)|$$

$$= \sum_{\gamma \in \Gamma} |\sum_{\lambda \in \Gamma} \mu(\gamma\lambda) g(\lambda)| \leq \sum_{\gamma \in \Gamma} (|\sum_{\lambda \in \Phi} (\mu(\gamma\lambda) - \mu(\gamma)) g(\lambda)| + |\sum_{\lambda \in \Gamma} \mu(\gamma) g(\lambda)|$$

$$\leq \sum_{\lambda \in \Phi} |g(\lambda)|\, ||D_\lambda \mu||_{l^1} + |\sum_{\lambda \in \Gamma} g(\lambda)| \leq ||D_\Phi \mu||\, ||g||_{l^1} + |\sum_{\lambda \in \Gamma} g(\lambda)|,$$

q.e.d.

(A) *Corollary.* — *Let the group* Γ *be amenable. Then for an arbitrary finite set* $S \subset X$ *and for any given* $\varepsilon > 0$ *there exists a probability measure* μ *on* Γ *with finite support such that all functions* $f(x)$ *with support in* S *satisfy*

$$||\mu * f||_{l^1} \leq |\sum_{x \in S} f(x)| + \varepsilon.$$

Proof. — As Γ is amenable, one can, for every finite set $\Phi \subset \Gamma$ and for any given $\varepsilon > 0$, find a finite measure μ on Γ for which $||D_\Phi \mu|| \leq \varepsilon$ (see [19]) and so the lemma applies.

(B) *Example.* — Let K be a simplicial multicomplex and let Γ be a group of simplicial automorphisms of K. Then Γ also acts on finite simplicial chains $c = \sum_i r_i \sigma_i$ in K,

$$\gamma c = \sum_i r_i . \gamma\sigma_i,$$

and we write $\mu * c = \sum_{\gamma \in \Gamma} \mu(\gamma)\gamma c$.

Suppose that for each simplex σ_i in some chain $c = \sum_i r_i \sigma_i$ there is a transformation $\gamma_i \in \Gamma$ which sends σ_i onto itself with the reversal orientation, $\gamma_i \sigma_i = \bar{\sigma}_i$. Then, if the group Γ is amenable one can " diffuse " c by some measure μ on Γ, that is, one can make $||\mu * c|| \leq \varepsilon$ for some $\mu = \mu(c, \varepsilon)$. Indeed, one can write c as a cochain c' on K which is supported by the set of the simplices $\{\sigma_i, \bar{\sigma}_i\}$ and such that $c'(\sigma_i) = -c'(\bar{\sigma}_i) = \frac{1}{2} r_i$. Then the sums of values of c' over the Γ-orbits of the oriented(!) simplices σ_i are zero. According to (A), there is a finite measure μ on Γ such that $||c' = \mu * c'||_{l^1} \leq \varepsilon$, and since $\mu * c = \sum_\sigma \tilde{c}'(\sigma)\sigma$, where σ runs over all oriented simplices $\sigma \subset K$, we also have $||\mu * c||_{l^1} \leq \varepsilon$. In particular, if c is an n-dimensional cycle whose homology class $[c]$ is invariant under Γ, then $||[c]|| = 0$. Thus we get an alternative proof of the vanishing theorem of section (3.1) (compare with (3.3)).

Let us generalize (B) to locally finite chains. We start with an arbitrary action of Γ on X with countably many orbits, called $X_j \subset X$, $j = 1, \ldots$ We say that the

action of Γ on X is " *locally finite* " if there exists subgroups $\Gamma_j \subset \Gamma$, $j = 1, \ldots$, with the following two properties:

(1) The group Γ_j is transitive on the set X_j for $j = 1, \ldots$

(2) The actions of the groups Γ_j are " asymptotically disjoint ": for every $j_0 = 1, \ldots$, there is an index $k = k(j_0)$ such that the *supports* of Γ_{j_0} and Γ_j do not intersect for $j \geq k$, that is, every point $x \in X$ is either kept fixed by all $\gamma \in \Gamma_{j_0}$ or x is kept fixed by all $\gamma \in \Gamma_j$.

Now, we define *locally finite* diffusion operators on X that are given by sequences of finite probability measures μ_j on Γ_j for all $j = 1, \ldots$ Namely, for a function $f(x)$ we inductively define $f_1 = \mu_1 * f, \ldots, f_j = \mu_j * f_{j-1}$. According to (2) above, the value $f(x)$ for every fixed point x does not depend on j for large j and so we put

$$\mu * f = \lim_{j \to \infty} f_j$$

for $\mu = (\mu_j)$ with $j = 1, \ldots$ Observe that this diffusion acts separately on each orbit X_j. Namely

$$\sum_{x \in X_j} (\mu * f)(x) = \sum_{x \in X_j} f(x), \quad \text{for all } j = 1, \ldots,$$

and

$$||\mu * f \mid X_j||_{l^1} \leq ||f \mid X_j||_{l^1},$$

where

$$||f \mid X_j||_{l^1} \overset{\text{def}}{=} \sum_{x \in X_j} |f(x)|.$$

Suppose that f is a " *locally finite* " function, that is, its supports on all orbits X_j are finite sets. Then, clearly, the diffused function $\mu * f$ is also " locally finite ". Furthermore, if $\sum_{x \in X_j} f(x) = 0$ for $j \geq j_0$ and if the groups Γ_j are amenable for $j \geq j_0$, then for an arbitrary positive sequence $\varepsilon_{j_0}, \varepsilon_{j_0+1}, \ldots$, there is a locally finite diffusion μ such that

$$||\mu * f \mid X_j||_{l^1} \leq \varepsilon_j \quad \text{for } j \geq j_0.$$

Indeed, this follows from (A) above.

Now, we are ready to extend the techniques of section 3 to open manifolds V. The modifications needed for the proofs of the theorems (1), (2) and (3) are quite similar and so we discuss below only the last case.

Proof of the Vanishing-finiteness theorem. — To each set $U_j \subset V$, we assign a large set $W_j \subset V$ which contains U_j in such a way that, for $j > j_0$, U_j is an amenable subset of W_j, and that $W_j \to \infty$. In section (3.2), we constructed a canonical multicomplex K(V) consisting of all singular simplices $\sigma : \Delta \to V$ which are injective on the sets of vertices of Δ. This set of vertices is sent by σ into the union of some sets U_j, call them $U_{j_0}, \ldots, U_{j_m}$, and we say that σ is an *admissible* simplex if the image $\sigma(\Delta) \subset V$

is contained in the union $W_{j_0} \cup \ldots \cup W_{j_m}$. Then, we consider a subcomplex $K \subset K(V)$ with the following four properties:

(1) All simplices in K are admissible.

(2) K is *complete*: for every admissible simplex $\sigma_0 : \Delta \to V$ with boundary in K there is a homotopy by admissible simplices with fixed boundary, $\sigma_t : \Delta \to V$ for $t \in [0, 1]$, such that σ_1 is contained in K.

(3) The vertices of K lie in the union of some fixed disjoint sets $V_j \subset U_j$, $j = 1, \ldots$

(4) K is minimal: it contains no proper subcomplexe which satisfies (1) and (2) and has the same vertices as K.

As every such K consists of some singular simplices $\sigma : \Delta \to V$, it is equipped with a canonical map $S : K \to V$. Furthermore, the direct product of the groups $\Pi(V_j, W_j)$ (see (3.3)) acts on the 1-skeleton of K and every such action extends (not uniquely!) to an automorphism $\gamma : K \to K$, such that the maps $S : K \to V$ and $S \circ \gamma : K \to V$ can be joined by an admissible homotopy, that is a homotopy of maps which are admissible on all simplices in K. We denote by $\Gamma(K)$ the group generated by all these automorphisms γ.

Now, let L be a triangulation of V such that each star of L is contained in one of the sets U_j, and let K be a multicomplex which contains L and satisfies the properties (1)-(4). Denote by $K' \subset K$ the $\Gamma(K)$-orbit of L,

$$K' = \bigcup_{\gamma \in \Gamma} \gamma(L).$$

The action of $\Gamma = \Gamma(K)$ is locally finite on the set of all oriented simplices in K'. Indeed, with every set U_j one first considers the union of all $U_{j'}$ which intersect U_j, call this union $\tilde{U}_j$, and then, one takes as $\Gamma_j \subset \Gamma$, for $j = 1, \ldots$, the group of all transformations which fix the simplices $\Delta \subset K'$ whose images in V do not intersect $\tilde{U}_j$.

Now, under the assumptions of the vanishing theorem, every m-simplex in L has an edge in one of the sets U_j and so some automorphism reverses the orientation of σ, that is $\gamma\sigma = \bar{\sigma}$. Therefore, for every m-chain

$$c = \Sigma r_i \sigma_i \in C_*(L) \subset C_*(K')$$

the corresponding cochain c', given by $c'(\sigma_i) = \frac{1}{2} r_i$ and $c'(\sigma_i) = -\frac{1}{2} r_i$, has zero sums over the orbits of Γ. Next, for the proof of the vanishing theorem, one may assume all U_j to be " amenable " subsets of W_j and then the groups Γ_j are amenable for all $j = 1, \ldots$ Finally, locally finite diffusions make the ℓ^1-norm of c, or rather of $\mu * c$, as small as one wishes, and now one must only show that the diffused *cycles* $\mu * c$ with $||\mu * c||_{\ell^1} \leq \varepsilon$ are, in fact, homologous to c, in the sense that $c - \mu * c = \partial \tilde{c}$ for some *locally finite* $(m+1)$-chain $\tilde{c}$. Recall that the diffusion $\mu * c$ for $\mu = \{\mu_j\}$ is the limit of chains $c_j = \mu_j * c_{i-1}$ where the operator $\mu_j *$ is obtained by combining some cycles γc_{j-1} with

$\gamma \in \Gamma_j$. The transformations in Γ_j are homotopies which have supports in some sets $\widetilde{W}_j$ such that $\widetilde{W}_j \to 0$ for $j \to \infty$. Therefore, one has $c_{j-1} - c_j = \partial \tilde{c}_j$ for some locally finite chains $\tilde{c}_j$ with supports in the sets $\widetilde{W}_j$, and then one takes

$$\tilde{c} = \sum_{j=1}^{\infty} \tilde{c}_j.$$

Thus the vanishing theorem is proved.

To prove the finiteness part of the theorem, one also applies diffusion operators but now only with the groups Γ_j for $j \geq j_0$. This makes the chains c as small as one wishes outside a fixed compact subset of V.

4.3. Diffusion of chains in Riemannian manifolds

We prove here the geometric inequalities announced in section (0.5) by combining the diffusion of chains with the smoothing operators of section 2.

For a singular simplex in a Riemannian manifold, $\sigma : \Delta \to V$, and an open set $U \subset V$, we denote by $\mathrm{mass}(\sigma \cap U)$ the mass of the "intersection" map $\sigma \mid \sigma^{-1}(U) : \sigma^{-1}(U) \to U \subset V$. For a singular chain $c = \sum_i r_i \sigma_i$, we put

$$\mathrm{mass}(c \cap U) = \sum_i |r_i| \, \mathrm{mass}(\sigma_i \cap U).$$

We denote by $B_U(R)$ the R-neighbourhood of U for any $R \geq 0$, and we abbreviate

$$\mathrm{mass}(c \cap U + R) \overset{\mathrm{def}}{=} \mathrm{mass}(c \cap B_U(R)).$$

Next, we restrict chains $c = \sum_i r_i \sigma_i$ to subsets $U \subset V$ by taking the sum $\sum r_i \sigma_i$ only over those indices i for which the images $\sigma_i(\Delta) \subset V$ intersect the set U, and we denote these restricted chains by $c \mid U$.

Finally, for the derivative of the growth function $\ell_v(R)$ of section (2.5), we put

$$[\ell'(R) \mid U] = \sup_{v \in B_U(R)} \ell'_v(R).$$

(A) *Theorem.* — *Let* V *be a complete Riemannian manifold of non-positive sectional curvature and let* $c = \sum_i r_i \sigma_i$ *be a locally finite m-cycle in* V. *Then, for arbitrary positive numbers* $R > 0$ *and* $\varepsilon > 0$, *there exists another locally finite m-cycle* $c' = c'(R, \varepsilon) = \sum_j r'_j \sigma'_j$ *in* V *with the following four properties*:

(1) c' *is homologous to* c.

(2) c' *is a straight cycle*: *the lifts* $\tilde{\sigma}'_j$ *of simplices* σ'_j *to the universal covering* $\tilde{V}$ *of* V *are straight geodesic simplices* (see (1.2)) *and each* $\tilde{\sigma}'_j$ *is contained in a ball of radius* $R + \varepsilon$.

(3) *For all subsets* U *of* V, *the cycle* c' *satisfies*

$$||c' \mid U||_{\ell^1} \leq (1 + \varepsilon) \, m! \, [\ell'(R) \mid U]^m \, \mathrm{mass}(c \cap U + R + \varepsilon).$$

(4) *Furthermore, if c represents the fundamental class of* V, *then the last inequality can be sharpened as follows*

$$||c' \mid U||_{l^1} \leq C_n(1+\varepsilon)\int_B (l'_v(R))^n\, dv,$$

for $B = B_U(R+\varepsilon)$ *and for* $C_n = \Gamma(n/2)/\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right) < 1$, $n = \dim V$.

Proof. — Let first $\mathscr{S} : Y \to \mathscr{M}^+$ be a Π-invariant smoothing operator (see (2.4)) on the universal covering $Y = \tilde{V} \to V$ for $\Pi = \pi_1(V)$. Suppose furthermore, that the smoothing $\mathscr{S}$ is locally finite: there is a discrete subset in Y which supports the measure $\mathscr{S}(y)$ for all $y \in Y$. Let also each measure $\mathscr{S}(y)$ be supported in the ball $B_y(R) \subset Y$. With such an $\mathscr{S}$, we have the following diffusion operator of straight Π-invariant chains $\tilde{c}$ in Y that is the dual to the smoothing operator on cochains (see (2.4)). Recall (see (1.2)) that every straight simplex $\tilde{\Delta}$ in Y is uniquely determined by the ordered set of its vertices, $y_0, \ldots, y_m \in Y$, and so finite chains of ordered simplices are interpreted as finite measures in the Cartesian product

$$\underbrace{Y \times Y \times \ldots \times Y}_{m+1}.$$

Now, we assign to the simplices $\tilde{\Delta} = \tilde{\Delta}(y_0, \ldots, y_m)$ the normalized Cartesian products of the measures $\mathscr{S}(y_0), \ldots, \mathscr{S}(y_m)$,

$$\mathscr{S} * \tilde{\Delta} = \prod_{i=0}^{m} ||\mathscr{S}(y_i)||_{l^1}^{-1}\, \mathscr{S}(y_i).$$

This diffusion extends by linearity to the chains of straight ordered simplices,

$$\tilde{c} = \sum_j r_j \tilde{\Delta}_j \to \mathscr{S} * \tilde{c} = \sum_j r_j\, \mathscr{S} * \tilde{\Delta}_j.$$

Finally, for *oriented* simplices $\tilde{\Delta}$, we take the $(m+1)!/2$ ordered simplices which have the same vertices as $\tilde{\Delta}$ with orders compatible with the orientation of $\tilde{\Delta}$, call them $\tilde{\Delta}_\nu$ with $\nu = 1, \ldots, \frac{(m+1)!}{2}$, and we put

$$\mathscr{S} * \tilde{\Delta} = (2/(m+1)!) \cdot \sum_\nu \mathscr{S} * \Delta_\nu.$$

From now on, we only deal with locally finite chains c of oriented simplices and the diffusion is denoted by $\tilde{c} \to \mathscr{S} * \tilde{c}$.

A small straight m-dimensional simplex $\tilde{\Delta}$ in Y is called *fat* if $\text{mass}\,\Delta \geq \text{const}(\text{diam}\,\tilde{\Delta})^m$, where for " const " one may use an arbitrarily chosen small positive number which only must be kept fixed as the diameter, diam $\tilde{\Delta}$, goes to zero.

Now, the proposition of section (2.4) implies the following dual

Proposition. — *For any given* $\delta > 0$ *all sufficiently small fat m-dimensional simplices* $\widetilde{\Delta}$ *satisfy*

$$||\mathscr{S} * \widetilde{\Delta}||_{\ell^1} \leq (1 + \delta)[\mathscr{S}]^m m! \operatorname{mass} \widetilde{\Delta}.$$

Furthermore, with the integral inequality of (2.4), *one has for* $m = n = \dim V$,

$$||\mathscr{S} * \Delta||_{\ell^1} \leq (1 + \delta)([\mathscr{S}]^*)^n n! \operatorname{mass} \widetilde{\Delta}.$$

Remarks. — *(a)* For non-fat small simplices of diameter $\leq \varepsilon$, we can only claim the inequality

$$||\mathscr{S} * \widetilde{\Delta}||_{\ell^1} \leq (1 + \delta)[\mathscr{S}]^m \varepsilon^m.$$

(b) The inequalities of the Proposition and of the remark *(a)* are purely local: if the simplices $\widetilde{\Delta}$ are supported in a subset $Y' \subset Y$, then one may replace the above " norms " $[\mathscr{S}]$ and $[\mathscr{S}]^*$ by the supremums $\sup_{y \in Y'} [\mathscr{S}(y)]$ and $\sup_{y \in Y'} [\mathscr{S}(y)]^*$.

Now, the theorem is reduced to a locally finite approximation of the smoothings $\mathscr{S}_{R,\varphi}$ in the theorem of (2.5). We construct such an approximation with a discrete, sufficiently dense subset Z in V, for example with a set $Z = Z_\varphi$ which intersects the balls $B_v(\varphi(v)) \subset V$ for all $v \in V$. We take the pullback of $Z \subset V$ under the covering map $Y \to V$, called $Z \subset Y$ and we assign to every point $\tilde{z} \in Z$ the measure of the set of those points $y \in Y$ for which $\operatorname{dist}(y, \tilde{z}) \leq \operatorname{dist}(y, \tilde{z}')$ for every $\tilde{z}' \neq \tilde{z}$ in $\tilde{Z}$. We call this measure $\mu(\tilde{z})$ and, finally, we send the points $y \in Y$ to the measures $\widetilde{\mathscr{S}}_{R,\varphi}(y)$ which are supported in $Z \subset Y$ for all $y \in Y$ and whose weight at $\tilde{z} \in Z$ is $\mu(\tilde{z})\mathscr{S}_{R,\varphi}(y, \tilde{z})$. When $\varphi \to 0$, this smoothing $y \to \widetilde{\mathscr{S}}_{R,\varphi}(y)$ satisfies in the limit the same relation as our old $\mathscr{S}_{R,\varphi}$ and so the corresponding diffusion of chains of *small fat* simplices satisfies the requirements (2)-(4).

Next, we modify the original cycle $c = \sum_i r\,\sigma_i$ as follows. First, we make all simplices $\sigma_i : \Delta \to V$ smooth by a small perturbation of c. Then, we subdivide the smoothed σ_i into very small simplices σ_{i_j} such that after straightening up these σ_{i_j} almost all of them become fat: " non-fatness " is allowed only for those σ_{i_j} which are adjacent to the boundary $\partial\sigma_i$ of σ_i. The mass of the straightened chain $\sum_{i_j} r_i \sigma_{i_j}$ is close to mass c and the lifts of non-fat simplices to $Y \to V$, called $\tilde{\sigma}_{i_k}$, satisfy

$$\sum_k (\operatorname{diam} \sigma_{i_j})^m \leq \varepsilon_i \operatorname{mass} \sigma,$$

where the numbers $\varepsilon_i > 0$ can be made arbitrarily small with sufficiently fine subdivisions of the chain c. Therefore, in the diffusion of the straightened chain $\sum_{i_j} r_i \sigma_{i_j}$ only the *fat* simplices substantially contribute to the ℓ^1-norm and thus our theorem is proved.

(B) *A Generalization to* $K(V) < K^2$. — Let the sectional curvature $K(V)$ be bounded from above, $K(V) < K^2$. Then, for every point $v \in V$ the exponential map $\exp_v : T_v \to V$ is an immersion on the ball $B_0(K^{-1}\pi) \subset T_v(V)$ (see [10]) and we equip this ball with the metric induced from V. The resulting (non-complete!) Riemannian manifold is denoted by $\widetilde{B}_v$. The exponential map sends this $\widetilde{B}_v$ to V locally isometrically, and the center $\tilde{v}$ of $\widetilde{B}_v$ is sent to v.

Now, every pair of points in the ball $B_{\tilde{v}}(R) \subset \widetilde{B}_v$, for $R \leq \frac{1}{2}K^{-1}\pi$, can be joined by a unique geodesic in $B_{\tilde{v}}(R)$ (see [10]), and so one has straight geodesic simplices in the balls $B_{\tilde{v}}(R)$ for all $v \in V$ and $R \leq \frac{1}{2}K^{-1}\pi$. The projections of these simplices to V are called *straight simplices of radius* $\leq R$.

Next, we modify the definition of $\ell'_v(R)$ by taking the balls $B_{\tilde{v}}(R)$ and by putting $\tilde{\ell}_v(R) = \log \mathrm{Vol}\, B_{\tilde{v}}(R)$. Then, we have the corresponding functions $\tilde{\ell}'_v(R)$ and $[\tilde{\ell}'(R) \mid U]$.

Finally, we claim that *the theorem* (A) *holds for all manifolds* V *as long as* $R < \frac{1}{2}K^{-1}\pi$. Namely, there is a cycle c' which is built of straight simplices of radius $\leq R$ and this c' satisfies the properties (1), (3) and (4) with $\tilde{\ell}'$ in place of ℓ'.

Proof. — The smoothing we used in (A) is, in fact, a map which assigns to the points $v \in V$ measures in the balls $B_{\tilde{v}}(R)$ in the universal covering $Y = \widetilde{V} \to V$, for some lifts v of $\tilde{v}$. With this new interpretation, the smoothing makes perfect sense at the points $v \in V$, with the maps $\exp_v : \widetilde{B}_v \to V$ instead of the universal covering. Finally, as the estimates we used in (A) were local, they extend to the balls $\widetilde{B}_v$, q.e.d.

(C) *Manifolds with* $|K| \leq 1$. — *If* $K(V) \geq -1$ *then Ricci* $V \geq 1 - n$, *and so* $\tilde{\ell}'_v(R) \leq \mathrm{const}_n R^{-1}$ *for all* $R \leq \pi$ *and* $v \in V$. *Therefore,* (B) *above implies the estimate* $\mathrm{Min\,Vol}(V) \leq \mathrm{const}_n \|V\|$ *claimed in section* (0.5).

Next, we turn to the

Proof of the injectivity radius estimate of section (0.5).

Lemma. — *For every* $n = 1, \ldots,$ *there exists a positive constant* $\varepsilon_n \in (0, 1)$, *called Margulis' constant, such that the balls* $B_v(\varepsilon)$ *in an n-dimensional manifolds* V *with* $|K(V)| \leq 1$ *are "amenable" subsets in the concentric unit balls* $B_v(1) \supset B_v(\varepsilon)$ *for all* $\varepsilon \leq \varepsilon_n$ *and* $v \in V$.

Proof. — Indeed, the image of $\pi_1(B_v(\varepsilon))$ in $\pi_1(B_v(1))$ contains a nilpotent subgroup of finite index (see [7]).

Next, for $|K(V)| \leq 1$, the exponential map $\exp_v B_0(1) \to V$ covers each point in the ball $B_v(1/2) \subset V$ approximately ρ_v^{-1} times for $\rho_v = \mathrm{Inj\,Rad}_v(V)$, and so $C\rho_v \leq \mathrm{Vol}(B_v(1)) \leq C^{-1}\rho_v$ for all $v \in V$ and for some universal constant $C = C_n > 0$ (see [11]).

Finally, for manifolds V with $|K(V)| \leq 1$ and for all $\varepsilon \geq 0$, the set $U_\varepsilon \subset V$ of those points $v \in V$ for which $\rho_v \geq \varepsilon$ contains the ε-neighbourhood of the set U_δ for $\delta \leq C'_n \varepsilon$ and for another universal constant $C'_n > 0$ (see [11]). We now apply the covering technique of section (3.4) and, for some universal positive $\varepsilon = \varepsilon(n) \in (0, \varepsilon_n)$, we find a cover of V by small open subsets U_j, $j = 1, \ldots,$ with the following two properties

(1) All U_j are "amenable" subsets in their respective $\frac{1}{2}$-neighbourhood $B_{U_j}(1/2) \supset U_j$.

(2) The inequality $\rho_v \leq \varepsilon$ implies, for all $v \in V$, that the ball $B_v(3\varepsilon) \subset V$ is contained in one of the sets U_j and that at most $n = \dim V$ sets U_j intersect $B_v(3\varepsilon)$.

We represent the fundamental class of V by a cycle c' as in Theorem (B), with $R = \varepsilon$, and we observe that with (1) and (2) above this c' splits into a sum, $c' = c_1 + c_2$, where the chain c_1 has its support in the set U_ε and $||c_1|| \leq \text{const}\, \text{Vol}(U_\varepsilon)$ for some universal const $= \text{const}(\varepsilon, n)$, while each simplex σ of the chain c_2 intersects no more than n subsets U_j and each σ is contained in one of the sets U_j. Therefore, the locally finite diffusion of section (4.2) makes the ℓ^1-norm of c_2 as small as we wish and so the norm of the diffused cycle c' is bounded from above by const $\text{Vol}(U_\varepsilon)$, q.e.d.

(D) *Manifolds with* $\text{Ricci}(V) \geq -1/(n-1)$. — We start with another generalization of the growth function $\ell'(R)$. Take a ball of radius $\widetilde{R}$ in $V > R$, call it $B_v(\widetilde{R}) \subset V$, and let $\widetilde{B}_v(\widetilde{R}) \to B_v(\widetilde{R})$ be the universal covering. Take a point $\tilde{v} \in \widetilde{B}_v(\widetilde{R})$ over v and consider the ball of radius R around $\tilde{v}$, called $B_{\tilde{v}}(R) \subset \widetilde{B}_v(\widetilde{R})$. Put

$$\ell_v(R, \widetilde{R}) = \log \text{Vol}(B_{\tilde{v}}(R)).$$

Observe that this function is decreasing in $\widetilde{R}$, but it is not continuous: the jumps may happen as the topology of the ball $B_v(\widetilde{R}) \subset V$ changes.

Next, we define the following "regularized" derivative $\frac{d\ell}{dR}$ (compare with (2.5)),

$$\ell'_v(R, \widetilde{R}) = \limsup_{\substack{\varepsilon \to 0,\ \tilde{\varepsilon} \to 0 \\ w \to v}} [\ell_w(R, \widetilde{R} \pm \tilde{\varepsilon}) - \ell_w(R - \varepsilon, \widetilde{R} \pm \tilde{\varepsilon})].$$

If $\text{Ricci}\, V \geq -1/(n-1)$ then Bishop's inequality gives a universal upper bound for $\ell'_v(R, \widetilde{R})$, namely $\ell'_v(R, \widetilde{R}) \leq \ell'_v(R)$, where ℓ'_0 is the corresponding function in the hyperbolic space. In particular, $\ell'_v(R, \widetilde{R}) \leq \text{const}_n R^{-1}$ for $R \leq 1$ and $\lim_{R \to \infty} \sup \ell'_v(R, \widetilde{R}) \leq 1$.

Then (compare with (A)), we put

$$[\ell'(R, \widetilde{R}) \mid U] = \sup_{v, \bar{R}} \ell'_v(R, \bar{R}),$$

where v runs over the R-neighbourhood $B_U(R) \subset V$ of the set U and $\bar{R}$ runs over the interval $[R, \widetilde{R}]$.

Finally, for a singular simplex $\sigma : \Delta \to V$, we write $\text{length}(\sigma) \leq \lambda$ if all edges of σ have length $\leq \lambda$. Furthermore, we write $\text{Rad}\,\sigma \leq \rho$ if the image $\sigma(\Delta) \subset V$ is contained in a ball $B_v(\rho) \subset V$ for some $v \in V$.

Now, we come to our most general relation between the growth function $\ell' = \ell'(R, \widetilde{R})$ and the ℓ^1-norm on the homology of locally finite chains $c = \sum_i r_i \sigma_i$ in V.

Main Technical Theorem. — *Let* V *be a complete Riemannian manifold and let* $c = \sum_i r_i \sigma_i$ *be a locally finite m-cycle in* V. *Then for arbitrary positive numbers,* $\varepsilon > 0$, $R > 0$ *and* $\widetilde{R} > R$, *there exists another locally finite m-cycle* $c' = \sum_j r'_j \sigma'_j$ *in* V *with the following four properties*:

(1) c' *is homologous to* c.

(2) $\text{length}\,\sigma'_j \leq 2R + \varepsilon$ *and* $\text{Rad}\,\sigma'_j \leq \widetilde{R} + 2R + \varepsilon$ *for all* $j = 1, \ldots$

(3) *For all open sets* $U \subset V$ *we have, with the notation* $c' \mid U$ *and* $c \cap U + R + \varepsilon$ *of* (A) *and with* $\ell' = [\ell'(R, \widetilde{R}) \mid U]$,

$$||c' \mid U||_{\ell^1} \leq (1 + \varepsilon)\, m!\, (\ell' + 4(\widetilde{R} - R)^{-1})\, \text{mass}(c \cap U + \widetilde{R} + 2R + \varepsilon).$$

(4) *If* c *represents the fundamental class of* V, *then, for*

$$\bar{\ell}_v = \bar{\ell}_v(R, \widetilde{R}) = \sup_{\bar{R} \in [R, \widetilde{R}]} \ell'_v(R, \bar{R})$$

and for the $(\widetilde{R} + 2R + \varepsilon)$*-neighbourhoods* $B = B_U(\widetilde{R} + 2R + \varepsilon)$ *of the sets* U, *we have*

$$||c' \mid U||_{\ell^1} \leq (1 + \varepsilon) n! \int_B (C_n \bar{\ell}_v + 4(\widetilde{R} - R)^{-1})^n\, dv$$

with

$$C_n = \Gamma(n/2)/\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right) < 1, \qquad n = \dim V.$$

Proof. — First, we formalize the local conception of smoothing (see (B)) as follows An *étale measure* μ over V is defined by the following data:

(i) A collection of étale domains $(\widetilde{U}, \widetilde{p})$ over V. These are n-dimensional manifolds U for $n = \dim V$ and immersions $\widetilde{p} : \widetilde{U} \to V$.

(ii) There are maps between some pairs of domains $(\widetilde{U}, \widetilde{p})$ and $(\widetilde{U}', \widetilde{p}')$ that are immersions $\widetilde{q} : \widetilde{U} \to \widetilde{U}'$ such that $\widetilde{p} = \widetilde{p}' \circ q$. If such a map $\widetilde{q}$ exists, we say that $\widetilde{U}$ *lies over* $\widetilde{U}'$. We assume that for any two domains U_1 and U_2 there is a third domain U which lies over both U_1 and U_2.

(iii) Finally, each domain $\widetilde{U}$ carries a positive measure, called $\widetilde{\mu}$. We require the measure $\widetilde{\mu}'$ in all domains $\widetilde{U}'$ over $\widetilde{U}$ to be lifts of $\widetilde{\mu}$, that is $\widetilde{\mu} = \widetilde{q}'_* \widetilde{\mu}'$, where $\widetilde{q}'_*$ denotes the push-forward map on measures for the immersion $\widetilde{q}' : \widetilde{U}' \to \widetilde{U}$. Observe that $\int_{\widetilde{U}'} \widetilde{\mu}' = \int_{\widetilde{U}} q'_* \widetilde{\mu}$ and so for *positive* measures $\widetilde{\mu}$ the L^1-norms $\int_{\widetilde{U}} |\widetilde{\mu}|$ are independent of the particular domain $\widetilde{U}$. Then, one may speak of the L^1-norm $||\mu||$ of the étale measure μ over V.

An *étale smoothing* $\mathscr{S} = \mathscr{S}(v)$ over V is a map which sends points $v \in V$ to étale measures $\mathscr{S}(v)$ over V. An instance of that are the measures defined in (B) on exponential balls $\widetilde{B}_v(R) \to V$.

We assume, furthermore, that for every point $v_0 \in V$ there is an étale domain $\widetilde{U} \to V$ such that the measures $\widetilde{\mathscr{S}}(v) = \mathscr{S}(v) \mid \widetilde{U}$ are simultaneously defined on $\widetilde{U}$ for all v in some neighbourhood $U_0 \subset V$ of v_0.

The map $\widetilde{\mathscr{S}}$ sends points of U_0 to ordinary measures on $\widetilde{U}$, and if this $\widetilde{\mathscr{S}}$ is smooth, then we define, as in section (2.4), the quantities $[\widetilde{\mathscr{S}}(v)] = [\mathscr{S}(v) \mid \widetilde{U}]$ and $[\widetilde{\mathscr{S}}(v)]^* = [\mathscr{S}(v) \mid \widetilde{U}]^*$ for all $v \in U_0$. We call these $\widetilde{U}$ *domains of definition* of $\mathscr{S}$ *near* v_0 and we finally define $[\mathscr{S}(v_0)]$ and $[\mathscr{S}(v_0)]^*$ as $\inf_{\widetilde{U}} [\mathscr{S}(v_0) \mid \widetilde{U}]$ and $\inf_{\widetilde{U}} [\mathscr{S}(v_0) \mid \widetilde{U}]^*$ where U runs over all domains of definition.

Next, we assign to each point $v \in V$ a neighbourhood of v, say $W_v \subset V$, and we call a singular simplex $\sigma : \Delta \to V$ *admissible* if it has distinct vertices, $v_0, \ldots, v_m \in V$, and if its image $\sigma(\Delta) \subset V$ is contained in the union $W_{v_0} \cup \ldots \cup W_{v_m}$. We take a subcomplex K in the canonical simplicial multicomplex K(V) (compare with the proof of the vanishing theorem in (3.3)) with the following three properties:

(1) All simplices in K are admissible.

(2) K is complete: every admissible simplex with boundary in K is " admissibly " homotopic to a simplex in K.

(3) K is minimal: it contains no proper subcomplex with the properties (1) and (2).

Every such complex K is uniquely determined, up to an admissible isomorphism, by its zero skeleton $K^\circ \subset V$: an isomorphism $I : K \to K'$ is admissible if the canonical map $S : K \to V$ (see (3.2)) is " admissibly " homotopic to $S' : K' \to V$ with the zero-skeleton kept fixed. Observe, that the group Γ_1 of all admissible automorphisms of K which keep the one-skeleton fixed, is amenable on all skeletons of K (compare (3.3)).

Finally, we associate to étale smoothings $\mathscr{S}$ over V diffusions of small simplices to chains in K. To do this, we need all domains of definition of the étale measures $\mathscr{S}(v)$ to be connected and simply connected, for $v \in V$, and we require the images $\widetilde{p}(\widetilde{U}) \subset V$ of all these domains to lie in the intersections $\bigcap_v W_v$ for v running over $\widetilde{p}(\widetilde{U})$.

Let Δ be a small straight simplex in V and let $\widetilde{\Delta} \subset \widetilde{U}$ be a lift of Δ to an étale domain, such that the measures $\widetilde{\mathscr{S}}(v) = \mathscr{S}(v) \mid \widetilde{U}$ are simultaneously defined for all $v \in \Delta$. Then the vertices $\widetilde{v}_0, \ldots, \widetilde{v}_m$ of $\widetilde{\Delta}$ are diffused to the points $\widetilde{v}_0', \ldots, \widetilde{v}_m'$ in $\widetilde{U}$ weighted with some measures according to $\widetilde{\mathscr{S}}$. Each vertex $\widetilde{v}_k$, $k = 1, \ldots, m$, is joined with $\widetilde{v}_k'$ by a path $\widetilde{e}_k$ and every edge $[\widetilde{v}_k, \widetilde{v}_l]$ of $\widetilde{\Delta}$ goes to the composition of paths

$$[\widetilde{v}_k', \widetilde{v}_l'] = \widetilde{e}_k * [\widetilde{v}_k, \widetilde{v}_l] * \widetilde{e}_l.$$

The projection of this path back to V, is an admissible singular one-simplex and so there is a unique one-simplex in K, say $[v'_k, v'_\ell]$, homotopic to the projected path $[\tilde{v}'_k, \tilde{v}'_\ell]$. Thus, we diffused Δ into a measure on the set of m-simplices in the quotient complex K/Γ_1, that is a "smeared chain" in K/Γ_1. It would be an actual finite chain if the measures $\mathscr{S}(v_k)$, $k = 0, \ldots, m$, were finite. We agree to normalize this smeared chain by dividing it by the product $\prod_{k=0}^{m} ||\mathscr{S}(v_k)||$ and the result is denoted by $\mathscr{S} * \Delta/\Gamma_1$.

Remark. — Among the diffused simplices, there may be some with coinciding vertices and those do not belong to K. One could add such degenerate simplices to K, but we just ignore them since their total mass is zero for measures $\mathscr{S}(v)$ with continuous densities and this mass remains negligibly small for our further approximations of continuous measures by discrete ones.

Now, let $c = \sum_i r_i \Delta_i$ be a chain of small simplices Δ_i in V. Then with an étale smoothing $\mathscr{S}$ over V, we have the diffusion

$$c \mapsto \mathscr{S} * c/\Gamma_1 = \sum_i r_i \mathscr{S} * \Delta_i/\Gamma_1.$$

We suppose that the measures $\tilde{\mathscr{S}}(v)$ have continuous densities on $\tilde{U}$ and we approximate the diffusion $\mathscr{S}^*$ by discrete diffusions $\mathscr{S}_0^*$. Namely, we consider discrete sets $Z \subset V$ with the weights $\mu(z)$ attached to all $z \in Z$ as in the proof of (A) and we replace the measures $\tilde{\mathscr{S}}(v)$ on the domains $\tilde{p}: \tilde{U} \to V$ by the measures $\mathscr{S}_0(v)$ with supports in $\tilde{Z} = \tilde{p}^{-1}(Z) \subset \tilde{U}$ and weights $\tilde{\mathscr{S}}_0(v, \tilde{z}) = \mu(z)\tilde{\mathscr{S}}(v, \tilde{z})$ for all $\tilde{z}$ in $\tilde{Z}$ over z. The diffused chains $c_0 = \mathscr{S}_0 * c/\Gamma$ are countable chains in K/Γ_1 and they lift (not uniquely) to chains in K, since the homotopies of projected edges $[\tilde{v}'_k, \tilde{v}'_\ell]$ extend to admissible homotopies of Δ to simplices in K. Furthermore, one can choose these extended homotopies such that *cycles* in K/Γ_1 lift to *cycles* in K.

Observe that the lifted chains $\bar{c}_0$ lie in the subcomplex K_Z in K spanned by the set Z as its zero skeleton, $K_Z^0 = Z$. If we assume that $W_z \to \infty$ for $z \to \infty$, then the chains $\bar{c}_0$ are locally finite, that is, their images in V under the canonical map $K_Z \to V$ are locally finite, provided the original chain c is locally finite.

Finally, for small fat m-simplices Δ near a point $v \in V$, the total mass of the chain $\mathscr{S}_0 * \Delta$ in K/Γ_1 is estimated from above by $(1 + \delta)[\mathscr{S}_0(v)]^m m!\,(\text{mass } \Delta)$ as in the Proposition of (A), provided that the set $Z \subset V$ is sufficiently dense, and for all ε-simplices we have $||\mathscr{S}_0 * \Delta||_{\ell^1} \leq (1 + \delta)[\mathscr{S}(v)]^m \varepsilon^m$ regardless of their fatness. We claim, that these estimates also hold for appropriate lifts of the chains c_0 to K. Indeed, since the group Γ_1 is amenable in dimension m (that is Γ_1/Γ_{m+1} is an amenable group, see (3.3)), we can construct locally finite diffusion operators in K_Z (see (4.2)), which send chains $\bar{c} = \sum_{j\nu} \bar{r}_{j\nu} \bar{\sigma}_{j\nu}$ in K_Z over chains $\sum_j r_j \sigma_j$ in K_Z/Γ_1 with $\sum_\nu \bar{r}_{j\nu} = r_j$, $j = 1, \ldots,$ to diffused chains $\hat{c} = \sum_{j\mu} \hat{r}_{j\mu} \hat{\sigma}_{j\mu}$ over $\sum_j r_j \sigma_j$ such that the norms $||\hat{c} \mid \sigma_j||_{\ell^1} = \sum_\mu |\hat{r}_{j\mu}|$ become as close to $|r_j| = |\sum_\mu \hat{r}_{j\mu}|$ as we want, for all $j = 1, \ldots$.

The proof of the theorem is now reduced to the construction of étale smoothing operators $\mathscr{S}$ whose " norms " $[\mathscr{S}]$ and $[\mathscr{S}]^*$ are controlled by the growth function $\ell'(R, \widetilde{R})$.

We take for étale domains the universal coverings of the open balls $B_v(R') \subset V$ for all $v \in V$ and $R' \in (R + \widetilde{R}, R + \widetilde{R} + \varepsilon/2)$. We fix a point $\tilde{v} \in \widetilde{B}_v(R')$ over v in every such covering $\widetilde{B}_v(R') \to B_v(R')$. Then, for pairs of balls $B_v(R')$ and $B_{v''}(R'') \subset B_v(R')$, with $R + \widetilde{R} < R'' < R' - \operatorname{dist}(v, v'')$, we allow as étale maps $\widetilde{B}_{v''}(R'') \to \widetilde{B}_v(R')$ only those which send the point $\tilde{v}'' \in \widetilde{B}_{v''}(R'')$ to $\tilde{v}' \in \widetilde{B}_v(R)$ with $\operatorname{dist}(\tilde{v}', \tilde{v}) = \operatorname{dist}(v'', v)$. We always assume these distances to be so small that this étale map is unique.

Next, we construct a family of measures $\bar{\mathscr{S}}_{R, \bar{R}}(v) = \bar{\mathscr{S}}_{\bar{R}}(v)$ in the étale domains $\widetilde{B}_v(R')$, for $\bar{R} \in [R, \widetilde{R}]$, by first taking the universal coverings of the balls $B_{\tilde{v}}(\bar{R}) \subset \widetilde{B}_v(R')$, then by taking balls $B_{\bar{v}}(R)$ in these coverings $\widetilde{B}_{\tilde{v}}(\bar{R})$ with the centers $\bar{v}$ over $\tilde{v}$, and finally by considering the measure $\bar{\bar{\mathscr{S}}}$ on $\widetilde{B}_{\tilde{v}}(\bar{R})$ which is given by the characteristic function ψ of the ball $B_{\bar{v}}(R) \subset \widetilde{B}_v(\bar{R})$, that is, ψ is equal to one on the ball $B_{\bar{v}}(R)$ and to zero on $\widetilde{B}_{\tilde{v}}(\bar{R}) \backslash B_{\bar{v}}(R)$. The push-forward of $\bar{\bar{\mathscr{S}}}$ to $\widetilde{B}_v(R')$ is our measure $\bar{\mathscr{S}}_{\bar{R}}(v)$. In order to make this $\bar{\mathscr{S}}$ into an *étale* measure, we also take all its push-forwards under the étale maps of the coverings of the balls. Thus, we get the étale " smoothing " $\bar{\mathscr{S}}_{\bar{R}}(v)$ for all $v \in V$; yet it is not smooth, not even continuous: the measures $\bar{\mathscr{S}}_{\bar{R}}(v)$ may jump as the topology of the balls $B_{\tilde{v}}(\bar{R})$ changes.

We regularize the " smoothings " $\bar{\mathscr{S}}_{\bar{R}}$ by the following averaging:

$$\mathscr{S}(v) = \mathscr{S}_{R, \widetilde{R}}(v) = (\widetilde{R} - R)^{-1} \int_R^{\widetilde{R}} \bar{\mathscr{S}}_{\bar{R}}(v)(a + b\bar{R}) d\bar{R},$$

for a and b chosen so that $a + bR = 1$ and $a + b\widetilde{R} = 0$. Observe that the measures $\bar{\mathscr{S}}_{\bar{R}}(v)$ are decreasing in $\bar{R}$ and so the measures $\mathscr{S}_{R, \widetilde{R}}(V)$ are decreasing in $\widetilde{R}$ (but increasing in R). Therefore,

$$\left\| \frac{d}{d\widetilde{R}} \mathscr{S}_{R, \widetilde{R}}(v) \right\|_{L^1} = -\frac{d}{d\widetilde{R}} \| \mathscr{S}_{R, \widetilde{R}}(v) \|_{L^1}$$

and a straightforward calculation yields

$$-\frac{d}{d\widetilde{R}} \| \mathscr{S}_{R, \widetilde{R}}(v) \|_{L^1} \leq 4(\widetilde{R} - R)^{-1} \| \mathscr{S}_{R, \widetilde{R}}(v) \|_{L^1}.$$

Then, we observe that $\| \bar{\mathscr{S}}_{\bar{R}}(v) \|_{L^1} = \operatorname{Vol} B_{\bar{v}}(R)$, and that

$$\| D\mathscr{S}_{R, \widetilde{R}}(v) \|_{L^1} \leq \left\| \frac{d}{dR} \mathscr{S}_{R, \widetilde{R}}(v) \right\|_{L^1} + \left\| \frac{d}{dR} \mathscr{S}_{R, \widetilde{R}}(v) \right\|_{L^1}.$$

Finally, we obtain the desired inequality

$$[\mathscr{S}_{R, \widetilde{R}}(v)] \leq 4(\widetilde{R} - R)^{-1} + \bar{\ell}_v(R, \widetilde{R})$$

(see (4) of the theorem for the notation), and in the same way we get

$$[\mathscr{S}_{R,\widetilde{R}}(v)]^* \leq 4(\widetilde{R} - R)^{-1} + C_n \bar{\ell}_v(R, \widetilde{R}).$$

This smoothing only needs a minor change: a regularization in R with a fast decaying function $\varphi(v)$ (see the proof of the theorem in (2.5)) and then we are ready to conclude the proof. We take the balls $B_v(R + \widetilde{R} + \varepsilon/2) \subset V$ as sets W_v, and we take a complex K such that every one-simplex σ in K which is admissibly homotopic to a simplex of length $2R + \varepsilon$ has length $< 2R + \varepsilon$. Then, diffusions of small $\varepsilon/2$-simplices Δ have length $< 2R + \varepsilon$ and the " admissibility " of simplices in K make their radii $\leq \widetilde{R} + 2R + \varepsilon$.

Next, before diffusing the cycle c, we first subdivide and then straighten its simplices, thus making the majority of them small and fat, and finally we diffuse them to the subcomplex $K_Z \subset K$ for a sufficiently dense discrete set $Z \subset V$.

Corollary: *Proof of the main inequality of section* (0.5). — *If* $Ricci \geq -1/(n-1)$, *then* $\limsup_{R\to\infty} \ell'(R, \widetilde{R}) \leq 1$; *if furthermore* $\widetilde{R} - R \to \infty$, *then, for* $U = V$ *and letting* $\varepsilon \to 0$, *the inequality* (4) *of the theorem implies*

$$||V|| \leq C_n\, n!\, \mathrm{Vol}(V) < n!\, \mathrm{Vol}(V).$$

4.4. Geometric simplicial volumes

Suppose, we have fixed a notion of " size " for singular simplices $\sigma : \Delta \to V$ in Riemannian manifolds V. We have already met such " sizes " as mass σ, Rad σ and length σ. Furthermore, one may consider lifts of Δ to some covering of V (or of open subsets of V which contain $\sigma(\Delta)$) and then we have " Rad " for these lifts. Another class of sizes appears if we fix a metric in the standard simplex Δ. Then we have the Lipschitz constant of $\sigma : \Delta \to V$, called Dil σ, that is the " sup "-norm of the differential of σ. One also has such norms of the exterior powers $\Lambda^k D\sigma$, $k = 1, \ldots, \dim \Delta$. Furthermore, there are the Dirichlet functionals

$$\int_\Delta ||\Lambda^k D\sigma(x)||^\ell\, dx.$$

For a chain $c = \sum_i r_i \sigma_i$, we put " size " $c = \sup_i$ " size " σ_i and for all real $s > 0$ we define

$$||V \mid \text{" size "} \leq s|| = \inf_c ||c||_{\ell^1}$$

where c runs over all locally finite cycles of " size " $\leq s$ which represent the fundamental class of V. We set

$$||V \mid \text{" size "} < \infty|| = \inf_{s>0} ||V \mid \text{" size "} \leq s||.$$

This simplicial volume is not a topogolical invariant any more; however it is invariant under uniformly Lipschitz homeomorphisms. Moreover, for proper Lipschitz maps

$f: V \to V'$ for which $\operatorname{dist}(f(x), f(y)) \leq \text{const} \operatorname{dist}(x, y)$ for all x and y in V the basic functorial inequality remains true

$$||V \mid \text{“ size ”} < \infty|| \geq |\deg f| \cdot ||V' \mid \text{“ size ”} < \infty||.$$

It is clear that $||V|| \leq ||V \mid \text{“ size ”} < \infty||$, and that for any reasonable “ size ”, we have the equality for compact V. However, these geometric simplicial volumes may lead to new topological invariant of compact manifolds. Namely, fix any sequence of positive numbers, $\Lambda = (\lambda_i)_{i=1,\ldots}$, and then for a compact manifold V with a metric g denote by V^Λ the disjoint union of the Riemannian manifolds $(V, \lambda_i g)$. Since the Lipschitz class of V^Λ only depends on Λ but not on g, we have the whole spectrum of invariants

$$||V^\Lambda \mid \text{“ size ”} < \infty||,$$

one invariant for every Λ and every “ size ”. In fact, these invariants tend to take only two values: zero and infinity, and then the actual invariant is the partition of all sequences Λ into the two classes.

The theorems (A) and (B) of the previous section give upper bounds not only for the topological simplicial volume but also for $||V \mid \text{“ size ”} \leq s||$ for “ size ” $\sigma = \operatorname{Dil} \sigma$, while the more general theorem (D) applies to a “ weaker size ”, namely to length + Rad.

Now, let us consider only the fundamental cycles $c = \sum_i r_i \sigma_i$ for which “ size ” $\sigma_i \to 0$ for $i \to \infty$. By taking the infimum of the norms $||c||_{l^1} = \sum_i |r_i|$ over all these cycles c, we now get another Lipschitz invariant simplicial volume

$$||V \mid \text{“ size ”} \to 0|| \geq ||V \mid \text{“ size ”} < \infty||.$$

The arguments of the previous section can be extended to this simplicial volume as well. Namely, instead of our smoothing $\mathscr{S}_R$ with a fixed R, one could take a function $R(v)$ with a small gradient and such that $R(v) \to 0$ for $v \to \infty$. We do not need this generalization here, but we are going to use the simplicial volume $||V \mid \operatorname{Rad} \to 0||$ for proving the existence of complete manifolds V of all dimensions n that are *extremal* in the following sense: extremal manifolds $V = (V, g)$ by definition have metrics g of class C^1 with measurable curvature $K(g)$ and $|K(V)| \overset{\text{def}}{=} ||K(g)||_{L^\infty} \leq 1$. Furthermore, $\operatorname{Vol}(V) < \infty$ and all complete manifolds V′ which admit proper Lipschitz maps $V' \to V$ of non-zero degrees satisfy the inequality, $\operatorname{Vol}(V') \geq \operatorname{Vol}(V)$. In particular, there is no bounded deformation of the metric g of V which keep $|K| \leq 1$ and which decreases the volume.

Theorem. — There exist extremal manifolds of all dimensions n.

Remarks. — (a) It is unclear whether there are compact, not just complete, extremal manifolds of all dimensions. Nor do we know how many different extremal manifolds there are for a given dimension.

(b) It appears more natural to define extremal manifolds with arbitrary homeomorphisms $f: V' \to V$ rather than with Lipschitz maps. This notion has been discussed for surfaces in section (0.1). In this discussion we omitted surfaces homeomorphic to $\mathbf{R}^2$. In fact, such surfaces do admit extremal metrics, but they are never smooth. Indeed, at every point where the curvature is continuous, it must itself be extremal, that is $K = \pm 1$, and so there must be a jump from $+1$ to -1.

It is unclear whether there are such topological extremal metrics on manifolds of dimensions > 2. Nor is it clear how to relax the condition $|K| \leq 1$ to Ricci ≥ -1.

Proof. — The theorem is an immediate corollary of the following

Sub-theorem. — For an arbitrary number $h > 0$ there exists a complete Riemannian manifold $\overline{V}$ of a given dimension n with the following three properties:

(a) $|K(\overline{V})| \leq 1$ *and* $\mathrm{Vol}(\overline{V}) < \infty$,

(b) $||\overline{V} \mid \mathrm{Rad} \to 0|| \geq h$,

(c) *all complete n-dimensional manifolds* V *with* $|K(V)| \leq 1$ *and with* $||V \mid \mathrm{Rad} \to 0|| \geq h$ *satisfy*

$$\mathrm{Vol}(V) \geq \mathrm{Vol}(\overline{V}).$$

Proof. — Consider all complete n-dimensional manifolds V for which

$$|K(V)| \leq 1 \quad \text{and} \quad ||V \mid \mathrm{Rad} \to 0|| \geq h$$

and let us show that the functional $V \mapsto \mathrm{Vol}(V)$ assumes its minimum on this set of manifolds. Let Vol_h denotes the infimum of this functional and let V_i be a minimizing sequence, that is $\mathrm{Vol}(V_i) \to \mathrm{Vol}_h$ for $i \to \infty$. As $|K(V_i)| \leq 1$ and $\mathrm{Vol}(V_i) \leq \mathrm{const}$, one may (by taking a subsequence if necessary) assume that the sequence V_i has a " limit " that is a complete manifold $\overline{V}$ with the following property (see [25]):

There is an increasing sequence of relatively compact open subsets $\overline{U}_i \subset \overline{V}$, such that $\bigcup_i \overline{U}_i = \overline{V}$, and a sequence of maps $\bar{g}_i: \overline{U}_i \to V_i$ whose distortions converge to zero,

$$\sup_{x, y \in U_i} \left| \frac{\mathrm{dist}(\bar{g}_i(x), \bar{g}_i(y))}{\mathrm{dist}(x, y)} - 1 \right| \xrightarrow[i \to \infty]{} 0,$$

and such that the injectivity radii of the complements of the images of the maps $\bar{g}_i$ go to zero,

$$\sup_{v \in W_i} \mathrm{Inj}\, \mathrm{Rad}_v(V_i) \xrightarrow[i \to \infty]{} 0, \quad \text{for } W_i = V_i \setminus \bar{g}_i(\overline{U}_i).$$

This limit manifold $\overline{V}$ clearly has $\mathrm{Vol}(\overline{V}) \leq \lim_{i \to \infty} \mathrm{Vol}(V_i) = \mathrm{Vol}_h$, and also $|K(V)| \leq 1$, though this $\overline{V}$ may not be C^2-smooth. Observe that the manifold $\overline{V}$ may be empty: this happens if $\mathrm{Inj}\, \mathrm{Rad}(V_i) \to 0$ for $i \to \infty$.

Our sub-theorem is now reduced to the inequality

$$||\overline{V} \mid \mathrm{Rad} \to 0|| \geq \lim_{i \to \infty} ||V_i \mid \mathrm{Rad} \to 0||. \qquad (*)$$

To prove (*), take a fundamental cycle $\bar{c} = \sum_k \bar{r}_k \bar{\sigma}_k$ in $\overline{V}$ for which $\mathrm{Rad}\, \sigma_k \xrightarrow[k\to\infty]{} 0$ and let us show that all manifolds V_i for $i \geq j = j(\bar{c})$ support fundamental cycles $c(i) = \sum_\ell r_\ell(i)\sigma_\ell(i)$, also with $\mathrm{Rad}\, \sigma_\ell(i) \xrightarrow[\ell\to\infty]{} 0$, such that $||c(i)||_{l^1} \leq ||\bar{c}||_{l^1} + i^{-1}$ for all $i \geq j$. Since all manifolds V_i have $\mathrm{Inj}\,\mathrm{Rad}_v(V_i) \xrightarrow[v\to\infty]{} 0$ they admit coverings by open subsets $U_\mu(i) \subset V_i$ with the following three properties (compare with (C) of (4.3)):

(1) Every set $U_\mu(i)$ is contained in a ball $B_\mu(i)$ in V_i of radius $R_\mu(i)$ and all $U_\mu(i)$ are amenable subsets in $B_\mu(i)$. Furthermore, for each i we have $R_\mu(i) \to 0$ as $\mu \to \infty$.

(2) For every $i \geq i_0 = i_0(\{\overline{U}_i, \bar{g}_i\})$ the multiplicity of the covering $\{U_\mu(i)\}$ outside the image $\bar{g}_i(U_{i_0}) \subset V_i$ does not exceed $n = \dim V = \dim V_i$, that is, every point $v \in V_i \backslash g_i(U_{i_0})$, for all $i \geq i_0$, is contained in at most n subsets $U_\mu(i)$.

(3) If a simplex $\bar{\sigma}_k$ of the chain $\bar{c}$ is not contained in the set $\overline{U}_{i_1} \subset \overline{V}$ for some large $i_1 = i_1(\bar{c})$, then, for $i \geq i_1$, the map $\bar{g}_i$ sends $\bar{\sigma}_k$ to one of the subsets $U_\mu(i) \subset V_i$ and the image $\bar{g}_i(\bar{\sigma}_k) \subset V_i$ intersects at most n subsets $U_\mu(i)$.

Now, take a set $\overline{U}_{i_2} \subset \overline{V}$ for some i_2 much greater than i_0 and i_1 and send the restricted chain $c \mid \overline{U}_{i_2}$ to the manifolds V_i, for large $i \geq i_2$, by the maps $\bar{g}_i$. We extend the resulting chains $\bar{g}_i(c \mid U_{i_2})$ to the fundamental *cycles* $c'(i)$ in V_i such that the new simplices of $c'(i)$, *i.e.* the simplices σ of the difference $c'(i) - g_i(\bar{c} \mid \overline{U}_{i_2})$, lie far away from $g_i(U_{i_0}) \subset V_i$, and such that every new simplex σ is contained in one of the sets $U_\mu(i)$ for all $i \geq i_2$ and that σ intersects no more than n sets $U_\mu(i)$.

We apply the locally finite diffusion (see (4.2)) to the cycle $c'(i)$ in V_i for each $i \geq i_2$. This diffusion " annihilates " all new simplices σ of the chain $c'(i)$ (compare (C) of (4.3)) while the norm of the chain $g_i(c, \mid U_{i_2})$ can only decrease under the diffusion. In other words, one can diffuse the cycle $c'(i)$ to a cycle $c(i)$ such that

$$||c(i)||_l \leq ||\bar{g}_i(c \mid U_{i_2})|| + \varepsilon_i$$

for an arbitrary $\varepsilon_i > 0$ and for all $i \geq i_2$. Since $||\bar{g}_i(\bar{c} \mid \overline{U}_{i_2})|| \leq ||\bar{c}||$, we make $||c(i)||_{l^1} \leq ||\bar{c}||_{l^1} + \varepsilon_i$ for $\varepsilon_i = i^{-1}$. Finally, according to (1) above, we have $R_\mu(i) \xrightarrow[\mu\to\infty]{} 0$ and so the radii of the simplices of all chains $c(i)$ for $i \geq i_2$ also go to zero, q.e.d.

4.5 Specific simplicial volumes

We now want to apply the results of section (4.3) to complete manifolds of *infinite* volume. First, for an open subset $U \subset V$ and for a positive number $R > 0$, we define the R-*interior* of U, denoted $U - R \subset U$, as the set of all those points $u \in U$ for which the closed balls $B_u(R) \subset V$ are contained in U. Observe, that this definition uses only the induced Riemannian structure of U and so one may speak of the R-interior of an abstractly defined nonnecessarily complete manifold U. If U is *complete*, then $U - R = U$ for all $R \geq 0$.

Next, a manifolds U of finite total volume is called R-*stable* if

$$\mathrm{Vol}(U - R) > (1 - R^{-1})\,\mathrm{Vol}\,U.$$

Furthermore, a sequence of manifolds U_i, $i = 1, \ldots,$ is called *stable* if the U_i are R_i-stable for suitable $R_i \to \infty$ as $i \to \infty$.

Examples. — *(a)* The sequences of Euclidean balls of radii $\to \infty$ are stable, while such sequences of balls in hyperbolic spaces are not stable.

(b) Let V be a manifold and let $\tilde{V} \to V$ be a regular covering with Galois group Π. When V is closed, $\tilde{V}$ admits a stable sequence of open subsets, $\tilde{U}_i \subset \tilde{V}$, if and only if the group Π is amenable (see [21]). For any V, if the group Π is amenable, then $\tilde{V}$ admits a stable sequence if and only if V does. In fact, if $\tilde{U} \to U$ is an arbitrary locally isometric map and if $\tilde{U}$ is R-stable, then U contains an R-stable open subset. Indeed, let $\tilde{U}_k \subset U$ be the sets of those points in U which have at least k points in their pullbacks in $\tilde{U}$. Then $(\tilde{U} - R)_k \subset \tilde{U}_k - R$ and so

$$\sum_k \mathrm{Vol}(\tilde{U}_k - R) \geq \sum_k \mathrm{Vol}(\tilde{U} - R)_k$$

$$= \mathrm{Vol}(\tilde{U} - R) \geq (1 - R^{-1})\,\mathrm{Vol}\,\tilde{U} = (1 - R^{-1}) \sum_k \mathrm{Vol}\,\tilde{U}.$$

Therefore, one of the sets $\tilde{U}_k \subset \tilde{U}$ is R-stable.

Now, for a given map between two n-dimensional manifolds, $f: V_1 \to V_2$, we want to define " the ratio of the volumes ", $\mathrm{Vol}(V_1)/\mathrm{Vol}(V_2)$, in the case of infinite volumes. First, for all relatively compact open subsets $U \subset V_2$, we put

$$(U, R) = \mathrm{Vol}(f^{-1}(U - R))/\mathrm{Vol}(U),$$

and then

$$\mathrm{Vol}_+(V_1 : V_2; f) = \inf_{R>0} \sup_{U \subset V} (U, R).$$

Next, we define $\mathrm{Vol}_-(V_1 : V_2; f)$ as the lower bound of the numbers $\nu \geq 0$ such that there exists a *stable* sequence $U_i \subset V$ of relatively compact open subsets with the property

$$\inf_{R \geq 0} \sup_{i=1,2,\ldots} (U_i, R) \leq \nu.$$

This definition is only meaningful if stable sequences do exist.

Examples. — If V_1 and V_2 are closed manifolds then

$$\mathrm{Vol}_+ = \mathrm{Vol}_- = \mathrm{Vol}(V_1)/\mathrm{Vol}(V_2).$$

For the identity map, $V \to V$, one has $\mathrm{Vol}_+ = \mathrm{Vol}_- = 1$ if V admits a stable sequence of open subsets, $U_i \subset V$. Otherwise, $\mathrm{Vol}_+ = 0$.

For linear maps $f: \mathbf{R}^n \to \mathbf{R}^n$ we have $\mathrm{Vol}_+ = \mathrm{Vol}_- = (\mathrm{Det} f)^{-1}$.

If $f: V_1 \to V_2$ is a proper locally diffeomorphic map (*i.e.* a finite covering) whose Jacobian is bounded everywhere from below by $J_0 > 0$, then $\mathrm{Vol}_+ \leq (\deg f)/J_0$.

Also observe that Vol_+ and Vol_- are stable under bounded perturbations of maps: if $\mathrm{dist}(f(x),f'(x)) \leq \text{const}$ for all $x \in V_1$ then the maps f and f' have equal " volumes ".

Our main purpose is to estimate Vol_+ and Vol_- from below in terms of appropriate specific (simplicial) volumes, which are defined as follows: First, for fundamental cycles $c = \sum_j r_j \sigma_j$ of a Riemannian manifold V, we define the quantity (see (4.3) for the notation)

$$||c : \mathrm{Vol}||_+ = \inf_{R \geq 0} \sup_{U \subset V} (||c \mid U - R||_{l^\infty} (\mathrm{Vol}\, U)^{-1}).$$

Then we define $||c : \mathrm{Vol}||_-$ by taking only the " sup " over some stable sequences of sets U, namely we take for $||\ ||_-$ the lower bound of the numbers ν for which there exists a stable sequence of relatively compact open subsets $U_i \subset V$ such that

$$\inf_{R \geq 0} \sup_i (||c \mid U_i - R||_{l^\infty} (\mathrm{Vol}\, U_i)^{-1}) \leq \nu.$$

Finally, we fix some notion of " size " (see (4.4)) and then we minimize the above norms over all cycles c with " size " $c = \sup_j$ " size " $\sigma_j < \infty$,

$$||V : \mathrm{Vol}||_+ = \inf_c ||c : \mathrm{Vol}||_+,$$

and

$$||V : \mathrm{Vol}||_- = \inf_c ||c : \mathrm{Vol}||_-.$$

Examples. — If V is a closed manifold then

$$||V : \mathrm{Vol}||_+ = ||V : \mathrm{Vol}||_- = ||V||(\mathrm{Vol}(V))^{-1}.$$

If V is a complete manifold of finite volume, then

$$||V : \mathrm{Vol}||_+ \geq ||V \mid \text{" size "} < \infty|| \, (\mathrm{Vol}(V))^{-1} \geq ||V : \mathrm{Vol}||_-.$$

Specific volumes of proper Lipschitz maps $f: V_1 \to V_2$ have the following functorial properties

$$|\deg(f)| \cdot ||V_2 : \mathrm{Vol}||_+ \leq \mathrm{Vol}_+(V_1 : V_2; f) \cdot ||V_1 : \mathrm{Vol}||_+$$

and

$$|\deg(f)| \cdot ||V_2 : \mathrm{Vol}||_- \leq \mathrm{Vol}_-(V_1 : V_2; f) \cdot ||V_1 : \mathrm{Vol}||_+.$$

If the manifolds V_1 and V_2 are closed, then both inequalities are equivalent to

$$|\deg(f)| \cdot ||V_2|| \leq ||V_1||.$$

To prove the above inequalities, we notice that Lipschitz maps f preserve chains of " size " $< \infty$ and that

$$f^{-1}(U - R) \subset f^{-1}(U) - LR$$

where $L = L_f$ is the Lipschitz constant of f.

Observe that the second inequality may not hold with $||\ ||_-$ on the left hand side.

With some estimates for specific volumes, these functorial inequalities yield, lower bounds for Vol_+ and Vol_-. For example, if a specific volume of a Riemannian manifold V

is different from zero as well as from infinity, then every bi-Lipschitz self-homeomorphism $f: V \to V$ with a constant Jacobian $J = J_f$ satisfies $|J| = 1$.

We start our estimates of $||\ ||_+$ and $||\ ||_-$ with a couple of elementary observations.

Let $g: V \to V'$ be a locally isometric covering map and suppose that lifts to V of chains c' in V' with "size" $c' < \infty$ also have finite "size" in V. For example, this happens to "size" = Dil. Then, clearly

$$||V : \mathrm{Vol}||_+ \leq ||V' : \mathrm{Vol}||_+.$$

Furthermore, if g is a regular covering with an amenable Galois group, then also

$$||V : \mathrm{Vol}||_- \leq ||V' : \mathrm{Vol}||_-.$$

Next, let V' be a closed manifold and let $g: V \to V'$ be a Lipschitz map whose Jacobian satisfies the following inequality for some stable sequence of subsets $U_i \subset V$:

$$\lim_{i\to\infty} \left((\mathrm{Vol}\, U_i)^{-1} \int_{U_i} J_g\right) = J_0 > 0.$$

Then for "size" = Dil *we claim*:

$$||V : \mathrm{Vol}||_+ \geq J_0 ||V'||/\mathrm{Vol}(V').$$

Furthermore, if all stable sequences U_i *satisfy*

$$\lim_{i\to\infty} \inf (\mathrm{Vol}(U_i))^{-1} \int_{U_i} J_g \geq J_0,$$

then also $$||V : \mathrm{Vol}||_- \geq J_0 ||V'||/\mathrm{Vol}(V').$$

In particular, for regular coverings $g: V \to V'$ *with amenable Galois groups, and choosing* "size" = Dil, *we get*

$$||V : \mathrm{Vol}||_+ = ||V : \mathrm{Vol}||_- = ||V'||/\mathrm{Vol}(V').$$

Proof. — Take a fundamental cycle $c = \sum_j r_j \sigma_j$ of V with $\sup_j \mathrm{Dil}\, \sigma_j < \infty$ and then restrict it to the sets $U_i - R$ for some large R. We send these restricted chains to V' and call the images $c'(i)$. Since the sequence U_i is stable and since all simplices of the chains $c'(i)$ have uniformly bounded "size" = Dil, we can extend these $c'(i)$ to *cycles* $c(i)$, also with $\sup_i \mathrm{Dil}(c(i)) < \infty$, such that

$$||c(i) - c'(i)||_{l^1}/\mathrm{Vol}(U_i) \underset{i\to\infty}{\longrightarrow} 0 \qquad (+)$$

Now, the homology classes $h_i = \left[\left(\int_{U_i} J_g\right)^{-1} c(i)\right]$ converge to $[V']/\mathrm{Vol}(V')$ and so

$$\lim_{i\to\infty} \inf ||h_i||_{l^1} \geq ||[V']||_{l^1}/\mathrm{Vol}(V') = ||V'||/\mathrm{Vol}(V').$$

Finally, since

$$[c(i)/\mathrm{Vol}(U_i)] = J_0 h_i,$$

and $||c \mid U_i||_{\ell^1} = ||c'(i)||_{\ell^1}$, we conclude, with (+) above, that

$$\lim_{i\to\infty} \inf ||c \mid U_i - R||_{\ell^1}/\mathrm{Vol}(U_i) \geq ||V'||/\mathrm{Vol}(V'),$$

and so, $||V : \mathrm{Vol}||_+ \geq ||V'||/\mathrm{Vol}(V')$. As the same argument applies to $||\ ||_-$, the proof is finished.

Let us indicate a generalization of the equality concerning the amenable coverings $g : V \to V'$. First, we call a manifold V *stable at infinity* if it contains a stable sequence of relatively compact open subsets $U_i \subset V$.

Example. — Let V be complete and let $B_v(R)$ be concentric balls around a fixed point $v \in V$. If

$$\lim_{R\to\infty} \inf R^{-1} \log B_v(R) = 0,$$

then V is stable at infinity. Indeed, $B_v(R - R') \subset B_v(R) - R'$ for all $R' < R$ and so some sequence of balls is stable.

Proportionality theorem. — If the universal coverings of two manifolds V *and* V′ *are isometric and if* V′ *is a closed manifold, then, for* " size " = Dil,

$$||V : \mathrm{Vol}||_+ \leq ||V'||/\mathrm{Vol}(V').$$

Furthermore, if V *is stable at infinity, then*

$$||V : \mathrm{Vol}||_+ = ||V : \mathrm{Vol}||_- = ||V'||/\mathrm{Vol}(V').$$

Proof. — Let G denote the isometry group of the universal covering $\tilde{V} = \tilde{V}'$. Then every *finite* singular chain in V as well as an V′ can be smeared (see [47]) with the Haar measure in G to a generalized G-invariant cycle in V (see (2.2)). In particular, fundamental cycles c' of V′ smear to cycles $\tilde{c}'$ in $\tilde{V}$ and these define smeared fundamental cycles c in V which can be approximated, as in (2.2), by usual cycles whose ℓ^1-norms on all sets $U \subset V$ are roughly proportional to $||c'|| \,\mathrm{Vol}(U)/\mathrm{Vol}(V')$. This proves the inequality $||V : \mathrm{Vol}||_+ \leq ||V'||/\mathrm{Vol}(V')$.

Next, we take a stable sequence $u_i \subset V$ and a fundamental cycle c of V for which the limit

$$\lim_{i\to\infty} (\mathrm{Vol}(U_i))^{-1} ||c \mid U_i - R||_{\ell^1}$$

is close to $||V : \mathrm{Vol}||_-$. Then, the chains $(\mathrm{Vol}(U_i))^{-1}(c \mid U_i - R)$ in V are first smeared to G-invariant chains in $\tilde{V}$ and then, as before, we have chains $c'(i)$ in V′ whose boundaries $\partial c'(i)$ satisfy $||\partial c'(i)||_{\ell^1} \xrightarrow[i\to\infty]{} 0$. As the chains $c(i)$ have uniformly bounded " size " = $\mathrm{Dil}(c'(i))$, one can extend them, as we did above (see also (2.2)), to cycles $c(i)$ such that $[c(i)] = [V']/\mathrm{Vol}(V')$ and such that the norms $||c(i)||_{\ell^1}$ are close to the norms $||c'(i)||_{\ell^1}$ which, in their turn, are close to the norms $||c \mid U_i - R||_{\ell^1}/\mathrm{Vol}(U_i)$. Thus, in the limit, we come to the required inequality $||V'||/\mathrm{Vol}(V') \leq ||V : \mathrm{Vol}||_-$, q.e.d.

Remarks and corollaries. — (1) If both manifolds V_1 and V_2 are stable at infinity and if their universal coverings are isometric, then we can conclude that

$$||V_1 : \mathrm{Vol}||_+ = ||V_2 : \mathrm{Vol}||_+ = ||V_1 : \mathrm{Vol}||_- = ||V_2 : \mathrm{Vol}||_-,$$

provided there is a third manifold V' which is *closed* and whose universal covering is isometric to those of V_1 and V_2. This last condition (the existence of V') cannot be dropped but probably it can be somewhat relaxed.

(2) The isometry of the universal coverings, $\widetilde{V} \simeq \widetilde{V}'$, can be replaced by a weaker condition, namely the existence of a locally isometric map of $\widetilde{V}$ to $\widetilde{V}'$. With this new condition, the proportionality theorem even holds for non-complete manifolds V. In fact, our proof for complete manifolds extends to the general case with only minor changes.

Digression: Euler's characteristic and signature

For a subset $U \subset V$, we denote by $H_c(U; F)$ the cohomology with compact supports for some coefficient field F, and we denote by $b^m(U \subset V; F)$ the ranks of the inclusion homomorphisms $H_c^m(U; F) \to H^m(V; F)$ for $m = 0, 1, \ldots, n = \dim V$. We denote by $b_m(U \subset V; F)$ the ranks of the homomorphisms $H_m(U; F) \to H_m(V; F)$ and we abbreviate: $b^m(U; F) = b(U \subset U; F)$ and $b_m(U; F) = b_m(U \subset U; F)$.

Now, we assume that the universal covering of the manifold V admits a locally isometric map into a closed manifold V' and we take a stable sequence of relatively compact open subsets $U_i \subset V$, $i = 1, \ldots$ We assume furthermore that the sequence U_i has *bounded geometry*: there is a number $\varepsilon \in (0, 1)$ such that $\mathrm{Inj\,Rad}_v(V) \geq \varepsilon$ for all $v \in U_i - \varepsilon^{-1}$ and all $i = 1, \ldots$

Proportionality Proposition. — The "*Euler characteristics*"

$$\chi(i) = \sum_{m=0}^{n} (-1)^m b_m((U_i - R) \subset V; F)$$

and

$$\bar{\chi}(i) = \sum_{m=0} (-1)^m b^m((U_i - R) \subset V; F)$$

satisfy

$$\lim_{i \to \infty} \chi(i)/\mathrm{Vol}(U_i) = \lim_{i \to \infty} \bar{\chi}(i)/\mathrm{Vol}(U_i) = \chi(V')/\mathrm{Vol}(V').$$

for all $R \geq 2\varepsilon^{-1}$.

Proof. — Since the geometry of U_i is bounded there are compact submanifolds U_i' in V such that $U_i - R \subset U_i' \subset U_i - R/2$, and such that the boundaries $\partial U_i'$ are smooth hypersurfaces in V whose principal curvatures are bounded in absolute value by a constant $C = C(V', \varepsilon)$. It follows that $\mathrm{Vol}(U_i')/\mathrm{Vol}(U_i) \to 0$ for $i \to \infty$ and, by the theorem of Gauss-Bonnet for manifolds with boundaries, we see that

$$\chi(U_i')/\mathrm{Vol}(U_i) \to \chi(V')/\mathrm{Vol}(V') \quad \text{as } i \to \infty.$$

Next, we observe that the geometry of the closed manifolds $\partial U_i'$ is bounded, and so, $b_m(\partial U_i'; F) \leq \text{const Vol}(\partial U_i')$ for some constant which only depends on V' and C. As the images of the inclusion homomorphism $H_m(\partial U_i'; F) \to H_m(U_i; F)$ contain the kernels of the homomorphisms $H_m(U_i'; F) \to H_m(V; F)$, we get

$$(b_m(U_i'; F) - b_m(U_i' \subset V; F))/\text{Vol}(U_i) \to 0,$$

and so, for every $R \geq 2\varepsilon^{-1}$,

$$[b_m(U_i'; F) - b_m((U_i - R) \subset V; F)]/\text{Vol}(U_i) \to 0.$$

This implies that $\lim_{i\to\infty} \chi(i)/\text{Vol}(U_i) = \chi(V')/\text{Vol}(V')$. In the same way, we obtain

$$[b_m(U_i'; F) - b^m((U_i - R) \subset V; F)]/\text{Vol}(U_i) \to 0$$

and deduce the second relation

$$\lim \bar{\chi}(i)/\text{Vol}(U_i) = \chi(V')/\text{Vol}(V').$$

Remark. — Our proof is a slight modification of an unpublished argument of Morgan and Phillips.

Corollaries. — Let V be an infinite Galois covering of V'. Denote by $\chi^{ev}(V)$ the sum of the Betti numbers $b^m(V; F)$ over all even numbers m and let χ^{odd} be the sum of the odd Betti numbers. Observe that the inequality $b^m(V; F) \neq 0$ for some m implies $b^m(V; F) = \infty$. Moreover, for any stable sequence U_i this inequality implies: $\lim_{i\to\infty} b^m(U_i \subset V; F)/\text{Vol}(U_i) > 0$.

Now, let the covering $V \to V'$ have an amenable Galois group. Then the proposition implies:

(A) *If* $\chi(V') > 0$, *then* $\chi^{ev}(V) = \infty$; *if* $\chi(V') < 0$, *then* $\chi^{odd}(V) = \infty$.

Furthermore, with the observation above we get

(A') *If* $\chi(V') = 0$ *and* $\chi^{ev}(V) \neq 0$, *then* $\chi^{odd}(V) = \infty$.

The corollaries (A) and (A') hold for an arbitrary finite polyhedron P' in place of the manifold V. In fact, the argument is even shorter in the combinatorial case. As an application we obtain a short proof of the following result of *Baumslag-Pride* [3].

(B) *Theorem.* — *Let an abstract group* Γ *be presented by* p *generators and* q *relations* $\Gamma = \{\gamma_1, \ldots, \gamma_p \mid w_1, \ldots, w_q\}$. *If* $q \leq p - 2$ *then the group* Γ *is large: it contains a subgroup of finite index,* $\Gamma' \subset \Gamma$, *such that* Γ' *admits a surjective homomorphism onto the free group with two generators.*

Proof. — Let P' be a two dimensional cell complex with one zero-cell, with p one-cells and with q 2-cells, such that $\pi_1(P') = \Gamma$. As $q < p$, there exists an infinite cyclic covering $P \to P'$ and as $\chi(P') < 0$ we have $b^1(P; \mathbf{R}) = \infty$ by the corollary (A). It follows, that some finite cyclic covering $\tilde{P} \to P$ possesses a pair of non-trivial (*i.e.* not

cohomologous to zero) integral cocycles with disjoint supports. This pair defines a map of $\widetilde{P}$ to the wedge of two circles such that the induced homomorphism $\pi_1(\widetilde{P}) \to \pi_1(\mathscr{S}^1 \vee \mathscr{S}^1)$ is surjective, q.e.d.

Next, we solve a problem posed by Baumslag and Pride in [4].

(B′) *Theorem.* — *If* $q \leq p - 1$ *and if the word* w_1 *is a proper power,* $w_1 = (w_0)^\ell$ *for some prime number* $\ell \geq 2$, *then the group* Γ *is large.*

Proof. — First, let us show that $H_2(P; \mathbf{Z}_\ell) \neq 0$. Indeed, the 2-cell in P′ which corresponds to the word w_1 gives us a $\mathbf{Z}_\ell$-cycle in P′ and this cycle lifts to P. Now, as $\dim P' = 2$, the translates of this lift generate an *infinite* subgroup in $H_2(P; \mathbf{Z}_\ell)$ and so $b^2(P; \mathbf{Z}_\ell) = \infty$. Therefore, $b^1(P; \mathbf{Z}_\ell) = \infty$ by corollary (A′), and so some finite cyclic covering $\widetilde{P}$ of P possesses a pair of non-trivial $\mathbf{Z}_\ell$-cocycles with disjoint supports. Then, we have a surjective homomorphism of $\pi_1(\widetilde{P})$ onto the free product $\mathbf{Z}_\ell * \mathbf{Z}_\ell$, q.e.d.

Remark. — In the algebraic theorems above, one could avoid any reference to infinite coverings. Indeed, let $\widetilde{P}_k \to P'$ be a sequence of cyclic coverings of orders $k = 1, 2, \ldots$, which converges to an infinite cyclic covering $P \to P'$. If P′ is a finite polyhedron and if $b_m(\widetilde{P}_k; F) \to \infty$ as $k \to \infty$ (for some fixed m), then all $\widetilde{P}_k$, for large k, possess pairs of non-trivial m-dimensional F-cocycles with disjoint supports. In particular, if $b_1(P_k; F) \to \infty$ for some field F, then the group $\pi_1(P')$ is large. As a corollary, we obtain the following

(B″) *Theorem.* — *Let* V′ *be a closed oriented* 4*-dimensional manifold whose Euler characteristic* χ *and signature* σ *satisfy* $\chi < |\sigma|$. *Then the fundamental group* $\pi_1(V')$ *is large.*

Proof. — The inequality $\chi \leq |\sigma|$ implies $\chi \leq b_2(V'; \mathbf{R})$ and so $b_1(V'; \mathbf{R}) > 0$. Then, we have our cyclic coverings $\widetilde{V}_k \to V'$. As $\sigma(\widetilde{V}_k) = k\sigma$ and $\chi(\widetilde{V}_k) = k\chi$, the inequality $\chi \leq |\sigma| - 1$ implies $\chi(\widetilde{V}_k) \leq b_2(\widetilde{V}_k; \mathbf{R}) - k$, and so $b_1(\widetilde{V}_k; \mathbf{R}) \geq k$, q.e.d.

Remarks. — (1) This proof is similar to an argument of *Winkelnkemper* who shows in [48] that the group $\pi_1(V)$ is non-abelian.

(2) Theorem (B) follows from (B″). Indeed, the two dimensional polyhedron P′ embeds into $\mathbf{R}^5$ and then (B″) applies to the boundary V′ of the regular neighbourhood of $P' \subset \mathbf{R}^5$.

The major drawback of our proportionality proposition for the Euler characteristic is the bounded geometry assumption. In the following special case this assumption can be omitted.

Let the manifold V be *complete* and let (U_i) be a stable *increasing* sequence which *exhausts* V, that is $U_1 \subset U_2 \subset \ldots \subset U_i \subset$ and $\bigcup_i U_i = V$. We assume as before that the universal covering of V is isometric to the universal covering of a compact manifold V′.

(C) *Theorem.* — *The proportionality relation* $\lim_{i\to\infty} \chi(i)/\mathrm{Vol}(U_i) = \chi(V')/\mathrm{Vol}(V')$ *holds if the following two conditions are satisfied,*

(i) *the manifold* V *is homotopy equivalent to a finite polyhedron,*
(ii) *the fundamental group* $\pi_1(V)$ *is residually finite.*

Remarks and Corollaries. — *(a)* It is unclear whether the conditions (i) and (ii) are essential (1).

(b) The theorem is equivalent, because of (i), to

$$\chi(V)/\mathrm{Vol}(V) = \lim_{i\to\infty} \chi(V)/\mathrm{Vol}(U_i) = \chi(V')/\mathrm{Vol}(V').$$

If $\mathrm{Vol}(V) = \infty$, then the theorem only claims that $\chi(V') = 0$.

Example. — *Let* V *be a complete manifold of constant negative curvature which satisfies the following three conditions*:

(1) V *is homotopy equivalent to a finite polyhedron.*

(2) V *has sub-exponential growth: balls* $B_v(R) \subset V$ *around a fixed point* $v \in V$ *satisfy* $\lim_{R\to\infty} \inf R^{-1} \log \mathrm{Vol}\, B_v(R) = 0$.

(3) *The dimension of* V *is even.*

Then $\mathrm{Vol}(V) < \infty$.

Notice that none of the conditions (1)-(3) can be dropped.

(c) If the universal covering V is homogeneous, then the condition $\mathrm{Vol}(V) < \infty$ implies (i) and (ii) (see [41]) and so we deduce the following result of *Harder* [28].

(C′) *Theorem.* — *If* V *is a locally homogeneous Riemannian manifold of finite volume, then* $\chi(V) = \int_V G(v)dv$, *where* $G(v)$ *is the Gauss-Bonnet integrand.*

Proof of (C). — According to (ii) there is a sequence of finite coverings $\tilde{V}(i) \to V$ for which the corresponding coverings $\tilde{U}_i(i) \subset \tilde{V}(i)$ of $U_i \subset V$ have bounded geometry. Then the proportionality proposition applies to the sequence $\{U_i(i)\}$ and with (i) we get $\lim_{i\to\infty} \chi(\tilde{V}(i)/\mathrm{Vol}(U_i(i)) = \chi(V')/\mathrm{Vol}(V')$. Therefore, $\lim_{i\to\infty} \chi(V)/\mathrm{Vol}(U_i) = \chi(V')/\mathrm{Vol}(V')$, q.e.d.

The proportionality proposition and its corollaries probably hold (2) replacing the Euler characteristic by the signature, which is understood, for *open* $4k$-dimensional manifolds U, to be the signature σ of the (possibly degenerate) intersection form on the homology $H_{2k}(U; \mathbf{R})$. The following fact supports this conjecture.

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(1) *Added in proof*: An application of the L^2-cohomology theory to the coverings of U_i allows one to drop condition (ii).

(2) *Added in proof*: They do hold, as shown in a forthcoming paper by J. Cheeger and the author.

Proposition. — *Let* V' *be a closed* $4k$*-dimensional manifold with non-zero signature and let* $V \to V'$ *be a regular covering with an infinite amenable Galois group* Π. *Then the homology* $H_{2k}(V; \mathbf{R})$ *is infinite dimensional.*

Proof. — By the index theorem for infinite coverings (see [1]) there exists a harmonic L_2-form on V of degree $2k$. Since the group Π is amenable, there also exists a non-exact closed $2k$-form on V with compact support. As the group Π is infinite, it follows that $b^{2k}(V; \mathbf{R}) = \infty$, q.e.d.

The Ricci curvature and volume provide the following upper bound for the signature and for the Euler characteristic.

(D) *Theorem.* — *Let* V *be a closed Riemannian manifold with* $\operatorname{Ricci} V \geq -1$. *Let* V' *be a closed manifold with contractible universal covering,* $\dim V' = n = \dim V$, *and let* $f: V \to V'$ *be a map of degree* $d \neq 0$. *If the fundamental group* $\Pi' = \pi_1(V')$ *is residually finite then*

$$|\chi(V')| \leq \text{const}_n \operatorname{Vol}(V)$$

as well as

$$|\sigma(V')| \leq \text{const}_n \operatorname{Vol}(V),$$

for some universal constant const_n.

Remarks. — *(a)* The " residually finite " condition is probably unnecessary.

(b) The topological assumptions of (D) imply, for all known examples, $|\chi(V')| + |\sigma(V')| \leq \text{const}'_n ||V'||$, and so all (known) specific applications of (D) also follow from the main inequality of (0.5).

Proof. — Since $\operatorname{Ricci} V \geq -1$, there exists (see (3.4)), for every positive $\delta \leq 1$, a map $h = h_\delta$ of V into some n-dimensional polyhedron $Q = Q_\delta$ (which is, in fact, a subpolyhedron of the nerve of some cover of V by ε-balls for $\varepsilon \approx \frac{1}{n}\delta$) with the following two properties:

(i) The pullback of every point, $h^{-1}(x) \subset V$, $x \in Q$, has diameter $< \delta$.

(ii) The total number of n-dimensional simplices in Q is at most $C_n \delta^{-n} \operatorname{Vol}(V)$, for some universal constant C_n. Moreover, the image $g_*[V] \in H_n(Q; \mathbf{Z})$ can be represented by an *integral* cycle with ℓ^1-norm $\leq C_n \delta^{-n} \operatorname{Vol}(V)$.

These properties imply the following

Sublemma. — *If all closed curves in* V *of* length $< 6\delta$ *are sent by the map* $f = V \to V'$ *into contractible curves, then the Betti numbers of* V' *satisfy*

$$\sum_0^n b_i(V'; \mathbf{R}) \leq \text{const}_n \delta^{-n} \operatorname{Vol}(V).$$

Proof. — Let us first show that the map f extends to the cylinder $Y = Y_h \supset V$ of the map h. We assume, without loss of generality, that the map $h: V \to Q$ is simplicial

in some subdivisions of V and Q, and then, we triangulate Y into very small simplices. Next, we send each vertex $y \in Y$ to the " nearest " point in V. For every edge in Y, the points of V corresponding to its extremities can be joined, according to (i), by a path of length $< 2\delta$. Thus, with the map $f: V \to V'$, we obtain a map of the one-skeleton of Y to V'. As the boundary of every 2-simplex in Y goes to a curve of length $< 6\delta$ in V, this map extends to the 2-skeleton of Y. Finally, as the manifold V' is aspherical, we get an extension to the all of Y.

Now, with (ii), we represent the d-times fundamental class of V' by an integral cycle of norm $\leq C_n \delta^{-n} \mathrm{Vol}(V)$ and by Poincaré duality

$$\Sigma b_i(V', \mathbf{R}) \leq 2^n C_n \delta^{-n} \mathrm{Vol}(V).$$

Remark. — If a prime number p does not divide d then this inequality also holds for $b_i(V'; \mathbf{Z}_p)$.

To prove (D), we take a sequence of subgroups $\Pi'_j \subset \Pi'$ whose intersection is the identity element and by applying the sublemma to the corresponding finite coverings $\tilde{V}'_j \to V'$ and $\tilde{V}_j \to V$ we get $\lim_{j\to\infty} \sup [\sum_i b_i(\tilde{V}_j; \mathbf{R})]/\mathrm{Vol}(\tilde{V}_j) \leq 2^n C_n \delta^{-n}$ for $\delta = 1$. Finally, since the numbers $\chi(V')$ and $\sigma(V')$ are multiplicative for finite coverings, they are estimated from above by $2^n C_n \mathrm{Vol}(V)$, q.e.d.

There is yet another characteristic number, the $\hat{A}$-genus $\hat{A}(V)$, which can also be estimated from above if $\mathrm{Ricci}\, V \geq -1$, but now with the diameter of V rather than with the volume (compare [15], [16], [17]).

(E) *Theorem.* — *Closed spin-manifolds* V (i.e. *the Stiefel class* $w_2 = 0$) *with* $\mathrm{Ricci}\, V \geq -1$ *satisfy*

$$\hat{A}(V) \leq 2^{n/2} + (\mathrm{Diam})^n (\mathrm{const}_n)^{1+\mathrm{Diam}}, \qquad (*)$$

for some universal constant const_n.

Proof. — By Lichnerowicz' formula (see [26] for details) the square of the Dirac operator is

$$D^2 = \Delta' + \frac{1}{4} S,$$

where Δ' is the Laplacian on the spin bundle and S is the scalar curvature of V. Therefore, the index $\mathrm{Ind}\, D = \hat{A}(V)$ is bounded from above by the number of those eigenvalues of the operator Δ' which do not exceed $-\inf_{v \in V} S(v)$. These eigenvalues λ'_i are related to the eigenvalues λ_i of the ordinary Laplace operator by *Kato's* inequality (see [29]),

$$\sum_i \exp(-\lambda'_i t) \leq 2^{n/2} \sum_i \exp(-\lambda_i t),$$

for all $t > 0$, and the application of the following inequality of [22] concludes the proof

$$\lambda_i \geq (\mathrm{Diam})^{-2} (C_n^{1+\mathrm{Diam}}) i^{2/n}.$$

Corollary. — *For* $n = \dim V = 4$ *the signature of a spin manifold* V *with* $\mathrm{Ricci}\, V \geq -1$ *satisfies*

$$|\sigma(V)| \leq 32 + (\mathrm{Diam})^4 (\mathrm{const})^{1+\mathrm{Diam}}.$$

Remark. — The argument above when applied to the Hodge-Laplace operator on forms gives a lower bound for the eigenvalues in terms of Diam V and of the lower bound on the curvature operator (compare [15], [16], [17]).

Finally, let V be a closed 4-dimensional *Einstein manifold* with $\mathrm{Ricci}\, V = -1$ (and so with $S(v) \equiv -2$). Then by a formula of Berger-Thorpe-Hitchin (see [32]),

$$\chi(V) \geq \theta\,|\sigma(V)| + \theta'\,\mathrm{Vol}\, V,$$

for every θ in the interval $[0, 3/2]$ and for $\theta' = 1/48\pi^2 - \theta^2/108\pi^2$.

Therefore, the main inequality of (0.5) implies the following

Theorem. — *All closed 4-dimensional Einstein manifolds satisfy*

$$\chi(V) \geq \theta\,|\sigma(V)| + \theta'\,\|V\|/216$$

for θ *and* θ' *as above.*

Example. — Take a Cartesian square $X \times X$ of a closed surface of genus $q \geq 2$. Delete p open balls from $X \times X$ and call V the double of the resulting manifold with boundary. This V has $\chi(V) = 8(q-1)^2 - 2p$ and $\sigma(V) = 0$, while, by Milnor-Sullivan-Thurston inequality of section (0.3),

$$\|V\| = 2\,\|X \times X\| \geq 32(q-1)^2.$$

It follows that for $8(q-1)^2 - 2p < (q-1)^2/324\pi^2$ the manifold V admits no Einstein metric.

Now, we return to our main topic and we extend Thurston's inequality of section (0.3) to the specific volume $\|\ \|_-$.

(A) *Let* V *be a Riemannian manifold with sectional curvature* ≤ -1. *If* V *is stable at infinity then it satisfies, for* "size" = Dil,

$$\|V : \mathrm{Vol}\|_- \geq \mathrm{const}_n^{-1} \geq (n-1)!/\pi.$$

Proof. — For all stable sequences $U_i \subset V$ and all fundamental cycles c with $\mathrm{Dil}(c) < \infty$, we obtain, by Thurston's straightening argument (see (1.2)),

$$\lim_{i\to\infty} \inf \mathrm{Vol}(U_i)/\|c \mid U_i - R\| \geq \mathrm{const}_n$$

for all $R \geq 0$, where const_n is the constant of section (1.2). Observe, that the manifold V does not even need to be complete.

Next, we observe the following estimate from above for $||\ ||_+$:

(B) *All n-dimensional manifolds* V *with* $\mathrm{Ricci}(V) \geq -1/(n-1)$ *satisfy, for* " size " = length + Rad,

$$||V : \mathrm{Vol}||_+ \leq C_n n! \qquad (*)$$

for the constant $C_n < 1$ *of* (D) *in* (4.3).

Proof. — For complete manifolds V this follows from (D) of (4.3) and Bishop's inequality [5]. If V is not complete, we just extend the induced metric in $V - R$, for some $R > 0$, to a complete metric in $V \supset V - R$, and then (D) of (4.3) applies.

(B′) *Remark.* — If V has non-positive sectional curvature, then according to (A) of (4.3) the inequality (*) holds with a sharper " size ", namely with " size " $\sigma = \mathrm{Dil}\,\sigma$, that is, the Lipschitz constant of singular simplices σ. Furthermore, the inequality (*) for " size " = Dil holds for all manifolds V with $|K(V)| \leq 1$, but now with another constant $C_n' > C_n$ in place of C_n (see *(c)* of (4.3)).

Finally, we come to the following

Volume comparison theorem. — *Let* V_1 *and* V_2 *be n-dimensional manifolds, such that* $\mathrm{Ricci}(V_1) \geq -1/(n-1)$ *and* $K(V_2) \leq -1$. *Suppose furthermore, that the manifold* V_2 *is stable at infinity. Then all proper Lipschitz maps* $f : V_1 \to V_2$ *satisfy*

$$\mathrm{Vol}_-(V_1 : V_2 ; f) \geq A_n \deg(f)$$

for $A_n = (\mathrm{const}_n C_n n!)^{-1} > (n\pi)^{-1}$.

Corollaries and examples

(1) *If* V_2 *is a complete manifold of subexponential growth, then the pullbacks of* R*-balls* $B(R) \subset V_2$ *around a fixed point satisfy, for* $R \to \infty$,

$$\lim_{R \to \infty} \sup \mathrm{Vol}(f^{-1}(B(R)))/\mathrm{Vol}(B(R)) \geq \mathrm{Vol}_- > (\deg f)/n\pi.$$

In particular, if V_2 *has finite volume, then*

$$\mathrm{Vol}(V_1)/\mathrm{Vol}(V_2) > (\deg f)/n\pi.$$

(2) *If both manifolds* V_1 *and* V_2 *are hyperbolic manifolds of sectional curvature* -1, *then, with the exact estimate of section* (2.2), *one gets the sharp inequality*

$$\mathrm{Vol}_-(V_1 : V_2 ; f) \geq \deg f.$$

It would be interesting to analyse the case of equality in the spirit of Thurston's rigidity theorem (see [47], [25]). Recall that, by a theorem of Sullivan [46], a bi-Lipschitz homeomorphism between hyperbolic manifolds of subexponential growths (in fact, between manifolds for which $\limsup R^{-1} \log \mathrm{Vol}\, B(R) < n - 1$) is homotopic to an isometry.

Proof. — Suppose first $K(V_1) \leq 0$. Then, by (B') we have the upper bound for $||V_1 : \mathrm{Vol}||_+$ with "size" = Dil, while (A) gives the lower bound for $||V_2 : \mathrm{Vol}||_-$ with the same "size". Therefore, the functorial properties of these specific volumes yield the theorem.

Now, if V_1 only has $\mathrm{Ricci}\, V_1 \geq -1$, this argument does not work since the theorem (B) applies to the "size" = length + Rad which is "weaker" then Dil. In order to close the gap, we first state a sharpening of the main technical theorem of section (4.3). To that effect, we introduce a new "size" called $\widetilde{\mathrm{Rad}}\,\sigma$: a singular simplex σ has $\widetilde{\mathrm{Rad}}\,\sigma < \rho$ if, first, it has $\mathrm{Rad}\,\sigma \leq \rho$ and so is contained in a ball $B_v(\rho) \subset V$, and, secondly, the lift of σ to the universal covering $\widetilde{B}_v(\rho)$ of $B_v(\rho)$, call it $\tilde{\sigma} : \Delta \to \widetilde{B}_v(\rho)$, has its image $\tilde{\sigma}(\Delta)$ in the ball $B_{\tilde{v}}(\rho) \subset \widetilde{B}_v(\rho)$ with center $\tilde{v}$ in $B_v(\rho)$ over $v \in V$. The minimal ρ for which σ lifts to such a ball $B_{\tilde{v}}(\rho)$ is our new "size" $\sigma = \widetilde{\mathrm{Rad}}\,\sigma$.

(1) *Claim.* — *The main technical theorem of* (D) *in* (4.3) *holds with* $\widetilde{\mathrm{Rad}}$ *substituted for* Rad.

We shall prove this later. Now, we sharpen the theorem (A) for manifolds V of negative curvature. If such V is complete, we denote by $\overline{\mathrm{Rad}}\,\sigma$, for $\sigma : \Delta \to V$, the radius of a lift of the simplex σ to the universal coverings V. If V is not complete, we only take into account the simplices contained in $V - R$ for large R and then lift them, whenever possible, to exponential balls $\exp_v : B(\rho) \to V$ for $v \in V - R$ and for $\rho \leq R$. The minimal ρ for which such a lift exists is called $\overline{\mathrm{Rad}}\,\sigma$. Then for a chain $c = \sum_i r_i \sigma_i$ we write $\overline{\mathrm{Rad}}\, c \leq \rho$ if all simplices σ_i which are contained in the R-interior $V - R$ for a sufficiently large R, have $\overline{\mathrm{Rad}}\,\sigma \leq \rho$.

(2) *Claim.* — *Thurston's theorem* (A) *holds for this weaker* "size" $c = \overline{\mathrm{Rad}}\, c$.

Indeed, simplices in exponential balls can be straightened as before, while the simplices outside $V - R$ do not matter anyway.

Now, let V_2 be a complete manifold. Then, for maps $f : V_1 \to V_2$ with the Lipschitz constant L_f we have $\overline{\mathrm{Rad}}(f_*(c)) \leq L_f \widetilde{\mathrm{Rad}}\, c$ for all chains c in V_1. Thus our theorem for complete manifolds V_2 is reduced to the first claim. We postpone the treatment of non-complete manifolds until the very end. Now we come to the

Proof of Claim (1). — We must diffuse all simplices σ_i of the chain $c = \sum_i r_i c_i$, or rather the small fat simplices Δ of a subdivision of c, to some simplices of $\widetilde{\mathrm{Rad}} \leq \widetilde{R} + 2R + \varepsilon$ (compare with proof in (D) of (4.3)). To do that, we modify the complex K only by admitting those simplices σ with vertices $v_0, \ldots, v_m$ in V, whose lifts $\tilde{\sigma}$ to the universal covering of the union of the balls $B_k = B_{v_k}(R + \widetilde{R} + \varepsilon/2)$, $k = 0, \ldots, m$, are contained in the union of the balls $B_{\tilde{v}_k}(R + \widetilde{R} + \varepsilon/2)$ in this covering $\widetilde{\bigcup_k B_k}$, where the centers $\tilde{v}_k$ of these new balls are exactly the vertices $\tilde{v}_0, \ldots, \tilde{v}_m$ of $\tilde{\sigma}$. With these new admissible

simplices $\tilde{\sigma}$ we build a new complex, called $\tilde{K}$ with the properties (i)-(iii) of (D) (4.3) and this $\tilde{K}$ has the same formal properties as the old K. But now, we have a problem with the operators $\mathscr{S}_{R,\bar{R}}$: these operators when applied to the edges of simplices Δ, are only defined up to homotopies in the coverings $\tilde{B}_v(R')$, rather than within some balls in these coverings, and so these homotopies may not be admissible in the new sense. Therefore, we must replace the étale domains $\tilde{B}_v(R')$ by some new domains which simultaneously enjoy the two properties: they are simply connected and they are " balls " of radius R'. Here is the construction. For an arbitrary open ball $B_v(R)$ in a Riemannian manifold, we first take the universal covering $\tilde{B}_v(R)$ of $B_v(R)$ and then the ball $B_{\tilde{v}}(R) \subset B_v(R)$ around some point $\tilde{v} \in \tilde{B}_v(R)$ over v, called $B'_v(R) = B_{\tilde{v}}(R) \to B_v(R)$. By iteration of that process, we get a projective system of balls and of locally isometric maps

$$B_v(R) \xleftarrow{p_1} B'_v(R) \xleftarrow{p_2} B''_v(R) \leftarrow \ldots \xleftarrow{p_i} B^{(i)}_v(R) \leftarrow \ldots$$

This projective system converges over all balls $B_v(R_0) \subset B_v(R)$ for $R_0 < R$. Namely for all $i \geq i_0 = i_0(R_0)$ the maps p_i become bijective on the concentric R_0-balls, that is, for every point $\tilde{w} \subset B^{(i-1)}(R)$ which projects on $B_v(R_0)$, there is only one point in the pullback $(p_i)^{-1}(\tilde{w})$. Therefore, the projective system $(p_i : B^{(i)} \to B^{(i-1)})$ has a limit $\hat{p} : \hat{B}_v(R) \to B_v(R)$, where $\hat{B}_v(R)$ is a *simply connected* Riemannian manifold and $\hat{p}$ is a locally isometric map. Furthermore, this $\hat{B}$ is *a ball* relative to a unique lift $\hat{v} \in \hat{B}$ of $v \in V$ as the center. Indeed, the exponential map at $\hat{v}$ is defined on the Euclidean ball $B_0(R) \subset T_{\hat{v}}(B)$ and the map $\exp_{\hat{v}} : B_0(R) \to \hat{B}_v(R)$ is surjective. Now, we replace the domains $\tilde{B}_v(R')$ by the balls $\hat{B}_v(R')$ and the rest of the proof of the technical theorem goes through without any problem. Thus the volume comparison theorem is established for complete manifolds V_2.

Now, let V_2 be an arbitrary manifold of negative curvature. Over all points $v \in V_2 - R$, we have the exponential (étale) balls, call them $p_v = \exp_v : \bar{B}_v(\rho) \to V_2$ for $\rho < R$, and with the map $f : V_1 \to V_2$, we lift this (étale) system of balls $\bar{B}_v(\rho)$ to a system of étale domains $\bar{B}^*_w$ over V_1 for all $w \in V_1$: the domain $\bar{B}^*_w$ is defined as the set of pairs $(w', v') \in V_1 \times B_v$ for which $f(w') = p_v(v')$ for all w in the pull-back $f^{-1}(v) \subset V_1$, while the étale map $p^*_w : \bar{B}^*_w \to V_1$ is induced by the projection $V_1 \times \bar{B}^*_w \to V_1$. Furthermore, all étale maps between the domains $\bar{B}_v$ lift to étale maps between the $\bar{B}^*_w$.

Next, we take a fundamental cycle $c = \sum_i r_i \Delta_i$ of V_1 which consists of small fat simplices Δ_i and we directly diffuse this c into a fundamental class of V_2. We do that with an étale smoothing $\mathscr{S}$ in V_1 over the domains $\bar{B}^*_w$ as follows. Simplices Δ in V_1 are lifted to simplices $\bar{\Delta}$ in $\bar{B}^*_w$ and the smoothing $\mathscr{S}$ sends the vertices of $\bar{\Delta}$ to some points $\bar{w}'_0, \ldots, \bar{w}'_n$ in $\bar{B}^*_w$, endowed with certain weights. Next, with the map f we have a unique map of $\bar{B}^*_w$ to an exponential ball $\bar{B}_v$ over V_2 for $v = f(w)$ and thus we send $\bar{w}'_0, \ldots, \bar{w}'_n$ to some points $\bar{v}'_0, \ldots, \bar{v}'_n$ in $\bar{B}_v$ which carry the same weight as $\bar{w}'_0, \ldots, \bar{w}'_n$. Finally, these points in $\bar{B}_v$ over V_2 span a unique straight simplex whose projection on V_2 is exactly what we need. So, we have sent Δ to a straight chain in V_2, say $\mathscr{S}^f * \Delta$, and the chain c is sent to the fundamental cycle $\mathscr{S}^f * c = \sum_i r_i \mathscr{S}^f * \Delta_i$ in V_2.

Now, we choose the étale smoothing $\mathscr{S} = \mathscr{S}(w)$ for all $w \in V_1$ in the same way as before. Namely, we use balls in the domains $\overline{B}^*_w$ over V_1: if $\overline{B}^*_w$ is sent by f to an exponential ball $\overline{B}_v(\rho)$ over V_2, for $v = f(w)$, we take the point $\overline{w} = (w, \bar{v}) \in \overline{B}^*_w$, where $\bar{v} \in \overline{B}_v(\rho)$ is the center of $\overline{B}_v(\rho)$. Next, we take the ball $B_{\overline{w}}(\rho') \subset \overline{B}^*_w$ of radius $\rho' < (L_f)^{-1}\rho$, where L_f is the Lipschitz constant of f, and, finally, we take our old measure with density one in $B_{\overline{w}}(\rho')$ and zero outside $B_{\overline{w}}(\rho')$. This measure is regularized with a fast decaying function φ on V_2 and then it is made discrete with a subset $Z \subset V_1$ (see (D) of (4.3)). We estimate the norm $[\mathscr{S}(w)]^*$ as before and for $\rho' \to \infty$ we get

$$||\mathscr{S}^i * c \mid U - R||_{\ell^1} \leq C_n\, n!\, \mathrm{Vol}(f^{-1}(U))$$

for all $U \subset V_1$ and for $R > \rho > L_f \rho'$. This estimate combined with (A) implies our theorem.

Final Remarks. — We want to indicate possible candidates for specific volumes for *compact* manifolds. Let us start with an example. Recall, that the singular homology theory can be built with singular cubes, $\delta : \square \to X$, rather than with simplices, and for this cubical singular homology we define as before the cubical ℓ^1-norm on the homology $H_*(X)$, called $||\ ||^{\square}$. Then, for all $h \in H_m(X)$, we consider the ratio $||h||^{\square}/||h||$, call it $||h;\ \square : \Delta||$. We cannot go further since we do not know the answer to the following

Question. — Does the "norm" $||h;\ \square : \Delta||$ depend on X and h, or is $||h;\ \square : \Delta|| = \mathrm{const} = \mathrm{const}(m)$?

In fact, there is a whole spectrum of norms on $H_*(X)$. Indeed, to each space X we associated in section (3.2) a complete minimal multicomplex K. This K is unique if all its components have countably many vertices and it comes with a homotopy equivalence $S : K \to X$.

All simplices in K can be identified with the faces of the standard infinite dimensional simplex Δ^∞ and thus for every metric G in Δ^∞ which is invariant under the automorphisms of Δ^∞ we obtain a metric in K. Now, we can speak of masses of homology classes in K and we put

$$||h;\ G|| = \mathrm{mass}(S_*^{-1}h)$$

for all $h \in H_*(X)$ and all spaces X.

Again, we do not know whether the ratio $||h;\ G||/||h||$ depends on h, or whether

$$||h;\ G||/||h|| = \mathrm{const}(G, \dim h).$$

Example. — Let Δ^∞ be realized by the "projective simplex" $P^+ \subset P(\mathscr{M})$ (see (2.4)) for $\mathscr{M}$ the space of measures on the set of all vertices of Δ^∞. The ℓ^1-metric in $\mathscr{M}$,

$$\mathrm{dist}(\mu_1, \mu_2) = ||\mu_1 - \mu_2||_{\ell^1}$$

induces a metric in the projective space $P(\mathcal{M})$: lengths of curves in $P(\mathcal{M})$ are defined as the lower bounds of the lengths of their lifts to $\mathcal{M}$. Then, with this metric G on P^+ (or rather with the mass defined in (2.4)) one can prove that, for all $h \in H_m(X)$,

$$||h; G|| = ||h||/m!.$$

Appendix 1. On Min Vol $\mathbf{R}^2$.

Let V be a complete simply connected surface with $K(V) \leq 1$.

Proposition. — *There exist two points x and y in V, with the following properties*:

(1) $\operatorname{dist}(x, y) = \pi$,

(2) $\operatorname{Inj Rad}_x(V) + \operatorname{Inj Rad}_y(V) \geq \pi$.

Proof. — If there is some point $v_0 \in V$, for which $R_0 = \operatorname{Inj Rad}_{v_0}(V) \geq \pi$, then the proof is finished. If $R_0 < \pi$, there is a simple geodesic loop, L_0 in V, at v_0 such that $\operatorname{length}(L_0) = 2R_0$ and a unique point $v_0' \in L_0$ with $\operatorname{dist}(v_0, v_0') = R_0$. The loop L_0 bounds a disk D_0, and the cut-locus of V relative to v_0 contains, by a theorem of Meyers (see [18]), a conjugate point v in D_0. Since $\operatorname{dist}(v_0, v) \geq \pi$, there is a point $v_1 \in D_0$ for which $\operatorname{dist}(v_0, v_1) = \pi$. Let R_1 denote the injectivity radius at v_1 and let L_1 be the corresponding loop at v_1. If the loops L_0 and L_1 intersect, then $L_0 + L_1 = 2(R_0 + R_1) \geq 2\pi$, and the proof is finished. If the intersection of these loops is empty, then L_1 bounds a disk D_1 inside D_0. Then again, we have a point $v_2 \in D_1$ with $\operatorname{dist}(v_1, v_2) = \pi$. By repeating this process, we arrive eventually at a pair of points $x = v_i$ and $y = v_{i+1}$, $i \leq \infty$, for which the loops L_i and L_{i+1} do intersect and the proof is finished.

Now, let γ be a minimal geodesic between x and y and let $z \in \gamma$ be a point for which $\rho' = \operatorname{dist}(z, x) \leq \operatorname{Inj Rad}_y(V)$ and $\rho = \operatorname{dist}(z, y) \leq \operatorname{Inj Rad}_x(V)$. Then the ball $B = B_z(\pi) \subset V$ contains two disjoint open injective balls $B_x(\rho)$ and $B_y(\rho')$ with $\rho + \rho' = \pi$. Therefore, $\operatorname{Vol}(B) \geq 4\pi$. Suppose that V is homeomorphic to $\mathbf{R}^2$ and let moreover $|K(V)| \leq 1$. Take points $w \in V$ far from B. Then by Bishop's inequality ([5]), one has, for increasing $R = \operatorname{dist}(w, B)$,

$$\lim_{R \to \infty} \sup \operatorname{Vol} B_w(R)/\operatorname{Vol} B_w(R + 2\pi) \geq e^{-2\pi},$$

and so $\operatorname{Vol}(V) \geq 4\pi(1 + e^{-2\pi})$. This implies our assertion in section (0.1):

$$\operatorname{Min Vol}(\mathbf{R}^2) \geq 4\pi(1 + e^{-2\pi}) > 4\pi + 0.01.$$

Finally, take the largest of the two balls in the standard sphere S^2 whose boundary circle has geodesic curvature -1 and then extend the metric of this ball B_0 to a complete C^1-metric in $\mathbf{R}^2$ with Gaussian curvature -1 outside B_0. This metric has $|K| = 1 \leq 1$, $\int K\, d\sigma = 2\pi$, $\int_{K>0} d\sigma = (2 + \sqrt{2})\pi$, $\int |K\, d\sigma| = \operatorname{Vol} = (2 + 2\sqrt{2})\pi$. Therefore,

$$\operatorname{Min Vol}(\mathbf{R}^2) \leq (2 + 2\sqrt{2})\pi.$$

Appendix 2. Manifolds with Min Vol = 0.

Let V be a compact Riemannian manifold with a non-vanishing Killing field X. The Riemannian metric g of V then splits into orthogonal components, $g = g_0 + g_1$, where the quadratic form g_0 of rank one is zero on the orthogonal complement to X, while the form g_1, of rank $n - 1$, is zero on X. Then the metrics $g = \varepsilon^2 g_0 + g_1$ have uniformly bounded curvatures, $|K(V, g_\varepsilon)| \leq \text{const}$ for $\varepsilon \to 0$, whereas $\text{Vol}(V, g_\varepsilon) \to 0$.

For a compact manifold V with a locally free S^1-action, there exists an S^1-invariant metric, hence a field X as above, and so Min Vol V = 0, as claimed in section (0.1). If V is not compact, then the above argument still applies to an appropriate S^1-invariant metric and to a suitable S^1-invariant *function* ε on V. Then again Min Vol V = 0.

Now, we introduce a notion of T-*manifold (or* T-*structure)* which generalizes manifolds with locally free S^1-action. Suppose that V is covered by open subsets U_i $(i = 1, \ldots)$, and let every manifold U_i be endowed with a locally free S^1-action, called A_i on U_i for $i = 1, \ldots$ The collection (U_i, A_i) is called a T-structure on V if all intersections $U_{i_1} \cap \ldots \cap U_{i_k}$ of the sets U_i are invariant under the actions $A_{i_1}, \ldots, A_{i_k}$, and if these actions commute whenever they are simultaneously defined.

Examples. — (1) A 3-manifold V is called a *graph manifold* if there are disjoint embedded 2-tori T_k in V, such that the complement of an open tubular neighbourhood of the union $\bigcup_k T_k$ admits a free circle action.

Clearly, graph manifolds are exactly all 3-dimensional T-manifolds.

Connected sums of graph manifolds are also graph manifolds (see [43], § 2, lemma 4). Furthermore, the argument of [43] applies to all odd dimensions:

(2) *Connected sums of odd dimensional* T-*manifolds are* T-*manifolds.*

(3) Observe that $\mathbf{R}^3$, being an infinite connected sum of 3-spheres, is a graph manifold. Since the product $V \times V_0$ of a T-manifold V and an arbitrary manifold V_0 is a T-manifold, products of the form $\mathbf{R}^3 \times V_0$ are T-manifolds.

(4) *Let* $f: W_0 \to W$ *be a proper immersion with transversal self-intersections. If* $\text{codim } W_0 = 2$ *then the boundary of a regular neighbourhood of the image* $f(W_0) \subset W$ *is a* T-*manifold.*

Indeed, local models of transversal intersections are systems of mutually orthogonal subspaces in $\mathbf{R}^n$ of codimensions 2. Rotations around these subspaces give commuting S^1-actions on some neighbourhoods U_i of our boundary, and small perturbations of these actions form a T-structure.

(5) *Corollary.* — *If the interior of a compact manifold* X *admits a complex quasi-projective structure, then, by Hironaka's theorem* (see [30]) *one has a* T-*structure on the boundary of* X.

Now, we claim:

Theorem. — *All* T-*manifolds* V *have* $\operatorname{Min} \operatorname{Vol}(V) = 0$.

Indeed, for an appropriate metric g on V which is A_i-invariant on the neighbourhoods U_i for all i, there are ε_i-squeezings of g in the A_i-directions that keep the curvature bounded and make $\operatorname{Vol}(V) \to 0$ for $\varepsilon_i \to 0$.

Let us now exhibit T-manifolds without any S^1-action. First, we recall some elementary facts (see [8] for more information) on actions of the group S^1 on closed connected manifold V. For each point $v \in V$, the orbit $S^1_v : S^1 \to V$ gives us an element in the fundamental group, called $\sigma \in \pi_1(V, v)$. Clearly, this σ is in the center of $\pi_1(V)$. Furthermore, it acts trivially on all homotopy groups $\pi_i(V)$. We denote by $C \supset V$ the mapping-cylinder of the quotient map $q : V \to V/S^1$ and, following an unpublished remark of Sullivan, we observe that, for *locally free* actions, the pair (C, V) is a rational homology manifold with boundary $\partial C = V$. In particular, all characteristic numbers of these manifolds V vanish. (Recall that, by section (0.1), this is also true for all T-manifolds V, since $\operatorname{Min} \operatorname{Vol}(V) = 0$.) Next, we denote by $\sigma'_v \in \pi_1(V, v)$ the homotopy class of the *image* of the orbit S^1_v in V and we observe that σ is a multiple of σ'_v for all $v \in V$.

Now, take a $K(\Pi, 1)$-space for some group Π and consider maps $f : V \to K(\Pi, 1)$. The following remark is also due to Sullivan:

If a map f sends all "geometric orbits" σ'_v *to the identity in* $\pi_1(K(\Pi, 1)) = \Pi$, *then it extends to the cylinder* $C \supset V$. *In particular, the fundamental class* $[V] \in H_n(V; \mathbf{R})$ *goes to zero under f.*

Indeed, the pullback of any point under the map $q : V \to V/S^1$ is path-connected: it is just a circle or a point. Therefore, the map q is surjective on the fundamental groups and the kernel of this surjective homomorphism is the normal subgroup N spanned by the fundamental groups of the pullbacks $q^{-1}(x)$ for all $x \in V/S^1$. This is seen by looking at the covering $\tilde{V} \to V$ which has Galois group $\pi_1(V)/N$.

Now, $\pi_1(C) = \pi_1(V/S^1)$, and so the inclusion homomorphism is surjective. Its kernel is spanned by $\sigma'_v \in \pi_1(V, v)$ for $v \in V$. It follows that the homomorphism $f : \pi_1(V) \to \Pi$ factors through $\pi_1(C)$ and so the map f extends to C.

Since Pontryagin numbers are bordism invariants for rational homology manifolds, we also have the

Corollary. — *Let the action of* S^1 *on* V *be locally free. Then all Pontryagin numbers of the map f vanish. That is, for all* $h \in H^*(K(\Pi, 1); \mathbf{R})$ *and for all Pontryagin classes* $p = p(V)$ *one has*

$$\langle p(V) \cup f^*(h), [V] \rangle = 0.$$

Example. — *Let* V_1 *be a* $K(\Pi, 1)$-*manifold such that the group* Π *has no center. If a manifold* V_2 *has a non-zero Pontryagin number, then the product* $V = V_1 \times V_2$ *admits no locally free* S^1-*action.*

Remark. — If we only require the (multiple!) orbit $\sigma \in \pi_1(V)$ to go to the identity (rather than all σ_v'), then f may not extend to C but the fundamental class of V goes to zero anyway, $f_*([V]) = 0$.

Indeed, as we work in homology with real coefficients, the difference between σ_v' and σ, which is a multiple of σ_v', is not essential. Geometrically speaking, one does not extend the map f itself, but a collection of lifts of f to finite coverings of S^1-invariant neighbourhoods of all S^1-orbits. These coverings unwrap all multiple orbits, so that we have coherent extensions of these lifts to the corresponding cylinders. These extensions add up algebraically to a chain in our $K(\Pi, 1)$, whose boundary is a multiple of $f_*[V]$.

Corollary (Sullivan, unpublished). — *If the* S^1*-action has a fixed point in* V, *then any map* f *of* V *to* $K(\Pi, 1)$*-spaces satisfies* $f_*([V]) = 0$.

Indeed, if there is a fixed point, then σ goes to the identity.

Let us call a manifold V *essential* if the classifying map $V \to K(\Pi, 1)$, for $\Pi = \pi_1(V)$, sends the fundamental class [V] to a non-zero element in $H_*(K(\Pi, 1); \mathbf{R})$. If V is not orientable, we apply the definition to an oriented double covering of V.

For example, all $K(\Pi, 1)$-manifolds V are essential. If $V_1 \to V_2$ is a map of non-zero degree and if V_2 is essential, then also V_1 is essential. On the other hand, manifolds with finite fundamental groups are not essential.

Now, if V *is an essential manifold which admits an* S^1*-action, then there is a (non trivial!) element* σ *in the center of the fundamental group and also the action of this* σ *on all homotopy groups* $\pi_i(V)$, $i = 1, 2, \ldots,$ *is trivial.*

Example. — A connected sum $V = V_1 \# V_2$, where V_1 is a $K(\Pi, 1)$-manifold and V_2 is not a homotopy sphere, has no nontrivial S^1-action.

Finally, observe that the above facts concerning S^1-actions are special cases of a general theorem due to Browder-Hsiang (see [50]).

There is a natural generalization of T-manifolds. Namely, instead of a covering $\{U_i\}$ we use *an étale covering*, that is a system of locally diffeomorphic maps $p_i : \tilde{U}_i \to V$ (see (D) of (4.3)) whose images cover V. The *intersection* of two étale domains, say of $p_i : \tilde{U}_i \to V$ and $p_j : \tilde{U}_j \to V$, is the set of pairs (u, u') for $u \in U_i$ and $u' \in U_j$ such that $p_i(u) = p_j(u')$.

We say that S^1-actions A_i on $\tilde{U}_i$ define an F-*structure* on V if for the intersections of all finite subsystems, $U_{i_1}, \ldots, U_{i_k}$, the actions $A_{i_1}, \ldots, A_{i_k}$ lift to these intersections and the lifted actions commute.

Example. — Let V be a flat manifold and let $U \to V$ be a covering torus. Then any isometric S^1-action on U defines an F-structure on V, although there is no obvious T-structure on flat manifolds.

One sees, as before, that F-manifolds V have $\operatorname{Min} \operatorname{Vol}(V) = 0$. Furthermore, interiors $V = \operatorname{Int} \overline{V}$ of compact manifolds whose boundaries $\partial \overline{V}$ admit F-structures,

have $\operatorname{Min}\operatorname{Vol}(V) < \infty$. Probably, all manifolds V with $\operatorname{Min}\operatorname{Vol}(V) = 0$ admit F-structures. This is not difficult to show for 3-manifolds. In fact, many 3-manifolds V can be cut along some 2-spheres and incompressible tori into simpler manifolds V_i, such that some of these V_i are T-manifolds and the remaining ones admit complete hyperbolic structures of finite volumes (see [47]). Let Vol_h denote the total volume of these hyperbolic pieces. Then an argument similar to the above one shows that $\operatorname{Min}\operatorname{Vol}(V) \leq \operatorname{Vol}_h$. Observe (see (0.4)) that $\operatorname{Vol}_h = R_3 ||V||$, where R_3 is the volume of the ideal regular simplex in the hyperbolic 3-space. Probably, $\operatorname{Min}\operatorname{Vol}(V) = R_3 ||V||$ for these manifolds V.

We conclude, as announced in section (0.1), with a simple *construction of complete metrics g on open manifolds* V, *such that the curvatures* $K_v(g)$ *at points* $v \in V$ *rapidly decrease for* $v \to \infty$, *while the volumes of balls* $B_{v_0}(R) \subset V$ *have slow growth for* $R \to \infty$.

We start with an arbitrary complete metric g on V and we indicate two closely related modifications of g. The first modification makes curvatures small at infinity, and the second takes care of $\operatorname{Vol}(B_{v_0}(R))$.

(1) *Conical telescope.* — Fix a reference point $v_0 \in V$ and let the smooth manifolds $S_1, \ldots, S_i, \ldots$ be smooth approximations of metric " spheres " of radius $1, \ldots, i, \ldots$, around v_0. Proceed by induction on $i = 1, 2, \ldots$ as follows.

Cut the manifold open along S_i and connect the two parts (sides) again by inserting a smooth differentiable cylinder

$$[0, \lambda_i \tau_i] \times S_i, \quad \lambda_i \geq 1, \quad \tau_i > 0.$$

By doing so and smoothing later, one does not change the diffeomorphism type of V.

Choose on this cylinder the following " conical " metric:

$$(\lambda_i \tau_i \, dt)^2 + [\lambda_i t + (1 - t)]^2 \, (g_{i-1} \mid S_i)$$

where $$g_{i-1} = \lambda_{i-1}^2 \lambda_{i-2}^2 \ldots \lambda_2^2 \lambda_1^2 g, \quad \lambda_i > 1.$$

Inside our deleted S_i, take $g_i = g_{i-1}$, *outside* take $g_i = \lambda_{i-1}^2 g_{i-1}$. The result is not (yet) smooth along the spheres.

If we choose τ_i big, then we can approximate the resulting metric $g' = \lim_{i \to \infty} g_i$ by a smooth metric g''. By choosing also $\lambda_i > 1$ big for increasing i we can make the sectional curvature at v tend to zero for $v \to \infty$ in the new metric g'' as rapidly as we please outside S_2. This operation makes Inj Rad(V) tend to ∞ for $v \to \infty$. The fastest possible decay of K_v in the new metric g'' is

$$|K_v| \leq \operatorname{const}(\operatorname{dist}''(v, v_0))^{-2},$$

for $v \to \infty$ and for some constant $= \operatorname{const}_n$, $n = \dim V$.

(2) *Cylindrical telescopes* are obtained for $\lambda_i = 1$. They give product metrics on the cylinders. By taking τ_i big we can obtain

$$\operatorname{Vol}(B_{v_0}(R)) \leq C(R)R$$

for $R > R_0$, for any given continuous positive function $C(R) = (\varepsilon(R))^{-1}$ converging to ∞ for $R \to \infty$ and R_0 some constant. Furthermore if the cylindrical telescope process is preceded by a suitable conical telescope we can obtain *for any positive function* $\varepsilon(R)$ *with* $\lim_{R=\infty} \varepsilon(R) = 0$:

$$\varepsilon(R) \cdot R^{-1} \operatorname{Vol}(B_{v_0}(R)) \to 0, \quad \text{for } R \to 0,$$

as well as

$$K_u(V) \to 0, \quad \text{for } u \to \infty,$$

and

$$\operatorname{Inj} \operatorname{Rad}_u(V) \to \infty, \quad \text{for } u \to \infty.$$

Finally, observe that the argument of the example of (0.1) also shows that, for topological products $V = V_0 \times \mathbf{R}$,

If the sectional curvatures at all points $v \in V$ *satisfy*

$$|K_v(V)| \leq \text{const}(\operatorname{dist}(v, v_0))^{-2-\varepsilon}$$

for a fixed point $v_0 \in V$ *and for some* $\varepsilon > 0$, *then all Pontryagin numbers of* V_0 *vanish.*

Appendix 3 : Manifolds of non-positive curvature.

First, let V be a complete Riemannian manifold of strictly negative curvature K(V), and $-k^2 \leq K(V) \leq -1$. If the manifold V has $\operatorname{Vol}(V) < \infty$, then the geometry of V at infinity is fairly simple. The complement of some compact submanifold V_0 with boundary is a union of finitely many *cusps*. These are described as follows. First one shows that the manifold V only has finitely many ends. Furthermore, the distance between any two rays going to one end is bounded. To get a better picture, we pass to the universal covering $\tilde{V} \to V$ and lift a ray corresponding to an end of V to a ray $\gamma \subset \tilde{V}$. Then, we consider the horofunction corresponding to γ, called $h(v) = h_\gamma(v)$ defined for all $v \in \tilde{V}$ by $h(v) = \lim_{x \to \infty} (\operatorname{dist}(v, x) - \operatorname{dist}(v_0, x))$, where x are points on γ and v_0 denotes the initial point of γ. There is a unique maximal subgroup Π_γ in the fundamental group of V which keeps this function h invariant. This group Π_γ necessarily contains a nilpotent subgroup of finite index whose rank equals $n - 1$ for $n = \dim V$. Next, we consider the sets $h^{-1}(-\infty, -t)$, for $t \in \mathbf{R}$, called *horoballs* $H(t) \subset \tilde{V}$. These horoballs are geodesically convex and invariant under Π_γ. One shows that for a sufficiently large t the quotient $H(t)/\Pi(\gamma)$ *embeds* isometrically into V under the covering map $\tilde{V} \to V$ and this is exactly the cusp which corresponds to γ. Now, it is clear that V is indeed concave at infinity as we claimed in section (1.2). For proofs of the properties above see [13].

Now, let $-k^2 \leq K(V) \leq 0$. Then, the geometry at infinity may be more complicated, even for products of manifolds of strictly negative curvature. However, if the volume Vol(V) is finite, and if the Riemannian metric in V is real analytic, then the manifold V has " finite topological type " [20]: it is homeomorphic to the interior of a

compact manifold with boundary, $V \approx \operatorname{Int} \overline{V}$. Furthermore, if V is a locally symmetric space, then the boundary $\partial\overline{V}$ has a natural F-structure, as is clear from the analysis performed in [41].

Finally, if V is real analytic and the universal covering $\tilde{V}$ of V has no Euclidean factor in the De Rham decomposition, $\tilde{V} \neq V_0 \times \mathbf{R}^1$, then

$$\sum_{i=0}^{n} b_i(V) \leq \text{const}_n \, k^n \operatorname{Vol}(V),$$

for Betti numbers b_i with arbitrary coefficients. The proof will appear elsewhere.

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