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The Relationship between Homology and Topological Manifolds via Homology Transversality

David E. Galewski* ** and Ronald J. Stern***

Department of Mathematics, University of Georgia, Athens, Georgia, 30602, USA
Department of Mathematics, University of Utah, Salt Lake City, Utah 84112, USA

1. Introduction

In this paper we show that every polyhedral homology n-manifold, $n \geqq 6$ ($n \geqq 5$ if $\partial M = \emptyset$ or if ∂M is a topological manifold), is canonically simple homotopy equivalent to a topological n-manifold. This is accomplished by first observing in Section 3 that any two homology manifolds PL embedded in a PL manifold can be ambient isotoped to be in transverse position. Then, using the work of N. Levitt and J. Morgan [6], as refined by G. Brumfiel and J. Morgan [1], we show in Section 4 that the Spivak normal fiber space of any homology manifold has a canonical topological reduction. Finally, in Section 5 we show that the resulting topological surgery problem has zero surgery obstruction, thus showing that any homology n-manifold M with $n \geqq 6$ ($n \geqq 5$ if $\partial M = \emptyset$) is simple homotopy equivalent to a topological manifold. We further show that if M is already a topological manifold, then the homotopy equivalence is homotopic to a homeomorphism.

In Section 6 we define natural maps $\underline{G_q}\colon BH(q) \to BG(q)$ for $q \geqq 3$, and in Section 7 we define maps $\theta_q\colon BH(q) \to B\widetilde{TOP}(q)$ for $q \geqq 3$, such that

$$\begin{array}{ccc} BPL(q) & \longrightarrow & B\widetilde{TOP}(q) \\ \downarrow & \overset{\theta_q}{\nearrow} & \downarrow \\ BH(q) & \xrightarrow[G_q]{} & B\tilde{G}(q) \end{array}$$

commutes up to homotopy. Here, $B\tilde{G}(q)$, $B\widetilde{TOP}(q)$, and $B\widetilde{PL}(q)$ classify spherical, topological, and PL block S^{q-1}-bundles respectively, and $BH(q)$ classifies homology cobordism S^{q-1}-bundles. In Section 8 we show that the (homotopic) fiber of $\theta_q\colon BH(q) \to B\widetilde{TOP}(q)$ is a $K(\ker(\alpha\colon \theta_3^H \to Z_2), 4)$, where θ_3^H is the group of

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oriented PL homology 3-spheres modulo those which bound PL acyclic 4-manifolds under the operation of connected sum, and $\alpha: \theta_3^H \to Z_2$ is the Kervaire-Milnor-Rochlin map onto Z_2. Finally, in Section 9, we give some applications of the above results.

We remark that by using different methods, T. Matumoto [10] has independently demonstrated the existence of maps $\theta_q: BH(q) \to B\widetilde{TOP}(q)$ with the same fiber and has demonstrated the existence of topological realizations of homology manifolds.

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We begin with Section 2 by giving the required background material on homology manifolds and homology cobordism bundles.

2. Background

In this section we recall some selected facts about homology manifolds and homology cobordism bundles. For further definitions and results we refer the reader to [7] and [9].

A polyhedron M is called a *homology n-manifold* if there exists a triangulation K of M such that for all $\chi \in |K|$, $H^*(lk(\chi, K))$ is isomorphic to $H^*(S^{n-1})$ or to H^*(point). Here, $lk(\chi, K)$ is the boundary of the simplicial neighborhood of χ in K. The *boundary* of M, denoted ∂M, is the set of all $\chi \in |K|$ such that $H^*(lk(\chi, K)) = H^*$(point) and is a homology $(n-1)$-manifold without boundary. Throughout this paper we assume that ∂M is collared in M.

In [9], N. Martin and C. Maunder introduce the notion of homology cobordism S^{q-1}-bundles over homology cell complexes. They also construct a classifying space $BH(q)$ for such bundles and construct as the fiber of the associated universal principal bundle the Kan Δ-set $H(q)$ of which a typical k-simplex is a homology cobordism S^{q-1}-bundle over $\Delta^k \times I$, which restricts over $\Delta^k \times \{0, 1\}$ to the product bundle $\Delta^k \times \{0, 1\} \times S^{q-1}$. Here, Δ^k denotes the standard k-simplex in $\mathbb{R}^k$.

Finally, we recall that the total space $E(\xi)$ of a homology cobordism S^{q-1} [respectively D^q] bundle ξ/M over a homology m-manifold M, is a homology $(m+q-1)$ [respectively $(m+q)$]-manifold.

3. Transversality for Homology Manifolds

Our goal in this section is to use the theory of transversality for polyhedra in PL manifolds, as developed by C. McCrory [13], to show that given two homology manifolds M and N embedded as proper subpolyhedra of a PL manifold Q, then there exists an arbitrarily small PL ambient isotopy of Q putting M in transverse position to N, so that their transversal intersection is a homology manifold. We then prove a transverse regularity theorem for maps.

We say that a subpolyhedron P of a PL manifold Q is *proper* if $(\partial Q, P \cap \partial Q)$ is collared in (Q, P). A proper *homology submanifold* M of Q is a proper subpolyhedron M of Q which is a homology manifold and with $M \cap \partial Q = \partial M$.

Let M be a proper homology submanifold of a PL manifold Q and let P be a proper subpolyhedron of Q. Then P is said to be *H-transverse* to M, denoted $P \perp M$, if there is a normal homology cobordism bundle ν/M of M in Q with blocks that are PL balls, such that P intersects $E(\nu)$ in blocks. We now reformulate this definition in terms of cone complexes.

Let X be a polyhedron. A *cone complex* C on X is a locally finite covering of X by compact subpolyhedra, together with a subpolyhedra $\partial\alpha$ of each element α of C, such that

(i) for each α in C, $\partial\alpha$ is a union of elements of C, denoted $\partial\alpha \sqrt{C}$,

(ii) if α and β are distinct elements of C, $(\alpha-\partial\alpha)\cap(\beta-\partial\beta)=\emptyset$,

(iii) for each α in C, there is a PL homeomorphism $\alpha \cong c(\partial\alpha)$ rel $\partial\alpha$.

A *structured* cone complex is a cone complex C such that for each cone α of C there is a prescribed homeomorphism $f_\alpha\colon \alpha \cong c(\partial\alpha)$.

A PL (homology) *cell complex* is a complex in which each cone α is a PL (homology) ball, and $\partial\alpha$ is its PL (homology) sphere boundary.

A structured cone complex C on a polyhedron X has a canonical derived subdivision C', which is a triangulation of X with vertices the cone points $\underline{\alpha}$ of the cones α in C (cf. Proposition 2.1 of [13]). Also C has a canonical dual structured cone complex C^* on X with $(C^*)^*=C$, $(C^*)'=C'$, and with $|\text{star}(\alpha, C')|$ canonically PL homeomorphic to $\alpha\times\alpha^*$ (cf. Theorem 2.2 of [13]).

(3.1) **Lemma.** *Let C be a cone complex on a PL (homology) manifold, such that $\partial M \sqrt{C}$. Then C is a PL (homology) cell complex.*

Proof. Let D be the restriction of C to ∂M. By an easy induction argument it suffices to show that if α is a cone of C (resp. D) such that α is not a proper subset of another cone of C (resp. D), then $\partial\alpha$ is a homology manifold which has the homology of a sphere. But the subdivision C' (resp. D') is a triangulation of M (resp. ∂M), and $lk(\alpha, C')$ (resp. $lk(\alpha, D')$) is just $\partial\alpha$. As M (resp. ∂M) is a PL or homology manifold, $\partial\alpha$ has required properties. □

(3.2) *Remark.* Observe that if C is a PL (homology) cell complex on a PL (homology) manifold, then the elements of C are always PL (homology) balls.

If P and Q are proper subpolyhedra of a PL manifold M, then P is *transverse* to Q, denoted $P \pitchfork Q$, if there is a structured cone complex C on M such that $Q \sqrt{C}$ and $P \sqrt{C^*}$. Observe that if P is a proper subpolyhedron of a PL manifold M and $P \sqrt{C^*}$, then there exists a subdivision β of C such that β is a cell complex and $P \sqrt{\beta^*}$ (Corollary 4.3 of [13]). Thus $P \pitchfork Q$ if only if there exists a structured PL cell complex C on M with $Q \sqrt{C}$ and $P \sqrt{C^*}$.

We have the following transversality theorem of C. McCrory [13].

(3.3) **Theorem.** *Let P and Q be proper subpolyhedra of a PL manifold M. Then there exists an arbitrarily small PL ambient isotopy h_t of M such that $h_1(P) \pitchfork Q$. If $(P\cap\partial M)\pitchfork(Q\cap\partial M)$ in ∂M, then h_t can be chosen so that $h_t|\partial M$ is the identity for all t. Furthermore, if Y is a subpolyhedron of P which is collared in P and $Y \pitchfork Q$, then h_t can be chosen so that $h_t|Y$ is the identity for all t.*

(3.4) **Lemma.** *Let N be a proper subhomology manifold of a PL manifold M and P is a proper subpolyhedron of M. Then $P \perp M$ if and only if $M \pitchfork P$.*

Proof. Suppose $P \perp M$. Let ν/M be a normal homology cobordism bundle of M in Q given by the fact that $P \perp M$. Let K be a triangulation of $\overline{Q - E(\nu)}$ such that $\partial(\overline{Q - E(\nu)})$ and all the sets $\beta \cap \partial E(\nu)$, β a block of ν, and $P \cap (\overline{Q - E(\nu)})$ are subcomplexes of K. Let C be the cone complex on Q consisting of all the blocks β of ν plus the simplices of K. Choose a cone structure for each cell β of C which is a block of ν so that the cone α over which β is defined is a subcone. Then $N \sqrt{C^*}$ and clearly $P \sqrt{C}$.

Now suppose $N \pitchfork P$. Then there is a PL cell complex C on M with $N \sqrt{C}$ and $P \sqrt{C^*}$. Let B be a PL cell complex on M with B a full subdivision of C and B^* a full subdivision of C^* (Lemma 2.6 of [13]). Then B restricts to a cone complex D on N. By (3.2) the cones of D^* are homology balls. Assign to each $\beta^* \in D^*$ the dual of β in B^*. This determines a normal homology cobordism bundle ν of N in Q with PL balls as blocks and with P intersecting $E(\nu)$ in blocks, so that $M \perp P$. □

(3.5) **Lemma.** *Let N^n be a proper homology submanifold of a PL manifold Q^{n+q}. If M^m is a homology manifold embedded in Q as a proper subpolyhedron with $N \perp M$, then $M \cap N$ is a proper homology $(m-q)$-submanifold of M.*

Proof. By (3.4) there is a structured PL cell complex C on Q, with $M \sqrt{C}$ and $N \sqrt{C^*}$. Let D be the restriction of C to M and let F be the restriction of C^* to N. Let $\alpha \in D$ be such that the dual of α in C^* is contained in N. Then the dual of α in D^* is contained in $M \cap N$. Thus, the cone complex $E = \{\beta \in D^* \mid \beta$ is the dual of an element α of D and the dual of α in C^* is contained in $N\}$ is a cone complex on $M \cap N$. Symmetrically, $\bar{E} = \{\beta \in F^* \mid \beta$ is the dual of an element of F and the dual of β in $(C^*)^* = C$ is contained in $M\}$ is a cone complex on $M \cap N$. By (3.1) and (3.2) the elements of both E and $\bar{E}$ are cones on homology spheres or balls. But $\bar{E} = E^*$ so that given $\gamma \in E$, $\gamma \times \gamma^*$ is PL homeomorphic to $|st(\underline{\gamma}, \bar{E}')|$, where $\underline{\gamma}$ is the cone point of γ. Thus, $M \cap N$ is a homology manifold. Note that $M \cap N$ has dimension $(m-q)$ as it is gotten by the transversal intersection of cones. Also, $M \cap N$ is a proper submanifold of M as $\partial N \sqrt{D}$ and $M \cap N \sqrt{D^*}$ so that $\partial N \pitchfork M \cap N$. □

Combining (3.3), (3.4), and (3.5), we have

(3.6) **Theorem.** *Let N^n be a proper homology submanifold of a PL manifold Q^{n+q} and let M^m be a homology manifold embedded in Q as a proper subpolyhedra. Then there exists an arbitrarily small PL ambient isotopy h_t of Q with $h_1(N) \perp M$ with transversal intersection a proper $(m-q)$ homology submanifold of N. Furthermore, if $\partial N \perp M$, then the isotopy can be chosen so that $h_t \mid \partial N$ is the identity for all t.*

We now discuss transversality for maps of homology manifolds. We first make precise the notions of bundle maps between homology cobordism bundles and transverse regularity.

Let K and L be homology cell complexes. A map f: $K \to L$ is called *cellular* if for each cone α of K there is a unique minimal cone α_f of L such that $f(\alpha) \subset \alpha_f$ and such that if $\alpha \subset \beta$, then $\alpha_f \subseteq \beta_f$.

Let ξ_0/K and ξ_1/L be homology cobordism S^{q-1}-bundles. A *bundle map* from ξ_0 to ξ_1 is a pair $(\bar{f}, f)$ of maps such that

(1) $f\colon K \to L$ is cellular.

(2) $\bar{f}$ covers f, i.e. for each $\alpha \in K$, $\bar{f}(E(\xi_0|\alpha)) \subset E(\xi_1|\alpha_f)$.

(3) $f^*(\xi_1)$ is stably isomorphic to ξ_0 as homology cobordism bundles over K, and

(4) for each $\alpha \in K, \bar{f}$ induces an isomorphism between $H_*(E(\xi_0|\alpha))$ and $H_*(E(\xi_1|\alpha_f))$.

There is a similar definition for bundle maps of homology disc bundles where there is the additional condition that a bundle map of disc bundles restricts to a bundle map of the associated sphere bundles.

A map $f\colon M \to Q$ between manifolds is called *admissable* if $f^{-1}(\partial Q) \subset \partial M$ and $f^{-1}(\partial Q)$ is collared in M. An admissable map $f\colon M^m \to Q^q$ between homology manifolds is said to be *t-regular to a proper submanifold* N^n of Q, if $f^{-1}(N)$ is an $(m+n-q)$-submanifold of M and if f induces a bundle map between a normal bundle of $f^{-1}(N)$ in M and a normal bundle of N in Q.

(3.7) **Theorem.** (a) *Let $f_0\colon M^m \to Q^{m+q}$ be an admissable map between a homology manifold M and a PL manifold Q with N^n a proper homology submanifold of Q. Then there is an arbitrarily small homotopy (through admissable maps) f_t of f_0 such that f_1 is t-regular to N.* (b) *Suppose $f_0|\partial M\colon \partial M \to Q$ is t-regular to N. Then we can choose the homotopy f_t so that $f_t|\partial M = f_0|\partial M$ for all $t \in I$.*

Proof. We consider case (a) when $\partial M = \partial N = \partial Q = \emptyset$, as case (b) and the bounded cases follow similarly. Choose an embedding $e\colon M \to \operatorname{int} I^r$, for some large r. Homotope f_0 to a PL map $\bar{f}_0$. Then $q = \bar{f}_0 \times e\colon M \to Q \times I^r$ is a PL embedding. Let $p\colon Q \times I^r \to Q$ be projection. By (3.4) and (3.6) there exists a PL cell complex C on $Q \times I^r$ and a small PL ambient isotopy h_t of $Q \times I^r$ such that $N \times I^r \sqrt{C}$ and $h_1 q(M) \sqrt{C^*}$. Let D be a PL cell complex on Q such that $N \sqrt{D}$ and let C' be a subdivision of C such that $p\colon C' \to D$ is cellular. Then there exists a small ambient isotopy g_t of $Q \times I^r$ such that $g_t(\alpha) = \alpha$ for all $\alpha \in C$, $N \times I^r \sqrt{C'}$ and $q_1 h_1 q(M) \sqrt{(C')^*}$ (cf. Lemma 5.2 of [13]). Define $f_t\colon M \to Q$ by $f_t = p g_t h_t q$. Then $P = f_1^{-1}(N)$ is a homology $(n-q)$-submanifold of M.

Let a normal bundle ν_N of N in Q be constructed as follows. Let D_0 be the restriction of D to N. By (3.1) each element of D_0^* is the cone on a homology sphere, so assign to each $\beta^* \in D_0^*$ the dual of β (an element of $D_0 \subset D$) in D^*.

We similarly construct a normal bundle $\nu_{N \times I^r}$ of $N \times I^r$ in $Q \times I^r$. Let C_0 be the restriction of C' to $N \times I^r$. Assign to each $\beta^* \in C_0^*$, the dual of β (an element of $C_0 \subset C'$) in $(C')^*$. Now $g_1 h_1 q(M)$ intersects $\nu_{N \times I^r}$ in blocks. Note that if α is an element of C_0 such that the dual of α in $(C')^*$ is contained in $g_1 h_1(M)$, then $\alpha^* \in C_0^*$ lies in $g_1 h_1 q(M) \cap N \times I^r$, and that all such α^* yield a homology cell complex on $g_1 h_1 q(M) \cap N \times I^r$. Thus $\nu_{N \times I^r} | g_1 h_1 q(M) \cap N \times I^r$ is a normal bundle $\bar{\nu}$ of $g_1 h_1 q(M) \cap N \times I^r$ in $g_1 h_1 q(M)$. If we let $\nu = (q^{-1} | q(M))\, h_1^{-1} g_1^{-1}(\bar{\nu})$, then ν is a normal bundle of P in M and f_1 induces a bundle map of $\bar{\nu}$ to ν_N. □

(3.8) *Remark.* There are more general versions of (3.6) and (3.7). Namely, if we consider a class of polyhedra whose links are closed under suspension and join, then any two such polyhedra in a PL manifold can be ambient isotoped to be transverse and so that their transversal intersection belongs to the given class. This will be developed in [2].

(3.9) *Remark.* Note that (3.7)(a) is not true when Q is just a homology manifold, for consider a non simply connected PL homology 3-sphere H^3 and let K be a regular neighborhood of the suspension circle C in the double suspension $\Sigma^2 H^3$ of H^3. If (3.7)(b) were true with $Q=K$, $M=D^2$, and $N=C$, then $\Sigma^2 H^3 - C$ would be 1-connected, a contradiction. This example shows that Theorem A of [8] is false.

(3.10) *Remark.* Note that homology transversality and PL block transversality are compatible in the sense that if in (3.6) and (3.7) M and N were PL manifolds, then the resulting isotopy of (3.6) would put M PL block transverse to N, or would in the case of (3.7), homotope f_0 to a map f_1 which is block transverse regular to N.

(3.11) *Remark.* Note that if in (3.7) M were a PL manifold, then the normal bundle ν of P in M in the proof of (3.7) can be chosen so that the blocks of ν are PL balls.

4. A Canonical Reduction of the Spivak Normal Fiber Space of a Homology Manifold

In this section we will give a canonical reduction of the Spivak normal fiber space of a homology to a topological bundle. The main tools used in exhibiting this reduction are Theorem (3.7) and the work of N. Levitt and J. Morgan [6], as refined by G. Brumfiel and J. Morgan [1], which relates transversality and topological reductions. We first summarize the relevant results of [1].

Let ξ^q/X be a spherical fibration over a space X and let $T(\xi^q)$ denote the Thom space of ξ^q. Note that $E(\xi^q)\subset T(\xi^q)$. A map $f\colon M^{q+n}\to T(\xi^q)$, M a closed PL manifold, is said to be *fiber homotopy transverse* (f.h.t) to X, if and only if

1) f is PL transverse to $E(\xi^q)\subset T(\xi^q)$, and

2) $$\begin{array}{ccc} f^{-1}(E(\xi^q)) & \xrightarrow{f|} & E(\xi^q) \\ \downarrow & & \downarrow \\ f^{-1}(M_\xi) & \xrightarrow{f|} & M_\xi \end{array}$$

is a map of spherical fiber spaces, where M_ξ denotes the mapping cylinder of the projection of ξ^q/X.

If M^{q+n} is a manifold with boundary, we require

3) $$\begin{array}{ccccc} f^{-1}(E(\xi))\cap\partial M & \subset & f^{-1}(E(\xi)) & \xrightarrow{f|} & E(\xi) \\ \downarrow & & \downarrow & & \downarrow \\ f^{-1}(M_\xi)\cap\partial M & \subset & f^{-1}(M_\xi) & \xrightarrow{f|} & M_\xi \end{array}$$

to be maps of fiber spaces.

Now let M^{q+n} be a PL manifold with a given triangulation and let ξ^q be as above. A map $f\colon M^{q+n}\to T(\xi^q)$ is *strongly fiber homotopy transverse* (s.f.h.t.) if and only if

1) $f|\Delta\colon\Delta\to T(\xi^q)$ is f.h.t. for each simplex Δ in the triangulation of M, and

2) $(f^{-1}(M_\xi)\cap\Delta^{q+2}, f^{-1}(M_\xi)\cap\partial\Delta^{q+2})$ is homotopy equivalent to a PL 2-manifold with boundary.

A *good cover* $\{V_\alpha\}$ of $T(\xi^q)$ is a cover of $T(\xi^q)$ induced by a cover $\{U_\alpha\}$ of the base X of ξ^q, such that every element of $\{U_\alpha\}$ contains the base point of X, all finite intersections are 1-connected, and $\xi^q | U_\alpha$ admits a PL reduction. If X is 1-connected good covers always exist (Lemma 1.5 of [1]).

Let $W(\xi^q)$ be the subcomplex of the singular complex $\tilde{T}(\xi^q)$ of $T(\xi^q)$ consisting of singular simplices $f: \Delta^{q+i} \to T(\xi^q)$ which are f.h.t. on Δ^{q+i} and all the faces of Δ^{q+i}, with $f(\Delta^{q+i}) \subset V_\alpha$ for some element V_α of a good cover of $T(\xi^q)$. If $i=2$, we further require that $f^{-1}(M_\xi) \subset \Delta^{q+i}$ be homotopy equivalent to a PL 2-manifold with boundary. There is a natural inclusion $W(\xi^q) \to \tilde{T}(\xi^q)$ and the obstruction to deforming a map $f: M \to T(\xi^q)$, M a PL manifold with a given triangulation, to be s.f.h.t. can be interpreted as the obstruction to lifting f to a map $\tilde{f}$ in the diagram

$$\begin{array}{ccc} & & W(\xi^q) \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \\ M & \longrightarrow & \tilde{T}(\xi^q). \end{array}$$

Let $MS\widetilde{PL}(q)$, $MS\widetilde{TOP}(q)$, and $MS\tilde{G}(q)$ denote the singular complexes of the Thom spaces of the universal bundles over $BS\widetilde{PL}(q)$, $BS\widetilde{TOP}(q)$, and $BS\tilde{G}(q)$, respectively. Also, let $WS\tilde{G}(q)$ denote $W(\gamma^q)$, where γ^q is the universal block q-fibration over $BS\tilde{G}(q)$.

Define $\tilde{T}(\xi^q)^{CP^2}$ to be the semi-simplicial complex whose k-simplices are maps $\Delta^k \times CP^2 \to \tilde{T}(\xi^q)$ which are contained in one of the sets in a good cover of X. Let the k-simplices of $W(\xi^q)^{CP^2}$ be all the above maps which are globally f.h.t. on $\Delta^k \times CP^2$ and on all the faces of $\Delta^k \times CP^2$. Also, whenever the preimage has dimension 2, we require that this Poincaré duality space with boundary be the homotopy type of a 2-manifold with boundary. We now quote some results of [1] which we will use.

(4.0) **Theorem.** [1] *Given* $j: X \to MS\tilde{G}(q)$, *then there is a commutative diagram*

$$\begin{array}{ccccc} & & WS\tilde{G}(q) & \xrightarrow{\times CP^2} & WS\tilde{G}(q)^{CP^2} \\ & & \downarrow & & \downarrow \\ X & \xrightarrow{j} & MS\tilde{G}(q) & \xrightarrow{\times CP^2} & MS\tilde{G}(q)^{CP^2} \end{array}$$

and there is a natural $1-1$ *correspondence between homotopy classes of liftings of* $(\times CP^2)\cdot j$ *to* $WS\tilde{G}(q)^{CP^2}$.

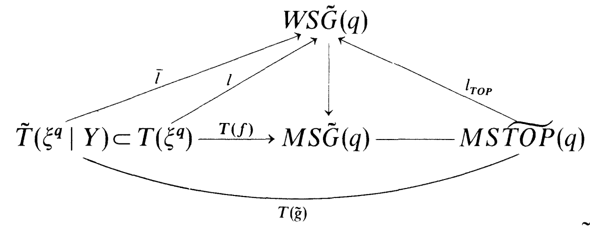
(4.1) **Theorem.** [1] *There is a lifting* l_{TOP} *of the natural map* $MS\widetilde{TOP}(q) \to MS\tilde{G}(q)$ *and a lifting* l_{PL} *of the natural map* $MS\widetilde{PL}(q) \to MS\tilde{G}(q)$ *such that the diagram*

$$\begin{array}{ccccc} & & & & WS\tilde{G}(q) \\ & \overset{l_{PL}}{\nearrow} & & \overset{l_{TOP}}{\nearrow} & \downarrow \\ MS\widetilde{PL}(q) & \to & MS\widetilde{TOP}(q) & \to & MS\tilde{G}(q) \end{array}$$

commutes up to homotopy.

The map l_{PL} is produced via PL transversality, and the map l_{TOP} is produced via topological transversality [4] after crossing with CP^2 to obviate the unknown validity of codimension 4 *TOP* transversality and then using (4.0).

(4.2) **Theorem.** [1] *Let ξ^q be a spherical fibration over a 1-connected complex X, $q \geqq 3$, which is classified by $f\colon X \to BS\tilde{G}(q)$. Let Y be a subcomplex of X such that $g = f \mid Y$ has a lifting $\tilde{g}$ to $BS\widetilde{TOP}(q)$. Then $\bar{l} = l_{TOP} \cdot T(\tilde{g})\colon \tilde{T}(\xi^q \mid Y) \to WS\tilde{G}(q)$ is a lifting of $T(\tilde{g})$ to $WS\tilde{G}(q)$ and there is natural bijection between homotopy classes of liftings l of $T(f)$ extending $\bar{l}$ in the diagram*



and vertical homotopy classes of liftings of $f\colon X \to BS\tilde{G}(q)$ to $BS\widetilde{TOP}(q)$ extending $\tilde{g}$. The correspondence is given by

$$\{\tilde{f}\colon X \to BS\widetilde{TOP}(q), \tilde{f} \text{ a lifting of } f, \tilde{f} \mid Y = \tilde{g}\} \to \{l_{TOP} T(\tilde{f})\colon T(\xi^q) \to WS\tilde{G}(q)\}.$$

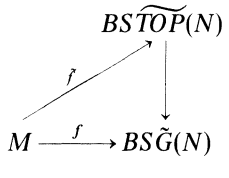
Remark. In [1] (4.2) is only proven when $Y = \emptyset$. But the proof of (4.2) is by a one simplex at a time relative to its boundary approach, so that the relative version clearly holds.

Let M^m be a homology manifold properly embedded in

$$R_+^{N+m} = \{(x_1, \ldots, x_{N+M}) \in R^{N+m} \mid x_1 \geqq 0\}.$$

Triangulate R_+^{N+m} so that M and ∂M are full subcomplexes of R_+^{N+m} and ∂R_+^{N+m}, respectively. For each simplex α of M, assign the dual cells $D(\alpha, M)$ and $D(\alpha, \partial M)$ the *PL* cells $D(\alpha, R_+^{N+m})$ and $D(\alpha, \partial R_+^{N+m})$, respectively. This determines a normal homology cobordism disk bundle ν_M of M on R_+^{N+m}, over a homology cell complex C_M, with blocks that are *PL* cells (cf. § 5 of [9]). Let $\bar{\nu}_M$ denote the associated homology cobordism sphere bundle and observe that $\bar{\nu}_M$ is a spherical block fibration and is a Spivak normal fiber space for M.

(4.3) **Theorem.** *Let M^m be an oriented homology manifold properly embedded in R_+^{N+m} and let $f\colon M \to BS\tilde{G}(N)$ classify $\bar{\nu}_M$. Then there exists a canonical lifting $\tilde{f}$ of f in the diagram*



Furthermore, if ∂M is a PL manifold, then the lifting $\tilde{f}$ can be chosen so as to extend the lifting $f'\colon \partial M \to BS\widetilde{PL}(N) \to BS\widetilde{TOP}(N)$ of $f \mid \partial M$ given by the PL structure on ∂M.

Proof. We inductively construct the lift $\tilde{f}$ over the skeleta of C_M. Let $C^{(k)}$ denote the k-skeleton of C_M, i.e. the complex of all dual cells of M of dimension $\leqq k$. Let $\{\alpha_i^k\}$ be the k-cells of C_M. Let K be a triangulation of $E(\nu_M)$ so that the blocks $E(\nu_M \mid \alpha_i^k)$, for all k and i, are subcomplexes. By inducting over the skeleta $C^{(k)}$ of C_M and the dimension of the simplices of K restricted to $E(\nu_M \mid C^{(k)})$ we can construct, using (3.7), a map $l_H: \tilde{T}(\nu_M) \to WS\tilde{G}(N)$ making the following diagram commute

$$\begin{array}{ccc} & & WS\tilde{G}(N) \\ & \overset{l_H}{\nearrow} & \downarrow \\ \tilde{T}(\nu_M) & \longrightarrow & MS\tilde{G}(N). \end{array}$$

Now, by inducting over the skeleta $C^{(k)}$ of C_M, (4.2) yields the first assertion.

The last statement of (4.3) follows from the compatibility of H-transversality and PL transversality (Remark 3.10) □

(4.4) *Remark.* The lift of (4.3) is canonical in the sense that it is the *unique* lift given by (4.2) using H-transversality (Theorem 3.7).

Our proof of (4.3) also proves the following relative version of (4.3).

(4.5) **Theorem.** *Let f: $M \to BS\tilde{G}(N)$ be as in (4.3). Suppose $f|\partial M$ has a lift $\tilde{g}$ to $BS\widetilde{TOP}(N)$ such that the following diagram commutes*

$$\begin{array}{ccccc} & & WS\tilde{G}(N) & & \\ & \overset{l_H}{\nearrow} & \downarrow & \overset{l_{TOP}}{\nwarrow} & \\ \tilde{T}(\nu_M|\partial M) & \xrightarrow{T(f|\partial M)} & MS\tilde{G}(N) & \longleftarrow & MS\widetilde{TOP}(N). \end{array}$$

Then there is a lift $\tilde{f}$ of f to $BS\widetilde{TOP}(N)$ extending $\tilde{g}$.

5. The Solution of the Surgery Problem

In this section we solve the surgery problem determined by the topological reduction given by (4.3). We first quote a lemma (Lemma 1.4 of [1]) which will be useful in our situation.

(5.1) **Lemma.** *Let ξ^q be a fibration over an oriented Poincaré duality pair $(X, \partial X)$. Let f: $D^{n+q} \to T(\xi^q)$ represent a homotopy class $\alpha \in \Pi_{n+q}(T(\xi^q), T(\xi^q|\partial X))$ with $h(\alpha) \cap U = [X] \in H_n(X, \partial X)$, where h is the Hurewicz homomorphism, $[X]$ the orientation class of X, and $U \in H^q(T(\xi))$ is the Thom class of ξ. Suppose there exists a topological bundle $\eta^q|X$ and a fiber homotopy equivalence g: $E(\xi^q) \to E(\eta^q)$. Then gf is homotopic to a map k which is topologically transverse to X and the following two surgery problems are equivalent*

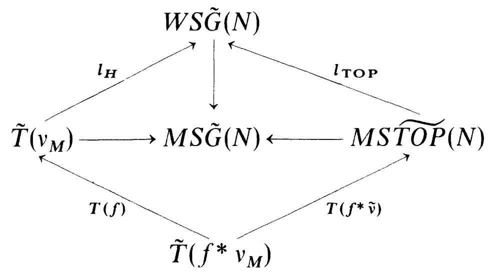
1)
$$\begin{array}{ccc} \nu_{Y \subset D^{n+q}} & \xrightarrow{k|} & E(\eta^q) \\ \downarrow & & \downarrow \\ k^{-1}(X, \partial X) = (Y, \partial Y) & \xrightarrow{k|} & (X, \partial X), \end{array}$$

2)
$$\begin{array}{ccc} \nu_{Y \subset D^{n+q}} & \longrightarrow & k^* j^*(\eta^q) \\ \downarrow & & \downarrow \\ (Y, \partial Y) & \hookrightarrow & (M_{\xi^q}, M_{\xi^q|\partial X}) \end{array}$$

where j is a deformation retraction of $(M_{\xi^q}, M_{\xi^q|\partial X})$ to $(X, \partial X)$.

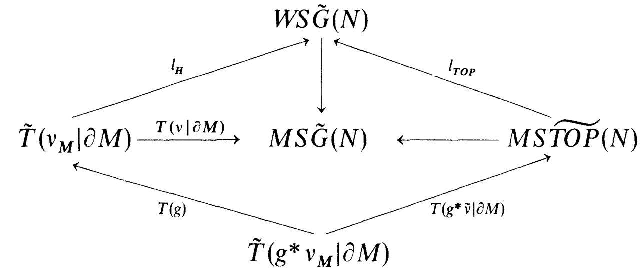
Now let M be an oriented compact homology m-manifold properly embedded in R_+^{N+m} and let ν_M be as in (5.4). Let $\nu: M \to BS\tilde{G}(N)$ classify $\bar{\nu}_M$.

(5.2). **Theorem.** (a) *If $m \geqq 6$ ($m \geqq 5$ if $\partial M = \emptyset$), then there exists a topological m-manifold P and a simple homotopy equivalence f: $(P, \partial P) \to (M, \partial M)$ such that if $\tilde{\nu}$: $M \to BS\widetilde{TOP}(N)$ is the lift of ν given by* (4.3), *then the diagram*

(∗)


commutes up to homotopy.

(b) *Suppose there is a topological $(m-1)$-manifold Q and a simple homotopy equivalence $g: Q \to \partial M$ such that if $\tilde{\nu}|\partial M: M \to BS\widetilde{TOP}(N)$ is the lift of $\nu|\partial M: \partial M \to BS\tilde{G}(N)$ given by* (4.3), *then the diagram*



commutes up to homotopy. Then, if $m \geqq 5$, there exists a topological m-manifold P, with $\partial P = Q$, and a simple homotopy equivalence f: $P \to M$ extending g such that if $\tilde{\nu}$: $M \to BS\widetilde{TOP}(N)$ is the lift of ν extending $\tilde{\nu}|\partial M$ given by (4.5), *then diagram* (∗) *commutes up to homotopy.*

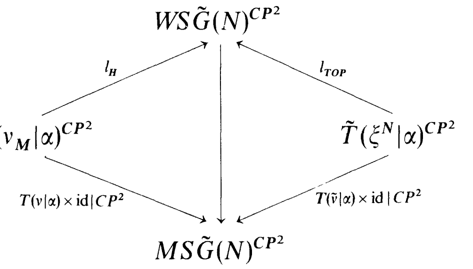
Proof. We first prove (5.2) (b). By (4.5), the lift $\tilde{\nu}|\partial M$ of $\nu|\partial M$ extends to a lift $\tilde{\nu}$ of ν to $BS\widetilde{TOP}(N)$, so let ξ^N be a topological block bundle over M given by this lift $\tilde{\nu}$ and let q: $E(\nu_M) \to E(\xi^N)$ be the resulting fiber homotopy equivalence. Let C_M be the homology cell complex on M over which ν_M is defined. Let K be a fine triangulation of $T(\nu_M)$ such that for each $\alpha \in C_M$, $T(\nu_M|\alpha)$ is a subcomplex. By inductive applications of (3.6), there exists an ambient isotopy h_t of $T(\nu_M)$ such that $h_t(T(\nu_M|\alpha)) = T(\nu_M|\alpha)$ for all $t \in I$ and $\alpha \in C_M$, and such that for each simplex σ of K, $h_1(\sigma) \perp M$. Let η be the homology cobordism bundle over C_M with total space $h_1 E(\nu_M)$. Then there is a fiber homotopy equivalence q_0: $E(\eta) \to E(\xi^N)$. Using topological transversality [4] homotope q_0 to a map q_1 : $T(\eta) \to T(\xi^N)$ such that q_1 is topologically transverse to M with $q_1^{-1}(\partial M) = Q$ and $q_1|Q = g$. We thus get a relative surgery problem

$$(P_0, \partial P_0) = (q_1^{-1}(M), q_1^{-1}(\partial M)) \xrightarrow{q_2} (M, \partial M) \quad \text{with } q_2|\partial P_0 = g: Q \to \partial M.$$

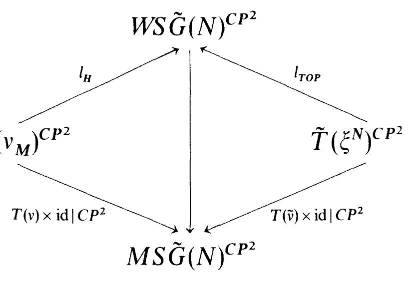
Let $\sigma(q_2)$ denote the surgery obstruction to normally cobording the normal map q_2, rel ∂P_0, to a simple homotopy equivalence. We now show that $\sigma(q_2)=0$.

By (5.1), the relative surgery problem $(P_0, \partial P_0) \xrightarrow{q_2} (M, \partial M)$ is equivalent to the relative surgery problem $(P_0, \partial P_0) \hookrightarrow (E(\eta), E(\eta|\partial M))$. We now cross the latter problem with CP^2 and show that it has zero surgery obstruction, thus implying that $\sigma(q_2)=0$ [18].

Consider $E(\eta)\times CP^2 \xrightarrow{\pi_1} E(\eta) \xrightarrow{q_1} E(\xi)$. We now homotope the map $q_3=(q_1|E(\eta))\pi_1$ to a map $\bar{q}\colon E(\eta)\times CP^2 \to E(\xi)$ so that $\bar{q}|\Delta \times CP^2$ is topologically transverse to M for every simplex Δ of $h_1(K)$, rel $(E(\eta)|\partial M)\times CP^2$. To do this we use topological transversality [4] inductively over the simplices of $h_1(K)$ of dimension $\leqq N+3$ and then cross with CP^2 to isotope $q_3^{-1}(M)=P_0\times CP^2$ transverse to these simplices crossed with CP^2. On the $k=N+4$ skeleton of $h_1(K)$ we have that $P_0\times CP^2$ is topologically transverse to $\partial\Delta^k\times CP^2$ and the transversal intersection is of dimension 7. Now apply relative topological transversality to isotope $P_0\times CP^2$ topologically transverse to $\Delta^k\times CP^2$ rel $\partial\Delta^k\times CP^2$. Then continue in this manner, using topological transversality, to obtain the desired map $\bar{q}$: $E(\eta)\times CP^2 \to T(\xi)$. By Corollary 2.3 of [5] $\bar{q}$ is f.h.t., so let ν be the bundle over $\bar{q}^{-1}(M)$ given by this fiber homotopy transversality. Then clearly the relative surgery problem $(P_1, \partial P_1)=(\bar{q}^{-1}(M), \bar{q}^{-1}(\partial M)=\partial M\times CP^2)\hookrightarrow(E(\nu), E(\nu|\partial M))$ has zero surgery obstruction. Note that $\bar{q}$ was constructed blockwise, so that $P_1\cap E(\eta|\alpha)$ is a topological m-manifold for each $\alpha\in C_M$. Since each $\alpha\in C_M$ is 1-connected, we see that by (4.0), (4.1) and (4.2) that the theory of transversality which isotoped the simplices of $K|E(\nu_M(\alpha))$ crossed with CP^2 H-transverse to $M\times CP^2$, rel $\partial M\times CP^2$, is equivalent to the theory of transversality which isotoped $P_0\times CP^2$ topologically transverse to the simplices of $K|E(\nu_M|\alpha)$ crossed with CP^2. That is to say, the following diagram commutes up to homotopy.



Thus, by inducting up the skeleta of C_M using (4.0) and (4.2) we have that the following diagram commutes up to homotopy.



Let τ be the fibration over $M \times CP^2 \times I$ given by the equivalence of these two transversality theories. Then $E(\tau) \subset E(\eta) \times CP^2 \times I$ and extends $E(\eta) \times CP^2$ and $E(\nu)$. Using relative topological transversality, we find a topological manifold $W \subset E(\tau)$ with $\partial W = P_1 \cup (\partial P_1 \times I) \cup P_0 \times CP^2$. The relative surgery problem $(W, \partial W) \hookrightarrow (E(\tau), E(\tau|\partial M) \times CP^2 \times I)$ exhibits a normal cobordism $\mathrm{rel}\, \partial M \times CP^2$ between our original problem crossed with CP^2 and a solved problem. Thus our original problem is normally cobordant to a simple homotopy equivalence $f\colon P \to M$, extending $g\colon \partial P \to \partial M$. The lift determined by f is the lift $\tilde\nu\colon M \to BS\widetilde{TOP}(N)$ of $\nu\colon M \to BS\tilde G(N)$ given by (4.5), so that (*) commutes up to homotopy.

Note that if $\partial M = \emptyset$, the above proof yields a proof of (5.2(a)). If $\partial M \neq \emptyset$, then the $\partial M = \emptyset$ version of (5.2(a)) and (5.2(b)) yield (5.2.(a)). □

We now discuss the "canonicality" of the simple homotopy equivalence provided by (5.2).

(5.3) **Corollary.** *Let M be a compact homology m-manifold, $m \geqq 6$ ($m \geqq 5$ if $\partial M = \emptyset$), and let $g\colon Q \to \partial M$ be a simple homotopy equivalence as in* (5.2(*b*)). *Let $f_0\colon P_0 \to M$ and $f_1\colon P_1 \to M$ be two simple homotopy equivalence extending g provided by* (5.2(*b*)). *Then there is a homeomorphism $h\colon (P_0, \partial P_0) \to (P_1, \partial P_1)$ with $h|\partial P_0 = \mathrm{id}|Q$ and with $f_1 h$ homotopic,* rel ∂P_0, *to f_0.*

Proof. We have a topological manifold $Q_0 = P_0 \cup (\partial P_0 \times I) \cup P_1$ and a simple homotopy equivalence $g_0\colon Q_0 \to \partial(M \times I)$ where $g_0 = f_0 \cup (g_0 \times \mathrm{id}\,|\,I) \cup f_1$. Also, if $\tilde\nu|\partial(M \times I)\colon \partial(M \times I) \to BS\widetilde{TOP}(N)$ is the lift of $\nu \times \mathrm{id}|\partial(M \times I)\colon \partial(M \times I) \to BS\tilde G(N)$ given by (4.3), then the diagram

$$
(5.4)\qquad
\begin{array}{ccccc}
 & & WS\tilde G(N) & & \\
 & \nearrow^{l_H} & \downarrow & \nwarrow^{l_{TOP}} & \\
\tilde T(\nu_{M\times I}\,|\,\partial(M\times I)) & \xrightarrow{T(\nu\times \mathrm{id}\,|\,\partial(M\times I))} & MS\tilde G(N) & \longleftarrow & MS\widetilde{TOP}(N)\\
 & \nwarrow_{T(g_0)} & & \nearrow_{T(g_0^*\tilde\nu|\partial(M\times I))} & \\
 & & \tilde T(g_0^*\,\nu_{M\times I}\,|\,\partial(M\times I)) & &
\end{array}
$$

commutes up to homotopy. Then (5.2(b)) yields a topological $(m+1)$-manifold N with $\partial N = Q_0$ and a simple homotopy equivalence $f\colon N \to M \times I$ extending g_0. The topological s-cobordism theorem then yields the result. □

(5.5) **Corollary.** *Let M be a closed homology m-manifold, $m \geqq 5$, which is also a topological manifold. Then the simple homotopy equivalence provided by* (5.2) *is homotopic to a homeomorphism.*

Proof. Note that if $f\colon P \to M$ is a simple homotopy equivalence provided by (5.2) and if $Q = P \cup M$ (disjoint union) and $g_0\colon Q_0 \to (M \times I)$ is given by $g_0 = f \cup \mathrm{id}\,|\,M$, then diagram (5.4) commutes up to homotopy. The proof of (5.3) then yields the desired result. □

(5.6) **Corollary.** *Let M be a homology m-manifold, $m \geqq 6$ ($m \geqq 5$ if $\partial M = \emptyset$). Then M is simple homotopy equivalent to a topological m-manifold.*

Proof. If M is compact, then (5.2) yields the result. If M is not compact, filter M by codimension zero compact homology manifolds $M_0 \subset M_0^+ \subset M_1 \subset M_1^+ \subset \cdots$ such that each $S_i = \mathrm{cl}(M_i^+ - M_i)$ is a compact codimension zero homology submanifold of M_i^+, $(S_i, \partial_- S_i)$ is PL homeomorphic to $\partial_- S_i \times ([0,1],0)$ where $\partial_- S_i = S_i \cap M_i$, and each $B_i = \mathrm{cl}(M_i - M_{i-1}^+)$ is a compact codimension zero homology submanifold of M_i. Such a filtration exists by considering a (homology) handelbody decomposition of M. Now apply (5.2) inductively to the B_i thus getting a topological manifold P which is filtered by compact codimension zero topological manifolds $P_0 \subset P_0^+ \subset P_1 \subset P_1^+ \subset \cdots$ such that each $R_i = \mathrm{cl}(P_i^+ - P_i)$ is a compact codimension zero topological submanifold of P_i^+, $(R_i, \partial_- R_i)$ is homeomorphic to $\partial_- R_i \times ([0,1],0)$ where $\partial_- R_i = R_i \cap P_i$, and each $C_i = (\mathrm{cl}(P_i - P_{i-1}^+))$ is a compact codimension zero topological submanifold of P_i. Furthermore there is a map $f: P \to M$ which restricts to simple homotopy equivalences of triads $f_i: (P_i; P_{i-1}^+, C_i) \to (M_i; M_{i-1}^+, B_i)$. Thus by Essay III of [4], f is a (infinite) simple homotopy equivalence. □

6. The Maps $G_q: BH(q) \to B\tilde{G}(q)$, $q \geqq 3$

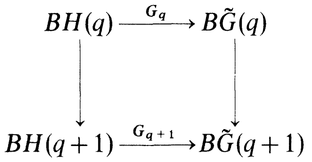
Let $\overline{PL}(q)$ be the Kan Δ-set of which a typical k-simplex is a PL S^{q-1}-block bundle over $\Delta^k \times I$ which restricts to the product bundle over $\Delta^k \times \{0,1\}$. According to N. Martin [7], $\overline{PL}(q)$ is homotopy equivalent to the structure Δ-group $\widetilde{PL}(q)$ of PL S^{q-1}-block bundles. Also $\pi_i(H(q), \overline{PL}(q)) = 0$ for $i \leqq 2$, $q \geqq 3$ (cf. [7, 11]).

Let $\bar{G}(q)$ be the Kan Δ-set of which a typical k-simplex is a PL S^{q-1}-block fibration over $\Delta^k \times I$ which restricts to the product bundle over $\Delta^k \times \{0,1\}$. Note that $\bar{G}(q)$ is homotopy equivalent to the Δ-structure set $\tilde{G}(q)$ (cf. [16]) of S^{q-1}-block fibrations.

As $\pi_i(H(q), \overline{PL}(q)) = 0$ for $i \leqq 2$, $q \geqq 3$, we can define a Δ-map $G_q: H(q)^{(k)} \to \bar{G}(q)^{(k)}$, for $k \leqq 1$, of the k-skeleton of $H(q)$ to the k-skeleton of $\bar{G}(q)$ by assigning an element ξ^q of $H(q)^{(k)}$ the underlying S^{q-1}-block fibration of an element of $\overline{PL}(q)^{(k)}$ which is connected to ξ^q by a $(k+1)$-simplex of $H(q)$.

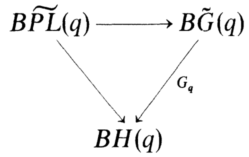
Now let ξ^q be an element of $H(q)^{(3)}$, i.e. a homology cobordism S^{q-1}-bundle over $\Delta^3 \times I$ which restricts to the product bundle over $\Delta^3 \times \{0,1\}$. By applying G_q to $\xi^q \mid \partial\Delta^3 \times I$ we have an element $\bar{\xi}^q$ of $H(q)^{(3)}$ which is connected to ξ^q by a 4-simplex of $H(q)$, and such that $\bar{\xi}^q \mid \partial\Delta^2 \times I$ is a S^{q-1}-block fibration. As $E(\xi^q)$ is a homology cobordism between $\Delta^3 \times S^{q-1}$ and itself and as this cobordism restricts to an h-cobordism between $\partial\Delta^2 \times S^{q-1}$ and itself, we can do surgery on $E(\bar{\xi}^q)$, rel boundary, to get a new block E' so that E' is an h-cobordism between $\Delta^3 \times S^{q-1}$ and itself (cf. Proposition 3.1 of [12]). Let $G_q(\xi^q)$ be the homology cobordism bundle over $\Delta^3 \times I$, which is also a S^{q-1}-block fibration over $\Delta^3 \times I$, with total space E'. Doing this for all $\xi^q \in H(q)^{(3)}$ we obtain a Δ-map $G_q: H(q)^{(3)} \to \bar{G}(q)^{(3)}$. Now by proceeding inductively up the skeleta of $H(q)$ in a similar fashion, we obtain a well-defined Δ-map $G_q: H(q) \to \bar{G}(q)$. Note that by our construction G_q is uniquely defined up to homotopy. Thus $G_q: H(q) \to \bar{G}(q)$ induces a map $G_q: BH(q) \to B\tilde{G}(q)$ which is uniquely defined up to homotopy. We note some elementary properties of this map.

1. *The following diagram commutes up to homotopy,*



where the vertical maps are induced by stabilization.

2. *The following diagram commutes up to homotopy,*

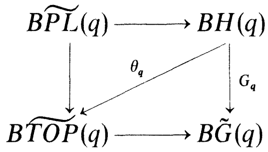


where the unlabeled maps are induced by the natural forgetful maps.

7. The Maps $\theta_q: BH(q) \to B\widetilde{TOP}(q)$, $q \geqq 3$

Let $\overline{TOP}(q)$ be the Kan Δ-set of which a typical k-simplex is a topological S^{q-1}-block bundle over $\Delta^k \times I$ which restricts to the product bundle over $\Delta^k \times \{0, 1\}$. Note that $\overline{TOP}(q)$ is homotopy equivalent to the structure Δ-group $\widetilde{TOP}(q)$ (cf. [15]) of topological S^{q-1}-block bundles. We now define a Δ-map $\theta_q: H(q) \to \overline{TOP}(q)$, $q \geqq 3$, which induces a map $\theta_q: BH(q) \to B\widetilde{TOP}(q)$, $q \geqq 3$, uniquely defined up to homotopy, such that the diagram

(7.1)



commutes up to homotopy.

As $\pi_i(H(q), \overline{PL}(q)) = 0$ for $i \leqq 2$, $q \geqq 3$, we can define a Δ-map $\theta_q: H(q)^{(k)} \to \overline{TOP}(q)^{(k)}$ by assigning an element ξ^q of $H(q)^{(k)}$ the underlying topological S^{q-1}-block bundle of an element of $\overline{PL}(q)^{(k)}$ which is connected to ξ^q by a $(k+1)$-simplex of $H(q)$.

Now let ξ^q be an element of $H(q)^{(3)}$. Then $G_q(\xi^q)$ is a homology cobordism S^{q-1}-bundle over $\Delta^3 \times I$ which restricts to the product bundle over $\Delta^k \times \{0, 1\}$ and which restrict to a PL block bundle over $\partial\Delta^3 \times I$. Then $E(G_q(\xi^q))$ is a homology 6-manifold with PL boundary. Then by (5.2), $E(\xi^q)$ is homotopy equivalent, rel $\partial E(G_q(\xi^q))$ to a topological 6-manifold E'. By the generalized Poincaré conjecture, there is a topological block bundle $\theta_q(\xi^q)$ over $\Delta^3 \times I$ with total space E'. Note that by (5.3) any two such bundles are connected in $\overline{TOP}(q)$ by a 4-simplex. Doing this for all $\xi^q \in H(q)^{(3)}$ we obtain a well-defined Δ-map $\theta_q: H(q)^{(3)} \to \overline{TOP}(q)^{(3)}$. Now by proceeding inductively up the skeleta of $H(q)$ in a similar fashion using (5.2), we obtain a Δ-map θ_q: $H(q) \to \overline{TOP}(q)$ uniquely defined up to homotopy by (5.3). Thus θ_q: $H(q) \to \overline{TOP}(q)$ induces a map θ_q: $BH(q) \to B\widetilde{TOP}(q)$, $q \geqq 3$, which is uniquely defined up to homotopy. By the construction of θ_q and (5.5), diagram (7.1) commutes up to homotopy.

8. The Fiber of $\theta_q: BH(q) \to B\widetilde{TOP}(q)$, $q \geqq 3$

We make the map θ_q: $BH(q) \to B\widetilde{TOP}(q)$ constructed in 6,7 into a Hurewicz fibration and compute the homotopy groups of its fiber $\widetilde{TOP}(q)/H(q)$.

For $q \geqq 3$, the fiber of the natural map $B\widetilde{TOP}(q) \to B\widetilde{PL}(q)$ is a $K(Z_2, 3)$ ([4, 15]) and the fiber of the natural map $B\widetilde{PL}(q) \to BH(q)$ is a $K(\theta_3^H, 3)$ ([7, 11]). Thus by considering the homotopy exact sequence of the triple ($B\widetilde{TOP}(q)$, $BH(q)$, $B\widetilde{PL}(q)$) we immediately have that $\pi_i(\widetilde{TOP}(q)/H(q)) = 0$ for $i \neq 3$, 4 and the following exact sequence

$$0 \to \pi_5(B\widetilde{TOP}(q), BH(q)) \to \pi_4(BH(q), B\widetilde{PL}(q)) \xrightarrow{\theta_q} \pi_4(B\widetilde{TOP}(q), B\widetilde{PL}(q)) \to \pi_4(B\widetilde{TOP}(q), BH(q)) \to 0.$$

Now there are isomorphisms $\pi_4(BH(q), B\widetilde{PL}(q)) \cong \theta_3^H$ ([7, 11]) and $\pi_4(B\widetilde{TOP}(q), B\widetilde{PL}(q)) \cong Z_2$ ([3, 15]). Let H^3 represent an element of θ_3^H. Then $c(H^3) \times S^{q-1}$ represents an element of $\pi_4(BH(q), B\widetilde{PL}(q))$ (cf. [7]) and θ_q assigns this a topological manifold M. If α: $\theta_3^H \to Z_2$ is the Kervaire-Milnor-Rochlin map, then $\alpha(H^3) = 0$ if and only if M possesses a PL manifold structure (cf. Lemma 1 of [17]). Thus θ_q is onto, so that $\pi_3(\widetilde{TOP}(q)/H(q)) = 0$ and

$$\pi_4(BH(q), B\widetilde{PL}(q)) = \text{kernel}\,(\theta_q\colon \pi_4(BH(q), B\widetilde{PL}(q)) \to \pi_4(B\widetilde{TOP}(q), B\widetilde{PL}(q))) \cong \text{kernel}\,(\alpha\colon \theta_3^H \to Z_2).$$

We have thus shown

(8.1) **Theorem.** *Let $\widetilde{TOP}(q)/H(q)$ denote the (homotopic) fiber of θ_q: $BH(q) \to B\widetilde{TOP}(q)$, $q \geqq 3$. Then $\widetilde{TOP}(q)/H(q)$ is a K*(kernel $(\alpha\colon \theta_3^H \to Z_2)$, 4).

9. Applications

In this section we give two elementary applications of the results of the preceeding sections.

(9.1) **Theorem.** *Let W^{n+1} be a compact homology $(n+1)$-manifold, $n \geqq 5$, and V, V' disjoint compact homology submanifolds of ∂W which are topological manifolds such that $\partial W -$*(int $V \cup$ int V') *is homeomorphic* ($\approx$) *to* $\partial V \times [0, 1] \approx \partial V' \times [0, 1]$. *If the inclusion $V \hookrightarrow W$ is a simple homotopy equivalence, then V and V' are homeomorphic.*

Proof. Apply [5.2] and the topological s-cobordism theorem. □

(9.2) **Theorem.** *Suppose there exists an oriented PL homology 3-sphere H^3 with $\alpha(H^3) = 1$ and with $H^3 \# H^3$ bounding an acyclic PL 4-manifold. Then $BH(q)$ is homotopy equivalent to $B\widetilde{TOP}(q) \times K$*(Kernel $(\alpha\colon \theta_3^H \to Z_2)$, 4).

Proof. We show that the fibration $\theta_q: BH(q) \to B\widetilde{TOP}(q)$ has a cross-section. Then, since by (8.1) θ_q has fiber an Eilenberg-MacLane space, it is a trivial fibration.

Let H^3 be the given PL homology 3-sphere. One can consider homology manifolds M whose 3-dimensional sphere links in M and ∂M are PL homeomorphic to connected sums of H^3, $-H^3$ (H^3 with the opposite orientation), and S^3. Call such manifolds H^3-manifolds. Following the construction of $BH(q)$ given by Martin and Maunder [9], one can develop a theory of homology cobordism bundles whose blocks are H^3-manifolds. (In fact there is a general theory for polyhedra whose links satisfy certain axioms and for cone bundles based on these polyhedra. This is developed in [2].) Then let $BH^3(q)$ denote the resulting classifying space. There are natural maps $h: BH^3(q) \to BH(q)$ and $j: B\widetilde{PL}(q) \to BH^3(q)$. By employing the techniques of [7] or [11] one has that the fiber of j is a $K(Z_2, 3)$. Then by the techniques of § 8, the composition

$$BH^3(q) \xrightarrow{h} BH(q) \xrightarrow{\theta_q} B\widetilde{TOP}(q)$$

is a homotopy equivalence. The result now follows. □

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