

AN INDEFINITE KÄHLER METRIC ON THE SPACE OF ORIENTED LINES

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ABSTRACT

The total space of the tangent bundle of a Kähler manifold admits a canonical Kähler structure. Parallel translation identifies the space \mathbb{T} of oriented affine lines in \mathbb{R}^3 with the tangent bundle of S^2 . Thus the round metric on S^2 induces a Kähler structure on \mathbb{T} which turns out to have a metric of neutral signature. It is shown that the identity component of the isometry group of this metric is isomorphic to the identity component of the isometry group of the Euclidean metric on \mathbb{R}^3 .

The geodesics of this metric are either planes or helicoids in \mathbb{R}^3 . The signature of the metric induced on a surface Σ in \mathbb{T} is determined by the degree of twisting of the associated line congruence in \mathbb{R}^3 , and it is shown that, for Σ Lagrangian, the metric is either Lorentz or totally null. For such surfaces it is proved that the Keller–Maslov index counts the number of isolated complex points of \mathbb{J} inside a closed curve on Σ .

1. Introduction

A $2n$ -manifold N has a Kähler structure (j, ω, g) if it admits a complex structure j , a symplectic structure ω and a metric g which are compatible. The total space of the tangent bundle TN of such a manifold itself carries a canonical Kähler structure $(\mathbb{J}, \Omega, \mathbb{G})$. If $n = 1$, this metric \mathbb{G} has neutral signature $(++--)$ and is conformally flat if and only if g is of constant curvature.

The minitwistor correspondence identifies the tangent bundle of the 2-sphere with the space \mathbb{T} of oriented affine lines in \mathbb{R}^3 . Thus, the round metric on S^2 induces a Kähler structure on \mathbb{T} with the above properties. While the metric \mathbb{G} has been noted before [10], it has not been studied in this context and the purpose of this paper is to consider its significance for differential geometry in \mathbb{R}^3 .

The complex structure on \mathbb{T} has proved crucial in understanding static monopoles which arise in theoretical physics [7]. Indeed, as early as 1866, Weierstrass [12] used this complex structure to construct minimal surfaces in \mathbb{R}^3 , while Whittaker used it to find solutions of the Laplace equation [13]. The symplectic structure Ω is equivalent to the canonical symplectic structure on T^*S^2 , which comes to the fore in geometric optics (see, for example, Arnold [2]).

The interpretation we give to \mathbb{G} is that of angular momentum of the Jacobi fields along an oriented line in \mathbb{R}^3 . This is invariant under the group of rotations and translations, and so the isometry group of \mathbb{G} contains the Euclidean group. In fact, we show the following.

THEOREM 1. *The identity component of the isometry group of the metric \mathbb{G} on \mathbb{T} is isomorphic to the identity component of the Euclidean isometry group.*

This is analogous to the classical result on the automorphism group of the Study sphere [11].

A curve in \mathbb{T} generates a ruled surface in \mathbb{R}^3 . We prove that the geodesics of the metric \mathbb{G} give particularly simple ruled surfaces.

THEOREM 2. *A geodesic of \mathbb{G} is either a plane or a helicoid in \mathbb{R}^3 , the former in the case when the geodesic is null, the latter when it is non-null.*

Of particular interest to us are surfaces $\Sigma \subset \mathbb{T}$, classically referred to as *line congruences*. By construction, Σ is *Lagrangian* if and only if the associated line congruence in \mathbb{R}^3 has zero twist, that is, it is integrable in the sense of Frobenius and there exist orthogonal surfaces to the lines in \mathbb{R}^3 . On the other hand, Σ is *complex* if and only if the shear of the line congruence vanishes.

We prove that the signature of the metric induced on Σ by \mathbb{G} is determined by the ratio of twist to shear of the congruence.

THEOREM 3. *The induced metric on a surface Σ in \mathbb{T} is Lorentzian (totally null, Riemannian) if and only if $\lambda^2 - |\sigma|^2 < 0$ ($= 0, > 0$), where λ and σ are the twist and shear of the line congruence Σ .*

As a consequence the induced metric on a Lagrangian surface is either Lorentz or totally null. The null directions on the orthogonal surface S in \mathbb{R}^3 are the eigen-directions of the second fundamental form, while the totally null points are umbilical points on S .

The Keller–Maslov index [2] associates an integer to closed oriented curves on a Lagrangian surface Σ in a symplectic 4-manifold. This is just the degree of the Gauss mapping which takes a point on the curve to the tangent plane of Σ , considered as a point in the Lagrangian Grassmanian $\Lambda = U(2)/O(2)$.

The reduction of $GL(2, \mathbb{C})$ to $U(2)$ is achieved by choosing a complex structure which is tamed by Ω , that is \mathbb{G} is positive definite. In our case, \mathbb{J} is not tamed by the symplectic structure and we look at the metric reduction of $GL(2, \mathbb{C})$ to $U(1, 1)$. Since $\pi_1(U(1, 1)/O(1, 1)) = \pi_1(S^1) = \mathbb{Z}$, we can still define an index on Lagrangian surfaces and we prove the following.

THEOREM 4. *The Keller–Maslov index of a curve on a Lagrangian surface in \mathbb{T} counts the number of isolated complex points of Σ inside the curve.*

This paper is organised as follows. The next section contains the general definition of the Kähler structures on tangent bundles. In Section 3 we describe the space \mathbb{T} of oriented affine lines in \mathbb{R}^3 and give the geometric interpretation of the neutral metric on \mathbb{T} .

Section 4 contains the proof that the isometry group of \mathbb{G} is isomorphic to the Euclidean group, while Section 5 investigates the geodesics in \mathbb{T} . In Section 6 we turn to the two-dimensional submanifolds of \mathbb{T} and their induced geometry. In the final section we relate the Keller–Maslov index on Lagrangian surfaces to the index of isolated complex points.

2. Canonical Kähler metric on tangent bundles

Throughout this paper we work solely with smooth maps and manifolds. Let M be a smooth even-dimensional manifold. An *almost complex structure* on M is an endomorphism $\mathbb{J} : T_p M \rightarrow T_p M$ at each $p \in M$ satisfying $\mathbb{J} \circ \mathbb{J} = -\text{Id}$. If, in addition, \mathbb{J} satisfies an integrability condition, it is said to be a *complex structure*.

A *symplectic structure* on M is a closed non-degenerate 2-form Ω . It is *compatible* with the almost complex structure if $\Omega(\mathbb{J}\cdot, \mathbb{J}\cdot) = \Omega(\cdot, \cdot)$.

Given a complex structure and compatible symplectic structure define a symmetric non-degenerate 2-tensor $\mathbb{G} : T_p M \times T_p M \rightarrow \mathbb{R}$ by $\mathbb{G}(\cdot, \cdot) = \Omega(\mathbb{J}\cdot, \cdot)$. The manifold M with this triple of structures is called a *Kähler manifold*. While \mathbb{G} is a metric, we make no assumption that it is positive definite.

Given a Kähler $2n$ -manifold (N, j, ω, g) we construct a canonical Kähler structure $(\mathbb{J}, \Omega, \mathbb{G})$ on the tangent bundle TN as follows. The Levi-Civita connection associated with g splits the tangent bundle $TTN \equiv TN \oplus TN$ and the complex structure is defined to be $\mathbb{J} = j \oplus j$. The integrability of \mathbb{J} follows from the integrability of j [8]. To define the symplectic form, consider the metric g as a mapping from TN to T^*N and pull back the canonical symplectic form Ω^* on T^*N to Ω on TN . Finally, the metric is defined as above by $\mathbb{G}(\cdot, \cdot) = \Omega(\mathbb{J}\cdot, \cdot)$. The triple $(\mathbb{J}, \Omega, \mathbb{G})$ determines a Kähler structure on TN .

We now find local coordinate descriptions of the above. Let ξ^k be local holomorphic coordinates on an open set of N such that the metric has the real potential Q :

$$g = \partial_k \bar{\partial}_l Q \, d\xi^k \otimes d\bar{\xi}^l \quad \text{where } \partial_i = \frac{\partial}{\partial \xi^i}.$$

Introduce coordinates on T^*N by identifying (ξ^k, α_l) with $\alpha_k d\xi^k + \bar{\alpha}_k d\bar{\xi}^k$, and coordinates on TN by identifying (ξ^k, η^l) with $\eta^k \partial_k + \bar{\eta}^k \bar{\partial}_k$.

The canonical symplectic structure on T^*N is $\Omega^* = d\alpha_k \wedge d\xi^k + d\bar{\alpha}_k \wedge d\bar{\xi}^k$. Considered as a map between TN and T^*N , the metric g takes (ξ^k, η^l) to $(\xi^k, \alpha_l = \bar{\eta}^m \bar{\partial}_m \partial_l Q)$. Thus, pulling back the canonical symplectic form, we get

$$\Omega = \text{Re} \left(\partial_k \bar{\partial}_l Q \, d\eta^k \wedge d\bar{\xi}^l + \eta^m \partial_m \partial_k \bar{\partial}_l Q \, d\xi^k \wedge d\bar{\xi}^l \right).$$

PROPOSITION 1. *The eigenspaces of \mathbb{J} are*

$$\mathbb{J} \left(\frac{\partial}{\partial \xi^k} \right) = i \frac{\partial}{\partial \xi^k}, \quad \mathbb{J} \left(\frac{\partial}{\partial \eta^k} \right) = i \frac{\partial}{\partial \eta^k}. \quad (2.1)$$

The complex structure \mathbb{J} and the symplectic structure Ω are compatible.

Proof. This follows from the fact that the metric connection of g is compatible with the complex structure j . \square

We now specialise to the case $n = 1$, where more can be said.

PROPOSITION 2. *Let (N, g) be a riemannian 2-manifold. Then $(TN, \mathbb{J}, \Omega, \mathbb{G})$ is a Kähler manifold. The metric \mathbb{G} has neutral signature $(++--)$ and is scalar-flat. Moreover, \mathbb{G} is Kähler–Einstein if and only if g is flat, and \mathbb{G} is conformally flat if and only if g is of constant curvature.*

Proof. Choose isothermal coordinates ξ on N so that $ds^2 = e^{2u} d\xi d\bar{\xi}$, and corresponding coordinates (ξ, η) on TN , as above. In such a coordinate system, the symplectic 2-form is

$$\Omega = 2\mathbb{R}e(e^{2u} d\eta \wedge d\bar{\xi} + \eta \partial(e^{2u}) d\xi \wedge d\bar{\xi}), \quad (2.2)$$

and the metric \mathbb{G} is

$$\mathbb{G} = 2\mathbb{I}m(e^{2u} d\eta d\bar{\xi} + \eta \partial(e^{2u}) d\xi d\bar{\xi}). \quad (2.3)$$

A direct computation shows that \mathbb{G} is neutral and scalar-flat.

The only non-vanishing component of the Ricci tensor is $R_{\xi\bar{\xi}} = -4\partial\bar{\partial}u$. Since the Gauss curvature of N is $\kappa = -4e^{-2u}\partial\bar{\partial}u$, \mathbb{G} is Kähler–Einstein if and only if $\kappa = 0$.

Furthermore, the only non-vanishing component of the conformal curvature tensor is

$$C_{\xi\bar{\xi}\eta} = \frac{1}{2}e^{2u}(\eta\partial\kappa + \bar{\eta}\bar{\partial}\kappa),$$

so \mathbb{G} is conformally flat if and only if κ is constant. \square

3. The Kähler metric on \mathbb{T}

Consider the space \mathbb{T} of oriented (affine) lines in \mathbb{R}^3 . Each oriented line can be uniquely described by two vectors: the unit direction vector of the line and the perpendicular distance vector of the line from the origin. If we think of the direction vector as a point ξ on the 2-sphere, the perpendicular distance vector gives a tangent vector to S^2 at ξ . Thus \mathbb{T} is diffeomorphic to the tangent bundle to S^2 , giving it the structure of a smooth 4-manifold [7].

Take local complex coordinates ξ on S^2 by stereographic projection from the south pole onto the plane through the equator. These coordinates yield canonical coordinates (ξ, η) on $\mathbb{T} = TS^2$, as in the last section.

PROPOSITION 3 [5]. *Consider the map $\Phi : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}^3$ defined by*

$$\Phi_z = \frac{2(\eta - \bar{\eta}\xi^2) + 2\xi(1 + \xi\bar{\xi})r}{(1 + \xi\bar{\xi})^2}, \quad (3.1)$$

$$\Phi_t = \frac{-2(\eta\bar{\xi} + \bar{\eta}\xi) + (1 - \xi^2\bar{\xi}^2)r}{(1 + \xi\bar{\xi})^2}. \quad (3.2)$$

Then Φ takes $(\xi, \eta) \in T_\xi S^2$ and $r \in \mathbb{R}$, to a point $(\Phi_z, \Phi_t) \in \mathbb{C} \oplus \mathbb{R} = \mathbb{R}^3$ which is a distance r along the line with direction ξ and minimal distance from the origin given by η (see Figure 1).

A null frame at a point p in \mathbb{R}^3 is a trio of vectors $e_{(0)}, e_{(+)}, e_{(-)} \in \mathbb{C} \otimes T_p \mathbb{R}^3$ such that $e_{(0)} = \overline{e_{(-)}}$, $e_{(+)} = \overline{e_{(-)}}$, $e_{(0)} \cdot e_{(0)} = e_{(+)} \cdot e_{(-)} = 1$, and $e_{(0)} \cdot e_{(+)} = 0$, where the Euclidean inner product is extended bilinearly over \mathbb{C} .

The derivative of the map Φ has the following description.

PROPOSITION 4. *The derivative $D\Phi : T_{(\xi, \eta, r)} \mathbb{T} \times \mathbb{R} \rightarrow T_{\Phi(\xi, \eta, r)} \mathbb{R}^3$ is given by*

$$D\Phi_{(\xi, \eta, r)} \left(\frac{\partial}{\partial \xi} \right) = \left(r - \frac{2\xi\bar{\eta}}{1 + \xi\bar{\xi}} \right) \frac{\sqrt{2}}{1 + \xi\bar{\xi}} e_{(+)} - \frac{2\bar{\eta}}{(1 + \xi\bar{\xi})^2} e_{(0)}, \quad (3.3)$$

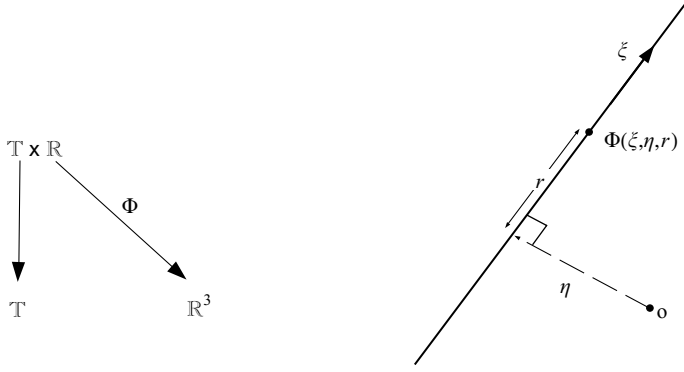


FIGURE 1.

$$D\Phi_{(\xi, \eta, r)} \left(\frac{\partial}{\partial \eta} \right) = \frac{\sqrt{2}}{1 + \xi \bar{\xi}} e_{(+)}, \quad D\Phi_{(\xi, \eta, r)} \left(\frac{\partial}{\partial r} \right) = e_{(0)}, \quad (3.4)$$

where we have introduced the null frame

$$e_{(0)} = \frac{2\xi}{1 + \xi \bar{\xi}} \frac{\partial}{\partial z} + \frac{2\bar{\xi}}{1 + \xi \bar{\xi}} \frac{\partial}{\partial \bar{z}} + \frac{1 - \xi \bar{\xi}}{1 + \xi \bar{\xi}} \frac{\partial}{\partial t},$$

$$e_{(+)} = \frac{\sqrt{2}}{1 + \xi \bar{\xi}} \frac{\partial}{\partial z} - \frac{\sqrt{2} \bar{\xi}^2}{1 + \xi \bar{\xi}} \frac{\partial}{\partial \bar{z}} - \frac{\sqrt{2} \bar{\xi}}{1 + \xi \bar{\xi}} \frac{\partial}{\partial t}.$$

Here and throughout we use $z = x^1 + ix^2$ and $t = x^3$, where x^1 , x^2 and x^3 are Euclidean coordinates on \mathbb{R}^3 . The unit vector $e_{(0)}$ determines the direction of the line given by (ξ, η) .

The map $D\Phi$ gives the identification of tangent vectors to \mathbb{T} at a line γ with the Jacobi fields orthogonal to the line in \mathbb{R}^3 . A Jacobi field along a line γ in \mathbb{R}^3 is a vector field X along the line which satisfies the equation

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X = 0.$$

Choosing an affine parameter r along the line, this has solution $X = X_1 + rX_2$, for constant vector fields X_1 and X_2 along γ . The Jacobi fields that are orthogonal to γ form a four-dimensional vector space, which $D\Phi$ identifies with the tangent space $T_{\gamma} \mathbb{T}$.

From the maps (3.3) and (3.4), along with equation (2.1), the complex structure \mathbb{J} acts on the Jacobi fields by multiplication of $e_{(+)}$ by i . This is equivalent to rotation about $e_{(0)}$ through 90° . This yields the following (see [7]).

PROPOSITION 5. *The complex structure \mathbb{J} on \mathbb{T} defined in the last section is given by rotation of the Jacobi fields through 90° about the corresponding line in \mathbb{R}^3 .*

PROPOSITION 6. *The symplectic 2-form Ω on \mathbb{T} , determined by the round metric on S^2 , is given by*

$$\Omega_{(\xi, \eta)}(\mathbb{X}, \mathbb{Y}) = \langle D\Phi(\mathbb{X}), \nabla_{(0)} D\Phi(\mathbb{Y}) \rangle - \langle D\Phi(\mathbb{Y}), \nabla_{(0)} D\Phi(\mathbb{X}) \rangle, \quad (3.5)$$

where $\mathbb{X}, \mathbb{Y} \in T_{(\xi, \eta)}\mathbb{T}$, $\langle \cdot, \cdot \rangle$ is the Euclidean metric on \mathbb{R}^3 and $\nabla_{(0)}$ is the covariant derivative in the ξ direction.

Proof. Taking (3.5) as the definition of Ω , we find, for example, using (3.3) and (3.4), that

$$\begin{aligned} \Omega\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \bar{\xi}}\right) &= g\left(\frac{\sqrt{2}}{1+\xi\bar{\xi}}\left(r - \frac{2\bar{\xi}\eta}{1+\xi\bar{\xi}}\right)e_{(+)}, \frac{\sqrt{2}}{1+\xi\bar{\xi}}e_{(-)}\right) \\ &\quad - g\left(\frac{\sqrt{2}}{1+\xi\bar{\xi}}\left(r - \frac{2\xi\bar{\eta}}{1+\xi\bar{\xi}}\right)e_{(-)}, \frac{\sqrt{2}}{1+\xi\bar{\xi}}e_{(+)}\right) \\ &= \frac{2}{(1+\xi\bar{\xi})^2}\left(r - \frac{2\bar{\xi}\eta}{1+\xi\bar{\xi}}\right) - \frac{2}{(1+\xi\bar{\xi})^2}\left(r - \frac{2\xi\bar{\eta}}{1+\xi\bar{\xi}}\right) \\ &= \frac{4(\xi\bar{\eta} - \bar{\xi}\eta)}{(1+\xi\bar{\xi})^3}, \end{aligned}$$

and this agrees with the canonical symplectic structure given in (2.2) in the case when g is the round metric on S^2 with $e^{2u} = 2(1+\xi\bar{\xi})^{-2}$. Similarly for the other components of the symplectic form. \square

The metric \mathbb{G} on \mathbb{T} is defined by $\mathbb{G}(\cdot, \cdot) \equiv \Omega(\cdot, \mathbb{J}\cdot)$. By Proposition 2 we have the following.

PROPOSITION 7. *The metric \mathbb{G} on \mathbb{T} is a neutral Kähler metric which is conformally flat, with zero scalar curvature, but is not Einstein.*

The local expression for the metric is

$$\mathbb{G} = \frac{2i}{(1+\xi\bar{\xi})^2} \left(d\eta d\bar{\xi} - d\bar{\eta} d\xi + \frac{2(\xi\bar{\eta} - \bar{\xi}\eta)}{1+\xi\bar{\xi}} d\xi d\bar{\xi} \right), \quad (3.6)$$

and it can be given the following geometric interpretation.

PROPOSITION 8. *The length of $\mathbb{X} \in T_\gamma\mathbb{T}$ with respect to \mathbb{G} is the angular momentum about γ of the line determined by the Jacobi field associated to \mathbb{X} .*

Proof. A direct computation using $D\Phi$ shows that the length of $\mathbb{X} = X_1 + rX_2$ is the oriented area of the parallelogram spanned by X_1 and X_2 , that is $\mathbb{G}(\mathbb{X}, \mathbb{X}) = (X_1 \times X_2) \cdot e_{(0)}$. \square

The symplectic 2-form on \mathbb{T} is globally exact $\Omega = d\Theta$. This global 1-form, locally pulled back in the above coordinates, is

$$\Theta = \frac{2\bar{\eta}d\xi}{(1+\xi\bar{\xi})^2} + \frac{2\eta d\bar{\xi}}{(1+\xi\bar{\xi})^2}. \quad (3.7)$$

We discuss this further in Section 6. The Kähler potential of the metric is

$$\Upsilon = \frac{2i(\xi\bar{\eta} - \bar{\xi}\eta)}{1+\xi\bar{\xi}}.$$

4. The isometry group of \mathbb{G}

The group of fibre-preserving holomorphic automorphisms of \mathbb{T} is isomorphic to $\mathrm{PSL}(2, \mathbb{C}) \ltimes \mathbb{C}^3$, where the action on \mathbb{T} is given by

$$\xi \rightarrow \xi' = \xi, \quad \eta \rightarrow \eta' = \eta + a_1 + b_1\xi - c_1\xi^2, \quad (4.1)$$

and

$$\xi \rightarrow \xi' = \frac{a_2\xi + b_2}{c_2\xi + d_2}, \quad \eta \frac{\partial}{\partial \xi} \rightarrow \eta' \frac{\partial}{\partial \xi'} = \frac{\eta}{(c_2\xi + d_2)^2} \frac{\partial}{\partial \xi'}, \quad (4.2)$$

for $a_1, a_2, b_1, b_2, c_1, c_2, d_2 \in \mathbb{C}$ with $a_2d_2 - b_2c_2 = 1$.

Here, $A \in \mathrm{PSL}(2, \mathbb{C})$ acts on the 2×2 complex symmetric matrices (to be identified with \mathbb{C}^3) by $M \rightarrow AMA^T$. This is the same action as that of $\mathrm{PSL}(2, \mathbb{C})$ on quadratic holomorphic transformations of \mathbb{T} as above.

The identity component of the Euclidean isometry group is double covered by $\mathrm{SU}(2) \ltimes \mathbb{R}^3$, which is a real form of the complex Lie group $\mathrm{PSL}(2, \mathbb{C}) \ltimes \mathbb{C}^3$, where the transformations (4.1) and (4.2) are restricted to those with $a_1 = \overline{c_1}$, $b_1 = \overline{b_1}$, $a_2 = \overline{a_2} = d_2$ and $b_2 = -\overline{c_2}$.

Since the above construction of \mathbb{G} is invariant under Euclidean motions, it is clear that the isometry group of \mathbb{G} contains the Euclidean group of translations and rotations. We now prove that \mathbb{G} admits no other continuous isometries.

Proof of Theorem 1. The Lie group of Euclidean motions is a subgroup of the Lie group of isometries of \mathbb{G} . This can be proved in local coordinates, or by noting that the angular momentum of a Jacobi field is invariant under Euclidean motions.

We prove in the following proposition that the associated Lie algebras are isomorphic, and, in particular, have the same dimension. Thus the connected component of the identity of the isometry groups of \mathbb{G} is isomorphic to the identity component of the Euclidean group. \square

PROPOSITION 9. *The Killing vectors of \mathbb{G} form a six-parameter Lie algebra given by*

$$\mathbb{K} = \mathbb{K}^\xi \frac{\partial}{\partial \xi} + \mathbb{K}^\eta \frac{\partial}{\partial \eta} + \mathbb{K}^{\bar{\xi}} \frac{\partial}{\partial \bar{\xi}} + \mathbb{K}^{\bar{\eta}} \frac{\partial}{\partial \bar{\eta}},$$

with

$$\mathbb{K}^\xi = \alpha + 2ai\xi + \bar{\alpha}\xi^2, \quad \mathbb{K}^\eta = 2(ai + \bar{\alpha}\xi)\eta + \beta + b\xi - \bar{\beta}\xi^2,$$

where $\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathbb{R}$. The Killing vectors given by α and a generate infinitesimal rotations while those given by β and b generate infinitesimal translations.

Proof. A vector field \mathbb{K}^i is a Killing vector for \mathbb{G} if and only if

$$\mathbb{K}^i \partial_i \mathbb{G}_{jk} + \mathbb{G}_{ki} \partial_j \mathbb{K}^i + \mathbb{G}_{ji} \partial_k \mathbb{K}^i = 0. \quad (4.3)$$

We will find the Killing vectors by solving these equations in a particular order. Our approach is to first integrate out the η and $\bar{\eta}$ dependence. Then the coefficients of the powers of η and $\bar{\eta}$ in the remaining equations must each be zero, which determines the ξ and $\bar{\xi}$ dependence.

To start then, consider the simplest equations, which are (j, k) equal to (η, η) , $(\eta, \bar{\eta})$ and (ξ, η) . These state that

$$\partial_{\eta} \mathbb{K}^{\bar{\xi}} = 0, \quad (4.4)$$

$$\partial_{\eta} \mathbb{K}^{\xi} - \partial_{\bar{\eta}} \mathbb{K}^{\bar{\xi}} = 0, \quad (4.5)$$

$$\partial_{\xi} \mathbb{K}^{\bar{\xi}} - \partial_{\eta} \mathbb{K}^{\bar{\eta}} = 0. \quad (4.6)$$

Differentiating equation (4.5) with respect to η and noting equation (4.4) we get $\partial_{\eta} \partial_{\eta} \mathbb{K}^{\xi} = 0$, with solution

$$\mathbb{K}^{\xi} = a_0(\xi, \bar{\xi}, \bar{\eta}) + a_1(\xi, \bar{\xi}, \bar{\eta})\eta, \quad (4.7)$$

where a_0 is complex-valued while a_1 is real-valued. Substituting (4.7) into (4.4) we get $\partial_{\eta} \bar{a}_0 + \bar{\eta} \partial_{\eta} a_1 = 0$. Looking at the $\bar{\eta}$ dependence, both terms must be zero. That is, $a_0 = a_0(\xi, \bar{\xi})$ and $a_1 = a_1(\xi, \bar{\xi})$ so that

$$\mathbb{K}^{\xi} = a_0(\xi, \bar{\xi}) + a_1(\xi, \bar{\xi})\eta. \quad (4.8)$$

Substituting this into (4.6) we get the equation $\partial_{\eta} \mathbb{K}^{\bar{\eta}} = \partial_{\xi} \bar{a}_0 + \bar{\eta} \partial_{\xi} a_1$, which we can solve to get

$$\mathbb{K}^{\eta} = b_0(\xi, \bar{\xi}, \eta) + (\partial_{\xi} a_0 + \eta \partial_{\xi} a_1) \bar{\eta}. \quad (4.9)$$

At this stage (4.8) and (4.9) solve five of the ten Killing equations. The remaining equations are (j, k) equal to $(\xi, \bar{\eta})$, (ξ, ξ) and $(\xi, \bar{\xi})$. The first of these reads

$$-(1 + \xi \bar{\xi})(\partial_{\bar{\eta}} \bar{b}_0 + \partial_{\xi} a_0 + 2\eta \partial_{\xi} a_1) + 2a_0 \bar{\xi} + 2\bar{a}_0 \xi + 4a_1 \xi \bar{\eta} = 0.$$

Now, the η term tells us that $\partial_{\xi} a_1 = 0$ while its conjugate implies that a_1 is constant. Integrating the remaining equation with respect to $\bar{\eta}$ we find that

$$\bar{b}_0 = \bar{b}_1(\xi, \bar{\xi}) - \left(\partial_{\xi} a_0 - \frac{2(a_0 \bar{\xi} + \bar{a}_0 \xi)}{1 + \xi \bar{\xi}} \right) \bar{\eta} + \frac{2a_1 \xi}{1 + \xi \bar{\xi}} \bar{\eta}^2.$$

The η and $\bar{\eta}$ dependence integrated out of the Killing equations, we turn now to the ξ and $\bar{\xi}$ dependence. Consider the Killing equation with (j, k) equal to (ξ, ξ) . The coefficient of $\bar{\eta}^2$ is simply $-2a_1$ which must therefore vanish. On the other hand the coefficient of the term independent of both η and $\bar{\eta}$ is $\partial_{\xi} \bar{b}_1$ which must also be zero. Turning to the η term we get

$$\partial_{\xi} \partial_{\xi} a_0 + \frac{2\bar{\xi}}{1 + \xi \bar{\xi}} \partial_{\xi} a_0 = 0,$$

with solution

$$a_0 = c_0(\xi) + \frac{c_1(\xi)}{\xi(1 + \xi \bar{\xi})}.$$

We now turn to the final equation, (j, k) equal to $(\xi, \bar{\xi})$. The η term of this gives $-2(1 + \xi \bar{\xi}) \partial_{\bar{\xi}} \bar{c}_1 + 8\xi \bar{c}_1 = 0$, which we solve to set $c_1 = c_2(1 + \xi \bar{\xi})^4$, for some constant c_2 . The remaining part of the $(\xi, \bar{\xi})$ equation is

$$\partial_{\xi} b_1 - \frac{2\bar{\xi}}{1 + \xi \bar{\xi}} b_1 = \partial_{\xi} \bar{b}_1 - \frac{2\xi}{1 + \xi \bar{\xi}} \bar{b}_1.$$

The general solution to this is $b_1 = c_3 + c_4 \xi - \bar{c}_3 \xi^2$, where c_3 is a complex constant while c_4 is a real constant.

Returning finally to the (ξ, ξ) Killing equation, the η term tells us that $c_2 = 0$. The last Killing equation is then

$$(1 + \xi\bar{\xi})^2 \partial_{\xi} \partial_{\xi} c_0 - 2\bar{\xi}(1 + \xi\bar{\xi}) \partial_{\xi} c_0 + 2\bar{\xi}^2 c_0 - 2\bar{c}_0 = 0. \quad (4.10)$$

Differentiating this three times with respect to $\bar{\xi}$ we find that $\partial_{\xi} \partial_{\xi} \partial_{\xi} \bar{c}_0 = 0$, so $c_0 = d_0 + d_1 \xi + d_2 \xi^2$. Substituting this back into (4.10) we get $d_0 = d_2$ and $d_1 = -\bar{d}_1$.

A relabelling of constants gives the result. \square

5. Geodesics in \mathbb{T}

Any curve in \mathbb{T} gives a one-parameter family of oriented lines in \mathbb{R}^3 . Classically, these are referred to as *ruled surfaces*, and we now determine the ruled surfaces that correspond to the geodesics of \mathbb{G} .

Proof of Theorem 2. Let $c : [0, 1] \rightarrow \mathbb{T}$ be a curve with tangent vector

$$\mathbb{X} = \dot{\xi} \frac{\partial}{\partial \xi} + \dot{\eta} \frac{\partial}{\partial \eta} + \dot{\bar{\xi}} \frac{\partial}{\partial \bar{\xi}} + \dot{\bar{\eta}} \frac{\partial}{\partial \bar{\eta}}.$$

By a translation and rotation we can set $\xi(0) = \eta(0) = 0$, so that the initial line $c(0)$ is the x^3 -axis. We still retain the freedom to rotate about, and translate along, the x^3 -axis.

The geodesic equations (with arc-length or affine parameter s) are

$$\ddot{\xi} - \frac{2\bar{\xi}}{1 + \xi\bar{\xi}} \dot{\xi}^2 = 0, \quad (5.1)$$

$$\ddot{\eta} - \frac{4\bar{\xi}}{1 + \xi\bar{\xi}} \dot{\xi} \dot{\eta} + \frac{2(\bar{\eta} + \bar{\xi}^2 \eta)}{(1 + \xi\bar{\xi})^2} \dot{\xi}^2 = 0, \quad (5.2)$$

where a dot represents differentiation with respect to s , and we have made use of the connection coefficients associated with \mathbb{G} .

These have first integral

$$\frac{2i}{(1 + \xi\bar{\xi})^2} \left(\dot{\eta} \dot{\xi} - \dot{\bar{\eta}} \dot{\bar{\xi}} + \frac{2(\xi\bar{\eta} - \bar{\xi}\eta)}{1 + \xi\bar{\xi}} \dot{\xi} \dot{\bar{\xi}} \right) = C_1, \quad (5.3)$$

where $C_1 \in \mathbb{R}$ vanishes if and only if the geodesic is null.

Suppose that $\dot{\xi}$ vanishes at $s = 0$, then by (5.1), it vanishes along the whole geodesic. In addition, (5.2) says that η is a linear function of the affine parameter and the ruled surface is a plane. By (5.3) the geodesic is null.

Suppose now that $\dot{\xi}(0) \neq 0$. Equation (5.1) is the geodesic equation for the round metric on S^2 and so the geodesic projects to a great circle on S^2 . We can integrate the geodesic equations on the sphere (with initial condition $\xi(0) = 0$) to determine the evolution of ξ :

$$\xi = \tan(C_2 s) e^{i\theta}, \quad (5.4)$$

for constants $C_2 \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. Substituting this in (5.3) we find that

$$\eta e^{-i\theta} - \bar{\eta} e^{i\theta} = \frac{C_1 s + C_3}{2i C_2 \cos^2(C_2 s)},$$

for some real constant C_3 . Equation (5.2) now simplifies to that of the forced harmonic oscillator.

$$\frac{d^2}{ds^2}(\cos^2(C_2s)\eta) + 4C_2^2 \cos^2(C_2s)\eta = -C_2(C_1s + C_3)ie^{i\theta},$$

with solution

$$\eta = \frac{C_4 \cos(2C_2s) + C_5 \sin(2C_2s) - (C_1s + C_3)i}{4C_2 \cos^2(C_2s)} e^{i\theta}.$$

Finally, since $\eta(0) = 0$ we have $C_3 = C_4 = 0$, so that

$$\eta = \frac{C_5 \sin(2C_2s) - C_1si}{4C_2 \cos^2(C_2s)} e^{i\theta}. \quad (5.5)$$

Equations (5.4) and (5.5) are the general solution to the geodesic equations in \mathbb{T} when the initial line in \mathbb{R}^3 is the x^3 -axis. The four real constants remaining, namely C_1, C_2, C_5 and θ , determine the initial direction of the geodesic in \mathbb{T} .

The associated ruled surface in \mathbb{R}^3 is a helicoid when $C_1 \neq 0$ and a plane when $C_1 = 0$. To see this we can put it in standard position as follows. By a rotation about the x^3 -axis we can fix $\theta = 0$, while a translation along the x^3 -axis allows us to put $C_5 = 0$ (cf. equation (4.1) with $a_1 = c_1 = 0$ and $b_1 = -C_5/2C_2$). The ruled surface can be explicitly determined using (3.1) and (3.2) and the result is

$$x^1 = t \sin(2C_2s), \quad x^2 = -\frac{C_1s}{2C_2}, \quad x^3 = t \cos(2C_2s).$$

This is a helicoid for $C_1 \neq 0$ and a plane for $C_1 = 0$, as claimed. \square

Given two oriented lines γ_1 and γ_2 , which neither intersect nor are parallel, the geodesics in \mathbb{T} joining them consist of the helicoids in \mathbb{R}^3 containing them. The \mathbb{G} -distance between the lines is $-l^2/d$, where l is the perpendicular distance between them in \mathbb{R}^3 , and d is the distance on \mathbb{P}^1 traversed by the direction vectors of the ruling of the helicoid. This is maximised (as appropriate for indefinite metrics) by the helicoid joining γ_1 and γ_2 with the minimum number of turns. The multiplicity of a non-maximising geodesic in \mathbb{G} is given by the number of turns of the helicoid.

6. Surfaces in \mathbb{T}

Consider now a *line congruence* Σ , that is, a two-parameter family of oriented lines in \mathbb{R}^3 . Equivalently, this is a mapping $f : \Sigma \rightarrow \mathbb{T}$ of a surface Σ into \mathbb{T} .

Away from crossings of the lines we can adapt a local null frame to Σ by aligning $e_{(0)}$ with the direction of the lines. The complex *spin coefficients* of the congruence are defined by

$$\Gamma_{mnp} = e_{(n)}^i e_{(p)}^j \nabla_j e_{(m)i},$$

where ∇ is the Euclidean covariant derivative and the indices m, n, p range over 0, +, −. Breaking covariance, introduce the complex optical scalars

$$\Gamma_{+0-} = \rho, \quad \Gamma_{+0+} = \sigma.$$

The complex scalar functions ρ and σ describe the first order geometric behaviour of the congruence of lines. The real part of ρ is the *divergence*, the imaginary part λ is the *twist* and σ is the *shear* of the congruence (see [4] and [9] for further details).

PROPOSITION 10. A surface Σ in \mathbb{T} is Lagrangian with respect to the symplectic structure Ω if and only if the associated congruence is integrable (twist-free), that is there exists a surface S in \mathbb{R}^3 orthogonal to the line congruence.

Proof. A surface Σ is Lagrangian if and only if Ω pulled back to Σ vanishes. Suppose that Σ is given parametrically by $f : \Sigma \rightarrow \mathbb{T}$, $\nu \mapsto (\xi(\nu, \bar{\nu}), \eta(\nu, \bar{\nu}))$. Then pulling Ω back to Σ , using (2.2),

$$f_*\Omega = 4\operatorname{Re} \left(\left(\partial\eta\bar{\partial}\bar{\xi} + \partial\bar{\eta}\bar{\partial}\xi - 2\frac{\bar{\xi}\eta}{1+\xi\bar{\xi}}(\partial\xi\bar{\partial}\bar{\xi} - \bar{\partial}\xi\partial\bar{\xi}) \right) \frac{d\nu \wedge d\bar{\nu}}{(1+\xi\bar{\xi})^2} \right),$$

where $\partial = \partial/\partial\nu$.

This is precisely the expression for the twist of a line congruence derived in [6], which vanishes if and only if there exist surfaces S in \mathbb{R}^3 orthogonal to the line congruence (by Frobenius' theorem). \square

The canonical 1-form Θ pulled back to a Lagrangian surface Σ is closed. Thus it defines an element in the real cohomology $H^1(\Sigma, \mathbb{R})$. Locally, $\Theta = dr$, where r is a real function on Σ . In fact, the surfaces in \mathbb{R}^3 orthogonal to the line congruence are obtained locally by substituting $r = r(\nu, \bar{\nu})$ in (3.1) and (3.2). The evaluation of Θ on a closed curve in Σ is equal to the jump of r as one goes around a curve on the orthogonal surfaces in \mathbb{R}^3 .

EXAMPLE 1. Consider the Lagrangian torus in \mathbb{T} given by

$$\xi = \tan \phi e^{i\theta}, \quad \eta = \pm a(1 - b \tan^2 \phi) e^{i\theta},$$

for $\theta, \phi \in [0, \pi)$. For $b = 1$ this is the normal congruence to the rotationally symmetric torus in \mathbb{R}^3 with core radius $2a$. For $b \neq 1$ it is a rotationally symmetric torus 'torn' along an equilateral. The jump in r as one traverses a meridian is equal to $2\pi a(1 - b)$.

Turning to the metric \mathbb{G} we have the following.

Proof of Theorem 3. Again, suppose that Σ is given parametrically by $f : \Sigma \rightarrow \mathbb{T}$, $\nu \mapsto (\xi(\nu, \bar{\nu}), \eta(\nu, \bar{\nu}))$. Then, pulling back \mathbb{G} to Σ ,

$$f_*\mathbb{G} = \operatorname{Im} \left(\frac{4}{(1+\xi\bar{\xi})^2} \left[\left(\partial\eta\bar{\partial}\bar{\xi} - \partial\bar{\eta}\bar{\partial}\xi + 2\frac{(\xi\bar{\eta} - \bar{\xi}\eta)}{1+\xi\bar{\xi}}\partial\xi\bar{\partial}\bar{\xi} \right) d\nu^2 \right. \right. \\ \left. \left. + \left(\partial\eta\bar{\partial}\bar{\xi} + \bar{\partial}\eta\bar{\partial}\bar{\xi} - 2\frac{\bar{\xi}\eta}{1+\xi\bar{\xi}}(\partial\xi\bar{\partial}\bar{\xi} - \bar{\partial}\xi\partial\bar{\xi}) \right) d\nu d\bar{\nu} \right] \right).$$

By the expressions derived in [6] this is equivalent to

$$f_*\mathbb{G} = \operatorname{Im} \left(\frac{4}{K(1+\xi\bar{\xi})^2} \left[((\rho - \bar{\rho})\partial\xi\bar{\partial}\bar{\xi} + \sigma(\partial\xi)^2 - \bar{\sigma}(\partial\bar{\xi})^2) d\nu^2 \right. \right. \\ \left. \left. + (\rho(\partial\xi\bar{\partial}\bar{\xi} + \bar{\partial}\xi\partial\bar{\xi}) + 2\sigma\partial\xi\bar{\partial}\bar{\xi}) d\nu d\bar{\nu} \right] \right), \quad (6.1)$$

where K is the (generalised) curvature of the congruence. The determinant of this matrix is proportional to $\lambda^2 - |\sigma|^2$, where λ is the imaginary part of ρ , and the theorem follows. \square

EXAMPLE 2. Suppose that Σ is a Lagrangian surface so that λ is zero, then Theorem 3 implies that the induced metric is either Lorentz or totally null. In this case the geometric significance of the metric is as follows.

Equation (6.1) implies that the null directions of the metric are given by the argument of the shear. According to the results in [6] these are the principal directions of the orthogonal surface S in \mathbb{R}^3 . Thus the null curves on Σ correspond to the lines of curvature on S . Similarly, the totally null points on Σ are the shear-free lines in the congruence and these are the umbilical points on S .

EXAMPLE 3. Suppose that Σ is holomorphic. Then the induced metric is either positive definite or totally null. To construct examples, note that a vector field on S^2 gives rise to a surface Σ in \mathbb{T} and consider the vector field generated on the unit sphere in \mathbb{R}^3 by a rotation about the x^3 -axis. This is the line congruence given by the global holomorphic section $\xi \mapsto (\xi, \eta = -bi\xi)$ for $b \in \mathbb{R}^3$. The congruence is twisting everywhere except along the equator $|\xi| = 1$. The induced metric is

$$\mathbb{G}_\Sigma = \frac{4b(1 - \xi\bar{\xi})}{(1 + \xi\bar{\xi})^3} d\xi d\bar{\xi},$$

which is positive definite except along the equator, where it is totally null. The northern and southern hemispheres are overtwisted discs [3].

EXAMPLE 4. The only line congruences that are both Lagrangian and holomorphic are the normal congruences to round spheres and flat planes in \mathbb{R}^3 .

7. The Keller–Maslov index

Given a surface Σ in a symplectic 4-manifold with compatible tamed complex structure, it is well known that one can define the Keller–Maslov index of a totally real curve on Σ [1]. We will now mirror this construction on Lagrangian surfaces in \mathbb{T} , where we will use the complex structure \mathbb{J} , despite the fact that it is not tamed by the symplectic structure Ω .

Let Λ_γ be the Grassmanian of 2-planes at $\gamma \in \mathbb{T}$. The Gauss map of Σ takes a point $\gamma \in \Sigma$ to an element $T_\gamma \Sigma \in \Lambda_\gamma$. Let the subgroup of matrices in $\mathrm{GL}(2, \mathbb{C})$ which preserve $T_\gamma \Sigma$ be denoted by G_γ .

A point $\gamma \in \Sigma$ is *complex* if \mathbb{J} preserves $T_\gamma \Sigma$ and such points, when isolated, have an associated multiplicity. In the case of \mathbb{T} , a point γ is complex with respect to \mathbb{J} if and only if the associated line in the congruence has vanishing shear [6].

Using the indefinite metric \mathbb{G} we reduce the group $\mathrm{GL}(2, \mathbb{C})$ to $U(1, 1) = \mathrm{PSL}(2, \mathbb{R}) \times S^1$ and, as proved in Theorem 3, at a Lagrangian point, the metric is either Lorentz or totally null, the latter occurring only at complex points. Thus, away from complex points, G_γ can be reduced to $O(1, 1)$ and $\pi_1(\Lambda_\gamma/G_\gamma)$ is isomorphic to \mathbb{Z} . The Keller–Maslov index of a closed oriented totally real curve on a Lagrangian surface Σ is the winding number of $T_\gamma \Sigma$ within Λ_γ .

THEOREM 4. *The Keller–Maslov index of a curve on a Lagrangian surface in \mathbb{T} counts the number of isolated complex points of Σ inside the curve.*

Proof. A complex point is a shear-free line, or, as Σ is Lagrangian, an umbilical point on the orthogonal surface S in \mathbb{R}^3 [6]. By the definition above, the

Keller–Maslov index measures the rotation of the null directions on the Lorentz surface Σ , or equivalently, the rotation of the principal foliation on S .

Thus the index counts (with multiplicity) the number of shear-free lines inside the curve on Σ . \square

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