ABSTRACT

AN ORTHONORMAL SYSTEM AND ITS LEBESGUE CONSTANTS -
(A lost and found manuscript).

In 1921 H. Rademacher wrote two papers on orthonormal systems. In the first one, the author discusses quite general systems and also defines the functions now commonly known as Rademacher functions; the paper appeared in 1922 (Math. Annalen, v. 87, p. 112-138). The second paper contains the completion of the system of Rademacher functions, theorems on expansions of arbitrary functions in series of the complete system and properties of the Lebesgue constants, both, for ordinary summation and for the first Cesaro means. This paper was never published and is discussed here. At Rademacher's death the manuscript had vanished from sight and only recently it miraculously reappeared.

Emil Grosswald

AN ORTHONORMAL SYSTEM AND ITS LEBESGUE CONSTANTS

(A lost and found manuscript)

E. Grosswald

Temple University

In 1921 Hans Rademacher wrote two papers on orthonormal functions. In the first [7], published in 1922, he discusses properties of fairly general systems; there he also introduces the functions known to-day under the name of Rademacher functions.

This system has much in common with the trigonometric functions, or the Haar functions. We recall that these are defined as follows (see [2]): Let $i_n^{(m)}$ be the interval $\frac{m-1}{2^n} \leq t < \frac{m}{2^n}$; then $h_n^{(m)}(t) = 2^{(n-1)/2}$ in $i_n^{(2m-1)}$, $h_n^{(m)}(t) = -2^{(n-1)/2}$ in $i_n^{(2m)}$, $h_n^{(m)}(t) = 0$ otherwise. The new system differs from both, the trigonometric and the Haar system, among others, by not being complete.

In the second paper, Rademacher completed that system and studied the problem of expanding arbitrary functions in series of the complete system, its Lebesgue constants, summability of those series by first Cesaro means and similar topics.

This second paper was never published. Immediate publication may have been postponed, due to a negative, critical appraisal of the manuscript by I. Schur. Then, just a few months later, Walsh's paper [9] on the same topic appeared and Rademacher set aside any idea of ever publishing his own.

Many years later, while he was teaching at the University of Pennsylvania, a graduate student was working on a related problem. With his well known generosity, Rademacher offered the student his own, old manuscript, in case it could be of some help. In this way, the manuscript came to light and some people had the opportunity to read it. In their opinion, the manuscript represented a very valuable work, of high esthetic appeal and they expressed their regret, that it had never been published.

Again, many years passed. Rademacher retired from the University of Pennsylvania in 1962 and went to New York. In 1967 he was stricken by a cruel illness and died in February 1969. Shortly afterwards, Professor Gian-Carlo Rota suggested to the present writer (a former student of Rademacher) to edit the collected papers of Hans Rademacher.

This suggestion was accepted. Also, remembering mentioned unpublished manuscript, every effort was made to locate it, in order to include it in the "Collected Papers" [8], but the manuscript had vanished from sight. It was not to be found among the posthumous papers, nor was it in the possession of the former student. The only thing that the editor of the "Collected Papers" could do was to mention the likely existence of that manuscript in the comments to Paper 13 in [8].

Again the years passed - in fact,more than ten. Then, during the summer of 1979, something surprizing happened. One day, the present author found in the mail a large envelope, without mention of a sender. Also, the envelope contained no letter; it did contain, however, the long lost manuscript (manu-script - not typed, but handwritten!), as well as an earlier draft of the same. The present author could only conjecture the identity of the anonymous sender(s) and offered him/them an easy opportunity to identify himself/themselves. This opportunity having been declined, it appears only fair to respect this wish to remain anonymous. The present writer wants to avail himself of this opportunity to thank the anonymous sender(s) for the service rendered to mathematics, by helping to rescue the manuscript from oblivion.

A rapid perusal of the manuscript seemed to confirm the impressions of those, who had seen it in the past; consequently, an attempt to have it published seemed justified. This would have been easy in the "Collected Papers", as a posthumous manuscript. It appears, however, almost impossible at present. Indeed, the problems discussed had all been solved most satisfactorily already some 60 years ago and later research went much beyond them.

In view of the fact that, if the manuscript was to be published, that may have to be in English, the present author undertook to translate it from the original German. While doing this, it became apparent that the manuscript is far from ready for publication. The meanings to be conveyed are clear enough, but some proofs are missing, others are barely sketched, the pages contain partly irrelevant computations, etc. For these reasons, it may well be the case that, instead of a publication of the manuscript as it is, there is more interest in following up on a certain suggestion by Professors Kőnig and Lamprecht, editors of the Archiv der Mathematik. This suggestion is to write up a short exposition of the contents of the manuscript and compare its results with those of Walsh and more recent work. It is the purpose of the present note to do just that.

In the first place, it is necessary to observe that Walsh obtained his results independently, even without knowledge of Rademacher's first paper. Indeed, he refers back to the older work of Haar [2], just like Rademacher himself in his first paper [7]; also, the point of view is completely different.

In fact, Walsh deduces most of his results, by representing his newly defined functions (now generally known as Walsh functions) as finite linear combinations of Haar functions and then invokes known theorems concerning the latter. In this way he obtains immediately the completeness of the new system. Also, if $F(x)$ is continuous on $(0,1)$ and if one sets $a_i^{(j)} = \int_0^1 F(x)\phi_i^{(j)}(x)dx$ ($j = 1,2,\ldots,2^{i-1}$; here $\phi_i^{(j)}(x)$ are the newly introduced functions of Walsh), then $\sum_{i,j} a_i^{(j)}\phi_i^{(j)}(x)$ converges uniformly to $F(x)$, provided that one groups the terms so that all superscripts j that correspond to a given i, are kept in the sum.

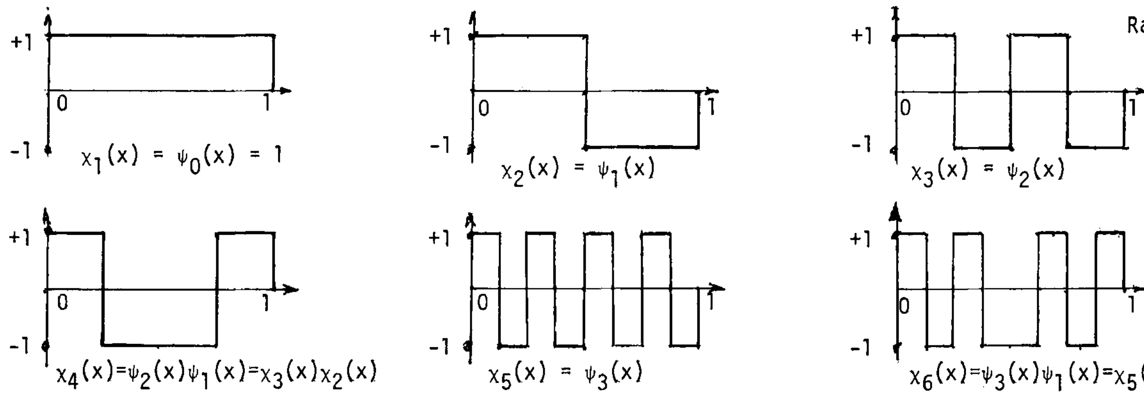
Rademacher, on the other hand, defines his functions, by using the binary expansion of the independent variable x. For $0 \le x < 1$ and $\nu = 1,2,\ldots$, one has $x = \sum_{\nu=1}^{\infty} e_\nu(x)/2^\nu$, $e_\nu(x) = 0$, or 1. If one sets $\psi_\nu(x) = 2e_\nu(x)-1$ for x and ν as before and completes the definitions by setting $\psi_\nu(1) = -1$ and $\psi_0(x) = 1$, then $x = \sum_{\nu=0}^{\infty} \psi_\nu(x)/2^{\nu+1}$. This series converges absolutely and uniformly on $0 \le x \le 1$; hence, one can square it, elevate it to the cube, etc. In this way one obtains representations of all successive powers of x by absolutely and uniformly convergent series of products of the functions $\psi_\nu(x)$. Specifically, if for $n = 2^r+s$, $0 < s \le 2^r$, we set $\chi_n(x) = \psi_{r+1}(x)\chi_s(x)$, then all powers of x are represented by absolutely and uniformly convergent series in the $\chi_n(x)$; this is sufficient to guarantee that the system of the $\chi_n(x)$ is complete. The orthonormality of the $\chi_n(x)$ easily follows from their definition.

In order to compare the results of Rademacher and Walsh, one has to observe that Rademacher's χ_{2^r+s} corresponds to Walsh's $\phi_r^{(s)}$ (up to an immaterial shift of index).

While Walsh, as already mentioned, obtains his convergence results from the analogous ones for the Haar functions, Rademacher considers directly the kernels $K_n(x,y) = \sum_{\nu=1}^{n} \chi_\nu(x)\chi_\nu(y)$. He uses only the most elementary, geometric reasonings in the (x,y)-plane (in fact, on the square $0 \le x \le 1$, $0 \le y \le 1$). Both authors observe that the convergence of the formal series for F(x) at a point x, depends only on the behaviour of F(x) in a neighborhood of that point. Rademacher states his results for (locally) monotonic functions; Walsh, equivalently, formulates the corresponding statements for functions of (locally) bounded variation. Both authors mention the special importance of the dyadic rationals, but Walsh stresses it more. Both authors realize the particular interest of the (C,1)-summability (by first Cesaro means), for which one has the theorem that if F(x) is (at least piecewise) continuous on (0,1), then its series in Rademacher-Walsh functions is uniformly (C,1)-summable.

While the condition of monotonicity (equivalently, of bounded variation) used in the proof of convergence in Rademacher's work implies that continuity alone is insufficient to guarantee said convergence, Walsh actually states this as a theorem and gives a construction for a continuous function, whose formal series diverges at a dyadic irrational (however, the partial sums s_{2^n} of order 2^n do converge to the function!).

Rademacher-Walsh Functions



$K_1(x,y)$

1

$K_2(x,y)$

0	2
2	0

$K_3(x,y)$

-1	1	1	3
1	-1	3	1
1	3	-1	1
3	1	1	-1

$K_4(x,y)$

0	0	0	4
0	0	4	0
0	4	0	0
4	0	0	0

$K_5(x,y)$

-1	1	-1	1	1	1	3	5
1	-1	1	-1	1	-1	5	3
-1	1	-1	1	3	5	-1	1
1	-1	1	-1	5	3	1	-1
-1	1	3	5	-1	1	-1	1
1	-1	5	3	1	-1	1	-1
3	5	-1	1	-1	1	-1	1
5	3	1	-1	1	-1	1	-1

$K_6(x,y)$

0	0	0	0	-2	2	2	6
0	0	0	0	2	-2	6	2
0	0	0	0	2	6	-2	2
0	0	0	0	6	2	2	-2
-2	2	2	6	0	0	0	0
2	-2	6	2	0	0	0	0
2	6	-2	2	0	0	0	0
6	2	2	-2	0	0	0	0

$K_7(x,y)$

1	-1	-1	1	-1	1	1	7
-1	1	1	-1	1	-1	7	1
-1	1	1	-1	1	7	-1	1
1	-1	-1	1	7	1	1	-1
-1	1	1	7	1	-1	-1	1
1	-1	7	1	-1	1	1	-1
1	7	-1	1	-1	1	1	-1
7	1	1	-1	1	-1	-1	1

$K_8(x,y)$

0	0	0	0	0	0	0	8
0	0	0	0	0	0	8	0
0	0	0	0	0	8	0	0
0	0	0	0	8	0	0	0
0	0	0	8	0	0	0	0
0	0	8	0	0	0	0	0
0	8	0	0	0	0	0	0
8	0	0	0	0	0	0	0

Kernels

Walsh gives additional theorems about the convergence of the formal series at $x = a$ to $\frac{1}{2}\{F(a+0)+F(a-0)\}$, if $F(x)$ is (locally) of bounded variation and a is a dyadic rational. If, in addition, $F(x)$ is actually continuous in a neighborhood of $x = a$, then the convergence is uniform in that neighborhood.

Walsh also gives a uniqueness theorem: If $\sum_{\nu} a_\nu \phi_\nu(x)$ converges to zero uniformly on (0,1), except, perhaps, in the neighborhoods of finitely many points, then $a_\nu = 0$ for all ν.

On the other hand, Rademacher states conditions for the convergence of the series of a piecewise continuous function,left continuous at only finitely many discontinuities, all at dyadic rationals. He also proves the uniform convergence of the formal series for functions that satisfy a Lipschitz type condition:
$|F(x_0)-F(x)| < A|\log(x_0-x)|^{-\alpha}$, $\alpha > 1$.

Next, he makes a detailed study of the kernels $K_n(x,y)$ for ordinary convergence and investigates the corresponding Lebesgue constants $\rho_n(x) = \int_0^1 |K_n(x,y)|dy$.

At the end of the paper, he discusses $K_n^{(1)}(x,y) = \frac{1}{n}\sum_{\nu=1}^{n} K_\nu(x,y)$ and $\rho_n^{(1)}(x) = \int_0^1 |K_n^{(1)}(x,y)|dy$, the kernels and Lebesgue constants of the first Cesaro means, respectively.

It is not possible - and may not even be appropriate - to quote here all the theorems obtained by Rademacher (those obtained by Walsh are, of course, easily accessible in [9]). But just to give the flavour of the paper, here is a sample of the results. In all cases it should be kept in mind that $n = 2^r+s$, with $0 < s \leq 2^r$ (observe the equal sign in the last, rather than the first equality).

Let D_r be the union of the set of squares of sides 2^{-r} along the main diagonal of the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$; then

(1) $K_{2^r}(x,y) = K_{2^{r-1}}(x,y)(1+\psi_r(x)\psi_r(y)) = \prod_{j=1}^{r} (1+\psi_j(x)\psi_j(y)) = \begin{cases} 2^r \text{ in } D_r, \\ 0 \text{ elsewhere;} \end{cases}$

(2) $K_n(x,y) = K_{2^r}(x,y)+\psi_{r+1}(x)\psi_{r+1}(y)K_s(x,y)$;

(3) $\rho_n(x)$ is independent of x (as in the trigonometric case !) and

(4) $\rho_n = 1+\rho_s - s/2^r$;

(5) $\rho_{2^r+s} = \rho_{2^r-s} + s/2^r$;

(6) $\rho_{2n} = \rho_n$ (so that each value of a Lebesgue constant occurs infinitely often);

(7) $\rho_{2^{r+1}+s} = \rho_{2^r+s} + s/2^{r+1}.$

Next, set $s_r = 2^{r-1}-2^{r-2}+2^{r-3}- \ldots + (-1)^{r-1} = \frac{1}{3}(2^r-(-1)^r)$;

then

(8) $\rho_{2^r+s_r}$ and $\rho_{2^r+s_{r+1}}$ $(= \rho_{2^{r+1}-s_r})$ are the largest Lebesgue constants for a given r; also, if $0 \leq \tau < s_r$, then

(9) $\rho_{2^r+s_r} - \rho_{2^r+\tau} > \frac{s_r-\tau}{2^{r+1}}$.

Furthermore,

(10) $\rho_{2^r+s_r} = \rho_{2^r+s_{r+1}} = \frac{10}{9} + \frac{r}{3} - \frac{1}{9}(-\frac{1}{2})^r$,

so that

(11) $\lim_{r \to \infty} (\rho_{2^r+s_r} - (\frac{10}{9} + \frac{r}{3})) = 0.$

From this follows rather easily

(12) $\overline{\lim_{n \to \infty}} (\rho_n - (\frac{4}{9} + \frac{\log 3n}{\log 8})) = 0,$

and

(13) $\underline{\lim_{n \to \infty}} \rho_n = 1.$

For the first Cesaro mean, the Lebesgue constants $\rho_n^{(1)}(x)$ are bounded. In fact

(14) $\rho_{2^r}^{(1)}(x) = 1$ identically in x, and, more generally

(15) $\rho_n^{(1)}(x) < 2.$

In order to put this work into perspective, it may be appropriate to make a brief mention of some further developments of this topic.

Kaczmarz [4] and Kaczmarz and Steinhaus [5] investigated the system of Rademacher-Walsh functions. (They call them Rademacher, or Rademacher-Kaczmarz functions, although they know, and in fact, quote Walsh). They study in detail properties and conditions on sets of constants $\{c_k\}$ that serve as coefficients for series $\sum c_k x_k(x)$, by taking as their model similar theorems on the coefficients of (trigonometric) Fourier series.

Paley [6] wrote a long paper in two parts on these functions, which he calls Walsh-Kaczmarz functions. He changes slightly the indexing of these functions. Like Walsh, he takes as his starting point the Haar functions. Among his numerous results, perhaps the most interesting ones deal with partial sums of order 2^n of the formal series of a function f(x) and with certain norm-type inequalities. The following are rather typical: For each k > 1, there exists a constant B_k, that depends only on k, such that, if n = n(t) is an arbitrary, integral valued function of t, $\int_0^1 |s_{2^{n(t)}}(t)|^k dt \leq B_k \int_0^1 |f(t)|^k dt$; while for k = 1

$\int_0^1 |s_{2^{n(t)}}(t)|dt \leq B \int_0^1 |f(t)|\log^+|f(t)|dt+B$; for fixed n, one has

$\int_0^1 |s_{2^n}(t)|dt \leq \int_0^1 |f(t)|dt$.

Perhaps the most important further developments are due to N. Fine [1]. He introduced into the study of these functions the dyadic group G and interprets the Rademacher-Walsh functions as characters on G. He computes the expansions of the fractional part x-[x] of a real x and of the "smoothed out" function $J_k(x) = \int_0^x \chi_k(u)du$ and uses these most effectively. He obtains bounds for the "Fourier"-coefficients of expansions for different categories of functions, finds again many of Rademacher's results on Lebesgue constants (in fact, he states that some were communicated to him by Rademacher), often with different, very elegant proofs, etc. He also computes the average value of these constants, $\frac{1}{x}\sum_{\nu=1}^{[x]}\rho_\nu = \frac{\log x}{\log 16} + O(1)$. He proves analogues of the theorems of Dini and of Lipschitz for the present system, considers also the (C,1)-summability and gives some results on the Abel kernel $\sum_{j=0}^{n-1} \chi_j(u)r^j$, apparently not discussed anywhere else. He defines the functions $\mu_n(t) = 2^n t-[2^n t] - \frac{1}{2}$ and $\lambda_n = 2^n t - [2^n t + \frac{1}{2}]$ and, in connection with a theorem of Kac [3], contrasts the behaviour of $\sum_{n=0}^{N} \mu_n(t)$ with that of $\sum_{n=0}^{N} \lambda_n(t)$. Finally, Fine proves a number of uniqueness and of localization theorems.

Although the elegance of Rademacher's manuscript and a feel for historic justice militate in favor of a publication in full of the manuscript, other considerations militate against it. Among these is the unfinished state of the manuscript(s) and the fact that, during his lifetime, Rademacher himself did not want to have it published. Instead, still following the suggestions of Professor König and Lamprecht, xeroxed copies of the original manuscript will be deposited in an easily accessible library, perhaps together with the previously mentioned edited and completed translation into English. Also, if a need should be perceived, an information, somewhat similar to the present note, could be made available in German.

BIBLIOGRAPHY

1. N.J. Fine - On the Walsh Functions - Transactions of the Amer. Math. Soc., vol. 65 (1949), pp. 372-414.

2. A. Haar - Zur Theorie der orthogonalen Funktionensysteme - Diss. Göttingen (1909) and Mathem. Annalen., vol. 69 (1910), pp. 331-371.

3. M. Kac - Sur les fonctions $2^n t-[2^n t]- \frac{1}{2}$ - Journal London Math. Soc., vol. 13 (1938), 131-134.

4. S. Kaczmarz- Über ein Orthogonalsystem - C.R. du premier Congrès des Mathematiciens des pays Slaves, Warsaw, 1929, pp. 189-192.

5. S. Kaczmarz and H. Steinhaus - Le systeme orthogonal de M. Rademacher - Studia Mathematica, vol. 2 (1930), pp. 231-247.

6. R.E.A.C. Paley - A remarkable series of orthogonal functions - Proc. London Math. Soc., vol. 34 (1932), pp. 241-279.

7. H. Rademacher - Einige Sätze über Reihen von allgemeinen Orthogonal - funktionen- Mathem. Annalen, vol. 87 (1922), pp. 112-138.

8. H. Rademacher - Collected Papers, 2 volumes (edited by E. Grosswald), The MIT Press (G.-C. Rota series editor), Cambridge, Mass., 1974.

9. J.L. Walsh - A closed set of normal, orthogonal functions - Amer. J. Math., vol. 55 (1923), pp. 5-24.