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Chapter IV. Symplectic Geometry

§1. The Darboux-Weinstein theorem.

In this chapter we shall collect various facts about the geometry of symplectic manifolds and of their Lagrangian submanifolds which will be of use to us later. Recall that a symplectic manifold is a manifold X together with a non-degenerate closed two form, ω. The first basic fact about symplectic manifolds is that, locally, all symplectic manifolds of the same finite dimension, n, look the same. A beautiful proof of this theorem, together with a strong generalization of it, has been recently given by Weinstein [8]. The method of proof is quite similar to the proof we gave for Morse's lemma and is one that we shall have occasion to use again several times.

Let X be a manifold and Y an embedded submanifold. If σ is a differential form on X, we shall let $\sigma_{|Y}$ denote the restriction of σ to $(\wedge\ TX)_{|Y}$. (Thus $\sigma_{|Y}$ can be evaluated on vectors which are not necessarily tangent to Y.)

THEOREM 1.1 (*Darboux-Weinstein*). *Let Y be a submanifold of X and let ω_0 and ω_1 be two non-singular closed two forms on X such that $\omega_{0|Y} = \omega_{1|Y}$. Then there exists a neighborhood, U, of Y and a diffeomorphism $f\colon U \to X$ such that*

(i) *$f(y) = y$ for all $y \in Y$,*

(ii) *$f^*\omega_1 = \omega_0$.*

If we take Y to be a point, the theorem asserts that if the two forms agree on the tangent space at a point, then, up to a diffeomorphism, f, they agree in a neighborhood of the point. For finite dimensional vector spaces, all non-degenerate anti-symmetric forms are equivalent up to linear transformation (see, for example, [2, Chapter 1]). Extend this linear transformation to some neighborhood, (by some exponential map, say, i.e., by using the linear coordinates in some neighborhood). Thus, the theorem implies that given any pair ω_0 and ω_1 defining a symplectic structure on X then near any point p there is a diffeomorphism g with $g(p) = p$ and $g^* \omega_1 = \omega_0$. This is the content of the Darboux theorem.

For the proof of Theorem 1.1 we need a basic formula of differential calculus which we now recall. Let W and Z be differentiable manifolds and let φ_t: $W \to Z$ be a smooth one parameter family of maps of W into Z. In other words the map φ: $W \times I \longrightarrow Z$ given by $\varphi_t(w) = \varphi(w, t)$ is smooth. Then we let ξ_t denote the tangent field along φ_t, i.e., ξ_t: $W \longrightarrow TZ$ is defined by letting $\xi_t(w)$ be the tangent vector to the curve $\varphi(w, \cdot)$ at t. If σ is a differential $k + 1$ form on Z, then $\varphi_t^*(\xi_t \lrcorner \sigma)$ is a well defined differential k form on W given by

$$\varphi_t^*(\xi_t \lrcorner \sigma)(\eta_1, \ldots, \eta_k) = (\xi_t(w) \lrcorner \sigma)(d\varphi_t \eta_1, \ldots, d\varphi_t \eta_k).$$

(Notice that since ξ_t is not a vector field on Z the expression $\xi_t \lrcorner \sigma$ does not define a differential form on Z.)

Let σ_t be smooth one parameter family of forms on Z. Then $\varphi_t^* \sigma_t$ is a smooth family of forms on W and the basic formula of the differential calculus of forms asserts that

$$\frac{d}{dt}\varphi_t^* \sigma_t = \varphi_t^* \frac{d\sigma_t}{dt} + \varphi_t^*(\xi_t \lrcorner d\sigma_t) + d\varphi_t^*(\xi_t \lrcorner \sigma_t). \tag{1.1}$$

For the sake of completeness we shall present a proof of this formula at the end of this section.

Let $Y \subset X$ be an embedded submanifold and suppose that there exists a smooth retraction, φ_t, of X onto Y. Thus we assume that φ_t is a smooth family of maps of $X \to X$ such that

$$\varphi_0 : X \to Y, \qquad \varphi_1 = \text{id}$$

and

$$\varphi_t y = y \qquad \text{for all } y \in Y \text{ and all } t.$$

(Notice that if X were a vector bundle and Y were the zero section then multiplication by t would provide such a retraction. Also if X were a convex open neighborhood of the zero section. By choosing a Riemann metric and using the exponential map on the normal bundle of Y, we can thus arrange that some neighborhood of Y has a differentiable retraction onto Y.) Then, for any form σ on X we have (in some neighborhood of Y)

$$\begin{aligned}\sigma - \varphi_0^* \sigma &= \int_0^1 \frac{d}{dt}(\varphi_t^* \sigma)\, dt = \int_0^1 (\varphi_t^*(\xi_t \lrcorner d\sigma))\, dt + d\int_0^1 (\varphi_t^*(\xi_t \lrcorner \sigma))\, dt \\ &= I d\sigma + d I\sigma\end{aligned}$$

where we have set

$$I\beta = \int_0^1 [\varphi_t^*(\xi_t \lrcorner \beta)]\, dt$$

for any form β on X. In other words I: $\wedge^k(X) \to \wedge^{k-1}(X)$ and

$$\sigma - \varphi_0^* \sigma = dI\sigma + Id\sigma. \tag{1.2}$$

PROOF OF THE DARBOUX-WEINSTEIN THEOREM. Set

$$\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + t\sigma \quad \text{where} \quad \sigma = \omega_1 - \omega_0.$$

Notice that

$$\sigma_{|Y} = 0$$

so that, in particular,

$$\varphi_0^* \sigma = 0 \qquad \text{and} \qquad d\sigma = 0.$$

Hence, by (1.2),

$$\sigma = d\beta \qquad \text{where } \beta = I\sigma.$$

Notice that

$$\beta_{|Y} = 0.$$

Now

$$\omega_{t|Y} = \omega_{0|Y} = \omega_{1|Y}$$

and so $\omega_{t|Y}$ is non-degenerate for all $0 \leqslant t \leqslant 1$. We can therefore find some neighborhood of Y on which ω_t is non-degenerate for all $0 \leqslant t \leqslant 1$. We can therefore find a vector field $\boldsymbol{\eta}_t$ such that

$$\boldsymbol{\eta}_t \lrcorner \omega_t = -\beta. \tag{1.3}$$

We can integrate the vector field $\boldsymbol{\eta}_t$ to obtain a one parameter family of maps, f_t, whose tangent vector is $\boldsymbol{\eta}_t$. Notice that $f_{t|Y} = \text{id}$. By restricting to a smaller neighborhood of Y we may assume that f_t is also defined for all $0 \leqslant t \leqslant 1$. (Strictly speaking, in proving this fact, we may want $\boldsymbol{\eta}_t$, etc. to be defined for some range of $t > 1$.) Then $f_0 = \text{id}$ and, by (1.1) and the fact that

$$\frac{d}{dt}\omega_t = \sigma,$$

we see that

$$f_1^* \omega_1 - \omega_0 = \int_0^1 \frac{d}{dt}(f_t^* \omega_t)\, dt = \int_0^1 f_t^*(\sigma + d(\eta_t \lrcorner \omega_t))\, dt = 0$$

since $d\omega_t = 0$. Thus f_1 provides the desired diffeomorphism, proving the theorem.

Let Λ be an embedded Lagrangian submanifold of a symplectic manifold X. As an example, let $X = T^* M$ and let Λ be the zero section of $T^* M$. It is an observation due to Kostant, that, locally, this is the only example.

PROPOSITION 1.1 (*Kostant*). *Let Λ be an embedded Lagrangian submanifold of a symplectic manifold X, whose sympectic form is ω. Let Λ also be regarded as the zero section of $T^* \Lambda$ and let ω' be the symplectic form on $T^* \Lambda$. Then there exists a neighborhood, U, of Λ in X and a diffeomorphism h of U into $T^* \Lambda$ such that $h_{|\Lambda} = \mathrm{id}$ and $h^* \omega' = \omega$.*

The proof of the proposition will use Theorem 1.1 and an algebraic fact concerning Lagrangian subspaces of a symplectic vector space which we shall prove in the next section. The algebraic fact is as follows: Let V be a symplectic vector space, and let Z be a Lagrangian subspace of V. Then the set of all Lagrangian subspaces, W, such that $W \cap Z = \{0\}$ is an affine space. For the precise statement, see Proposition 2.3 below. For us this fact has the following consequence.

We can find a smooth bundle, E, of Lagrangian subspaces of $TX_{|\Lambda}$ such that $E_\lambda \cap T\Lambda_\lambda = \{0\}$ for all $\lambda \in \Lambda$.

In fact the bundle of all Lagrangian subspaces of $TX_{|\Lambda}$ which have zero intersection with $T\Lambda$ is an affine bundle by the above algebraic fact, and hence has a smooth section. (Just choose sections locally and patch together by averaging, using a partition of unity, i.e., give s_i locally and let $s = \sum \phi_i s_i$ where (ϕ_i) is a suitable partition of unity. Averaging makes sense in an affine space.) Now once we have fixed E, this determines an isomorphism, for each $\lambda \in \Lambda$, of TX_λ with $T\Lambda_\lambda \oplus T^* \Lambda_\lambda$, since E_λ is naturally dual to $T\Lambda_\lambda$. Now if we regard Λ as the zero section of $T^* \Lambda$ then the tangent space to $T^* \Lambda$ at λ splits into a direct sum of the tangent to the fiber and the tangent to the zero section. Now the fiber is a vector space, so we may identify the tangent to the fiber with $T^* \Lambda_\lambda$. In this way we have an identification

$$T(T^* \Lambda)_\lambda = T\Lambda_\lambda \oplus T^* \Lambda_\lambda .$$

We have thus an isomorphism of TX_λ with $T(T^* \Lambda)_\lambda$ which clearly preserves the symplectic structure and varies smoothly with λ. In other words we have a map of vector bundles $TX_{|\Lambda} \to T(T^* \Lambda)_{|\Lambda}$ which is an isomorphism of symplectic structures. We now choose some diffeomorphism, g, of some neighborhood of Λ in X into $T^* \Lambda$ such that $g_{|\Lambda} = \mathrm{id}$ and dg is the isomorphism constructed above on $TX_{|\Lambda}$. (This is always possible by using some exponential map. Notice that we do not yet require g to have any properties relative to the symplectic structure.) Then let $\omega_1 = g^*\omega'$. By construction

$$\omega_{1|\Lambda} = \omega_{|\Lambda}$$

and therefore by Theorem 1.1 there exists an f mapping some neighborhood of Λ in X into X with $f^*\omega_1 = \omega$. Thus $f^*g^*\omega' = \omega$ and $h = g \circ f$ is the desired diffeomorphism.

Let us now give a proof of (1.1). We first prove the formula in the special case where $W = Z = M \times I$ and φ_t is the map $\psi_t\colon M \times I \to M \times I$ given by

$$\psi_t(x, s) = (x, s + t).$$

The most general differential form on $M \times I$ can be written as

$$ds \wedge a + b$$

where a and b are forms on M which may depend on t and s. (In terms of local coordinates, $s, x^1, \ldots, x^n$, these forms are sums of terms which look like

$$c dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where c is a function of t, s and x.) To show the dependence on x and s we shall rewrite the above expression as

$$\sigma_t = ds \wedge a(x, s, t)dx + b(x, s, t)dx.$$

With this notation it is clear that

$$\psi_t^*\sigma_t = ds \wedge a(x, s + t, t)dx + b(x, s + t, t)dx$$

and therefore

$$\frac{d\psi_t^*\sigma_t}{dt} = ds \wedge \frac{\partial a}{\partial s}(x, s + t, t)dx + \frac{\partial b}{\partial s}(x, s + t, t)dx$$
$$+ ds \wedge \frac{\partial a}{\partial t}(x, s + t, t)dx + \frac{\partial b}{\partial t}(x, s + t, t)dx,$$

so that

$$\frac{d\psi_t^*\sigma_t}{dt} - \psi_t^*\left(\frac{d\sigma_t}{dt}\right) = ds \wedge \frac{\partial a}{\partial s}(x, s + t, t)dx + \frac{\partial b}{\partial s}(s, s + t, t)dx. \qquad \text{(a)}$$

It is also clear that in this case the tangent to $\psi_t(x, s)$ is $\partial/\partial s$ evaluated at $(x, s + t)$.

In this case $\partial/\partial s$ is a vector field and

$$\frac{\partial}{\partial s} \lrcorner \sigma_t = adx$$

so

$$\psi_t^*\left(\frac{\partial}{\partial s}\lrcorner\sigma_t\right) = a(x, s + t, t)dx$$

and therefore

$$d\psi_t^*\left(\frac{\partial}{\partial s}\lrcorner\sigma_t\right) = \frac{\partial a}{\partial s}(x, s + t, t)ds \wedge dx + d_x a(x, s + t, t)dx \tag{b}$$

(where d_x denotes the exterior derivative of the form $a(x, s + t, t)dx$ on the manifold M, holding s fixed). Similarly,

$$d\sigma_t = -ds \wedge d_x adx + \frac{\partial b}{\partial s}ds \wedge dx + d_x bdx$$

so

$$\frac{\partial}{\partial s}\lrcorner d\sigma_t = -d_x adx + \frac{\partial b}{\partial s}dx$$

and

$$\psi_t^* \frac{\partial}{\partial s}\lrcorner d\sigma_t = -d_x a(x, s + t, t)dx + \frac{\partial b}{\partial s}(x, s + t, t)dx. \tag{c}$$

Adding (a), (b), (c) proves (1.2) for ψ_t.

Now let φ: $W \times I \to Z$ be given by

$$\varphi(w, s) = \varphi_s(w).$$

Then the image under φ of the lines parallel to I through w in $W \times I$ is just the curves $\varphi_s(w)$ in Z. In other words

$$d\varphi\left(\frac{\partial}{\partial s}\right)_{(w,t)} = \xi_t(w).$$

If we let ι: $W \to W \times I$ be given by

$$\iota(w) = (w, 0)$$

then we can write the map φ_t as

$$\varphi \circ \psi_t \circ \iota.$$

Thus

$$\varphi_t^* \sigma_t = \iota^* \psi_t^* \varphi^* \sigma_t$$

and, since ι and φ do not vary with t,

$$\frac{d}{dt}\varphi_t^* \sigma_t = \iota^* \frac{d}{dt}\psi_t^* (\varphi^* \sigma_t).$$

At the point w, t of $W \times I$, we have

$$\frac{\partial}{\partial s} \lrcorner \varphi^* \sigma_t = (d\varphi)^* \left\{ \left(d\varphi \frac{\partial}{\partial s} \lrcorner \sigma_t \right) \right\} = (d\varphi)^* \xi_t \lrcorner \sigma_t$$

and thus

$$\iota^* \psi_t^* \left(\frac{\partial}{\partial s} \lrcorner \varphi^* \sigma_t \right) = \iota^* \psi_t^* \varphi^* (\xi_t \lrcorner \sigma_t) = \varphi_t^* (\xi_t \lrcorner \sigma_t).$$

Substituting into the formula for

$$\frac{d\psi_t^*}{dt} \varphi^* \sigma_t$$

yields (1.1).

§2. Symplectic vector spaces.

In this section we list various facts concerning the geometry of symplectic vector spaces. Let V be a vector space and $(,)$ an antisymmetric bilinear form on V. If the form $(,)$ is non-singular, then V, together with $(,)$, is called a *symplectic vector space*. If $(,)$ is singular, then we set

$$\begin{aligned} V^\perp &= \{v \mid (v, w) = 0 \quad \text{all } w \in V\} \\ &= \{v \mid (w, v) = 0 \quad \text{all } w \in V\} \end{aligned}$$

and it is clear that we get an induced bilinear form on $V/V^\perp$ and that $V/V^\perp$ is a symplectic vector space.

Let V be a symplectic vector space. The symplectic group $Sp(V)$ consists of all non-singular linear transformations, B, such that

$$(Bu, Bv) = (u, v)$$

for all u, $v \in V$. The conformal symplectic group $CSp(W)$ consists of those non-singular linear transformations satisfying

$$(Bu, Bv) = \mu_B(u, v) \qquad \forall u, v \in V,$$

where μ_B is some scalar depending on B. The corresponding Lie algebras are the symplectic algebra, $sp(V)$, consisting of those $A \in \text{Hom}\,(V, V)$ satisfying

$$(Au, v) + (u, Av) = 0 \qquad \forall u, v \in V$$

and the conformal symplectic algebra, $csp(V)$, consisting of those A which satisfy

$$(Au, v) + (u, Av) = \mu_A(u, v) \qquad \forall u, v \in V. \tag{2.1}$$

If $A \in csp(V)$ then $A - \frac{1}{2}\mu_A I \in sp(V)$, and so $csp(V) = sp(V) + Z$, where the center, Z, consists of all multiples of the identity transformation.

If V is a real symplectic vector space then its complexification, $V^{\mathbf{C}} = V \otimes \mathbf{C}$ is easily seen to be a complex symplectic vector space with the obvious bilinear form:

$$(x + iy, u + iv) = (x, u) - (y, v) + i\{(x, v) + (y, u)\}.$$

Let $A \in sp(V)$, where V is a finite dimensional symplectic vector space. Then for any scalar λ we have

$$([A - \lambda]u, v) = -(u, [A + \lambda]v) \qquad \forall u, v \in V$$

and therefore

$$([A - \lambda]^k u, v) = (-1)^k (u, [A + \lambda]^k v) \qquad \forall u, v \in V.$$

Let V_λ denote the generalized eigenspace of A corresponding to the eigenvalue λ. Thus V_λ consists of those $u \in V$ such that

$$[A - \lambda]^k u = 0$$

for sufficiently large k. Thus $u \in V_\lambda$ if and only if $u \in ([A + \lambda]^k V)^\perp$. In particular $\dim V_\lambda = \dim([A + \lambda]^k V)^\perp = \dim V_{-\lambda}$. Also, if $0 \neq u$ is an eigenvector:

$$(A - \lambda)u = 0$$

then $u \in ((A + \lambda)V)^\perp$, and the set of eigenvectors with eigenvalue λ is paired, under $(\,,)$, with the eigenvectors corresponding to the eigenvalues $-\lambda$. If λ is complex and V is real, we can apply the same results to A acting on $V^{\mathbf{C}}$. If $A \in csp(V)$ we can apply the above to $A - \frac{1}{2}\mu_A I \in sp(V)$. We thus obtain:

PROPOSITION 2.1. *Let $A \in csp(V)$. Then the eigenvalues of A are symmetric about $\frac{1}{2}\mu_A$. That is, if λ is an eigenvalue of A then so is $\mu_A - \lambda$ and*

$$\dim(V_\lambda^{\mathbf{C}}) = \dim(V_{\mu_A - \lambda}^{\mathbf{C}}).$$

In fact V_λ and $V_{\mu_A - \lambda}$ are non-singularly paired under $(\,,)$. Also the eigenspaces corresponding to λ and $\mu_A - \lambda$ are non-singularly paired under $(\,,)$.

A subspace $X \subset V$ is called *Lagrangian* if it is maximally isotropic. Thus $(u_1, u_2) = 0$ if $u_i \in X$ and X is maximal with respect to this property.

Let X be a fixed Lagrangian subspace and let Y be a second Lagrangian subspace such that $X \cap Y = \{0\}$. Then X and Y are non-singularly paired by $(\,,)$. Let P denote the projection of V onto Y with kernel X so that

$$0 \to X \to V \xrightarrow{P} Y \to 0.$$

Then $P \in csp(V)$ and $\mu_P = 1$. Indeed, we must show that

$$(Pu, v) + (u, Pv) = (u, v).$$

As X and Y span V we need only consider three cases: $u, v \in X$; $u \in X, v \in Y$; and $u, v \in Y$. If $u, v \in X$ then $Pu = Pv = 0$ and the right hand side vanishes since X is isotropic. If $u \in X$ and $v \in Y$ the equation becomes $(u,v) = (u,v)$ while if u and v both lie in Y both sides are zero. Conversely, let $P \in csp(V)$ satisfy $\mu_P = 1$ and $P_{|X} \equiv 0$. Then according to Proposition 2.1, P must have an eigenspace Y corresponding to the eigenvalue $\lambda = 1$ whose dimension is equal to dim X. Obviously $X \cap Y = \{0\}$ and, if $u, v \in Y$ we have $(u,v) = (Pu,v) + (u,Pv) = 2(u,v)$ so $(u,v) = 0$; thus Y is Lagrangian. Thus, the set of Lagrangian subspaces Y with $Y \cap X = \{0\}$ is in one to one correspondence with the set of $P \in csp(V)$ such that $\mu_P = 1$ and $P_{|X} = 0$.

Given any element, P, of $csp(V)$ we obtain a symmetric bilinear form Q_P on V by setting

$$Q_P(x,y) = (Px,y) - \tfrac{1}{2}\mu_P(x,y). \tag{2.2}$$

Indeed

$$\begin{aligned} Q_P(y,x) &= (Py,x) - \tfrac{1}{2}\mu_P(y,x) = -(x,Py) + \tfrac{1}{2}\mu_P(x,y) \\ &= (Px,y) - \tfrac{1}{2}\mu_P(x,y) = Q_P(x,y). \end{aligned}$$

Conversely, given Q and μ_P the equation defines $P \in csp(V)$.

If $Px = 0$ for $x \in X$ then $Q_P(x,x) = 0$ for $x \in X$, while $Q_P(x,y) = -\tfrac{1}{2}(x,y)$ is a non-singular pairing between X and Y if $X \cap Y = \{0\}$ and $\mu_P = 1$. Thus Q_P has rank n. Conversely, let Q_P be any symmetric quadratic form such that

$$Q_P(x,v) = -\tfrac{1}{2}(x,v) \qquad \forall x \in X, v \in V.$$

Then we get a $P \in csp(V)$ with $\mu_P = 1$ and $(Px,y) \equiv 0$ for $x \in X$ and y arbitrary so that $Px = 0$ for $x \in X$. We have thus established

PROPOSITION 2.2. *Let X be a fixed Lagrangian subspace. Then the following sets are in one to one correspondence*:

(i) *The set of all Lagrangian subspaces Y such that $Y \cap X = \{0\}$.*

(ii) *The set of all $P \in csp(V)$ such that $\mu_P = 1$ and $P_{|X} \equiv 0$.*

(iii) *The set of all symmetric quadratic forms, Q, on V, such that*

$$Q(x,v) = -\tfrac{1}{2}(x,v) \qquad \forall x \in X,\ v \in V. \tag{2.3}$$

Here $Y = \ker(P - I)$ *while P and Q are related by* (2.2). *We shall denote the space* (i) *by* $\mathfrak{L}_X$.

The third description shows that the space in question has the structure of an affine space whose associated vector space is $S^2(V/X)$. Indeed, let Q_1 and Q_2 be two symmetric forms on V which satisfy (2.3). Then $Q_1 - Q_2 = H$ is a symmetric form on V such that $H(x,v) \equiv 0$ for $x \in X$ and all v. Thus H defines a symmetric bilinear form on V/X. Conversely, $S^2(V/X)$ can be considered as the space of symmetric bilinear forms H on V such that $H(x,v) \equiv 0$ for $x \in X$. Then $Q + H$ satisfies (2.3) if Q does. Thus we have proved:

PROPOSITION 2.3. *Let X be a fixed Lagrangian subspace. Then the space of all Lagrangian subspaces transversal to X is an affine space whose associated linear space is* $S^2(V/X)$. *In particular, if we fix a transversal Lagrangian subspace Y then* $\mathfrak{L}_X$ *becomes identified with* $S^2(Y)$, *since we may identify* V/X *with Y. If W is some other element of* $\mathfrak{L}_X$ *then the quadratic form associated with W on Y is given by*

$$H(y_1,y_2) = (P_W y_1, y_2) \tag{2.4}$$

where P_W *is the projection of V onto W along X.*

The element P_W described in Proposition 2.3 is the projection described by the exact sequence

$$0 \to X \to V \xrightarrow{P_W} W \to 0.$$

The quadratic form, Q_Y, associated to Y by Proposition 2.2 vanishes on Y so that the H defined by (2.4) does indeed satisfy

$$H = (Q_W - Q_Y)_{|Y}.$$

Notice that

$$H \text{ is non-singular if and only if } Y \cap W = \{0\}. \tag{2.5}$$

Indeed if $y \in W \cap Y$ then $P_W y = P_Y y = y$ and thus $Q_W(y,v) = Q_Y(y,v)$ for all $v \in V$ and hence $H(y,v) \equiv 0$. On the other hand, since $X \cap Y = \{0\}$ we know that P: $Y \to W$ is a isomorphism. If $W \cap Y = \{0\}$ then W and Y are non-singularly paired under (,). Thus (2.4) defines a non-singular pairing.

Since $S^2(Y)$ has plenty of non-singular elements we conclude that there always is a Lagrangian subspace W transversal to two given Lagrangian subspaces X and Y, at least if X and Y are transversal to each other. Of course, if X and Y are not transversal to each other it should be even "easier" to pick a W transversal to both. Indeed consider the subspace $X + Y$ and the restriction of $(\,,)$ to it. The only elements of V which are orthogonal to X and to Y must be in $X \cap Y$ so that $(X + Y)/X \cap Y$ is a symplectic vector space. We can therefore find a subspace $W_0 \subset X + Y$ whose dimension equals that of $X/(X \cap Y)$ such that W_0 is totally isotropic and $W_0 \cap X = W_0 \cap Y = \{0\}$. If we choose a basis, $q_1, \ldots, q_r$ of $X \cap Y$ and a basis $p_{r+1}, \ldots, p_n$ of W_0 then the $q_1, \ldots, q_r, p_{r+1}, \ldots, p_n$ span a Lagrangian subspace. We can choose a dual basis $p_1, \ldots, p_r, q_{r+1}, \ldots, q_n$. Then the space W spanned by $p_1, \ldots, p_n$ clearly has the desired properties. We have thus proved:

PROPOSITION 2.4. *Given any pair X and Y of Lagrangian subspaces it is always possible to find a third Lagrangian subspace transversal to both.*

Let us return to the situation described by the pair of transversal Lagrangian subspaces X and W. If Y is a second Lagrangian subspace transversal to X then

$$P_Y - P_W \in sp(V)$$

since both P_Y and $P_W \in csp(V)$ and $\mu_{P_Y} = \mu_{P_W} = 1$. Furthermore, we claim that

$$(P_Y - P_W)^2 = 0. \tag{2.6}$$

Indeed, for $x \in X$ we have $P_Y x = P_W x = 0$; while for $w \in W$ we have

$$(P_Y - P_W)w = P_Y w - w \in X.$$

Since X and W span V this proves (2.6). Thus

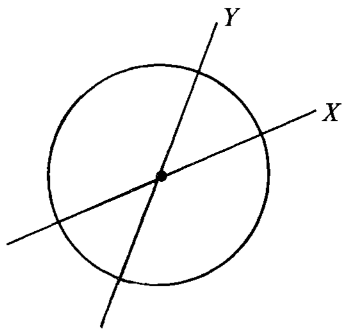
$$\exp(P_Y - P_W) = 1 + (P_Y - P_W) \in Sp(V) \tag{2.7}$$

is the transformation which is the identity on X and maps W into Y. In (2.7) we have used the exponential map in the symplectic group. For reasons that will become clear later on, we will want to consider covering groups of the symplectic group (in particular the double covering). The exponential map again is well defined (but not given by the right hand side of (2.7)). We record these results as

PROPOSITION 2.5. *Let X be a Lagrangian subspace. If W and Y are two Lagrangian subspaces transversal to X then $P_Y - P_W \in sp(V)$ and $(P_Y - P_W)^2 = 0$ where P_Z denotes projection onto Z along X (with $Z = Y$ or W). The map $1 + P_Y - P_W$ is the identity on X, it carries W into Y, and lies in $Sp(V)$. If G is any covering group of $Sp(V)$ then* Exp $(P_Y - P_W)$ *lies in G and covers $1 + P_Y - P_W$, where* Exp *is the exponential map: $sp(V) \to G$ for the group G. In this way we have associated an element of G to each pair of Lagrangian subspaces transversal to X.*

Let us now examine the structure of the space of all Lagrangian subspaces of a real symplectic space, V. We will denote the set of all Lagrangian subspaces by $L(V)$. If $V = 2$, then any one dimensional subspace is Lagrangian. Thus $L(V)$ is just the one dimensional projective space, which is, topologically, a circle. In particular, $H^1(L(V)) = \mathbf{Z}$.

Notice that if we pick one Lagrangian subspace, X, then $L(V) - \{X\} = \mathfrak{L}_X$ consists of the projective line with a point omitted: If we consider X as the "point at infinity" $\mathfrak{L}_X$ becomes the affine line.



Fixing another Y determines the origin of the affine line, and hence a linear structure. To actually visualize $L(V)$ as a circle, we put a Riemann metric and an orientation on V. This makes V into a one dimensional complex vector space. Then each line will determine two points on the unit circle. If we let $U(1)$, the one dimensional unitary group, decribing the unit circle, and $O(1) = \{+1, -1\}$ be the subgroup of $U(1)$ consisting of the orthogonal group of the line (i.e., the real unitary group) then

$$L(\mathbf{R}^2) \sim U(1)/O(1).$$

The identification, of course, depends on the choice of Riemann metric.

We can perform the same construction in general: Suppose that V is a complex vector space with a Hermitian scalar product $\langle\,,\rangle$. Let

$$\langle v, w\rangle_R = \mathrm{Re}\,\langle v, w\rangle$$

and

$$\langle v, w\rangle_I = \mathrm{Im}\,\langle v, w\rangle.$$

Then V is a $2n$ dimensional real space and $\langle v, w\rangle_I$ is clearly an anti-symmetric form which is non-singular, hence a symplectic form. If Z is a Lagrangian subspace of V then we claim that Z is orthogonal to iZ with respect to $\langle\,,\rangle_R$. Indeed, for v and $w \in Z$ we have

$$\mathrm{Re}\langle v, iw\rangle = -\mathrm{Re}\langle iv, w\rangle = -\langle v, w\rangle_I = 0.$$

If U is any unitary transformation, then, by definition, it preserves $\langle\,,\rangle$ and hence $\langle\,,\rangle_I$ and so defines a symplectic transformation. Thus U acts on $L(V)$. We claim that it acts transitively on $L(V)$. Indeed, let X and X' be Lagrangian subspaces and let $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ be orthonormal bases of X and X' with respect to $\langle\,,\rangle_R$. Since $\langle e_i, e_j\rangle_I = 0$ we conclude that $\{e_1, \ldots, e_n\}$ is an orthonormal basis for the Hermitian structure as well, and the same for $\{e'_1, \ldots, e'_n\}$. Hence there is a unitary transformation with $U(e_i) = e'_i$, for all i, and therefore $UX = X'$. The set of unitary transformations keeping X fixed will be those unitaries with $Ue_i = \sum a_{ij} e_j$ where the a_{ij} are real; thus $U \in O(n)$. Thus

$$L(V) = U(n)/O(n). \tag{2.8}$$

We have derived this result starting from a Hermitian structure on V. Let us now show how to put a Hermitian structure on any symplectic vector space V (whose imaginary part gives the symplectic form) and, indeed, parametrize all such Hermitian structures.

Fix a Lagrangian subspace X. Suppose we are given a transversal Lagrangian subspace Y and a positive definite scalar product $\langle\,,\rangle_R$ on X. Then $\langle\,,\rangle_R$ determines an isomorphism of $X \to X^*$ and Y can be identified with X^* via the symplectic form. Thus we are given a map

$$X \to Y.$$

Call this map multiplication by i. Then $i(ix) = -x$ determines multiplication by i on all of V, making V into a complex vector space. Also set

$$\begin{aligned} \langle u, v\rangle &= \langle u, v\rangle_R, & u, v &\in X,\\ \langle u, y\rangle &= i(u, y), & u &\in X, y \in Y,\\ \langle y, z\rangle &= \langle iy, iz\rangle_R, & y, z &\in Y, \end{aligned}$$

and extending by linearity defines a Hermitian form, $\langle\,,\rangle$, with

$$\langle\,,\rangle_I = (\,,).$$

Conversely, starting with a Hermitian form $\langle\,,\rangle$ on V then we have already observed that iX is a Lagrangian subspace transversal to X and $\langle\,,\rangle_R$ restricted to X is positive definite. We have thus proved:

PROPOSITION 2.6. *Let V be a finite dimensional real symplectic space with form $(\,,)$. Let H be the space of Hermitian forms $\langle\,,\rangle$ (and complex structures) with $\langle\,,\rangle_I = (\,,)$. Then*

$$H \cong \mathfrak{L}_X \times P_X$$

where X is a Lagrangian subspace of X and P_X denotes the space of positive definite quadratic forms on X. In particular, since $\mathfrak{L}_X$ and P_X are both diffeomorphic to cells, we conclude that H is a cell.

Let det be the determinant function mapping $U(n) \to S^1$. It maps $O(n) \to \{\pm 1\}$ which we shall denote by S^0. We then get a well defined function $\det^2$: $U(n)/O(n) \to S^1$. Let $S[U(n)/O(n)] = (\det^2)^{-1}(1)$. Now if $\det U = \pm 1$ we can find an $O \in O(n)$ such that $\det UO = 1$. Thus $SU(n)$ acts transitively on $S[U(n)/O(n)]$ and the isotropy group is $SO(n)$. Thus we have a fibration of

$$U(n)/O(n) \xrightarrow{\det^2} S^1$$

where the fiber over each point is diffeomorphic to $SU(n)/SO(n)$. Now $SU(n)$ is simply connected and $SO(n)$ is connected. Therefore $SU(n)/SO(n)$ is also simply connected. (Indeed, any curve starting and ending at $SO(n)$ in $SU(n)/SO(n)$ is homotopic to the image of a curve starting and ending at 1 in $SU(n)$, since $SO(n)$ is connected. But any closed curve in $SU(n)$ is homotopic to the trivial curve, and thus so is its image.) Thus,

$$\pi_1(L(V)) = \mathbf{Z} \tag{2.9}$$

and, in particular,

$$H^1(L(V), \mathbf{Z}) = \mathbf{Z}. \tag{2.10}$$

We shall present an independent proof of these facts in the next section. Now $dz/2\pi iz$ is a form on S^1 which generates $H^1(S^1)$. Hence

$$(\det^2)^* \frac{dz}{2\pi iz}$$

defines a form on $L(V)$, generating $H^1(L(V))$. Now $\det^2$ is defined on $U(n)/O(n)$ and hence becomes a map on $L(V)$ only after we have identified $L(V)$ with $U(n)/O(n)$ by a choice of Hermitian metric (and Lagrangian subspace). However, by Proposition 2.6, all such choices can be smoothly deformed into one another and hence the cohomology class is independent of the choice. This class is called the Maslov class.

We now give an explicit description of the universal covering space, $\tilde{L}(V)$, of the space of all Lagrangian subspaces, $L(V)$, of a symplectic vector space, V. We shall show, following Leray, that there exists an invariant, $m(u, u')$, for any two transversal elements, u and u' of $\tilde{L}(V)$. (Here u and u' are called transversal if πu and $\pi u'$ are transversal Lagrangian subspaces, where π denotes the projection of $\tilde{L}(V)$ onto $L(V)$.) We shall relate the invariants of three transversal elements, u, u' and u'' to the signature of the quadratic form associated to the three Lagrangian subspaces, πu, $\pi u'$ and $\pi u''$, and use the invariant, m, to give an alternative description of the Maslov class. For the purposes of obtaining these results, we continue in our choice of some Hermitian metric on V (and a choice of an orthonormal basis) which allows us to identify V with $\mathbf{C}^n$. Any Lagrangian subspace, X, is of the form $X = A\mathbf{R}^n$ for some $A \in U(n)$ and

$$A\mathbf{R}^n = B\mathbf{R}^n \quad \text{if and only if } A\bar{A}^{-1} = B\bar{B}^{-1},$$

where $\bar{A}$ denotes the matrix whose entries are the complex conjugates of the entries of A. We can thus define a map v: $L(V) \Rightarrow U(n)$ by

$$v(X) = A\bar{A}^{-1} \quad \text{if } X = A\mathbf{R}^n. \tag{2.11}$$

Notice that if $B \in U(n)$ then

$$v(BX) = Bv(X)\bar{B}^{-1}. \tag{2.12}$$

If $z \in \mathbf{C}^n$ then $z = Ar$ for $r \in \mathbf{R}^n$ if and only if $z = A\bar{A}^{-1}\bar{z}$. Thus

$$z \in X \quad \textit{if and only if } z = v(X)\bar{z}. \tag{2.13}$$

If $X \cap Y \neq \{0\}$ the pair of equations $z = v(X)\bar{z}$ and $z = v(Y)\bar{z}$ has a nontrivial solution, so $v(X) - v(Y)$ is not invertible. Conversely: by applying a suitable element of $U(n)$ we may assume that $X = \mathbf{R}^n$ and $Y = A\mathbf{R}^n$. If $u = A\bar{A}^{-1}\bar{u}$ then $\bar{u} = A\bar{A}^{-1}u$ so we can find a $v \in \mathbf{R}^n$ with $v = A\bar{A}^{-1}\bar{v}$ so $v \in X \cap Y$. Thus,

$$X \textit{ and } Y \textit{ are transversal if and only if } v(X) - v(Y) \textit{ is invertible.} \tag{2.14}$$

Let $\tilde{U}(n)$ denote the space of all pairs

$$(A, \varphi),\ A \in U(n),\ \varphi \in \mathbf{R} \quad \textit{satisfying } \det A = e^{i\varphi}. \tag{2.15}$$

The multiplication $(A, \varphi) \cdot (A', \varphi') = (AA', \varphi + \varphi')$ makes $\tilde{U}(n)$ into a group and the map $\tilde{U}(n) \to U(n)$ sending (A, φ) into A makes $\tilde{U}(n)$ into a covering group of $U(n)$. The map

$$SU(n) \times \mathbf{R} \to \tilde{U}(n) \text{ sending } (B, \psi) \text{ into } (Be^{i\psi}, n\psi)$$

is easily seen to be an isomorphism. Since $SU(n)$ is simply connected, it follows that $\tilde{U}(n)$ is the universal covering group of $U(n)$. This shows that the fundamental group of $U(n)$ is $\mathbf{Z}$. This also implies that the fundamental group of $Sp(V)$ is $\mathbf{Z}$; we shall sketch a proof of this fact here, referring the reader ahead to Chapter V, §5, for the proof of some of the group theoretical facts that we will use. We first observe the following fact about the algebra $sp(V)$: Let J denote multiplication by i (in the complex structure we have introduced on V).

We have the vector space direct sum decomposition $sp(V) = u(n) \oplus \mathcal{P}$ *where,* $u(n) = \{A \in sp(V) | JAJ^{-1} = A\}$ and $\mathcal{P} = \{B \in sp(V) | JBJ^{-1} = -B\}$. *Every element of* $\mathcal{P}$ *is of the form* $B = SC$, *where C denotes complex conjugation and S is a symmetric complex $n \times n$ matrix, i.e.* $Bz = S\bar{z}$ *for all* $z \in V \sim \mathbf{C}^n$.

PROOF. Since $J^2 = -1$, the operator, Θ on $sp(V)$ consisting of conjugation by J, i.e. the operator $\Theta(A) = JAJ^{-1}$ satisfies $\Theta^2 = 1$. Thus the direct sum in question is the decomposition of $sp(V)$ into $+1$ and -1 eigenspaces for Θ. The fact that $u(n)$ consists of the $+1$ eigenspace is just the characterization of $U(n)$ as the subgroup of the symplectic group preserving the complex (and hence the Hermitian) structure. The complex conjugation, C, clearly satisfies $JCJ^{-1} = -C$. Since S is complex linear, $JSJ^{-1} = S$ and hence $JBJ^{-1} = -B$ if $B = SC$. Let us show that all such B belong to $sp(V)$. We must show that

$$\operatorname{Im}(\langle Bu, v\rangle + \langle u, Bv\rangle) = 0.$$

Now $\langle Bu, v\rangle = \langle S\bar{u}, v\rangle$ and $\langle u, Bv\rangle = \langle u, S\bar{v}\rangle = \langle S^* u, \bar{v}\rangle = \langle \bar{S}u, \bar{v}\rangle$ since S is symmetric. But $\langle \bar{S}u, \bar{v}\rangle$ is the complex conjugate of $\langle S\bar{u}, v\rangle$ so that the above equality holds. Now the dimension, over the real numbers, of the space of complex symmetric matrices is $n(n+1)$, while $\dim u(n) = n^2$ and $\dim sp(V) = n(2n+1)$. Thus, by dimension count, we see that all elements of $\mathcal{P}$ have the desired form.

It now follows (cf. §5 of Chapter V) that we have the polar decomposition: every element of $Sp(V)$ can be uniquely written as the product

$$a = u \cdot \exp B \quad u \in U(n),\ B \in \mathcal{P},$$

where exp: $sp(V) \to Sp(V)$ is the exponential map. Since $\mathcal{P}$ is contractible, this shows that $Sp(V)$ and $U(n)$ have the same fundamental group. In fact, if $\tilde{S}p(V)$ denotes the universal covering group of $Sp(V)$ the polar decomposition theorem for $\tilde{S}p(V)$ implies that every element of $\tilde{S}p(V)$ can be written in the form

$$\tilde{a} = \tilde{u} \cdot \operatorname{Exp} B \quad \tilde{u} \in \tilde{U}(n),\ B \in \mathcal{P}$$

where Exp: $sp(V) \Rightarrow \tilde{S}p(V)$ is the exponential map for $\tilde{S}p(V)$. We shall identify the element

$$g = (I, 2\pi) \in \tilde{U}(n)$$

with the generator of the fundamental group of $Sp(V)$.

Let $\tilde{L}(V)$ denote the set of all pairs (X, θ), $X \in L(V)$, $\theta \in \mathbf{R}$ satisfying

$$\det v(X) = e^{i\theta}. \tag{2.16}$$

The group $\mathbf{Z}$ acts on $\tilde{L}(V)$ by

$$k(X, \theta) = (X, \theta + 2\pi k) \quad k \in \mathbf{Z},$$

and

$$\tilde{L}(V)/\mathbf{Z} = L(V),$$

making $\tilde{L}(V)$ into a covering space of $L(V)$. The group $\tilde{U}(n)$ acts transitively on $\tilde{L}(V)$ by

$$(A,\varphi)\cdot(X,\theta) = (AX,\theta + 2\varphi). \tag{2.17}$$

In view of (2.12) this is well defined. The subgroup which leaves the element $(\mathbf{R}^n,0) \in \tilde{L}(V)$ fixed consists of all $(A,0)$ where

$$A = \bar{A} \quad \text{and} \quad \det A = 1$$

i.e. $A \in SO(n)$. Thus,

$$\tilde{L}(V) = \tilde{U}(n)/SO(n).$$

Since $\tilde{U}(n)$ is simply connected and $SO(n)$ is connected, this implies that $\tilde{L}(V)$ is simply connected. Thus

$\tilde{L}(V)$ is the universal covering space of $L(V)$.

Let $\tilde{\gamma}$ be a path in $\tilde{L}(V)$ such that $\tilde{\gamma}(0) = (X,\theta)$ and $\tilde{\gamma}(1) = (X,\theta + 2\pi)$. Let γ be the corresponding curve on $L(V)$, so that γ is a closed path starting and ending at X. It follows from (2.13) and (2.15) that

$$\int_\gamma (\det{}^2)^* \frac{dz}{2\pi iz} = 1. \tag{2.18}$$

Let $u = (X,\theta)$ and $u' = (X',\theta')$ be two elements of $\tilde{L}(V)$. We say that u and u' are transverse if X and X' are transverse Lagrangian subspaces. We now wish to define the Maslov index, $m(u,u')$, associated to a pair, u and u' of transverse elements of $\tilde{L}(V)$. For this purpose, we define the logarithm of an element A of Gl $(n,\mathbf{C})$ by the formula

$$\operatorname{Log} A = \int_{-\infty}^{0} \{(sI - A)^{-1} - (s-1)^{-1}I\}\,ds$$

where I is the unit matrix. This definition is valid for any: $A \in$ GL $(n,\mathbf{C})$ which does not have any eigenvalue on the negative real axis. It is easy to check that

$$\exp(\operatorname{Log} A) = A \quad \text{wherever } \operatorname{Log} A \text{ is defined,}$$

$$e^{tr(\operatorname{Log} A)} = \det A,$$

and

$$\operatorname{Log} A^{-1} = -\operatorname{Log} A.$$

If $u = (X,\theta)$ and $u' = (X',\theta')$ are transversal, then following Souriau [24], we define

$$m(u,u') = \frac{1}{2\pi}\{\theta - \theta' + itr \operatorname{Log}(-\upsilon(X)\upsilon(X')^{-1})\}. \tag{2.19}$$

By (2.14), $\upsilon(X) - \upsilon(X')$ is invertible, thus $-\upsilon(X)\upsilon(X')^{-1}$ does not have -1 as an eigenvalue, and hence does not have any negative real eigenvalues since it is unitary. Thus (2.19) is defined if u and u' are transverse. Since $e^{tr \operatorname{Log} A} = \det A$, it follows that

$$e^{2\pi i m(u,u')} = (-1)^n = e^{in\pi} \quad \text{where } \dim V = 2n.$$

Thus

$$m(u,u') \in \mathbf{Z} \quad \textit{if } n \textit{ is even and } m(u,u') \in \mathbf{Z} + \tfrac{1}{2} \quad \textit{if } n \textit{ is odd.} \tag{2.20}$$

Notice that

$$m(k \cdot u, k' \cdot u') = k - k' + m(u,u') \quad \textit{for } k, k' \in \mathbf{Z} \tag{2.21}$$

so that all values permitted by (2.20) are in fact taken on.

The group $Sp(V)$ is connected and acts on $L(V)$, and hence this action is covered by a unique action of $\tilde{S}p(V)$ on $\tilde{L}(V)$. If u and u' are transverse, then so are $\tilde{a}u$ and $\tilde{a}u'$, for any $\tilde{a} \in \tilde{S}p(V)$. The map $\tilde{a} \to m(\tilde{a}u, \tilde{a}u')$ is well defined, continuous, and takes values in a discrete set, and hence is constant. Thus

$$m(\tilde{a}u, \tilde{a}u') = m(u,u') \quad \text{for all } a \in \tilde{S}p(V). \tag{2.22}$$

In other words, $m(u,u')$ does not depend on the choice of complex structure, but is an invariant of $\tilde{S}p(V)$. It is clear from the definition that

$$m(u,u') + m(u',u) = 0. \tag{2.23}$$

Let X, X' and X'' be three transversal Lagrangian subspaces. By Proposition 2.3, the spaces X, X'' determine a quadratic form on X' given by

$$H(y_1,y_2) = (P_{X''}y_1,y_2) \quad y_1, y_2 \in X'$$

where $P_{X''}$ denotes projection onto X'' along X. We define

$$i(X,X',X'') = \tfrac{1}{2}\operatorname{sig} H. \tag{2.24}$$

This is a symplectic invariant assigned to any triple of transverse Lagrangian subspaces, that is

$$i(aX,aX',aX'') = i(X,X',X'') \tag{2.25}$$

for any $a \in Sp(V)$. (Recall that there is *no* invariant for pairs of transverse Lagrangian subspaces; the group $Sp(V)$ acts transitively on the set of all pairs of transverse Lagrangian subpsaces.)

Let $u = (X,\theta)$, $u' = (X',\theta')$ and $u'' = (X'',\theta'')$ be points of $\tilde{L}(V)$ sitting over X, X', and X''. We claim that the following formula

$$m(u,u') + m(u',u'') + m(u'',u) = i(X,X',X''), \tag{2.26}$$

due to Leray [23], holds.

To prove this formula we may apply any $a \in \tilde{S}p(V)$ to the u, u', and u'' on left hand sides with the corresponding $a \in Sp(V)$ to the X, X', X'' on the right. We may thus assume that $X = \mathbf{R}^n$ and that $X' = i\mathbf{R}^n$, and that $X'' = b\mathbf{R}^n$ where $b \in Sp(V)$ has the form

$$b = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}$$

relative to the basis determined by $V \sim \mathbf{R}^n \oplus i\mathbf{R}^n$, where S is a symmetric matrix. It follows from (2.4) and (2.24) that

$$i(X,X',X'') = \tfrac{1}{2}\operatorname{sig} S.$$

Finally, by applying an element $c \in Sp(n)$ of the form

$$c = \begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix} \quad A \in \mathrm{Gl}\,(n)$$

we may assume that S is a diagonal matrix with $+1$'s and -1's on the diagonal, i.e.

$$S = \operatorname{diag}(+1,\ldots,+1,-1,\ldots,-1)$$

where there are k +'s and $(n-k)$ −'s. Thus $\operatorname{sig} S = 2k - n$ and

$$i(X,X',X'') = \tfrac{1}{2}(2k - n).$$

If $\delta_1, \ldots, \delta_n$ denotes the standard basis of $\mathbf{R}^n$, then $i\delta_1, \ldots, i\delta_n$ is a basis of X', and the vectors $(1 \pm i)\delta_j$ form a basis of X'', where the choice of sign is + for the first k vectors and − for the last $n-k$. Since $\sqrt{2}\cdot e^{\pm\pi i/4} = 1 \pm i$, we may, as well, take $e^{\pm\pi i/4}\delta_j$ as the basis vectors. Thus $X'' = A\mathbf{R}^n$ where A is the diagonal unitary matrix

$$A = \operatorname{diag}(e^{\pi i/4},\ldots,e^{\pi i/4},e^{-\pi i/4},\ldots,e^{-\pi i/4}).$$

Therefore, by (2.11),

$$v(X'') = \operatorname{diag}(e^{\pi i/2},\ldots,e^{\pi i/2},e^{-\pi i/2},\ldots,e^{-\pi i/2}),$$

where, as always, there are k +'s and $(n-k)$ −'s. It is obvious that

$$v(X) = I \quad\text{and}\quad v(X') = -I.$$

Then if $u'' = (X'', \theta'')$ we must have, by (2.16)

$$\theta'' = 2\pi[q'' + \tfrac{1}{4}(2k - n)]$$

for some integer q''. Similarly,

$$\theta = 2\pi q \quad \text{and} \quad \theta' = 2\pi[q' + n/2]$$

where q and q' are integers. Thus, by (2.19),

$$m(u, u') = q - q' - n/2.$$

Also

$$\begin{aligned}\operatorname{Log}\,(-v(X')v(X'')^{-1}) &= \operatorname{Log}\operatorname{diag}\,(e^{-\pi i/2}, \ldots, e^{-\pi i/2}, e^{\pi i/2}, \ldots, e^{\pi i/2}) \\ &= \operatorname{diag}\,(-\pi i/2, \ldots, -\pi i/2, \pi i/2, \ldots, \pi i/2)\end{aligned}$$

since our choice of the Log function is the one which is given by analytic continuation from the positive real axis in both the upper and lower half planes, i.e. $\operatorname{Log} e^{\pi i/2} = \pi i/2$ and $\operatorname{Log} e^{-\pi i/2} = -\pi i/2$. Thus

$$itr\,\operatorname{Log}\,(-v(X')v(X'')^{-1}) = \tfrac{1}{2}\pi(2k - n)$$

and

$$m(u', u'') = q' + n/2.$$

Similarly,

$$itr\,\operatorname{Log}\,(-v(X'')v(X)^{-1}) = \tfrac{1}{2}\pi(2k - n)$$

so that

$$m(u'', u) = q'' - q + \tfrac{1}{2}(2k - n).$$

Adding up the three expressions for m proves Leray's formula, (2.26). Notice that it follows from (2.26) that $i(X, X', X'')$ is an antisymmetric function of its three variables. It also follows that if we define the Hormander cross index of four transverse Lagrangian subspaces X, Y, Z, and W by

$$(X, Y, Z, W) = i(X, Y, Z) - i(X, Y, W)$$

that

$$(X, Y, Z, W) = -(Z, W, X, Y).$$

Neither of these facts is immediately obvious from the definition (2.24). We shall discuss them further in the next section.

We are now in a position to discuss the relation between the Maslov class as defined earlier in this section, and the Maslov cycle as introduced in §7 of Chapter II. Let Y be a fixed Lagrangian subspace. (Eventually, Y will play the role of the "tangent to the vertical" in T^*M in a local coordinate system.) Let $X(t)$ be a curve of Lagrangian subspaces defined for $0 \leqslant t \leqslant 1$, and suppose that $X(0)$ and $X(1)$ are both transversal to Y. Let $\tilde{Y}$ be a point of $\tilde{L}(V)$ covering Y and let $u(t)$ be a curve in $\tilde{L}(V)$ covering $X(t)$. It follows from (2.21) that the integer

$$l = m(\tilde{Y}, u(1)) - m(\tilde{Y}, u(0))$$

is independent of our choices of liftings. The function $m(\tilde{Y}, u(\cdot))$ fails to be defined at precisely those values of t for which $X(t)$ is not transversal to Y. Suppose that s is an isolated point where this happens. That is, suppose $X(t')$ is transversal to Y for all $t' < s$ and sufficiently close to s and also that $X(t'')$ is transversal to Y for all $t'' > s$ and sufficiently close to s. We can use (2.26) to evaluate the jump in $m(\tilde{Y}, u(t))$ as we cross the value s in terms of the difference of signatures of quadratic forms defined downstairs on $L(V)$: In fact, let us choose some other Lagrangian subspace, Z, which is transverse to Y and to $X(t)$ for all $t' \leqslant t \leqslant t''$. This is clearly possible if t' and t'' are sufficiently close. Then, by (2.26)

$$\begin{aligned} i(Z, Y, X(t'')) - i(Z, Y, X(t')) &= m(\tilde{Y}, u(t'')) - m(\tilde{Y}, u(t')) \\ &\quad - (m(\tilde{Z}, u(t'')) - m(\tilde{Z}, u(t'))) \end{aligned}$$

for some choice of $\tilde{Z}$ sitting over Z. But $m(\tilde{Z}, u(t))$ is a continuous function on the interval $t' \leqslant t \leqslant t''$ with discrete values and hence $m(\tilde{Z}, u(t'')) = m(\tilde{Z}, u(t'))$. Hence

$$m(\tilde{Y}, u(t'')) - m(\tilde{Y}, u(t')) = i(Z, Y, X(t'')) - i(Z, Y, X(t')). \tag{2.27}$$

Now suppose that $X(1) = X(0)$, i.e. that the curve is closed. It follows from (2.21) that $l \cdot u(0) = u(1)$ and hence from (2.18) that

$$l = \int (\det{}^2)^* \frac{dz}{2\pi i z}. \tag{2.28}$$

Suppose that Λ is a Lagrangian submanifold of V. We may identify the tangent space, $T\Lambda_\lambda$, with a Lagrangian subspace of V, when we identify TV_λ with V, under the usual identification of the tangent space of a vector space with the vector space itself. Any curve, γ, on Λ then gives rise to a curve of Lagrangian subspaces defined by $X(t) = T\Lambda_{\gamma(t)}$. If γ is a closed curve then (2.28) defines an integer associated to γ; in fact, we have defined an element of $H^1(\Lambda, \mathbf{Z})$ which can be computed as an integral, (2.28) once a complex structure has been chosen, or as sum of "crossing numbers" (2.27) in terms of a fixed Lagrangian subspace,

Y. The actual class is independent of the choice of Y or of the complex structure. We shall call this the Leray class of Λ.

Similarly, suppose that M is a differentiable manifold. For each $z \in T^*M$, the tangent space $T(T^*M)_z$ is a symplectic vector space and is equipped with a preferred Lagrangian subspace, Y_z, where Y_z is tangent to the vertical, i.e. Y_z is the subspace consisting of those vectors, ζ, which satisfy $d\pi_z\zeta = 0$, where π denotes the standard projection of T^*M onto M. A choice of a Riemann metric on M puts a positive definite scalar product on Y_z, giving an identification of $L(T(T^*M))$ with $U(n)/O(n)$ so that $(\det^2)^*(dz/2\pi iz)$ is a well-defined differential form on the bundle $L(T^*M)$, where $L(T^*M)$ denotes the bundle over T^*M whose fiber over z consists of the set of all Lagrangian subspaces of $T(T^*M)_z$. If Λ is a Lagrangian submanifold of T^*M, then, once again, any curve, γ, on Λ determines a curve on $L(T^*M)$ and we can use (2.28) to define an integer, i.e. we have defined an element of $H^1(\Lambda, \mathbf{Z})$ which is the Maslov class of Λ. This class does not depend on the choice of Riemann metric, since we can continuously deform any two Riemann metrics into one another. In terms of local coordinates on M, we get a local identification of a neighborhood of T^*M with an open subset of a symplectic vector space, V, in which all the Y_z are identified with a fixed Lagrangian subspace, Y. We can then use (2.27) for the computation of this class in terms of local crossing numbers. In the next section we shall describe the definition of the Maslov class due to Hörmander using Čech theory.

§3. The cross index and the Maslov class.*

Let us begin this section by giving a somewhat different presentation of the computation of $\pi_1(L(V))$ and of the class introduced in the preceding section. We will use induction on dim V rather than the introduction of a complex structure, and we will find some applications for this alternative approach in what follows.

Let R be an isotropic subspace of V. Then $R^\perp \supset R$ and since $R = (R^\perp)^\perp$, we see that $W = R^\perp/R$ is again a symplectic vector space with

$$\begin{aligned}\dim R^\perp/R &= \dim R^\perp - \dim R \\ &= \dim V - 2\dim R\end{aligned}$$

since

$$\dim R + \dim R^\perp = \dim V.$$

Let X be any Lagrangian subspace of V. Then $X \cap R^\perp/X \cap R$ is clearly an isotropic subspace of W. We claim that it is Lagrangian, i.e., that

$$\dim(X \cap R^\perp/X \cap R) = \tfrac{1}{2}\dim W = \tfrac{1}{2}\dim V - \dim R.$$

To prove this, notice that

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* This section should be omitted on first reading.

$$\begin{aligned}\dim X \cap R^{\perp} &= \dim V - \dim(X \cap R^{\perp})^{\perp} \\ &= \dim V - \dim(X + R)\end{aligned}$$

and

$$\dim(X + R) + \dim(X \cap R) = \dim X + \dim R$$

so that

$$\begin{aligned}\dim X \cap R^{\perp}/X \cap R &= \dim X \cap R^{\perp} - \dim X \cap R \\ &= \dim V - [\dim(X + R) + \dim(X \cap R)] \\ &= \dim V - [\dim X + \dim R] \\ &= \tfrac{1}{2} \dim V - \dim R\end{aligned}$$

as $\dim X = \frac{1}{2} \dim V$. We have thus proved:

PROPOSITION 3.1. *Let R be an isotropic subspace of V. Then $W = R^{\perp}/R$ is a symplectic vector space and the map ρ defined by*

$$\rho(X) = X \cap R^{\perp}/X \cap R$$

sends $L(V) \to L(W)$.

Unfortunately, the map ρ is not continuous. For example, let us examine the map ρ for the case $V = \mathbf{R}^4$ with basis $\{e_1, e_2, f_1, f_2\}$ where

$$(e_1, e_2) = (f_1, f_2) = 0 \qquad \text{and} \qquad (e_i, f_j) = \delta_{ij},$$

and where we take

$$R = \{e_1\}, \qquad \text{so that} \qquad R^{\perp} = \{e_1, e_2, f_2\}.$$

We will describe the map ρ locally, in the coordinate chart consisting of those

$$X \in \mathfrak{L}_{\{f_1, f_2\}}.$$

For such X the projection onto $\{e_1, e_2\}$ is non-singular and X determines a symmetric map of $\{e_1, e_2\}$ into $\{f_1, f_2\}$. In particular, X is spanned by vectors

$$e_1 + a_{11}f_1 + a_{12}f_2 \qquad \text{and} \qquad e_2 + a_{21}f_1 + a_{22}f_2$$

where $a_{12} = a_{21}$ and the matrix $A = (a_{ij})$ determines, and is determined by X. We shall therefore denote X by X_A. Now $X \cap R^{\perp}$ consists of combinations of the above vectors having 0 as the coefficient of f_1. Thus, there are two cases to consider:

(i) $a_{11} = a_{21} = 0$. This amounts to the assumption that $X_A \subset R^{\perp}$. In this case $a_{12} = 0$ and

$$\rho(X_A) = ([e_2] + a_{22}[f_2])$$

where $[e_2] = e_2/R$. We write this for short as

$$\rho(X_A) = (1, a_{22})$$

(ii) $(a_{11}, a_{21}) \neq (0,0)$. Then $\dim X_A \cap R^{\perp} = 1$ and

$$\rho(X_A) = (a_{11}, \det A).$$

If, in the above formulae we let

$$A = \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}$$

we see that

$$\rho(X_A) = \begin{cases} (1,0) & \text{for } s = 0, \\ (0,s) = (0,1) \text{ projectively} & \text{for } s \neq 0, \end{cases}$$

so that ρ is not continuous.

From this example we see that we can expect trouble from ρ at those X satisfying $X \cap R \neq 0$. In fact, this is indeed the case, and indeed the only troublesome locus for ρ for general V and R. By choosing a basis of R, we may proceed inductively on the dimension of R. Let us therefore analyse, in some detail, the map ρ in the case where $R = \{e\}$ is one dimensional. Thus $\dim V = 2n$ and letting $W = R^{\perp}/R$, we get $\dim W = 2n - 2$. Recall that $\dim L(V) = \dim(S^2(\mathbf{R}^n)) = n(n+1)/2$. We will let

$$S_R = \{X \in L(V) \mid X \supset R\} = \{X \in L(V) \mid X \subset R^{\perp}\}$$

since $X = X^{\perp}$.

PROPOSITION 3.2. *The set S_R is a submanifold of codimension n in $L(V)$. The map ρ restricted to S_R is a diffeomorphism of S_R onto $L(W)$. The map ρ, when restricted to $L(V) - S_R$, is a smooth map making $L(V) - S_R$ into a fiber bundle over $L(W)$ with fiber $\mathbf{R}^n$.*

Notice that if $n > 2$, then the proposition implies that the inclusion of $L(V) - S_R$ into $L(V)$ induces an isomorphism on π_1. Indeed any smooth curve can be deformed so as to avoid S_R and so can any smooth homotopy. Since $L(V) - S_R$ is asserted to be a fiber bundle over $L(W)$ with homotopically trivial fiber, we conclude that ρ induces an isomorphism of $\pi_1(L(V))$ with $\pi_1(L(W))$. We shall see by direct calculation that the same is true for the case that $\dim V = 4$. This will provide an alternative proof of the fact that $\pi_1(L(V)) = \mathbf{Z}$. Also, the generator in the plane will then determine the generator of $\pi_1(L(V))$. We will see in the course of a subsequent calculation, that this generator coincides with the one previously obtained from the Hermitian structure.

PROOF OF THE PROPOSITION. The fact that S_R has codimension n is pretty obvious. It suffices to check that $S_R \cap \mathcal{L}_X$ is a submanifold of codimension n for each $X \in L(V)$. Now choose a complementary $Y \in \mathcal{L}_X$, giving a direct sum decomposition $V = X \oplus Y$ with corresponding projections π_X and π_Y. Now if $S_R \cap \mathcal{L}_X \neq \varnothing$ we conclude that $\pi_Y e \neq 0$, where $R = \{e\}$. Any $Z \in \mathcal{L}_X$ corresponds to a symmetric map, A, from Y to X, and $Z \in S_R$ if and only if $A\pi_Y e = \pi_X e$. This clearly represents n linear conditions on A.

To see that $\rho_{|S_R}$ is a bijection notice that if $R \subset X \subset R^\perp$ then $\rho(X) = X/R$. If X' is a Lagrangian subspace of $R^\perp/R$ then its inverse image in $R^\perp$ is a Lagrangian subspace, the unique X with $\rho(X) = X'$.

Let us now examine $L(V) - S_R$. For $X \not\subset R^\perp$ it is clear that the map $X \to X \cap R^\perp$ is smooth, and $X \cap R^\perp$ does not contain R. The map $X \cap R^\perp \longrightarrow X \cap R^\perp/X \cap R$ is thus also smooth, proving that ρ is smooth on $L(V) - S_R$. Let us examine the inverse image. Let Z'_1 and Z'_2 be two $(n-1)$ dimensional isotropic spaces of $R^\perp$ with $Z'_1/R = Z'_2/R = Z''$. Then given $z \in Z''$ if we get $z_1 \in Z'_1$ and $z_2 \in Z'_2$ and so $z_1 - z_2 \in R$. In this way it is clear that the inverse image of Z'' is an affine space whose associated linear space is $\operatorname{Hom}(Z'', R)$. Now for a given $n-1$ dimensional isotropic Z' lying in $R^\perp$ we must determine all possible Z's in $L(V)$ with $Z \cap R^\perp = Z'$. Such a Z must lie in $(Z')^\perp$ which is $n+1$ dimensional. We are thus looking for all lines in $(Z')^\perp/Z'$, with the exclusion of the line $\{e + Z'\}$. Since $\dim(Z')^\perp/Z' = 2$ we are essentially adding the affine line. Thus the entire inverse image of Z'' in $L(V) - S_R$ will be diffeomorphic to $\mathbf{R}^n$.

We now wish to show how to associate an integer (A, B, C, D) to a quadruplet of Lagrangian subspaces where

$$C \cap A = \{0\} = C \cap B$$

and

$$D \cap A = \{0\} = D \cap B.$$

It is defined as follows: Let

$$R = A \cap B.$$

Then $\rho(A)$ and $\rho(B)$ are transversal Lagrangian subspaces of $R^\perp/R$. By Proposition 3.4, any third Lagrangian subspace of $R^\perp/R$ which is transversal to $\rho(A)$ and

$\rho(B)$ corresponds to a quadratic form on $\rho(B)$. Since C and D are both transversal to A and B we conclude that $\rho(C)$ and $\rho(D)$ each determine non-singular quadratic forms, Q_C and Q_D, on $\rho(B)$. The quadratic form, Q_C, on $\rho(B)$ is given by $Q_C(b) = (P_A^C b, b)$ where P_A^C is the projection of $\rho(B)$ onto $\rho(A)$ along $\rho(C)$. Let

$$(A, B, C, D) = \tfrac{1}{2}[\operatorname{sig} Q_C - \operatorname{sig} Q_D] = \operatorname{ind} Q_D - \operatorname{ind} Q_C.$$

We shall write $i(A, B, C) = \frac{1}{2} \operatorname{sig} Q_C$ so that

$$(A, B, C, D) = i(A, B, C) - i(A, B, D).$$

It is clear that

$$(A, B, C, D) = -(A, B, D, C) \tag{3.1}$$

and

$$(A, B, C, D) + (A, B, D, E) + (A, B, E, C) = 0. \tag{3.2}$$

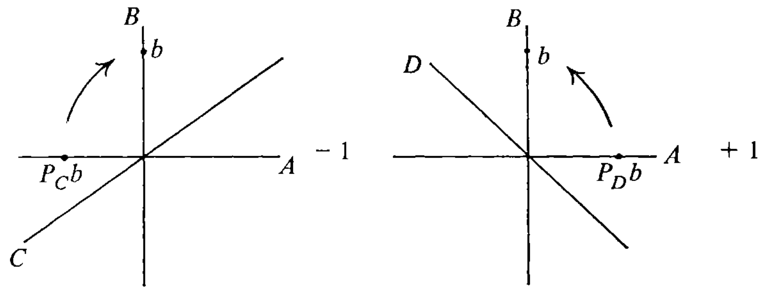
If $A \cap B = \{0\}$ then for all nearby A, B, C and D it is clear that the transversality conditions will still be satisfied and, since the signatures of Q_C and Q_D will not change, we conclude that (A, B, C, D) is locally constant. We wish to prove that this remains true even if $A \cap B \neq \{0\}$, provided that C and D each remain transversal to A and B. For this purpose, we will give an alternative definition of (A, B, C, D). Recall that $\mathfrak{L}_A$ is a cell. Since C and D both belong to $\mathfrak{L}_A$ there is a curve, γ_{CD}, joining C to D in $\mathfrak{L}_A$, and two such curves are homotopic. Similarly there is a unique curve (up to homotopy), γ_{DC}, joining D to C in $\mathfrak{L}_B$. This then defines a closed curve γ_{CDC} in $L(V)$ up to homotopy, i.e., an element of $\pi_1(L(V))$. It is some multiple of the basic generator. Our claim is that this multiple is exactly (A, B, C, D). In other words that

$$(A, B, C, D) = \int_{\gamma_{CDC}} (\det{}^2)^* \frac{dz}{2\pi i z} \tag{3.3}$$

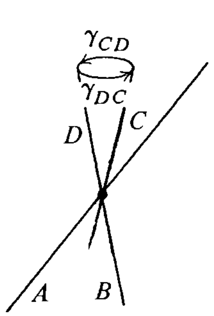
in terms of some choice of identification of $L(V)$ with $U(n)/O(n)$. Since the curve γ_{CDC} can be made to vary smoothly with A, B, C and D so long as C and D remain transversal to A and to B we see that the left hand side is indeed a continuous (and hence constant) function of its arguments. The proof of (3.3) that we present below is essentially due to Kostant.

Before presenting the proof let us look at the various possibilities for the curve γ_{CDC} when $\dim V = 2$. We begin with the geometric picture of $i(A, B, C)$. For

$A = B$ we have $i(A, B, C) = 0$ by definition. For $A \neq B$ we have, for $b \in B$ and $P_A^C b \in A$, that the signature of Q_C will be ± 1 according as to whether the vectors $P_A^C b$, b form a positive or negative orientation of the plane.

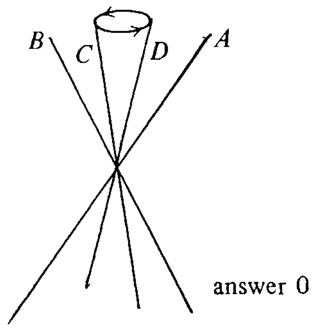


Now the various cases for (A, B, C, D):



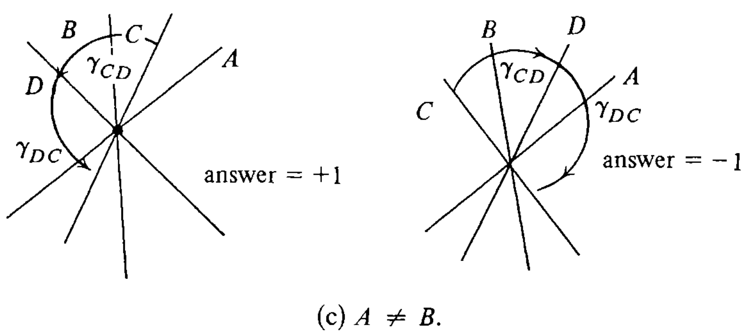
(a) $A = B$.

$$\int_{\gamma_{CDC}} (\det^2)^* \frac{dz}{2\pi i z} = 0$$



(b) $A \neq B$.

D, C lie in the same component of $\mathfrak{L}_A \cap \mathfrak{L}_B$.



(c) $A \neq B$.

D, C lie in different components of $\mathfrak{L}_A \cap \mathfrak{L}_B$.

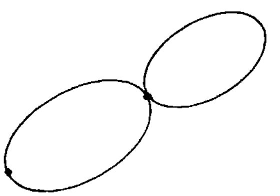
The formula is thus clear from the diagrams when dim $V = 2$. (We shall give an analytical proof soon.) To prove the formula in general let us observe that it suffices to prove the formula for the case that $A \cap B = \{0\}$. Indeed, suppose that

$$A \cap B = R.$$

Since γ_{DC} lies in $\mathfrak{L}_A$ and γ_{CD} lies in $\mathfrak{L}_B$ we see that γ_{CDC} lies in $L(V) - S_R$. Now ρ: $L(V) - S_R \to L(R^{\perp}/R)$ determines an isomorphism between $\pi_1(L(V))$ and $\pi_1(L(R^{\perp}/R))$. Thus if the class of γ_{CDC} is k times the generator of $\pi_1(L(V))$ then $\rho(\gamma_{CDC})$ will determine the same multiple, k, of the generator of $\pi_1(L(R^{\perp}/R))$. But $\rho(A)$ and $\rho(B)$ are transversal in $R^{\perp}/R$. Hence we are reduced to the transversal case.

Now both sides of (3.3) don't change under deformations so long as transversality is maintained. Our object will be to deform the spaces until the formula becomes obvious.

First choose a Hermitian structure with $B = iA$, and choose an orthonormal basis, $e_1, \ldots, e_n$ in A. This determines a dual basis $f_1, \ldots, f_n$ in B. With respect to the basis $e_1, \ldots, e_n$, the subspace C determines a non-singular symmetric matrix $C = (C_{ij})$. Now we can find an orthogonal matrix, $O \in SO(n)$, such that $O(C)O^{-1}$ is diagonal. Since $SO(n)$ is connected we can find a curve $O(t)$ with $O(0) = \text{id}$ and $O(1) = O$. Then $O(t)CO(t)^{-1}$ deforms C into a diagonal matrix. By further deformation we may assume that the entries of C are ± 1, and similarly for D. It is clear from (3.2) and from the fact that we can choose $\gamma_{CE} = \gamma_{CD} \circ \gamma_{DE}$, etc. (see the figure)



that it suffices to prove the formula when C and D differ in at most one position. If $C = D$ there is nothing to prove. Suppose that

$$C = \begin{pmatrix} +1 & & & & 0 \\ & \pm 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \pm 1 \end{pmatrix} \quad \text{and } D = \begin{pmatrix} -1 & & & & 0 \\ & \pm 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \pm 1 \end{pmatrix}$$

so that

$$\operatorname{ind} D - \operatorname{ind} C = 1.$$

Thus C is spanned by the vectors

$$e_1 + f_1, g_2, \ldots, g_n \qquad \text{where } g_i = e_i \pm f_i$$

or, if we like, by

$$\cos\frac{\pi}{4}e_1 + \sin\frac{\pi}{4}f_1, g_2, \ldots, g_n.$$

Similarly D is spanned by

$$-e_1 + f_1, g_2, \ldots, g_n$$

or by

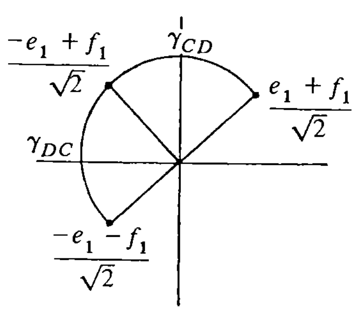
$$\cos\frac{3\pi}{4}e_1 + \sin\frac{3\pi}{4}f_1, g_2, \ldots, g_n.$$

Now we define the curve γ_{CD} by

$$\gamma_{CD}(\theta) = (\cos\theta e_1 + \sin\theta f_1, g_2, \ldots, g_n), \qquad \frac{\pi}{4} \leqslant \theta \leqslant \frac{3\pi}{4}.$$

Throughout this range of θ the coefficient of f_1 does not vanish so that γ_{CD} lies in $\mathcal{L}_A$. Similarly define

$$\gamma_{DC}(\theta) = (\cos\theta e_1 + \sin\theta f_1, g_2, \ldots, g_n), \qquad \frac{3\pi}{4} \leqslant \theta \leqslant \frac{5\pi}{4}.$$



It is now clear that the projection of this curve onto the e_1, f_1 plane describes the projective line once in the proper orientation. In terms of $U(n)/O(n)$ it is clear that the curve γ_{CDC} is given by the equivalence class of the curve of unitary matrices

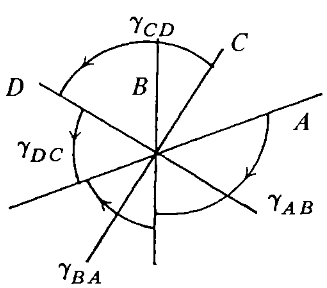
$$\begin{pmatrix} e^{i\theta} & & & & 0 \\ & c_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & c_n \end{pmatrix}, \qquad \frac{\pi}{4} \leqslant \theta \leqslant \frac{5\pi}{4},$$

and hence the function $\det^2$ goes around the circle once in the counter clockwise direction. This proves (3.3) and, incidentally, the consistency of the inductive definition of the generator of H^1 with the definition coming from $\det^2$.

PROPOSITION 3.3. *The symbol* (A, B, C, D) *satisfies*

$$(A, B, C, D) = -(C, D, A, B). \tag{3.4}$$

For the case $\dim V = 2$ we can get the result by examining the cases described in the figures given above: If $A = B$ then we can deform C into D remaining transversal to both A and B so that both sides of (3.4) vanish. If C and D lie in the same component of $\mathcal{L}_A \cap \mathcal{L}_B$ then we can deform C into D and then A into B so we are back in the preceding case. If C and D lie in different components of $\mathcal{L}_A \cap \mathcal{L}_B$ then it is clear from the figure that γ_{CDC} is oriented in the opposite direction from γ_{ABA}.



For the general case we shall use the following observation: Let V_1 and V_2 be two symplectic vector spaces. Then $V_1 + V_2$ is again a symplectic vector space and if $X_1 \in L(V_1)$ and $X_2 \in L(V_2)$ then $X_1 + X_2 \in L(V_1 + V_2)$. Thus we have a map of

$$L(V_1) \times L(V_2) \to L(V_1 + V_2).$$

We can clearly choose the Hermitian structure on $V_1 + V_2$ to be consistent with this direct sum decomposition and thus the map corresponds to the block diagonal embedding of $U(n) \times U(m) \to U(m+n)$. Thus we obtain

$$\frac{U(n) \times U(m)}{O(n) \times O(m)} \xrightarrow{f} \frac{U(n+m)}{O(n+m)} \xrightarrow{\det^2_{m+n}} S^1$$

where

$$\det^2_{m+n} \circ f = \det^2_m \det^2_n$$

with the obvious notation. Thus

$$f^*(\det^2_{m+n})^* \frac{d \log z}{2\pi i} = df^* \log \det^2_{m+n}/2\pi i = (\det^2_m)^* \frac{dz}{2\pi i z} + (\det^2_n)^* \frac{dz}{2\pi i z}.$$

Therefore in computing (A, B, C, D), if we could arrange that $A = A_1 + A_2$, etc., we would conclude that

$$(A, B, C, D) = (A_1, B_1, C_1, D_1) + (A_2, B_2, C_2, D_2).$$

Now by the deformation argument presented in the proof of the preceding proposition we know that we can arrange that $A = \{e_1, \ldots, e_n\}$, $B = \{f_1, \ldots, f_n\}$, $C = \{g_1, \ldots, g_n\}$, $D = \{h_1, \ldots, h_n\}$ where $g_i = e_i \pm f_i$ and $h_j = e_j \pm f_j$ for suitable choices of $\pm$. Thus we are reduced to the two dimensional case, which has already been established.

Now let $E \to N$ be a symplectic vector bundle. (Thus E is a vector bundle such that each fiber, E_n, has a symplectic structure varying smoothly with n.) The main application we have in mind will be the situation where $N = \Lambda$ is a Lagrangian submanifold of some cotangent bundle T^*M and where $E_n = T_n(T^*M)$. Of course, we then get a fiber bundle $L(E) \to N$ where $L(E)_n$ consists of all Lagrangian subspaces of E_n. Suppose that we are given two sections, A and B, of $L(E)$. (For example, if $N = \Lambda \subset T^*M$ then we could take $A_\lambda = T_\lambda(\Lambda)$ and take B_λ to be the tangent to the fiber of the projection $T^*M \to M$.) With this data, $\{E; A \text{ and } B\}$, Hormander has introduced an element of $H^1(N)$, which is defined, in the Čech theory, as follows: We can always locally find sections, C, D, etc. defined on open sets U_C, U_D, etc. such that C_x is transversal to both A_x and B_x for each $x \in U_C$. The U's form an open cover of N and we define a Čech 1-cocycle

$$\mathfrak{L}(U_C, U_D) = (A, B, C, D),$$

where the right hand side is taken to mean the function $x \to (A_x, B_x, C_x, D_x)$ defined on $U_C \cap U_D$. Since (A, B, C, D) is continuous and integer valued, it defines a Čech cochain $z(A,B)$ on N. Since for each $x \in U_C \cap U_D$ we have

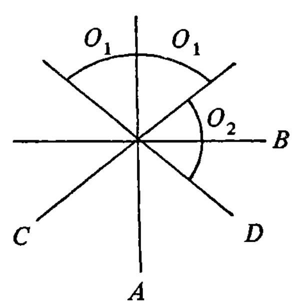
$$(A_x, B_x, C_x, D_x) = i(A_x, B_x, C_x) - i(A_x, B_x, D_x)$$

we see that $\delta z(A, B) = 0$. (This is just (3.2).) Notice that $i(A,B,C)$ need not be continuous so that $z(A,B)$ is not a coboundary, in general. We will denote the corresponding cohomology class by α or $\alpha(E; A, B)$.

Let us compute the cohomology class α for the following situation. Let V be a symplectic vector space, and let $N = L(V)$. We define the symplectic vector bundle $E \to N$ by assigning a copy of V to each point of N. (In other words, E is the pull back to N of the vector bundle $V \to$ pt. under the constant map.) Then $L(E)$ has a canonical (tautologous) section, B, namely $B(n) = n$ where n is considered as a subspace of $E_n = V$. Let A be a constant section of E (i.e., the pull back of a "section" of $L(V) \to$ pt.) We then obtain an element $\alpha(A,B) \in H^1(L(V))$. We claim that

PROPOSITION 3.4 (HÖRMANDER). *The class $\alpha(A,B)$ coincides with the fundamental generating class of $H^1(L(V))$ introduced above.*

As before, it suffices to verify the proposition for the case $\dim V = 2$. (Indeed, we need only check that the two classes coincide when evaluated over some nontrivial cycle, since we know that $H^1(L(V)) = \mathbf{Z}$. We can then choose this cycle as a curve $n(t)$ such that $n(t) \cap A = F$ is a fixed space of dimension $n - 1$ where $\dim V = 2n$. Then all formulas are obtained by projecting onto a two dimensional space.) Let us choose a fixed vector space A. Then if we pick two lines, C and D, transversal to A, then C is transversal to A and B on $U_C = L(V) - \{C\}$ and D is transversal to A and B on $U_D = L(V) - \{D\}$. (See the figure.)



Now $U_C \cap U_D$ has two components, one, O_1, containing A, and the other, O_2, not containing A. We clearly have

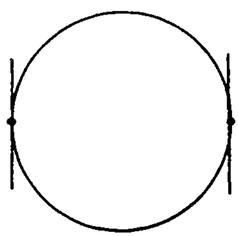
$$z(C,D) = (A,B,C,D) = \begin{cases} 0 & \text{on } O_1, \\ -1 & \text{on } O_2. \end{cases}$$

It is easy to check directly, that this defines the generating cohomology class. It is instructive to compute this class via the passage from the Čech to the de Rham theory. Recall that this is done as follows: We choose (smooth) functions f_C defined on U_C (and f_D defined on U_D) such that $f_C - f_D = z(C, D)$ on $U_C \cap U_D$. Then $df_C = df_D$ on $U_C \cap U_D$ and so defines a one form, β, on the whole space. The integral of this one form over the cycle is the value of α on the cycle. In our case, let us take

$$f_C(B) = \tfrac{1}{2}\operatorname{sig}(A, B, C) \qquad \text{and} \qquad f_D(B) = \tfrac{1}{2}\operatorname{sig}(A, B, D).$$

Notice that f_C isn't quite smooth—it has a jump, from $-\frac{1}{2}$ to $+\frac{1}{2}$ as B goes through the point A. We could replace f_C by any smooth function which agrees with it in some neighborhood of A. It is simpler to allow differential forms with distribution coefficients, in which case $df_C = \delta_A\, ds$ where δ_A is the δ-function at A and ds is the fundamental form on S^1. Since $\int \delta_A\, ds = +1$ we see that $\alpha(A, B)$ is indeed the fundamental class.

If $N = \Lambda \subset T^*M$ is a Lagrangian manifold and if we take $A(\lambda) = T_\lambda(\Lambda)$ and $B(\lambda) =$ tangent space to the fiber, then the corresponding class is called the *Maslov class* of Λ. For example, the same computation as we just gave shows that if Λ is a simple closed curve in $\mathbf{R}^2 = T^*(\mathbf{R}^1)$ then the Maslov class is exactly twice the fundamental generator: If Λ has only isolated (non-degenerate)



tangencies with the vertical, then a δ-function contribution occurs (with the appropriate orientation) at each point of tangency, i.e., at each point where $d\pi$ is not injective, or, what amounts to the same thing, at each point where $A \cap B \neq \{0\}$.

We can give such a geometric interpretation of the Hormander class $\alpha(A, B)$ in general. For this purpose we need the following fact, which we shall occasion to use quite a bit later on.

PROPOSITION 3.5. *Let Y be a Lagrangian subspace of V. Then the set*

$$L_k(V, Y) = \{W \mid \dim(W \cap Y) = k\}$$

is a submanifold of $L(V)$ of codimension $k(k+1)/2$.

PROOF. It suffices to verify the proposition locally, so that we may assume that Y and W are in $\mathcal{L}_X$. Then, by Proposition 2.3, all $W \in L(V)$ are parametrized by $S^2(Y)$, and, it is clear from the proof of Proposition 4.3, that $L_k(V,Y) \cap \mathcal{L}_X$ corresponds to symmetric matrices of corank k. Thus, we are reduced to proving

PROPOSITION 3.6. *The set of symmetric matrices of corank k is a submanifold of codimension $k(k+1)/2$ in the space of all symmetric matrices.*

Let

$$\begin{pmatrix} P_0 & Q_0 \\ R_0 & S_0 \end{pmatrix}$$

be a symmetric matrix, where, with no loss of generality, we may assume that the upper left hand $(n-k)\times(n-k)$ block is non-singular. Then all nearby matrices have the form

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

with P non-singular. Now

$$\begin{pmatrix} P^{-1} & 0 \\ -RP^{-1} & I \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I & P^{-1}Q \\ 0 & S - RP^{-1}Q \end{pmatrix},$$

and this matrix has rank k if and only if the symmetric $k \times k$ matrix $S - RP^{-1}Q$ vanishes. This imposes $k(k+1)/2$ conditions, proving Proposition 3.6 and hence also Proposition 3.5.

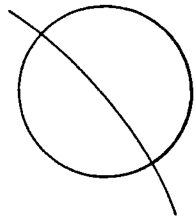
Let E be a symplectic vector bundle over N, with two sections, A and B, of $L(E)$. Then we obtain a subbundle, $L_k(E,A)$ for each integer, k, whose codimension in $L(E)$ is $k(k+1)/2$. We will say that $(E;A,B)$ is *in general position*, if B intersects each of these subbundles transversally. (Notice that by the Thom transversality theorem, we can modify B by an arbitrarily small amount, and hence not change $\alpha(A,B)$, so that $(E;A,B)$ is in general position.) We let $S_k(E,A,B) = B^{-1}(L_k(E,A))$, so that, if $(E;A,B)$ is in general position then $S_k(E,A,B)$ is a submanifold of N of codimension $k(k+1)/2$. Notice that

$$\overline{S}_k = S_k \cup S_{k+1} \cup \cdots \cup S_n.$$

In particular, if $(E;A,B)$ is in general position, then

$$\begin{aligned} \overline{S}_1 &= \{x \mid A(x) \cap B(x) \neq \{0\}\} \\ &= S_1 \cup \overline{S}_2 \end{aligned}$$

where $\overline{S}_2$ is a union of submanifolds of codim ≥ 3. In particular, every (smooth) curve can be deformed into a curve intersecting S_1 transversally, and also every homotopy of curves. We now claim that S_1 is also oriented in N, i.e. has a $+$ and $-$ side in N. In fact, let x be any point of S_1. In some neighborhood, U_C, of x, we can find a C transversal to both A and B. Then at all points of U_C not on S_1 the spaces A and B intersect transversally, and hence C defines a quadratic form, Q_C,



on B. We may assume that U_C is connected and that $U_C - U_C \cap S_1$ has two components.
Then sig Q_C is clearly constant on each component and the difference in sig Q_C between the two components is independent of the choice of C. Indeed, if D were another section transverse to A and B, then

$$\tfrac{1}{2}[\operatorname{sig} Q_C - \operatorname{sig} Q_D] = (A,B,C,D)$$

off S_1, but (A,B,C,D) is well defined and continuous even across S_1. Now we can clearly choose a trivialization of E near x such that $A = \{e_1, \ldots, e_n\}$ and $C = \{f_1, \ldots, f_n\}$ where the e_i and f_j form dual bases. We may also arrange that the basis is chosen so that $B \cap A = \{e_1\}$ on S_1 near x. Then

$$B = \{e_1 + \varphi_1 f_1, e_2 + \varphi_2 f_2, \ldots, e_n + \varphi_n f_n\}$$

where $\varphi_2, \ldots, \varphi_n$ are all non-zero near x and S_1 is given locally by $\varphi_1 = 0$. Transversality requires that $d\varphi_1 \neq 0$, so that φ_1 changes sign across S_1 and sig Q_C clearly changes by exactly 2 as we cross S_1. Now if γ is any smooth oriented closed curve which intersects S_1 transversally we can apply the argument used in the proof of Proposition 3.4 to conclude

PROPOSITION 3.7. *If $(E;A,B)$ is in general position and γ is a smooth closed curve intersecting S_1 transversally, then the class $\alpha(A,B)$, when evaluated on γ, is given by the number of intersections of γ with S_1, each counted with sign ± 1 according to whether the crossing is in the positive or negative direction.*

In the case of $\Lambda \subset T^*M$ the set $\overline{S}_1$ consists of exactly the Maslov cycle introduced in Chapter II, and the Maslov class is the cohomology class discussed there. (We must still establish that by a slight perturbation we can bring every Lagrangian manifold into general position, i.e. that the vertical section and the tangential section be in general position.)

We close this section with an alternative description, also due to Hormander, of the class $\alpha(A,B)$ on a symplectic bundle, E. Consider the pullback, $\tilde{E}$, of E to $L(E) \to N$. We also obtain a bundle $L(\tilde{E}) \to L(E)$ and a natural section, S, where $S(x) = x$ where $x \in L(E)_n$ and we have identified $\tilde{E}_x$ with E_n. A section A of $L(E)$ pulls back to a section $\tilde{A}$ of $L(\tilde{E})$. Notice that if $s: N \to L(E)$ is a section of $L(E)$ then

$$s^*\tilde{A} = A \qquad \text{and} \qquad s^*S = s,$$

Finally, we have

$$\alpha(\tilde{A},\tilde{B}) = \alpha(\tilde{A},S) - \alpha(\tilde{B},S) \in H^1(L(E)).$$

Now if $g: N_1 \to N_2$ is a continuous map and E_1 is the pullback of a symplectic bundle E_2 over N_2 then $\alpha(g^*A, g^*B) = g^*\alpha(A,B)$; in other words the assignment of α to $(E;A,B)$ is functorial. Therefore, taking $B = s$ we get

$$s^*\alpha(A,B) = B^*\alpha(A,S)$$

since $\alpha(B,B) = 0$. Now each section, A, of $L(E)$ determines the class

$$\alpha(\tilde{A},S) \overset{\text{def}}{=} \alpha_A$$

on $L(E)$. This class has the property

(i) that its restriction to each fiber is exactly the generating class of the fiber. This is the content of Proposition 3.4.

It further has the property that

(ii) $A^*\alpha_A = 0$.

Now it is a consequence of a standard theorem in the topology of fiber bundles (the Leray-Hirsch theorem, cf. for example Spanier [19, p. 258]) that any one form, β, on $L(E)$ must be of the form $\beta = \pi^*c + k\alpha_A$. Now if β satisfies (ii) then $A^*\beta = (\pi \circ A)^*c = c = 0$ and if it satisfies (i) clearly $k = 1$. *Thus* (i) *and* (ii) *characterize* α_A.

Thus if β_A is any form satisfying (i) and (ii) then

$$\alpha(A,B) = B^*\beta_A.$$

For example, given A, we can always choose a section everywhere transversal to A and a Riemann metric on A. This determines a Hermitian structure on E, which together with A allows us to identity $L(E)$ with $U(n)/O(n)$ for each N. Thus

$$\sigma = (\det^2)^*\left(\frac{dz}{2\pi iz}\right)$$

is a well defined form on E which clearly satisfies (i) and (ii). Thus the form

$$B^*\sigma$$

will define the class $\alpha(A,B)$ on N.

As above let $E \to N$ be a symplectic vector bundle and A and B sections of $L(E)$. Let $\{\mathcal{U}\}$ be a contractible open cover of N. We recall again how $\alpha(A, B)$ is defined. We choose sections

$$C_{\mathcal{U}}: \mathcal{U} \to L(E) \tag{3.5}$$

transversal to A and B. The Cech cocycle defining $\alpha(A, B)$ has the value

$$\mathcal{C}(\mathcal{U}, V) = (A, B, C_{\mathcal{U}}, C_v)$$

on the pair of open sets $(\mathcal{U}, V)$. Let us set

$$\tau(\mathcal{U}, V) = e^{(\pi i/2)\mathcal{C}(\mathcal{U}, V)}. \tag{3.6}$$

We can regard (3.6) as defining transition functions for a line bundle on N; i.e., we can define a line bundle on N by requiring that it have local trivializations, $S_{\mathcal{U}}$, on $\mathcal{U}$ and that these be related by $S_{\mathcal{U}} = \tau(\mathcal{U}, v)S_v$ on $\mathcal{U} \cap v$. The line bundle defined this way is called the *Maslov bundle* associated with (E, A, B) and denoted $\mathfrak{M} = \mathfrak{M}_{A,B}$. It is a *locally constant* bundle: i.e., we define a section of $\mathfrak{M}$ over a subset Z of N to be *constant* if for each $\mathcal{U}$ intersecting Z it is a constant multiple of $S_{\mathcal{U}}$ on $\mathcal{U} \cap Z$. Since the transition functions (3.6) are constant this definition is independent of the choice of $\mathcal{U}$. Moreover, locally constant sections exist (e.g. the $S_{\mathcal{U}}$'s).

In [3] Hörmander gives an alternative definition of $\mathfrak{M}$ which avoids the use of transition functions. This definition goes as follows. First given a fixed symplectic vector space V and fixed Lagrangian subspaces A and B, we attach to this data a one dimensional vector space $\mathfrak{M}_{A,B}(V)$; Let $\mathcal{O}$ be the open subset of $L(V)$ consisting of all C such that $C \cap A = C \cap B = \{0\}$. Then $\mathfrak{M}_{A,B}(V)$ is the space of all functions

$$f: \mathcal{O} \to \mathbf{C}, f(C) = e^{i\pi/2(A,B,C,D)}f(D) \tag{3.7}$$

for C, $D \in \mathcal{O}$. Such a function is determined by its value at one point, so the space $\mathfrak{M}_{A,B}(V)$ is one dimensional.

Now let $E \to N$ be a symplectic vector bundle and A and B sections of $L(E) \to N$. We define a line bundle $\mathfrak{M} \to N$ by defining its fiber at $p \in N$ to be the vector space $\mathfrak{M}_{A,B}(E_p)$. Let us show that $\mathfrak{M}$ is identical with the bundle defined by the transition functions (3.6). Over the open set $\mathcal{U}$ a section $S_{\mathcal{U}}$ is defined by choosing $S(p)$ to be the function (3.7) taking the value 1 at the point $C_{\mathcal{U}}(p)$ of $L(E_p)$, $C_{\mathcal{U}}$ being the section (3.5). It is clear from (3.6) and (3.7) that $S_{\mathcal{U}} = \tau(\mathcal{U}, v)S_v$; so $\mathfrak{M}_{A,B}$ is the bundle defined by (3.6) as claimed.

We will conclude this section with a brief description of the Lagrangian Grassmannian $L(V)$ when $\dim V = 4$. We will determine, *inter alia*, its diffeotype and prove directly that $\pi_1(L(V)) = \mathbf{Z}$, thus justifying the inductive derivation of this fact given at the beginning of this section.

The space $\bigwedge^2(V)$ is equipped with a canonical bilinear map into $\bigwedge^4(V)$ given by exterior multiplication:

$$\bigwedge^2(V) \otimes \bigwedge^2(V) \to \bigwedge^4(V), \qquad \mu \otimes \nu \rightsquigarrow \mu \wedge \nu.$$

If V Is four dimensional, then $\dim \bigwedge^4(V) = 1$, so we may identify $\bigwedge^4(V)$ with $\mathbf{R}$ by fixing a volume form, $\Omega \in \bigwedge^4(V)^*$, on V. We then get a symmetric bilinear form

$$q \colon \bigwedge^2(V) \otimes \bigwedge^2(V) \to \mathbf{R}, \qquad q(\mu, \nu) = \Omega[\mu \wedge \nu].$$

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of V satisfying $\Omega[e_1 \wedge e_2 \wedge e_3 \wedge e_4] = 1$, and let us define

$$u_1 = e_1 \wedge e_2, \qquad u_2 = e_1 \wedge e_3, \qquad u_3 = e_1 \wedge e_4,$$

and

$$v_1 = e_3 \wedge e_4, \qquad v_2 = e_4 \wedge e_2, \qquad v_3 = e_2 \wedge e_3.$$

Then $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ form a basis of the six dimensional space $\bigwedge^2(V)$ and satisfy

$$q(u_i, u_j) = q(v_i, v_j) = 0 \quad \text{and} \quad q(u_i, v_j) = \delta_{ij}.$$

This shows that the form q is nondegenerate and has signature $(3, 3)$.

A two vector $\mu \in \bigwedge^2(V)$ is a null vector for q if and only if $\mu \wedge \mu = 0$, i.e. if and only if μ is decomposable, i.e. if and only if

$$\mu = u_1 \wedge u_2,$$

where u_1 and u_2 are vectors in V. Let U denote the two dimensional subspace of V spanned by u_1 and u_2. It is clear that U depends only on μ, and not on the choice of u_1 and u_2. It is also clear that if λ is any nonzero real number, then μ and $\lambda\mu$ determine the same subspace, U. Conversely, given U, we can choose a basis u_1, u_2, and hence a $\mu = u_1 \wedge u_2$ determine up to scalar multiple.

In other words, we have a bijective map of the Grassmannian, $G_2(V)$, of all two dimensional subspaces of V, onto the set of null lines in $\bigwedge^2(V)$ relative to the quadratic form q. If we use the basis $\{u_1, v_1, u_2, v_2, u_3, v_3\}$ of $\bigwedge^2(V)$, where the u's and v's were chosen above,to identify $\bigwedge^2(V)$ with $\mathbf{R}^6$, then q becomes the quadratic form $x_1x_2 + x_3x_4 + x_5x_6$. We can then view the correspondence between U and $\{\lambda\mu\}$ as establishing a bijective map between $G_2(V)$ and the projective quadric

$$x_1x_2 + x_3x_4 + x_5x_6 = 0$$

in $\mathbf{RP}^5$.

Suppose that V has a symplectic structure. The symplectic form can be viewed as a linear map

$$\ell : \bigwedge^2(V) \to \mathbf{R}.$$

Conversely, any such linear map $\ell \in \bigwedge^2(V)^*$ defines a symplectic form on V if and only if $q^*(\ell,\ell) \neq 0$, where q^* is the quadratic on $\bigwedge^2(V)^*$ dual to the quadratic form q on $\bigwedge^2(V)$. In fact, if we start with a symplectic form ℓ then we can choose the Ω that we used to define q as $\Omega = \ell \wedge \ell$, in which case we would have

$$q^*(\ell,\ell) = 1.$$

Let us make this choice of Ω. Let W be the five dimensional subspace of $\bigwedge^2(V)$ annihilated by ℓ. Since $q^*(\ell,\ell) = 1$, the restriction of q to W is nondegenerate, and has signature $(2,3)$. Let us denote this restriction by q_W. If μ is a decomposable element of $\bigwedge^2(V)$ corresponding to the two dimensional subspace U of V, then U is Lagrangian if and only if $\ell(\mu) = 0$, i.e. if and only if $\mu \in W$. Thus the projective embedding of $G_2(V)$ into $\mathbf{RP}^5$ described above restricts to a projective imbedding of $L(V)$, the space of Lagrangian planes in V, into $\mathbf{RP}^4$ whose image is a projective quadric of type $(2,3)$. In fact, by choice of an appropriate basis we can arrange that q_W be given as

$$q_W = -y_1^2 - y_2^2 + y_3^2 + y_4^2 + y_5^2.$$

We have thus proved:

THEOREM 3.8. *If V is a real four dimensional symplectic vector space, then the space $L(V)$ of Lagrangian two planes of V can be imbedded in $\mathbf{RP}^4$ as the projective quadraic*

$$-y_1^2 - y_2^2 + y_3^2 + y_4^2 + y_5^2 = 0. \tag{3.8}$$

To see what this quadric looks like as a topological space, let us identify $\mathbf{RP}^4$ as the four sphere with antipodal points identified. That is, let S^4 denote the four sphere in $\mathbf{R}^5$ given by

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 = 2. \tag{3.9}$$

Let us identify $\mathbf{RP}^4$ as $S^4/\mathbf{Z}_2$. Then (3.8) and (3.9) reduce to

$$y_1^2 + y_2^2 = 1 \quad \text{and} \quad y_3^2 + y_4^2 + y_5^2 = 1.$$

This proves

THEOREM 3.9. *$L(V)$ is diffeomorphic to $S^1 \times S^2$ with antipodal points, (x,y) and $(-x,-y)$, identified.*

COROLLARY. $\pi_1(L(V)) = \mathbf{Z}$.

The action of $Sp(V)$ on V induces a representation of $Sp(V)$ on $\bigwedge^2(V)$ which preserves ℓ and hence q. Thus W is an invariant subspace, and hence we get a representation of $Sp(V)$ on W which preserves q_W. If we identify

W with $\mathbf{R}^5$ as above, so that q_W becomes identified with $-y_1^2-y_2^2+y_3^2+y_4^2+y_5^2$, then this representation can be viewed has a Lie group homomorphism of $Sp(V)$ onto $O(2,3)$. It is clear that $\pm I$ is in the kernel of this homomorphism as $-I$ induces the identity transformation on $\bigwedge^2(V)$, and it is easy to check that this is the entire kernel. Since $\dim Sp(V) = 10 = \dim O(2,3)$, and $Sp(V)$ is connected it follows that the above described homomorphism is a double covering,

$$Sp(V) \to SO(2,3)_0,$$

where $SO(2,3)_0$ denotes the identity component of $SO(2,3)$. In other words

$Sp(V)$ is the spin double cover of the identity component of $O(2,3)$.

Now for any signature (p,q) the Lie algebra $o(p+1,q+1)$ can be identified as the Lie algebra of infinitesimal conformal transformations of $\mathbf{R}^{p,q}$, so the Lie algebra $o(2,3)$ can be identified as the infinitesimal conformal transformations of three dimensional Minkowski space $\mathbf{R}^{1,2}$. Furthermore, the space $\mathbf{R}^{p,q}$ has a natural "conformal compactification" which can be viewed as the projective quadric in $\mathbf{RP}^{p+q+1}$ coming from a quadratic form of signature $(p+1,q+1)$ on $\mathbf{R}^{p+q+2}$, and $O(p+1,q+1)$ acts as conformal transformations of this conformal compactification. We shall verify these facts presently in the special case at hand. So we will prove

THEOREM 3.10. *$L(V)$ carries an intrinsically defined conformal structure of type $(2,3)$ and $Sp(V)$ acts as the double cover of the connected component of the group of all conformal transformations of $L(V)$. Indeed, for any Lagrangian subspace X let $\mathscr{L}_X$ denote the set of Lagrangian subspaces transverse to X so $\mathscr{L}_X$ is an affine space according to Proposition 2.3. Then $\mathscr{L}_X$ is equipped with a flat Lorentz metric which is intrinsically determined up to a conformality factor.*

PROOF. According to Proposition 2.3, $\mathscr{L}_X$ is, in an intrinsic way, an affine space whose associated linear space is $S^2(V/X)$. If we choose a basis of V/X, we may identify $S^2(V/X)$ with the space of all symmetric two by two matrices

$$\left\{A = \begin{pmatrix} a & b \\ b & c \end{pmatrix},\ (a,b,c) \in \mathbf{R}^3\right\} \tag{3.10}$$

and this space has an intrinsically defined quadratic form

$$A \rightsquigarrow \det A = ac - b^2 \tag{3.11}$$

which has signature $(2,1)$. (Or, to conform with the notation preceding the theorem, we could use $-\det A$ which has signature $(1,2)$.) The identification of elements of $S^2(V/X)$ with matrices depends on the choice of basis. But changing the basis has the effect of replacing A by BAB^t, and hence of replacing $\det A$ by $(\det B)^2 \det A$, in other words of changing the metric (3.11) by a conformal factor. Q.E.D.

§4. Functorial properties of Lagrangian submanifolds.

Let X and Y be symplectic manifolds with forms ω_X and ω_Y. Then $X \times Y$ becomes a symplectic manifold with two form

$$\omega_{X \times Y} = \rho_X^* \omega_X + \rho_Y^* \omega_Y$$

where $\rho_X: X \times Y \to X$ is the projection onto X and ρ_Y the projection onto Y. Let $\Lambda_X \subset X$ be a Lagrangian submanifold, and let us set

$$H = \Lambda_X \times Y.$$

Let $\iota_H: H \to X \times Y$ be the immersion of H into $X \times Y$. Notice that since Λ_X is Lagrangian we have

$$\iota_H^* \omega_{X \times Y} = \iota_H^* \rho_Y^* \omega_Y. \tag{4.1}$$

Now suppose that $\Lambda_{X \times Y}$ is a Lagrangian submanifold of $X \times Y$, and suppose that $\Lambda_{X \times Y}$ intersects H transversally. Notice that since the codimension of H is $\frac{1}{2} \dim X$ and $\dim \Lambda_{X \times Y} = \frac{1}{2}(\dim X + \dim Y)$ we see that $\dim(H \cap \Lambda_{X \times Y}) = \frac{1}{2} \dim Y$. We claim that the *projection* ρ_Y *makes this intersection into an immersed Lagrangian submanifold of* Y. Let us first show that the map is an immersion. Suppose that ξ is a tangent vector to $H \cap \Lambda_{X \times Y}$ and that $d\rho_Y \xi = 0$. Then $\xi \lrcorner \rho_Y^* \omega_Y = 0$. By (4.1) this implies that

$$\xi \lrcorner \iota_H^* \omega_{X \times Y} = 0,$$

i.e., that

$$\langle \xi \wedge \eta, \omega_{X \times Y} \rangle = 0$$

for all η tangent to H. On the other hand, since ξ is tangent to $\Lambda_{X \times Y}$ which is Lagrangian, the above equation must hold for all η tangent to $\Lambda_{X \times Y}$. Since $T(\Lambda_{X \times Y})$ and $T(H)$ span all of $T(X \times Y)$, then the above equation holds for all η, which implies that $\xi = 0$. By (4.1) we see that $\rho_Y^* \omega_Y = 0$ on $\Lambda_{X \times Y} \cap H$ since $\omega_{X \times Y}$ vanishes on $\Lambda_{X \times Y}$. We shall now give several examples of this construction.

(i) *Composition of canonical relations.* Let U, W and Z be symplectic manifolds with corresponding forms ω_U, ω_W and ω_Z. Let us consider $W \times U$ as a symplectic manifold with the two form $\omega_W - \omega_U$ (where, for example, ω_W is considered as a form on $W \times U$ via projection). A Lagrangian submanifold, Λ_1, of $W \times U$ is called a canonical relation. For example, if f: $U \to W$ were a canonical transformation, then $f^* \omega_W - \omega_U = 0$, so that $\omega_W - \omega_U$ would vanish on graph f. Since dim graph $f = \dim U = \frac{1}{2}(\dim W \times U)$ we see that graph f is a Lagrangian submanifold. The concept of a canonical relation is thus a generalization of the notion of a canonical map. Let $Z \times W$ have the symplectic form $\omega_Z - \omega_W$ and let $\Lambda_2 \subset Z \times W$ be a canonical relation. We would like, in favorable circumstances, to know that $\Lambda_2 \circ \Lambda_1$ is a canonical relation in $Z \times U$. As a set the composition $\Lambda_2 \circ \Lambda_1$ consists of all pairs (z,u) such that there is some w with $(z,w) \in \Lambda_2$ and $(w,u) \in \Lambda_1$. This can be described as follows, let Δ denote the diagonal in $W \times W$ and let π denote the projection

$$\pi\colon Z \times W \times W \times U \to Z \times U.$$

Then $Z \times \Delta \times U$ and $\Lambda_2 \times \Lambda_1$ are submanifolds of $Z \times W \times W \times U$ and

$$\Lambda_2 \circ \Lambda_1 = \pi(\Lambda_2 \times \Lambda_1 \cap Z \times \Delta \times U).$$

Let us take $X = W \times W$ with the two forms $\omega_{W_2} - \omega_{W_1}$ (where ω_{W_1} is the pullback to X of the ω_W of the first factor and similarly for ω_{W_2}) . Then Δ is a Lagrangian submanifold of X. Let us take $Y = Z \times U$ with form $\omega_Z - \omega_U$. We can now apply our construction to conclude that

if $\Lambda_2 \times \Lambda_1$ intersects $Z \times \Delta \times U$ transversally then $\Lambda_2 \circ \Lambda_1$ is a canonical relation in $Z \times U$.

For example, if Λ_2 is the graph of a map then for any $\eta \in TW$ there is ζ such that $(\zeta,\eta,0,0)$ is tangent to $\Lambda_2 \times \Lambda_1$. Thus it is easy to see that in this case the intersection will be transversal.

(ii) *Pullback of a Lagrangian submanifold of the cotangent bundle.* Let M and N be differentiable manifolds and $f\colon M \to N$ a differentiable map. Let $\Lambda \subset T^*N$ be a Lagrangian submanifold. Then

$$df^*\Lambda = \{(m,\xi) \in T^*M \mid \exists(n,\eta) \in \Lambda \text{ with } f(m) = n \text{ and } df_m^*\eta = \xi\}$$

is a subset of T^*M and we would like to know whether it is a Lagrangian submanifold. Here let us take $X = T^*N$ and $Y = T^*M$. Let us write graph $f \subset N \times M$ as $\{(f(x),x) \mid x \in M\}$, and let us take $\Lambda_{X\times Y} = \mathfrak{N}(\text{graph } f)$. Thus a point of $\Lambda_{X\times Y}$ is of the form

$$(f(x),x,\gamma,-df^*\gamma) \qquad \gamma \in T^*N_{f(x)}.$$

We take $\Lambda_X = \Lambda$ so that H consists of all points of the form

$$(y,x,\eta,\xi) \qquad (y,\eta) \in \Lambda.$$

It is clear that

$$\pi_Y(H \cap \mathfrak{N}(\text{graph } f)) = -df^*\Lambda.$$

Of course, $-df^*\Lambda$ is a Lagrangian submanifold if and only if $df^*\Lambda$ is. Rewriting the graphs in the usual way we have proved

Let $f\colon M \to N$ be a smooth map and $\Lambda \subset T^*N$ a Lagrangian manifold. If $\mathfrak{N}(\text{graph } f)$ and $T^*M \times \Lambda$ intersect transversally in $T^*(M \times N)$ then $df^*\Lambda$ is a Lagrangian submanifold of T^*M.

Examining the above form of H and $\mathfrak{N}(\text{graph } f)$ we see that all values of the last three components can be achieved for any f and Λ, and that the intersection will be transversal if and only if the maps $f\colon M \to N$ and $\pi_N\colon \Lambda \to N$ are transversal. Thus

PROPOSITION 4.1. *Let $f\colon M \to N$ be a smooth map and $\Lambda \subset T^*N$ a Lagrangian manifold. Let $\pi\colon \Lambda \to N$ be the restriction to Λ of the projection of $T^*N \to N$. If f and π are transversal then $df^*\Lambda$ is a Lagrangian submanifold of T^*M.*

For example, if $\pi\colon \Lambda \to N$ is locally a diffeomorphism then the hypothesis is fulfilled. Here $\Lambda = \text{graph } d\varphi$ (locally) and $df^*\Lambda = \text{graph } df^*\varphi$.

As a second example, suppose that $\Lambda = \mathfrak{N}(S)$ where S is a submanifold of N. Then $\pi\Lambda = S$ and the hypothesis becomes that f intersects S transversally. In this case $f^{-1}S$ is a submanifold of M and

$$df^*\Lambda = \mathfrak{N}(f^{-1}S).$$

(iii) *Pushforward of Lagrangian manifolds.* Let $f\colon M \to N$ be a smooth map and let Λ be a Lagrangian submanifold of T^*M. Then

$$df_*\Lambda = \{(y,\eta) \mid y = f(x), (x, df^*\eta) \in \Lambda\}.$$

Take $X = T^*M$ and $Y = T^*N$ and

$$\Lambda_{X\times Y} = \mathfrak{N}(\text{graph } f) = \{(x, f(x), df^*\eta, -\eta) \mid \eta \in T^*N_{f(x)}\}.$$

Then, with $\Lambda_X = \Lambda$,

$$H = \{(x,y,\xi,\eta) \mid (x,\xi) \in \Lambda\}$$

so that $\pi_Y(H \cap \mathfrak{N}(\text{graph } f)) = -df_*\Lambda$. Thus, if *$\Lambda \times T^*N$ intersects $\mathfrak{N}(\text{graph } f)$ transversally, then $df_*\Lambda$ is a Lagrangian submanifold of T^*N.*

Notice that if df has constant rank then this condition takes on a somewhat simpler form. In this case the dimension of $df_x^*T^*N_{f(x)}$ does not vary so that df^*T^*N is a sub-bundle of T^*M. The transversality condition is then clearly the condition that this subbundle intersect Λ transversally. Thus

PROPOSITION 4.2. *Let $f\colon M \to N$ be a smooth map with df of constant rank and let Λ be a Lagrangian submanifold of T^*M. If Λ intersects df^*T^*N transversally then $df_*\Lambda$ is a Lagrangian submanifold of T^*N.*

For example, if f is an immersion (so that df^* is surjective everywhere and thus $df^*T^*N = T^*M$) the conditions are verified and $df_*\Lambda$ is a Lagrangian submanifold of T^*N for any Λ.

At the other extreme, suppose $f\colon M \to N$ is a fibration. Then $df^*T^*N = H$ is the bundle of those covectors which vanish on vectors tangent to the fiber. If Λ

intersects H transversally then $df_* \Lambda$ is a Lagrangian submanifold. For example, if $\Lambda = \text{graph } d\varphi$ then $\Lambda \cap H$ consists of those points $(m, d\varphi(m))$ on Λ where the vertical derivative, $d_V\varphi$, vanishes. At such points $d\varphi$ clearly defines a covector at $n = f(m)$ and thus gives a map from $\Lambda \cap H \to T^*N$. According to the general theory this map is a Lagrangian immersion. If we consider the Lagrangian manifold, Λ_1, of $T^*\mathbf{R}^1 = \mathbf{R}^1 \times \mathbf{R}^1$ where $\Lambda_1 = \{(x, 1)\}$ then $\Lambda = d\varphi^* \Lambda_1$ and thus we can think of the pushforward of Λ as described by the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & \mathbf{R} \\ {\scriptstyle f}\big\downarrow & & \\ N & & \end{array}$$

We shall soon see that the most general Lagrangian manifold on T^*N can locally be described as $df_* d\varphi^* \Lambda_1$, where M, φ, and f are suitably chosen.

Suppose that instead of Λ_1 on $\mathbf{R}$ we take $\mathfrak{N}(\{0\})$, the normal bundle to the origin. Then, if in the above diagram φ is transversal to $\{0\}$, i.e., if 0 is a regular value of φ, then $d\varphi^*(\mathfrak{N}\{0\}) = \mathfrak{N}(\varphi^{-1}(0))$. If $\mathfrak{N}(\varphi^{-1}(0))$ intersects H transversally then $df_* \mathfrak{N}(\varphi^{-1}(0))$ is a Lagrangian submanifold of N. This construction generalizes the classical notion of an *envelope*: Suppose that $M = N \times S$ where S is some auxiliary parameter space. We have assumed that the map

$$\varphi \colon N \times S \to \mathbf{R}$$

has zero as a regular value, so that $\varphi^{-1}(0) = Z$ is a hypersurface (of codimension one) in $N \times S$. Let $\varphi_s \colon N \to \mathbf{R}$ be defined by $\varphi_s(x) = \varphi(x, s)$. We can make the stronger hypothesis that φ_s has zero as a regular value for each s. If we set $N_s = \varphi_s^{-1}(0)$ then N_s is a hypersurface in N for each s, and $N_s = Z \cap N \times \{s\}$ so that, as a set, $Z = \bigcup_s N_s$. Now the Lagrangian manifold $\mathfrak{N}(Z)$ consists of all points of the form $\{(x, s, t d_N\varphi, t d_S\varphi) \mid \varphi(x, s) = 0, t \in \mathbf{R}\}$ and our transversality condition asserts that the rank of $d(d_S\varphi)$ be equal to dim S on Z. The Lagrangian manifold $\Lambda = df_*(\mathfrak{N}(Z))$ then consists of all covectors $t d_N\varphi$ where

$$\varphi(x, s) = 0 \qquad d_S\varphi(x, s) = 0.$$

These represent $p + 1$ equations in $p + n$ variables where $p = \dim S$ and $n = \dim N$. Our transversality assumption asserts that these equations define a submanifold of $N \times S$, i.e., that $0 \in \mathbf{R}^{p+1}$ is a regular value for $(\varphi, d_S\varphi)$. If we make the stronger assumption that the equations $d_S\varphi(x, s) = 0$ can be solved for s as a function of x the first equation becomes

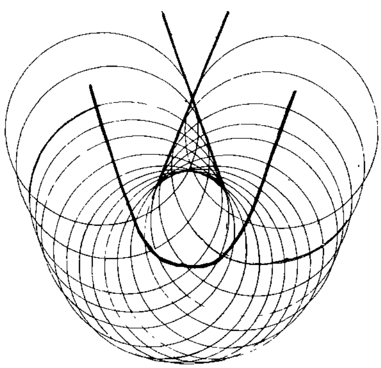
$$\varphi(x, s(x)) = 0$$

which defines a hypersurface, $\mathcal{E}$, called the envelope of the hypersurfaces N_s.

Furthermore

$$\begin{aligned} d\varphi(\cdot, s(\cdot)) &= d_N\varphi(\cdot, s(\cdot)) + d_S\varphi(\cdot, s(\cdot)) \\ &= d_N\varphi(\cdot, s(\cdot)) \end{aligned}$$

since $d_S\varphi = 0$. Thus $\Lambda = \mathfrak{N}(\mathcal{E})$. From our point of view it is more natural to consider the Lagrangian manifolds than the hypersurfaces. But even classically one is obliged to consider the Lagrangian manifold rather than the hypersurfaces if one wants to avoid singularities. For example, let S be a submanifold of $N = \mathbf{R}^n$ and let N_s be the sphere of radius r centered at $s \in S$. (We are thus in the situation envisaged in our treatment of Huygens' principle in Chapter I.) Then the (classical) envelope of the spheres N_s will develop singularities, if r is larger than the minimum radius of curvature of S. However, the Lagrangian submanifold $\Lambda \subset T^*(\mathbf{R}^n)$ will still be perfectly well defined. It just won't project onto a hypersurface in $\mathbf{R}^n$.



Later on we shall show how to associate to any distribution, μ, on M, a subset of T^*M, describing the singularities of μ. A particular class of distributions will have their singularities along (the normal bundle to) hypersurfaces. We will be able to show that when we superimpose distributions concentrated along N_s we obtain a distribution concentrated along the envelope, $\mathcal{E}$. In this way we shall be able to give a purely geometric version of Huygens' principle.

§5. Local parametrizations of Lagrangian submanifolds.

In this section we discuss local presentations of Lagrangian submanifolds of T^*M. The first basic observation is that, locally, we can represent any Lagrangian manifold as the push forward of graph $d\varphi$. More precisely

PROPOSITION 5.1. *Given $\Lambda \subset T^*M$ we can find, near each point $\lambda \in \Lambda$, a fibered* *manifold* $N \xrightarrow{\pi} M$ *and a function* $\varphi\colon N \to \mathbf{R}$ *such that*

$$\Lambda = d\pi_* \text{ graph } d\varphi$$
$$= d\pi_* d\varphi^* \{(x, 1)\}.$$

PROOF. Let $x^1, \ldots, x^n, \xi^1, \ldots, \xi^n$ be dual local coordinates. By a *linear* change of variables, we can arrange that $\xi^1, \ldots, \xi^k, x^{k+1}, \ldots, x^n$ are independent on Λ for some suitable k. (Indeed let X and Y be two complementary Lagrangian subspaces of a symplectic vector space V. For any Lagrangian W we know that $W \cap X$ and $PW \subset Y$ are orthogonal to each other under the symplectic form where P is the projection on Y through X. We can choose $e^1, \ldots, e^k$ as a basis of $W \cap X$ and $f^{k+1}, \ldots, f^n$ as a basis of PW and extend to get a dual basis. Applying this result to $T_\lambda \Lambda$ gives the desired result.)

We can thus find functions $f^1, \ldots, f^k, f^{k+1}, \ldots, f^n$ such that Λ is described by the equations

$$x^1 = -f^1(x^{k+1}, \ldots, x^n; \xi^1, \ldots, \xi^k)$$
$$\vdots$$
$$x^k = -f^k(x^{k+1}, \ldots, x^n; \xi^1, \ldots, \xi^k)$$
$$\xi^{k+1} = f^{k+1}(x^{k+1}, \ldots, x^n; \xi^1, \ldots, \xi^k)$$
$$\vdots$$
$$\xi^n = f^n(x^{k+1}, \ldots, x^n; \xi^1, \ldots, \xi^k).$$

Now the fundamental two form, ω, is given by

$$-d(x^1 d\xi^1 + \cdots + x^k d\xi^k) + d(\xi^{k+1} dx^{k+1} + \cdots + \xi^n dx^n).$$

Thus on Λ

$$0 = d(f^1 d\xi^1 + \cdots + f^k d\xi^k + f^{k+1} dx^{k+1} + \cdots + f^n dx^n).$$

We can thus find a function $F = F(x^{k+1}, \ldots, x^n; \xi^1, \ldots, \xi^k)$ such that

$$f^1 = \frac{\partial F}{\partial \xi^1} \quad \cdots \quad f^n = \frac{\partial F}{\partial x^n}.$$

Consider φ, defined on T^*M, by the formula

$$\varphi(x^1, \ldots, x^n, \xi^1, \ldots, \xi^n) = \xi^1 x^1 + \cdots + \xi^k x^k + F + \sum_{k+1}^{n} (\xi^j - f^j)^2.$$

Now (x, ξ) are coordinates on T^*M. Let (x, ξ, z, ζ) be the corresponding coordinates on $T^*(T^*M)$ so that, for example, the subbundle $H = d\pi^* T^*M$ is given by $\zeta = 0$. Then graph $d\varphi$ consists of all points of the form (x, ξ, z, ζ) where

$$
\begin{aligned}
z^1 &= \xi^1 \\
&\vdots \\
z^k &= \xi^k \\
z^{k+1} &= \frac{\partial}{\partial x^{k+1}}[F + \textstyle\sum (\xi^j - f^j)^2] \\
&\vdots \\
z^n &= \frac{\partial}{\partial x^n}[F + \textstyle\sum (\xi^j - f^j)^2] \\
\zeta^1 &= x^1 + \frac{\partial F}{\partial \xi^1} + \frac{\partial}{\partial \xi^1} \sum_{k+1}^{n} (\xi^j - f^j)^2 \\
&\vdots \\
\zeta^k &= x^k + \frac{\partial F}{\partial \xi^k} + \frac{\partial}{\partial \xi^k} \sum_{k+1}^{n} (\xi^j - f^j)^2 \\
\zeta^{k+1} &= 2(\xi^{k+1} - f^{k+1}) \\
&\vdots \\
\zeta^n &= 2(\xi^n - f^n).
\end{aligned}
$$

Now graph $d\varphi \pitchfork H$ implies that $\zeta^j = 0$ so that

$$\xi^{k+1} = f^{k+1}, \ldots, \xi^n = f^n, \qquad x^1 = -\frac{\partial F}{\partial \xi_1}, \ldots, x^k = -\frac{\partial F}{\partial \xi_k},$$

and

$$z^i = \xi^i, \qquad i = 1, \ldots, n.$$

This, of course, gives Λ.

Notice that it is a peculiarity of the particular representation that we have chosen that $N = T^*M$ and, setting $\Lambda_N =$ graph $d\varphi$, that in the diagram

$$
\begin{array}{ccccccc}
T^*M = N & \xleftarrow{\pi_N} & T^*N & \leftarrow & \Lambda_N & \leftarrow & \Lambda_N \cap H \\
\big\downarrow \pi_M & & & \swarrow & & & \\
M & \leftarrow & T^*M & & & &
\end{array}
$$

we have $d\pi_{M^*}\Lambda_N = \Lambda$. We shall discuss the significance of this particular type of representation in a later section.

It is of interest to know when by a possibly non-linear change of coordinates, we can arrange that $k = n$, and so eliminate the unpleasant quadratic terms that appear in the expression for φ. Thus, we wish to choose coordinates so that $\xi^1, \ldots, \xi^n$ are linearly independent on Λ. Now this is certainly not always going to be possible. For example, if Λ is the zero section, then $\xi^1 = \cdots = \xi^n \equiv 0$ in any coordinate system. Suppose, on the other hand, that $\lambda \in \Lambda$ with $\lambda \neq 0$. Let us choose a Lagrangian subspace of $T_\lambda(T^*X)$ which is transversal both to $T_\lambda\Lambda$ and to the vertical. This is always possible by Proposition 2.4. Let us pass a Lagrangian submanifold, K, tangent to this subspace. Since K is transversal to the vertical, it is of the form graph $d\psi$. If $x = \pi\lambda$ then $d\psi(x) = \lambda \neq 0$. We can thus introduce a coordinate system $x^1, \ldots, x^n$ with $\psi = x^1$. Then, in terms of these coordinates $K = \{(x^1, \ldots, x^n, 1, 0, \ldots, 0)\}$. At λ the tangent space to K is exactly the kernel of the projection onto the $\xi^1, \ldots, \xi^n$. Since Λ is transversal to K we conclude that the ξ_i are independent on Λ near λ. We thus have

PROPOSITION 5.2. *If* $0 \neq \lambda \in \Lambda$ *then near* λ *we can parametrize* Λ *as follows: We can introduce coordinates* $x^1, \ldots, x^n$ *near* $\pi\lambda$, *with corresponding coordinates* $x^1, \ldots, x^n, \xi^1, \ldots, \xi^n$ *on* T^*M *near* λ, *and find a function* $f(\xi^1, \ldots, \xi^n)$ *such that, near* λ,

$$\Lambda = d\pi_* \operatorname{graph} d\varphi$$

where

$$\varphi = x \cdot \xi - f. \tag{5.1}$$

If Λ *is homogeneous, i.e., invariant under multiplication by* $\mathbf{R}^+$, *then we can choose* φ *to be homogeneous of degree one in* ξ.

The only assertion that remains to be proved is the last one. If Λ is invariant under $\mathbf{R}^+$ then $\xi\partial/\partial\xi$ is tangent to Λ and hence $\alpha = \xi dx = (\xi\partial/\partial\xi) \lrcorner\, \omega$ must vanish when restricted to Λ. Also the f^i are homogeneous of degree zero so that on Λ

$$0 = \alpha = \sum \xi^i dx^i = d(\sum f^i \xi^i) - \sum f^i d\xi^i$$

so for f we may take $f = \sum f^i \xi^i$ which is homogeneous of degree one.

Let us call φ: $N \to \mathbf{R}$ a local *phase function* for $\Lambda \subset T^*M$ if Λ is locally of the form $d\pi_*$ (graph $d\varphi$) as above. Here $N \xrightarrow{\pi} M$ is a submersion. We shall set $\Lambda_\varphi = \operatorname{graph} d\varphi$.

We shall set $C_\varphi = \pi_N(\Lambda_\varphi \cap H)$ where π_N: $T^*N \to N$. Thus we have the commutative diagram

$$\begin{array}{ccc} N \leftarrow C_\varphi & \xleftarrow{\pi_N} & \Lambda_\varphi \cap H \\ \pi \Big\downarrow & & \swarrow \\ M & \xleftarrow[\pi_M]{} & T^* M. \end{array}$$

Now $\pi_N\colon \Lambda_\varphi \to N$ is a diffeomorphism so its restriction to $\Lambda_\varphi \cap H$ is an immersion. Thus, for $\lambda \in \Lambda_\varphi \cap H = \Lambda$ let

$$k_\lambda = \ker d\pi_M\colon T_\lambda \Lambda \to M$$

and let $n = \pi_N \lambda$ and

$$l_n = \ker d\pi\colon T_n(C_\varphi) \to T_{\pi n}(M)$$

so that $l_n = T_n C_\varphi \cap T_n F$ where F is the fiber through n. Then

$$d\pi_N k_\lambda = l_n$$

and in particular

$$\dim k_\lambda = \dim l_n.$$

Let $x^1, \ldots, x^n$ be coordinates on M and $x^1, \ldots, x^n, \theta^1, \ldots, \theta^k$ be coordinates on N. Then $C_\varphi = \{(x,\theta) \mid \partial\varphi/\partial\theta^i = 0, i = 1, \ldots, k\}$. The tangent to the fiber is spanned by $\partial/\partial\theta^i$. Notice that $\eta = \sum \alpha^i \partial/\partial\theta^i$ is tangent to C_φ if and only if

$$\sum_i a^i \frac{\partial^2\varphi}{\partial\theta^i \partial\theta^j} = 0 \qquad \text{all } j$$

i.e., if η is in the null space of the Hessian

$$H_\theta(\varphi) = d_\theta^2 \varphi = \left(\frac{\partial^2\varphi}{\partial\theta^i \partial\theta^j}\right).$$

Thus

$$\dim k_\lambda = \text{nullity } H_\theta(\varphi) \qquad \text{at } \pi_N \lambda. \tag{5.2}$$

Notice that this proves that

$$\text{fiber dim } N \geqslant \dim k_\lambda \tag{5.3}$$

(where fiber dim N is the dimension of the fibers of π, i.e., fiber dim N $=$ dim N $-$ dim M) .

Suppose $H_\theta(\varphi)$ is non-singular. Then $\Lambda_\varphi \cap H$ projects diffeomorphically on M and is of the form graph $d\psi$. Notice that then

$$\begin{array}{ccc} C_\varphi & \leftrightarrow & \Lambda_\varphi \cap H \\ \downarrow & \swarrow & \\ M & & \end{array}$$

are all diffeomorphisms. In particular there is a map $s\colon M \to C_\varphi$, i.e., a section of N and $d\varphi \circ ds = d\psi$ so we may take $\psi = s^* \varphi$.

We will say that the phase function φ is *reduced* (at λ) if fiber dim N $=$ dim k_λ. Notice that if φ is reduced at λ then $H_\theta \varphi$ is identically zero at $\pi_N \lambda$.

PROPOSITION 5.3. *We can always factor N and φ through a reduced parametrization of Λ. In other words, we can locally find an intermediate fibration*

$$\begin{array}{ccl} N & & \\ \downarrow \pi_1 & & \\ N^1 & & \pi\colon N \to M \\ \downarrow \pi_2 & & \\ M & & \end{array}$$

and a function φ^1 on N^1 such that $d\pi_{1*}\Lambda_\varphi = \Lambda_{\varphi^1}$ and $d\pi_{2*}\Lambda_{\varphi^1} = \Lambda$, with φ^1 a reduced phase function at λ.

PROOF. Choose a complement in the tangent space to the fiber at $\pi_N \lambda$ to the null space of $H_\theta(\varphi)$. Extend this to an integrable foliation in the fibers. This makes N fibered over some N^1, locally, where $H_\theta \varphi$ is non-degenerate when restricted to the tangent space to the fibers of π_1.

Let $H_1 = d\pi_1^* T^* N^1$ so that $H \subset H_1$ where $H = d\pi^* T^* M = d\pi_1^*(d\pi_2^* T^* M)$. Since Λ_φ intersects H transversally, it certainly intersects H_1 transversally and we have

$$\begin{array}{ccc} N & \leftarrow & \Lambda_\varphi \cap H_1 \\ \downarrow \pi_1 & & \\ N^1 & & \end{array}$$

Since $H_\theta \varphi$ is non-degenerate on fibers of N over N_1, $\Lambda_\varphi \cap H_1$ projects diffeomorphically onto N^1, and hence is of the form graph $d\varphi^1 = \Lambda_{\varphi^1}$, with $\varphi^1 = s^* \varphi$ where $s\colon N^1 \to N$ is a section of π_1. The rest follows easily.

PROPOSITION 5.4 (*Hormander-Morse lemma*). *Let* (N_1, φ_1) *and* (N_2, φ_2) *be two parametrizations of* Λ *near* λ *with* $\dim N_1 = \dim N_2$. *Let* $z_1 = \pi_{N_1}\lambda$ *and* $z_2 = \pi_{N_2}\lambda$. *Suppose that*

$$\operatorname{sig} H_\theta(\varphi_1)(z_1) = \operatorname{sig} H_\theta(\varphi_2)(z_2).$$

Then there exists a fiber preserving diffeomorphism f *of* $N_1 \to N_2$ *defined near* z_1 *with* $f(z_1) = z_2$ *and*

$$\varphi_2 \circ f = \varphi_1 + \text{const} .$$

In other words, Λ, $\dim N$ *and* $\operatorname{sig} H_\theta \varphi$ *determine* φ *up to a diffeomorphism.*

(If we take M to be a point, this reduces to the standard Morse lemma. Indeed H is the 0 section of T^*N in this case, and hence the transversality condition is that φ have a non-degenerate critical point. Then we are clearly in the situation described by Morse's lemma.)

We first point out that it is sufficient to consider the case where N_1 and N_2 are reduced. Indeed, suppose that the proposition has been established in the case of reduced parametrizations. Then there are fibrations $N_1 \to N_1^1$ and $N_2 \to N_2^1$ with non-degenerate phase functions. By applying a diffeomorphism, we may assume that $N_1^1 = N_2^1$. Then we have sections s_1 and s_2 with $\varphi_1 \circ s_1 = \varphi_2 \circ s_2 + \text{const}$.

$$\begin{array}{ccc} N_1 & \dashrightarrow & N_2 \\ {\scriptstyle s_1}\nwarrow & & \nearrow{\scriptstyle s_2} \\ & N^1 & \end{array}$$

given by $d_\theta \varphi_i = 0, i = 1, 2$. Now $H_\theta(\varphi_i)$ is non-singular, so we may apply the standard Morse's lemma to each fiber. This then gives the desired diffeomorphism from N_1 to N_2.

We thus may assume that (N_1, φ_1) and (N_2, φ_2) are reduced. Our next step will be to show that we can assume that $N_1 = N_2$, $C_{\varphi_1} = C_{\varphi_2}$, and the first derivatives of φ_1 and φ_2 agree on $C_{\varphi_1} = C_{\varphi_2}$ and so that h maps $d\varphi_2 \,|C_{\varphi_2}$ onto $d\varphi_1|\, C_{\varphi_1}$. For this, we need only exhibit a fiber diffeomorphism h: $N_1 \to N_2$ with $h(C_{\varphi_1}) = C_{\varphi_2}$. Then replace φ_2 by $\varphi_2 \circ h$. For this purpose, let (x, θ) be local coordinates on N_1 and consider the fiber map $g_1 : N_1 \to T^*M$ given by

$$g_1(x, \theta) = \left(x, \frac{\partial \varphi_1}{\partial x}(x, \theta)\right).$$

On C_{φ_1} this gives the diffeomorphism with Λ. Since $T_{z_1}(\text{fiber}) \subset T_{z_1}(C_{\varphi_1})$ we conclude that g_1 is an immersion at z preserving the fiber over M. The same is true for a similar map g_2. By the implicit function theorem, there is some map ρ: $T^*M \to N_2$ commuting with the fibrations over M such that $\rho \circ g_2 = \text{id}$. Then $\rho \circ g_1 : N_1 \to N_2$ is a diffeomorphism, and since $g_1 C_{\varphi_1} = g_2 C_{\varphi_2} = \Lambda$ we have $(\rho \circ g_1) C_{\varphi_1} = C_{\varphi_2}$. Moreover, from the definition of ρ, $d(\rho \circ g_1)^* \varphi_2 = \varphi_1$ on C_{φ_1}.

We are thus in the situation of Appendix I. For convenience we repeat the relevant proofs:

φ_1 and φ_2 are reduced phase functions for Λ on N with $C_{\varphi_1} = C_{\varphi_2}$, with

$$\frac{\partial\varphi_1}{\partial x} = \frac{\partial\varphi_2}{\partial x} \qquad \text{and} \qquad \frac{\partial\varphi_1}{\partial\theta} = 0 = \frac{\partial\varphi_2}{\partial\theta}$$

on C_φ. Thus $\varphi_1 = \varphi_2$ on C_φ (up to a constant which we shall ignore) and $\varphi_1 - \varphi_2$ vanishes to second order. We set $\varphi_t = (1 - t)\varphi_1 + t\varphi_2$ and seek a fiber diffeomorphism f_t with $f_t^* \varphi_t = \varphi_1, f_t = I$ on C_φ. We look for the corresponding vector field ξ_t. It should satisfy

$$\dot{\varphi}_t + D_{\xi_t}\varphi_t = 0.$$

Now $\varphi_2 - \varphi_1$ vanishes to second order on C_φ and $d(\partial\varphi_1/\partial\theta^i)$ are independent by the transversality of Λ_φ with H. Thus

$$\varphi_2 = \varphi_1 + \sum b_{ij}\frac{\partial\varphi_1}{\partial\theta^i}\frac{\partial\varphi_1}{\partial\theta^j} \qquad b_{ij} = b_{ij}(x,\theta)$$

so

$$\dot{\varphi}_t = \sum b_{ij}\frac{\partial\varphi_1}{\partial\theta^i}\frac{\partial\varphi_1}{\partial\theta^j}.$$

We seek

$$\xi_t = \sum \mu_t^i(x,\theta)\frac{\partial}{\partial\theta^i}$$

with $\mu_t^i = 0$ on C_φ so

$$\mu_t^i = \sum \mu_{ij}\frac{\partial\varphi_1}{\partial\theta^j}$$

and

$$\dot{\varphi}_t + D_{\xi_t}\varphi_t = 0$$

becomes

$$0 = \sum b_{ij}\frac{\partial\varphi_1}{\partial\theta^i}\frac{\partial\varphi_1}{\partial\theta^j} + \sum \mu_{ij}\frac{\partial\varphi_1}{\partial\theta^j}\frac{\partial}{\partial\theta^i}\left[\varphi_1 + t\sum b_{kl}\frac{\partial\varphi_1}{\partial\theta^k}\frac{\partial\varphi_1}{\partial\theta^l}\right].$$

Equating the coefficients of $(\partial\varphi_1/\partial\theta^i)(\partial\varphi_1/\partial\theta^j)$ this becomes

$$B + U(I + S) = 0 \qquad B = (b_{ij}) \qquad U = (\mu_{ij})$$

where $S = 0$ at z_1 since φ_1 is reduced. We can thus solve for U near z_1, find ξ_t and integrate to get f_t, proving the proposition.

In many applications we will be dealing with homogeneous Lagrangian manifolds. For these applications we will require a slight modification of Proposition 5.4.

Let $N^+ = N \times \mathbf{R}^+$ and consider parametrizations of the form

$$\begin{array}{ccc} N^+ & \xrightarrow{\varphi^+} & \mathbf{R} \\ {\scriptstyle\pi}\downarrow & & \\ M & & \end{array}$$

which are equivariant with respect to the action of $\mathbf{R}^+$ in the obvious way, namely $\varphi^+(z,sa) = a\varphi^+(z,s)$ and $\pi(z,sa) = \pi(z,s)$. It is clear that the resulting Lagrangian manifold $\Lambda = \pi_*(\text{graph } d\varphi)$ will be invariant under the action of $\mathbf{R}^+$ on T^*M and hence homogeneous. By Proposition 5.2 every homogeneous Lagrangian manifold admits an $\mathbf{R}^+$ equivalent parametrization, at least locally. Moreover, it is easy to modify the proof of Proposition 5.3 so as to show:

PROPOSITION 5.3 (*homogeneous version*). *Let* (Λ,λ) *be a homogeneous Lagrangian manifold (germ) in* T^*M. *Then every* $\mathbf{R}^+$ *equivariant parametrization of* (Λ,λ) *can be factored through a reduced* $\mathbf{R}^+$ *equivariant parametrization.*

PROOF. Let $\varphi(z) = \varphi^+(z,1)$. Then $\varphi^+(z,a) = a\varphi(z)$. Now if $z^+ = (z,a)$ is in the critical set of φ^+, then

$$\partial\varphi^+/\partial z = 0 = a(\partial\varphi/\partial z);$$

so $\partial\varphi/\partial z = 0$ since $a \neq 0$. Moreover,

$$d^2\varphi^+\left(\frac{\partial}{\partial t}, w\right) = \left(d\left(\frac{\partial\varphi^+}{\partial t}\right), \eta\right) = \langle d_z\varphi, \eta\rangle = 0.$$

Here $\partial/\partial t$ is the unit vector tangent to $\mathbf{R}^+$ at z^+ and η is any vector tangent to N. This shows that $\partial/\partial t$ is in the null-space of the Hessian of φ^+ at z^+ so in the proof of Proposition 5.3 we can choose the fibration π_1 so that its fibers lie in the sets $N \times \text{const.}$; that is, so that π_1 commutes with the action of $\mathbf{R}^+$. Thus the reduced parametrization is also an $\mathbf{R}^+$ equivariant parametrization. Q.E.D.

We will use this result to prove:

PROPOSITION 5.4 (*homogeneous version*). *Let* (N_1^+,φ_1^+) *and* (N_2^+,φ_2^+) *be two parametrizations of* (Λ,λ) *with* $\dim N_1^+ = \dim N_2^+$. *Let* $z_1^* = \pi_{N_1^+}(\lambda)$ *and* $z_2^+ = \pi_{N_2^+}(\lambda)$. *Suppose that*

$$\operatorname{sig} H_\theta(\varphi_1^+)(z_1^+) = \operatorname{sig} H_\theta(\varphi_2^+)(z_2^+).$$

Then there exists an $\mathbf{R}^+$ *equivariant fiber preserving diffeomorphism* $f\colon N_1^+ \to N_2^+$ *defined near* z_1^+ *with* $f(z_1^+) = z_2^+$ *and* $\varphi_2^+ \circ f = \varphi_1^+$.

PROOF. As before it is sufficient to consider the reduced case. Define $g_1^+: N_1^+ \to T^*M$ and $g_2^+: N_2^+ \to T^*M$ exactly as we defined g_1 and g_2 above, i.e.,

$$g_1^+(x,\theta,s) = \left(x, \frac{\partial \varphi_1^+}{\partial x}\right).$$

It is clear that g_1^+ and g_2^+ are $\mathbf{R}^+$ equivariant , so we can choose $h: C_{\varphi_1^+} \to C_{\varphi_2^+}$ as before, but so that it is equivariant, and replacing φ_2^+ by $\varphi_2^+ \circ h$ reduce to the case: $N_1^+ = N_2^+$, $C_{\varphi_1}^+ = C_{\varphi_2}^+$, and $\varphi_1^+ = \varphi_2^+$ on $C_{\varphi_1}^+ = C_{\varphi_2}^+$. Note that $\varphi_1^+(z,t) = t\varphi_1(z)$; so at a critical point

$$z^+ = (z,t), \quad \frac{\partial \varphi_1^+}{\partial t} = 0 = \varphi_1(z);$$

and hence $\varphi^+(z^+) = 0$. Therefore, in the homogeneous case $\varphi_1^+ = 0$ on $C_{\varphi_1}^+$; so not only do the first derivatives of φ_1^+ and φ_2^+ agree on $C_{\varphi_1}^+ = C_{\varphi_2}^+$ but the functions themselves do as well. (Recall that in the inhomogeneous case we had to adjust by an arbitrary constant at this point in the argument.) The rest of the proof now goes as before. A little care must be exercised in choosing the matrix B. Namely it has to be homogeneous of degree 1 in $\mathbf{R}^+$. To do this just define B on a set where the $\mathbf{R}^+$ variable is constant and extend by linearity.

We can reformulate some of the discussion surrounding equation (5.3) of Chapter II in our current framework, together with a generalization of the exponential map and the computations of Chapter I.

PROPOSITION 5.5. *Let Λ be a homogeneous Lagrangian submanifold of T^*M and let a be a function defined on $T^*M - \{0\}_M$ which is homogeneous of degree one, and such that a is nowhere zero on Λ. Then we can find a homogeneous φ defined on T^*M such that*

$$d\pi_*(\text{graph } d\varphi) = \Lambda$$

and

$$\{a,\varphi\} = a.$$

Notice that if χ is any homogeneous phase function for Λ, we have since $\alpha = d\chi$ on Λ, and by Euler's theorem,

$$\{a,\chi\} = \sum \frac{\partial a}{\partial \xi}\frac{\partial \chi}{\partial x} - \frac{\partial a}{\partial x}\frac{\partial \chi}{\partial \xi}$$

$$= \sum \frac{\partial a}{\partial \xi}\xi = a.$$

Thus $\{a,\chi\} = a$ will always hold on Λ for any homogeneous phase functions. Our problem is to adjust χ off Λ. Notice also that

$$\xi_a = \frac{\partial a}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial a}{\partial x}\frac{\partial}{\partial \xi}$$

is nowhere tangent to Λ. Indeed if ξ_a were tangent to Λ at z, then $\langle \xi_a, \alpha \rangle_z = 0$ since $\alpha \mid_\Lambda = 0$ for homogeneous Λ. But then $0 = (\partial a/\partial \xi) \cdot \xi = a$ by Euler's theorem again, contradicting the assertion that a does not vanish on Λ. Now choose a homogeneous submanifold C, of codimension one transversal to ξ_a and containing Λ and choose φ to be the solution of the non-singular first order partial differential equation with initial condition

$$\{a, \varphi\} = \xi_a \varphi = a \qquad \varphi = \chi \qquad \text{on } C.$$

This defines φ uniquely in a neighborhood of C. Since a, ξ_a and Λ are all homogeneous we conclude that φ is homogeneous. Finally, $\iota_C^* d\varphi = \iota_C^* d\chi$ since $\varphi = \chi$ on C, and $\xi_a \varphi = \xi_a \chi = a$ at points of Λ. Thus $d\varphi = d\chi$ at points of Λ and hence φ is a phase function for Λ. Q.E.D.

Now consider the flow, f_t, generated by ξ_a, and consider the set of points

$$(f_t(z), t, -a) \subset T^*(M \times \mathbf{R}).$$

This consists exactly of the set swept out by the isotropic submanifold

$$\{\Lambda; 0, -a\}$$

under the Hamiltonian flow generated by the vector field $\xi_a + \partial/\partial t$ on $T^*(M \times \mathbf{R})$ and is thus a (homogeneous) Lagrangian submanifold, which we shall denote by Λ^a. Notice that for small values of t we have

$$\left(\frac{\partial}{\partial t} + \xi_a\right)(\varphi - ta) = 0$$

and, at Λ^a with $t = 0$, letting τ denote the dual variable to t,

$$\begin{aligned} d(\varphi - ta) &= d\varphi - a dt \\ &= \xi dx + \tau dt \mid_{\Lambda^a, t=0} \\ &= (\alpha_{M \times \mathbf{R}}) \mid_{\Lambda^a, t=0} . \end{aligned}$$

Now $D_{\xi_a} \alpha_M = 0$ and $D_{\partial/\partial t}(\tau dt) = 0$ so

$$D_{(\partial/\partial t + \xi_a)}(\alpha_{M \times \mathbf{R}}) = 0.$$

(In fact if b is any homogeneous function on $T^* N$ then it follows from Euler's equation that $D_{\xi_b} \alpha_N = 0$.) Thus

$$d(\varphi - ta) = \alpha_{M \times \mathbf{R}}$$

along Λ^a for sufficiently small t, i.e., $\varphi - ta$ is a phase function for Λ^a.

Associated with the same picture we can consider an inhomogeneous Lagrangian submanifold Λ_1 of $T^* M$, defined as follows. Let $S_1(\Lambda)$ consist of those points of Λ on which $a = 1$. (Notice that da does not vanish on Λ. Indeed

$\langle \xi\partial/\partial\xi, da\rangle = a \neq 0$ and $\xi(\partial/\partial\xi)$ is tangent to Λ. Thus $S_1(\Lambda)$ defines a submanifold of Λ of codimension one.) Then $S_1(\Lambda)$ is an isotropic submanifold which is transversal to ξ_a (since ξ_a is in fact transversal to Λ). Let us map $S_1(\Lambda)\times \mathbf{R} \to T^*M$ by $f(\lambda, t) = f_t(\lambda)$ where f_t is the flow generated by ξ_a. Then $\Lambda_1 = f(S_1(\Lambda)\times\mathbf{R})$ is a Lagrangian manifold of T^*M. Notice that since $\xi_a a = 0$, we can conclude that $a \circ f \equiv 1$ on $S_1(\Lambda)\times\mathbf{R}$. We can now define the exponential map, exp: $S_1(\Lambda)\times\mathbf{R}\to M$ by setting $\exp = \pi \circ f$. For example, let a denote length relative to a Riemann metric and let Λ be the cotangent space at a point y. Then $S_1(\Lambda)$ is the unit sphere at the point in question, and exp is the usual exponential map. Similarly, we can let Λ be the normal bundle to a submanifold.

Let us set $S_1 = \{z \in T^*M \mid a(z) = 1\}$ so that, in the preceding notation $S_1(\Lambda) = \Lambda \cap S_1$. Let us assume that S_1 is a submanifold of T^*M, let $\pi_1 : S_1 \to M$ be the restriction of π to S_1 and let φ_1 be the restriction of φ to S_1, where φ is the previously constructed phase function for Λ. We claim that

$$\Lambda_1 = d\pi_1(\text{graph } d\varphi_1).$$

Indeed the set C_{φ_1} is given (by the method of Lagrange multipliers) by the equations

$$\begin{aligned} d_\xi\varphi - \mu d_\xi a &= 0\\ a &= 1 \end{aligned}$$

for some μ. Thus $(x,\xi) \in C_{\varphi_1}$ if and only if $(x,\xi,\mu,-1)\in\Lambda^a$. Now on Λ^a we have $d(\varphi - ta) = \alpha + \tau dt$ so

$$d\varphi - tda = \alpha.$$

On S_1 this becomes

$$d\varphi_1 = \iota^*\alpha \qquad \iota\colon S_1 \to T^*M$$

so, on C_{φ_1}

$$\frac{\partial\varphi_1}{\partial x^i} = \xi^i$$

which is exactly the required assertion.

Notice that the value of the Lagrange multiplier μ has a very simple geometric interpretation. Indeed, on the one hand, since φ is homogeneous, we have

$$\varphi = \sum \frac{\partial\varphi}{\partial\xi_i}\xi_i = \mu\sum\frac{\partial a}{\partial\xi_i}\xi_i = \mu a = \mu$$

on S_1. If $S_1(x)$ denotes $S_1 \cap T^*M_x$ and $x \in \pi\Lambda_1$ then $x = \pi(z)$ where z is a critical point of φ_1, i.e. μ is a critical value of φ_1. Notice that μ is also the time parameter such that $\dot{f}_t(\lambda) = z$, where $\lambda \in S_1(\Lambda)$. Thus μ is a critical value of $\varphi_{1|S_1(x)}$ and represents the length of time for an extremal from $S_1(\Lambda)$ to reach T^*M_x. In the Riemannian case the values of μ are exactly the length of the geodesics joining x to the point y.

[1]We close this section by examining the behavior of the Maslov cochain under functorial operations on Lagrangian submanifolds of cotangent bundles. Let $f\colon M \to N$ be a submersion. Then df^*T^*N is a subbundle of T^*M and we have the diagram

$$\begin{array}{ccccc} M & \xleftarrow{\pi_M} & T^*M & \xleftarrow{\iota} & df^*TN = \mathcal{K} \\ f\downarrow & & \downarrow g & & \\ N & \xleftarrow{\pi_N} & T^*N & & \end{array}$$

Let $p \in \mathcal{K}$ and $x = \pi_M p \in M$, while q is the image of p in T^*N and $y = f(x)$ is the image of q in N. Thus

$$p = df_x^* q \qquad q = gp$$

$$x = \pi_M p \qquad y = \pi_N q.$$

Let α_M and α_N denote the fundamental one forms on T^*M and T^*N. We claim that

$$\iota^*\alpha_M = g^*\alpha_N.$$

Indeed, let ξ be tangent to $\mathcal{K}$ at p. Then

$$\begin{aligned} \langle \xi, g^*\alpha_N \rangle &= \langle dg\xi, \alpha_N \rangle \\ &= \langle d\pi_N \circ dg\xi, q \rangle \\ &= \langle df \circ d\pi_M \xi, q \rangle \\ &= \langle d\pi_M \xi, df^* q \rangle \\ &= \langle d\pi_M \xi, p \rangle = \langle \xi, \alpha_M \rangle. \end{aligned}$$

If $\omega_M = d\alpha_M$ and then, *a fortiori*, we have (as in (4.1))

$$g^*\omega_N = \iota^*\omega_M. \tag{5.4}$$

If Λ_M is a Lagrangian submanifold of T^*M which intersects $\mathcal{K}$ transversally, then $df_*\Lambda_M$ is defined to be $\Lambda_M \cap \mathcal{K}$ (when mapped in T^*N by g). Let

[1]The rest of this section should be omitted on first reading.

$j\colon \Lambda_M \cap \mathcal{H} \to \Lambda_M$, and let $\mathfrak{M}_{\Lambda_M}$ be the Maslov class of Λ_M. We claim that

$$j^* \mathfrak{M}_{\Lambda_M} = \mathfrak{M}_{df_* \Lambda_M} \tag{5.5}$$

where $\mathfrak{M}_{df_* \Lambda_M}$ is the Maslov class of $df_* \Lambda_M$.

Similarly, suppose that Λ_N is a Lagrangian submanifold of $T^* N$. Then $df^* \Lambda_N = g^{-1} \Lambda_N$ and we claim that

$$g^* \mathfrak{M}_{\Lambda_N} = \mathfrak{M}_{df^* \Lambda_N} \tag{5.6}$$

where $g\colon df^* \Lambda_N \to \Lambda_N$. (We have used the same letter, g, to denote the restriction of g to $g^{-1} \Lambda_N$.)

We first prove (5.5):
Let us set

$$\begin{aligned} h &= T_p \mathcal{H} \\ v_M &= \text{tangent space to fiber (of } \pi_M\text{) at } p \\ v_N &= \text{tangent space to fiber (of } \pi_N\text{) at } q \\ T_p &= T_p(T^* M) \\ T_q &= T_q(T^* N) \end{aligned}$$

and k the kernel of $dg\colon h \to T_q$ so that

$$0 \to k \to h \to T_q \to 0$$

is exact. We claim that k is an isotropic subspace and that, in fact $h = k^\perp$ under ω_M so that $T_q = k^\perp / k$. Indeed, if $\xi \in k$ and $\eta \in h$ then

$$\langle \xi \wedge \eta, \omega_M \rangle = \langle dg\xi \wedge dg\eta, \omega_N \rangle = 0$$

so that $h \subset k^\perp$ on the other hand $\dim k = \dim M - \dim N$ while $\dim h = \dim M + \dim N$ so that we must have $h = k^\perp$.

In particular, we have a map, ρ, from $L(T_p) \to L(T_q)$ sending any $u \subset T_p$ into $u \cap k^\perp / u \cap k \subset T_q$. Let Λ be a Lagrangian submanifold of $T^* M$ intersecting $\mathcal{H}$ transversally. We have

$$T_p(\Lambda \cap \mathcal{H}) = T_p \Lambda \cap h$$

while

$$\begin{aligned} (T_p \Lambda \cap k)^\perp &= T_p \Lambda + k^\perp \\ &= T_p \Lambda + h \\ &= T_p(T^* M), \end{aligned}$$

by transversality, so that

$$T_p\Lambda \cap k = 0.$$

Thus

$$T_q(\Lambda \cap \mathcal{H}) = \rho T_p\Lambda.$$

Of course this holds at all of points of $\Lambda \cap \mathcal{H}$ so we can write

$$T(\Lambda \cap \mathcal{H}) = \rho T\Lambda.$$

Since v_M is transversal to h, the same argument shows that

$$v_N = \rho(v_M).$$

Now let C_N and D_N be sections of $L(\Lambda \cap \mathcal{H})$. Then $C_N(p)$ is a subspace of $k^\perp/k$ and hence determines a unique Lagrangian subspace of $k^\perp$ which we denote by $C_M(p)$. We claim that if $C_N(p) \cap T_p(\Lambda \cap \mathcal{H}) = 0$ then $C_M(p) \cap T_p(\Lambda) = 0$. Indeed

$$C_M(p) \cap T_p(\Lambda) \subset k$$

since $C_M(p) \subset k^\perp$ and $C_M(p) \cap T_p(\Lambda)/k = 0$. But then

$$C_M(p) \cap T_p(\Lambda) \subset C_M(p) \cap T_p(\Lambda) \cap k = 0$$

since $T_p(\Lambda) \cap k = 0$. The same argument works for v_N.

Thus, if C_N is transversal to v_N and $T(\Lambda \cap \mathcal{H})$, then C_M is transversal to v_M and $T(\Lambda)$ along $\Lambda \cap \mathcal{H}$, and hence can be extended to a section of $L(\Lambda)$ defined near $\Lambda \cap \mathcal{H}$ which is still transversal to v_M and $T(\Lambda)$. We claim that

$$(T(\Lambda), v_M, C_M, D_M) = (T(\Lambda \cap \mathcal{H}), v_N, C_N, D_N),$$

if D_N is a second such section. Indeed,

$$(T(\Lambda), v_M, C_M, D_M) = -(C_M, D_M, T(\Lambda), v_M)$$

and since $C_M \cap D_M \supset k$ we have

$$\begin{aligned}(C_M, D_M, T(\Lambda), v_M) &= (C_N, D_N, \rho T(\Lambda), \rho v_M)\\ &= (C_N, D_N, T(\Lambda \cap \mathcal{H}), v_N).\end{aligned}$$

This shows explicitly how to relate the Čech cochains on Λ and $\Lambda \cap \mathcal{H}$ so that the Maslov class on Λ pulls back to the Maslov class on $\Lambda \cap \mathcal{H}$, proving (5.5).

We leave (5.6) as an exercise for the reader. The crucial point is to observe that at any point of $g^{-1}\Lambda_M$ all of the intersection $v_M \cap T(g^{-1}\Lambda_N)$ arises from N, in other words comes from $v_N \cap \Lambda_N$.

The functoriality of the Maslov class implies a corresponding functoriality for the Maslov line bundle. This follows from general nonsense (the isomorphism between cohomology classes and equivalence classes of line bundles) but can also be seen directly as follows. Let $f\colon M \to N$ be a submersion, let $y = f(x)$ and $p = df_x^* q$, $p \in \Lambda_M$ and $q \in \Lambda_N = df_* \Lambda_M$, the notations being as in the preceding paragraph. We recall from Section 3 that the fiber of the Maslov bundle at p is just the set of all functions

$$f\colon \mathcal{O} \to \mathbf{C},\, f(C) = e^{(i\pi/2)\alpha(C,D)} f(D),$$

$\alpha(C, D)$ being the cross-ratio $(T_p(\Lambda), v_M(p), C, D)$, and $\mathcal{O}$ being the subset of $L(T_p)$ consisting of those Lagrangian spaces which are transversal both to $T_p(\Lambda)$ and $v_M(p)$. Now the map ρ defined in the preceding paragraph maps $\mathcal{O}$ into the corresponding subset of $L(T_q)$ and preserves cross-ratios. Thus the fiber of the Maslov bundle at q gets mapped bijectively onto the fiber of the Maslov bundle at p by ρ^*.

§6. Periodic Hamiltonian systems

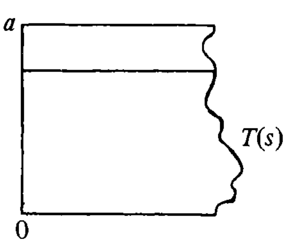
In this section we depart briefly from the study of general properties of symplectic manifolds and examine the special case of a Hamiltonian flow with manifolds of periodic solutions. We have already encountered such a situation in conjunction with the phenomenon of perfect focusing in Chapter III. We shall elaborate on some of the properties of such systems in this section and describe some important examples. The reader who prefers to continue with the general theory can skip to the next section.

Let ξ be a Hamiltonian vector field on a symplectic manifold X so that

$$\xi \lrcorner \omega = -dH.$$

Suppose we are given a one parameter family of trajectories of ξ. That is, suppose we have a map F of the region $0 \leqslant t \leqslant T(s)$, $0 \leqslant s \leqslant a$ of the (t, s) plane into X such that

$$dF\left(\frac{\partial}{\partial t}\right)_{(t,s)} = \xi_{F(t,s)}.$$



Thus, for each fixed s, the curve $\gamma_s = F(\cdot, s)$ is a trajectory of ξ. Now

$$\begin{aligned}\left\langle \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial s}, F^*\omega \right\rangle &= \left\langle dF\frac{\partial}{\partial t} \wedge dF\frac{\partial}{\partial s}, \omega \right\rangle \\ &= \left\langle dF\frac{\partial}{\partial s}, \xi \lrcorner \omega \right\rangle \\ &= -\left\langle dF\frac{\partial}{\partial s}, dH \right\rangle \\ &= -\left\langle \frac{\partial}{\partial s}, d(H \circ F) \right\rangle \\ &= \left\langle \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \lrcorner (d(H \circ F) \wedge dt) \right\rangle \\ &= \left\langle \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial s}, d(H \circ F) \wedge dt \right\rangle\end{aligned}$$

or, in short, we obtain Cartan's formula,

$$F^*\omega - d(H \circ F) \wedge dt = 0. \tag{6.1}$$

Now suppose there is a form α on X with $d\alpha = \omega$. Then the above equation becomes

$$dF^*(\alpha + t dH) = 0.$$

Let b_0 denote the curve $t = 0$ and b_1 the curve $s \rightsquigarrow (T(s), s)$. We can now apply Stokes' theorem to conclude, since H is constant along γ_0 and γ_a, that

$$\int_{\gamma_a} \alpha - \int_{\gamma_0} \alpha = \int_{b_1} F^*\alpha - \int_{b_0} F^*\alpha + \int_{b_1} T(s)\, d(H \circ F). \tag{6.2}$$

We have already encountered versions (and applications) of this formula in Chapter II and Chapter III. Let us now apply this formula to the case where each trajectory γ_s is periodic, i.e., that

$$F(0, s) = F(T(s), s).$$

Then

$$\int_{b_1} F^*\alpha = \int_{b_0} F^*\alpha$$

and (6.2) becomes

$$\int_{\gamma_a} \alpha = \int_{\gamma_0} \alpha + \int_0^a T(s)\, d\overline{H} \tag{6.3}$$

where $\overline{H}(s) = H(F(T(s), s))$. In particular, we obtain the following result, due to W. B. Gordon [4],

If γ_s is a one parameter family of periodic orbits lying on fixed energy surface H $=$ const *then $\int_{\gamma_s} \alpha$ is independent of s.*

Now suppose that all the trajectories of ξ are periodic and that the surfaces $H =$ const. are connected. Then the integral $\int \alpha$ over any periodic trajectory depends only on H so we can write it as $I(H)$. The equation (6.3) shows that

$$I'(H) = T(s) \tag{6.4}$$

showing that all orbits on the same energy surface have the same period. (The above argument is also due to Gordon, [4]. The actual result in various special cases had been known for some time, cf. Wintner [5] and the references cited in Moser [6] and Gordon [4].) We now present a generalization of this result due to Weinstein [8]. Let f: $X \times \mathbf{R} \to X$ be the map giving the flow generated by ξ (so that $f(x, \cdot)$ is the trajectory through x). If the flow is not globally defined then the domain of definition of f is understood to be the appropriate subset of $X \times \mathbf{R}$. We shall consider ω and H to be defined on $X \times \mathbf{R}$ via the projection of $X \times \mathbf{R}$ onto X. Let η be any tangent vector to X, considered as a tangent vector to $X \times \mathbf{R}$. Then the computation we did in proving (6.1) shows that

$$\left\langle \frac{\partial}{\partial t} \wedge \eta, f^* \omega \right\rangle = \left\langle \frac{\partial}{\partial t} \wedge \eta, dH \wedge dt \right\rangle.$$

If η and ζ are two tangent vectors to X then

$$\langle \eta \wedge \zeta, f^* \omega \rangle = \langle \eta \wedge \zeta, \omega \rangle$$

since the map $x \to f(x, t)$ is a symplectic diffeomorphism for each fixed t. We can combine these last two equations into the formula

$$f^* \omega = \omega + dH \wedge dt \tag{6.5}$$

Now let g: $X \times \mathbf{R} \to X \times X$ be the graph of f so that $g(x, t) = (x, f(x, t))$. (Again g may not be defined on all of $X \times \mathbf{R}$, it has the same domain of definition as f.) On $X \times X$ we put the symplectic structure $\omega_{X \times X} = \pi_2^* \omega - \pi_1^* \omega$ where π_1 and π_2 denote the projections of $X \times X$ onto the first and second factors, so that, for example $\pi_2 \circ g = f$. Then, by (6.5)

$$\begin{aligned} g^* \omega_{X \times X} &= g^* \pi_2^* \omega - g^* \pi_1^* \omega \\ &= f^* \omega - \omega \end{aligned}$$

or

$$g^* \omega_{X \times X} = dH \wedge dt. \tag{6.6}$$

In particular, if $M \subset X \times \mathbf{R}$ is a submanifold such that $g_{|M}$ is an isotropic submanifold of $X \times X$, so that $(g_{|M})^* \omega_{X \times X} = 0$, we conclude that $dH \wedge dt$ vanishes on M. For example, suppose that all the trajectories of ξ are periodic, and that the period of the trajectory through x is $T(x)$. Then take M

$= \{(x, T(x))\}$ and we have $g(x, T(x)) = (x, x)$. Since the diagonal is a Lagrangian submanifold of $X \times X$ we conclude that

$$dH \wedge dT = 0$$

showing once again that the period and the energy are "functionally dependent".

Suppose that $H = c$ defines an energy surface, Z_c, and that the trajectories of ξ make this energy surface into a fiber bundle over Y. Thus Y is the "space of orbits" of ξ. Notice ω, when restricted to Z_c, is of rank $2n - 2$ (where $\dim X = 2n$) and on this surface $\xi \lrcorner \omega = dH_{|\{H=c\}} = 0$. We can identify TY_y with the space $TZ_x/\{\xi_x\}$ where x is any point in the orbit given by y. Since ω induces a nondegenerate antisymmetric bilinear form on $TZ_x/\{\xi_x\}$, we obtain an induced form $\bar{\omega}_y$ on TY_y. As the flow generated by ξ preserves ω this induced form does not depend on x. We thus get a form, $\bar{\omega}$, on Y and it is easy to check that this makes Y into a symplectic manifold. If h is any symplectic automorphism preserving ξ it clearly induces a symplectic automorphism of Y.

For example, let us consider the geodesic flow on the sphere S^n (considered as the unit sphere in $\mathbf{R}^{n+1}$) . The energy surfaces correspond to tangent (or cotangent) vectors of constant length and all trajectories are great circles, traversed at different speeds, the period being inversely proportional to the speed, and hence constant on each energy hypersurface. The group $O(n + 1)$ acts as automorphisms of the flow, and hence also as automorphisms of the associated orbit spaces. A most important special case is where $n = 3$. In this case S^3 is itself a group, and the metric invariant under right and left translations. Since for any group G we can identify $T(G)$ with $G \times g$, where g is the Lie algebra of G, we can identify the unit tangent vectors with $S^3 \times S^2$. As the metric is invariant, the corresponding geodesic flow is just the exponential map and its translates. See, Sternberg [2]. The trajectories are hence of the form $\gamma \times v$ where γ is a great circle determined by the unit vector v and v is a constant unit vector in g. For each fixed v this induces the Hopf fibration of S^3 and we thus see that the space of orbits is $S^2 \times S^2$. What is remarkable about this example is the beautiful result, due to Moser [6], (see also Souriau [9]) that, up to compactification and "regularization", the geodesic flow on a fixed energy surface is the same as the flow on a fixed surface of constant negative energy for the Kepler problem, i.e., for a particle moving according to the inverse square law of attraction from a fixed center. In particular the orbit spaces are the same, i.e., $S^2 \times S^2$. As we shall see, this accounts for the role of the group $O(4)$ in the quantization of the hydrogen atom.

We first make a preliminary remark about "regularizing" a Hamiltonian vector field. Suppose ξ is a Hamiltonian vector field with $\xi \lrcorner \omega = -dH$, and f is a smooth function. Then on any fixed energy hypersurface $H = c$ the vector field $f\xi$ coincides with the restriction to $H = c$ of a Hamiltonian vector field. Indeed let η be the Hamiltonian vector field determined by

$$\eta \lrcorner \omega = -d((H-c)f) = -(H-c)df - fdH.$$

On $H = c$ we get $\eta \lrcorner \omega = -fdH$ so $\eta = f\xi$, and thus $f\xi$ coincides with the Hamiltonian vector field η on $H = c$. (Of course $f\xi$ will not be Hamiltonian everywhere if f and H are not functionally dependent.)

Now multiplying a vector field, ξ, by a function f can be thought of as making a "change of independent variable" $t = \int f ds$ (i.e., $dt/ds = f(z(t))$ where $z(t)$ is a trajectory of ξ). Now if f is nowhere zero, nothing much happens. If f is allowed to become zero, then η might have nicer properties than ξ. For example, ξ might be defined on an open submanifold, U, of a manifold W, becoming "infinite" at ∂U, while $\eta = f\xi$ is defined on all of W. This is precisely what happens in the "collision orbits" of the Kepler problem, i.e., those orbits on which the particle moves on a straight line course to the origin. The velocity becomes infinite as the particle approaches the origin, and so ξ blows up. On the other hand, after multiplication by an appropriate f, these orbits will turn out to be well behaved, indeed to correspond to the tangent vectors to great circles passing through the north pole, while the energy surface will correspond to those tangent vectors lying over the sphere with the north pole removed. Furthermore by suitable normalization it turns out that the relation between t and s will be given by the celebrated Kepler equation $t = s - e \sin s$ where e is the eccentricity of the ellipse. For $e = 1$ the ellipse degenerates into a straight line collision orbit with collision at $t = 0$. For t near zero $t = s - \sin s = s^3/3! + \cdots$ yielding the result that $t^{1/3}$ is a "regularizing parameter".

Let us now give the details. We follow Moser's elegant discussion almost verbatum, with the exception of minor notational changes. Let $x = (x_0, \ldots, x_n)$ be Euclidean coordinates on $\mathbf{R}^{n+1}$ and $\xi = (\xi_0, \ldots, \xi_n)$ the dual coordinates so that (x, ξ) are coordinates on $\mathbf{R}^{n+1} + (\mathbf{R}^{n+1})^* = T^*\mathbf{R}^{n+1}$. The Hamiltonian flow generated by the function $\Phi = \frac{1}{2}\|\xi\|^2\|x\|^2$ gives the differential equations

$$\frac{dx}{ds} = \frac{\partial \Phi}{\partial \xi} = \|x\|^2 \xi \qquad \frac{d\xi}{ds} = -\frac{\partial \Phi}{\partial x} = -\|\xi\|^2 x.$$

It is clear that

$$\frac{d(x \cdot \xi)}{ds} = 0, \qquad \frac{d\|x\|^2}{ds} = 2\|x\|^2 x \cdot \xi,$$

and

$$d\|\xi\|^2/ds = -2\|\xi\|^2(x \cdot \xi).$$

Thus, the (co)tangent bundle to the unit sphere, T^*S^n, given by

$$\|x\|^2 = 1, \qquad x \cdot \xi = 0,$$

is preserved under the flow and on T^*S^n the trajectories are given by

$$\frac{dx}{ds} = \xi \qquad \frac{d\xi}{ds} = -\|\xi\|^2 x$$

or

$$\frac{d^2x}{ds^2} + \|\xi\|^2 x = 0$$

which is clearly the equation of a great circle. Identifying $T^* S^n$ as a submanifold of $T^* \mathbf{R}^{n+1}$ as indicated above thus yields the fact that the restriction of Φ to $T^* S^n$ generates the geodesic flow on $T^* S^n$. We shall continue to denote the restriction of Φ by Φ. The unit tangent vectors constitute the "energy" hypersurface $\Phi = \frac{1}{2}$.

We now let $y = (y_1, \ldots, y_n)$ denote Euclidean coordinates on $\mathbf{R}^n$, with dual coordinates η, and consider stereographic projection from the north pole

$$y_k = \frac{x_k}{1 - x_0}, \qquad k = 1, \ldots, n$$

so that

$$\|y\|^2 = \frac{1 - x_0^2}{(1 - x_0)^2} = \frac{1 + x_0}{1 - x_0}$$

and hence

$$x_0 = \frac{\|y\|^2 - 1}{\|y\|^2 + 1} \qquad x_k = \frac{2y_k}{\|y\|^2 + 1}$$

gives the inverse map. It is easy to check the well-known fact that this map is conformal, more particularly that

$$dx^2 = \sum_0^n dx_k^2 = \frac{4}{(\|y\|^2 + 1)} dy^2.$$

Let S_0^n denote the sphere with the north pole removed. The stereographic projection is a diffeomorphism of S_0^n onto $\mathbf{R}^n$ and hence induces a diffeomorphism of $T^* S_0^n$ with $T^* \mathbf{R}^n$ (which carries the canonical one form of $T^* S_0^n$ onto the canonical one form of $T^* \mathbf{R}^n$) . Thus we will get $\xi = \xi(y, \eta)$ and $\eta = \eta(x, \xi)$ with $\xi \cdot dx = \eta \cdot dy$, and where, in fact, the $\xi = \xi(y, \eta)$ and the $\eta = \eta(x, \xi)$ are determined by this equation. Using the fact that $\xi \cdot x = 0$ and $\sum x_i^2 = 1$ it is easy to check that $\eta(x, \xi)$ is given by $\eta_k = (1 - x_0)\xi_k + \xi_0 x_k$ and since $\eta \cdot y = \sum \eta_k y_k = \xi_0$ we get

$$\xi_0 = \eta \cdot y \qquad \xi_k = \frac{\|y\|^2 + 1}{2} \eta_k - (\eta \cdot y) y_k$$

so that

$$\|\xi\| = \frac{\|y\|^2 + 1}{2}\|\eta\|.$$

We can now use the diffeomorphism of $T^* S_0^n$ with $T^*\mathbf{R}^n$ to transfer the Hamiltonian which becomes

$$F(y,\eta) = \frac{(\|y\|^2 + 1)^2 \|\eta\|^2}{8}.$$

If u is any function of a real variable with $u'(\frac{1}{2}) = 1$, then $u(F)$ and F will define the same trajectories on the energy surface $F = \frac{1}{2}$. Let us take $G = u(F) = \sqrt{2F} - 1$. Then

$$G = \frac{(\|y\|^2 + 1)\|\eta\|}{2} - 1$$

and the corresponding Hamiltonian vector field, $\boldsymbol{\eta}$, is defined by $\boldsymbol{\eta} \lrcorner \omega = -dG$. Let us now introduce the vector field $\boldsymbol{\xi}$ given by

$$\boldsymbol{\eta} = \|\eta\|\boldsymbol{\xi}$$

so that $\boldsymbol{\xi}$ is defined only for $\|\eta\| \neq 0$. Now $F = \frac{1}{2}$ corresponds to $G = 0$ and so the vector field $\boldsymbol{\xi}$ is generated by the Hamiltonian

$$H = \|\eta\|^{-1}G - \tfrac{1}{2} = \|\eta\|^{-1}(\sqrt{2F} - 1) - \tfrac{1}{2} = \frac{\|y\|^2}{2} - \frac{1}{\|\eta\|},$$

Let us set $p = -y$ and $q = \eta$ so that $dp \wedge dq = d\eta \wedge dy$. Then

$$H = \tfrac{1}{2}\|p\|^2 - 1/\|q\|$$

which is the Hamiltonian for the Kepler problem. The energy surface $\Phi = \frac{1}{2}$ corresponds to $H = -\frac{1}{2}$. We may get a more general energy hypersurface as follows: Let T_λ denote the transformation of (q,p) space given by

$$T_\lambda \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \lambda^2 q \\ \lambda^{-1} p \end{pmatrix}.$$

Then T_λ is not symplectic, but $T_\lambda^* \, dp \wedge dq = \lambda \, dp \wedge dq$ so that

$$\{f,g\} \circ T_\lambda = \lambda\{f \circ T_\lambda, g \circ T_\lambda\},$$

while

$$H \circ T_\lambda = \lambda^{-2}.$$

This shows that if we change the time as well according to $t \rightsquigarrow \lambda^3 t$ so that we consider the transformation

$$S_\lambda \begin{pmatrix} q \\ p \\ t \end{pmatrix} = \begin{pmatrix} \lambda^2 q \\ \lambda^{-1} p \\ \lambda^3 t \end{pmatrix},$$

then S_λ carries solutions of the equations of motion into themselves. This is essentially the content of Kepler's third law which he formulated as saying that the period of an orbit varies as the 3/2 power of its major axis.

Now consider the set of all (x, ξ) with $\|\xi\| = \rho$, and its image under stereographic projection, i.e. the hypersurface $F = \frac{1}{2}\rho^2$. Set

$$G = (2\rho^2 F)^{\frac{1}{2}} - \rho^2 = \frac{1}{2}\rho(\|y\|^2 + 1)\|\eta\| - \rho^2,$$

so that the surface is given by $G = 0$. Then

$$\rho^{-3}\|\eta\|^{-1}G = \frac{\|y\|^2}{2\rho^2} - \frac{1}{\rho\|\eta\|} + \frac{1}{2\rho^2}.$$

Make the symplectic change of coordinates $p = -y/\rho$, $q = \rho\eta$. The surface $G = 0$ is carried into the energy surface $H = 1/2\rho^2$ for the Kepler problem.

We can summarize the discussion as follows: Each of the energy surfaces $\|\xi\|^2 = \frac{1}{2}\rho^2$ is a manifold of dimension $2n - 1$ which is fibered by circles (the geodesics) over a symplectic manifold of dimension $2n - 2$. Under stereographic projection, the $2n - 2$ dimensional manifold (less the points sitting over the north pole) goes over into a manifold that can be identified with the space of orbits of the Kepler problem with energy $H = -1/2\rho^2$. In particular, the induced map of the $2n - 2$ dimensional manifold onto the $2n - 2$ dimensional space of orbits of the Kepler problem is a symplectic diffeomorphism. For $n = 3$, the "completed" space of orbits is topologically $S^2 \times S^2$.

Let us analyze, in terms of more classical terminology, the parameters that enter into the Kepler orbits in terms of the stereographic projection, $(x, \xi) \to (y, \eta) \to (-p, q)$. A given orbit corresponds to a great circle. Since the whole picture is invariant under all rotations about the x_0 axis, we may assume that the orbit lies in the subspace $x_3 = \cdots = x_n = 0$ and is described by the equations

$$x_0 = \sin\alpha\cos s \qquad x_1 = \sin s \qquad x_2 = -\cos\alpha\cos s$$

so

$$\xi_0 = -\sin\alpha\sin s \qquad \xi_1 = \cos s \qquad \xi_2 = \cos\alpha\sin s$$

and thus setting $e = \sin\alpha$

$$p_1 = -\frac{\sin s}{1 - e\cos s} \qquad p_2 = \frac{\sqrt{1 - e^2}\cos s}{1 - e\cos s}$$

$$q_1 = \cos s - e \qquad q_2 = \sqrt{1 - e^2}\sin s$$

so that the p's describe a circle while the q's describe an ellipse of eccentricity e

parametrized by the eccentric anomaly, s; (see Sternberg [2]). Now $\|q\|^2 = (1 - e \cos s)^2$ so

$$t = \int_0^s |q|\, ds = \int_0^s (1 - e \cos s)\, ds = s - e \sin s$$

showing that our regularization procedure does indeed reduce to Kepler's equation.

We refer the reader to Moser [6] and Weinstein [8] for applications of these results to the existence of periodic orbits for the three body problem under conditions which can be regarded as small perturbation of Kepler motion. More generally these papers discuss the problem of establishing periodic orbits for systems which are perturbations of a system having manifolds of periodic orbits. Our main use of these examples will be to the study of a "quantization procedure" to be described in the next chapter and to the study of the asymptotic distribution of eigenvalues of elliptic operators.

The preceding considerations show why the group $O(4)$ enters as a symmetry group of the Kepler problem, on the set of orbits of some fixed (negative) energy. (This fact is strikingly confirmed in the study of atomic spectra. Each energy level of the hydrogen atom occurs with a multiplicity corresponding to an irreducible representation of $O(4)$. In comparison, the alkali metals exhibit a spectrum corresponding only to $O(3)$ symmetry. However for the higher energy levels of the outer electron, where the potential is approximately of the form k/r, an approximate $O(4)$ degeneracy appears.) If we consider the set of orbits of all negative energies, it is known that a still larger group, the group $SO(2,4)$, of all orthogonal transformations in six dimensions preserving a quadratic form of signature $++----$ acts as a group of symmetries. That is, let T^+S^3 denote the subset of T^*S^3 consisting of the non-zero covectors. We shall show that the group $SO(2,4)$ acts transitively as a group of symplectic diffeomorphisms on T^+S^3. (In all that follows we could replace 3 by n and $SO(2,4)$ by $SO(2, n+1)$ etc.)

Let G be a Lie group and g its Lie algebra. If $\xi \in g$, then $\exp t\xi$ is a one parameter subgroup of G, where exp: $g \subset G$ denotes the exponential map, and the most general one parameter subgroup of G is of this form. If a is some element of G, then $a(\exp t\xi)a^{-1}$ is again a one parameter subgroup, and so is of the form $\exp t\zeta$. The map $\xi \rightsquigarrow \zeta$ is linear–we write $\zeta = \mathrm{Ad}_a\, \xi$. This assigns to each $a \in G$ a linear transformation; in this way we get a linear representation of G on G known as the adjoint representation. Since G acts on g, it has a contragredient action on the dual space, g^*: for any $l \in g^*$ we define the element $a \cdot l$ by

$$\langle \xi, a \cdot l \rangle = \langle \mathrm{Ad}_{a^{-1}}\, \xi, l \rangle.$$

This representation is known as the coadjoint representation. For any $\eta \in g$ and any $l \in g^*$ we can consider the curve $(\exp t\eta) \cdot l$. We denote the tangent to this curve at $t = 0$ by η_l. Since

$$\frac{d}{dt}\,\mathrm{Ad}_{(\exp t\eta)}\,\xi_{|t=0} = [\eta, \xi],$$

where [,] denotes Lie bracket, we have

$$\langle \xi, \eta_l \rangle = \frac{d}{dt}\langle \mathrm{Ad}_{(\exp - t\eta)}\,\xi, l\rangle_{|t=0} = -\langle [\eta, \xi], l\rangle.$$

The vectors η_l span the tangent space to the orbit, $G \cdot l$, at the point l. In the next section we shall prove that for any Lie group, G, and any $l \in g^*$, the orbit $G \cdot l$ is always a symplectic manifold, and the symplectic form ω is given by

$$\langle \xi_l \wedge \eta_l, \omega\rangle = -\langle [\xi, \eta], l\rangle.$$

(In fact, we shall prove that all symplectic manifolds on which G acts transitively as a group of symplectic diffeomorphisms can be obtained by a slight modification of the preceding construction.) We shall now show how T^+S^3 can be regarded as a particular orbit in the dual of the Lie algebra of $SO(2,4)$. For this purpose, it is convenient to have the following description of the Lie algebra: Let V be a real vector space equipped with a non-degenerate scalar product, (,). Let $o(V)$ denote the Lie algebra of the orthogonal group of this scalar product. We can identify $\wedge^2(V)$ with $o(V)$ by letting $u \wedge v$ act as the linear transformation

$$(u \wedge v)w = (v, w)u - (u, w)v.$$

It is easy to check that this does indeed define a map which is equivariant with respect to the action of $O(V)$ on both sides. Similarly, the metric on V induces an isomorphism of V with V^*, and hence of $\wedge^2(V)$ with its dual space. Hence we may identify $o(V)^*$ with $\wedge^2(V)$ as well. In particular, we have the identification of $o(2,4)$ with $\wedge^2(\mathbf{R}^{2,4})$.

Let us consider the set of all elements of $o(2,4) \sim \wedge^2(\mathbf{R}^{2,4})$ of the form

$$0 \neq f \wedge f' \quad \text{where } \|f\|^2 = \|f'\|^2 = 0 \text{ and } (f, f') = 0.$$

(As linear transformations, these elements can be characterized as those nilpotent elements of rank two whose range (which is two dimensional) is totally isotropic.) By Witt's theorem, any two such elements are conjugate under $O(2,4)$. We shall now show that under the connected group $SO(2,4)$ they split into two components. Let us choose an orthonormal basis, $e_{-1}, e_0, e_1, e_2, e_3, e_4$ of $\mathbf{R}^{2,4}$, with e_{-1} and e_0 positive definite and the rest negative. We claim that for any such element $f \wedge f'$ we must have

$$(e_{-1} \wedge e_0, f \wedge f') \neq 0. \tag{6.7}$$

Indeed, suppose that $(e_{-1} \wedge e_0, f \wedge f') = 0$. Then the space spanned by f and f', i.e. the range of $f \wedge f'$, would have to contain some non-zero vector orthogonal to both e_{-1} and e_0. But such a vector would lie in $\mathbf{R}^4$ and could not be isotropic. Thus, under the connected group, $SO(2,4)$, the set of all $f \wedge f'$ splits into two components according to whether the scalar product in (6.7) is positive or negative. Let $\mathcal{V}$ denote the set of $f \wedge f'$ for which the scalar product is positive. Without changing the element $f \wedge f'$, we can choose f and f' so that $(f, e_0) = 0 = (f', e_{-1}) = 0$, while (f, e_{-1}) and (f', e_0) are both positive. This shows that every element of $\mathcal{V}$ has a unique representation as

$$s(e_{-1} + p) \wedge (e_0 + q) \quad \text{where } s > 0, p, q \in \mathbf{R}^4, (p,q) = 0 \quad \text{and } \|p\|^2 = \|q\|^2 = -1. \tag{6.8}$$

Now we can think of p as ranging over the unit sphere in $\mathbf{R}^4$, and then for fixed p, the q ranges over the unit tangent vectors to the sphere. Thus (6.8) becomes identified with a non-zero tangent vector to the three sphere. Using the Riemann metric we may identify TS^3 and T^*S^3, and so we see that $\mathcal{V}$ is diffeomorphic to $T + S^3$. We shall soon see that this is a symplectic diffeomorphism, when we put the canonical symplectic form on T^*S^3.

Let us first examine the action of $SO(2)$ and $SO(4)$. Any $A \in SO(4)$ preserves e_{-1} and e_0 and sends p into Ap and q into Aq. This is clearly the induced action of $SO(4)$ on TS^3. The element R_θ in $SO(2)$ sends e_{-1} to $\cos\theta e_{-1} + \sin\theta e_0$ and e_0 to $-\sin\theta e_{-1} + \cos\theta e_0$. Thus

$$\begin{aligned} R_\theta[(e_{-1} + p) \wedge (e_0 + q)] &= (\cos\theta e_{-1} + \sin\theta e_0 + p) \wedge (-\sin\theta e_{-1} + \cos\theta e_0 + q) \\ &= (e_{-1} + \cos\theta p - \sin\theta q) \wedge (e_0 + \sin\theta_p + \cos\theta q). \end{aligned}$$

In other words, from the point of view of TS^3, the map R_θ moves (p, q) through angle θ along the circle determined by p and q. But this is precisely the image of (p, q) under the geodesic flow. Thus for *unit* vectors, the group $SO(2)$ acts as geodesic flow. Let F_t denote the geodesic flow. The above computation shows that for vectors of length s, we have $R_\theta(p,q) = F_{s^{-1}\theta}(p,q)$. To express this slightly differently, let $H = \frac{1}{2}\|q\|^2 = -\frac{1}{2}s^2$. The geodesic flow has, as its infinitesimal generator, the vector field ξ_H corresponding to $-dH$ under the identification of differential forms and vector fields on T^*S^3 given by the symplectic form. Suppose we consider the function $\sqrt{-2H}$. It gives rise to the vector field $-(-2H)^{-1/2}\xi_H = -s^{-1}\xi_H$. We thus see that the action of R_θ is the same as that generated by a vector field corresponding to the function $(-2H)^{1/2}$.

Let $O(1,4)$ be the subgroup of $O(2,4)$ which fixes the vector e_{-1}. It is clear that $SO(1,4)$ acts transitively on $\mathcal{V}$. Indeed, let $\mathbf{R}^{1,4}$ be the subspace of $\mathbf{R}^{2,4}$ spanned by e_0, e_1, e_2, e_3, and e_4. By Witt's theorem, $O(1,4)$ acts transitively on the set of all pairs of the form (p,f') where $\|p\|^2 = -1$, $\|f'\|^2 = 0$ and $\langle p,f'\rangle = 0$, with p and $f' \neq 0$ in $\mathbf{R}^{1,4}$. Now f' cannot be orthogonal to e_0, since it would then lie in $\mathbf{R}^4$ and could not be isotropic. Thus any such f' can be written as $f' = s(e_0 + q)$

where $\|q\|^2 = -1$ and $\langle p, q\rangle = 0$, $s \neq 0$. The group $SO(1,4)$ acts transitively on those pairs for which $s > 0$, and these can clearly be identified with points of $\mathcal{V}$. The map of $\mathcal{V} \to o(1,4) \sim o(1,4)^*$ sending $s(e_{-1} + p) \wedge (e_0 + q)$ to $p \wedge s(e_0 + q)$ is manifestly equivariant with respect to $SO(1,4)$ and sends $\mathcal{V}$ onto one of the two components of the set of (non-zero) nilpotent elements in $o(1,4)$.

For the group $SO(1,4)$ we can give a rather direct interpretation of its action on $T^* S^3$. Indeed, $SO(1,4)$ can be identified as the (connected component of the) group of all conformal transformations of S^3, and hence has an induced action on $T^* S^3$, and we claim that this induced action is exactly the orbit action described above. To make the identification, we regard S^3 as the set of "forward light-like directions" in $\mathbf{R}^{1,4}$. That is, we consider the "forward light cone" in $\mathbf{R}^{1,4}$ consisting of all null vectors f' with $\langle e_0, f'\rangle > 0$. The group $SO(1,4)$ acts transitively on all such vectors, and hence on the set of all rays, i.e. on equivalence classes of such vectors where two vectors are identified if they differ by a positive scalar multiple. The set of such rays is topologically S^3, and we can parametrize them explicitly by S^3 once we choose the unit vector e_0, namely each such ray has a unique representative of the form $e_0 + q$ with $\|q\|^2 = -1$. We thus obtain an action of $SO(1,4)$ on S^3 which is easily checked to be conformal. The cotangent space at f' to S^3 can be identified as the quotient space of the space of all vectors orthogonal to the light cone, modulo the line spanned by f'. If $f' = e_0 + q$, this space can be identified with the set of all vectors of the form sp where $p \in \mathbf{R}^4$, $\langle p, q\rangle = 0$, and $\|p\|^2 = -1$. It now follows readily that the action of $SO(1,4)$ on $\mathcal{V}$ can be identified with the induced action of $SO(1,4)$ on $T^+ S^3$ coming from the conformal action on S^3. Furthermore, if we think of p as a covector, and if $\boldsymbol{\xi}$ is any element of $o(1,4)$, whose corresponding tangent vector at $[(e_0 + q)]$ is denoted by $\boldsymbol{\xi}_{[(e_0+q)]}$, then the value of the covector p on the tangent vector $\boldsymbol{\xi}_{[(e_0+q)]}$ is given by

$$-(p, \boldsymbol{\xi} \cdot (e_0 + q)) = -\langle p \wedge (e_0 + q), \boldsymbol{\xi}\rangle$$

(where the scalar product on the left is between two vectors in $\mathbf{R}^{1,4}$ and the scalar product on the right is between two vectors in $o(1,4)$).

We can now use the above remarks to show that the three different identifications that we have made of $\mathcal{V}$–as nilpotent elements in $o(2,4)$, as nilpotent elements of $o(1,4)$ and as $T^+ S^3$ are symplectic diffeomorphisms. Since the passage from $o(2,4)^*$ to $o(1,4)^*$ is the restriction map, it follows from general considerations, or directly from the formula

$$\langle \boldsymbol{\xi}_l \wedge \boldsymbol{\eta}_l, \omega_{\mathcal{V}}\rangle = \langle [\boldsymbol{\xi}, \boldsymbol{\eta}], l\rangle$$

for the value of the symplectic form, $\omega_{\mathcal{V}}$, of $\mathcal{V}$ evaluated at the image vectors, $\boldsymbol{\xi}_l$ and $\boldsymbol{\eta}_l$ of $\boldsymbol{\xi}$ and $\boldsymbol{\eta} \in o(2,4)$ at $l \in \mathcal{V}$ that the identification of $\mathcal{V}$ either as a subvariety in $o(2,4)^*$ or in $o(1,4)^*$ is a symplectic diffeomorphism. We now show that the identification of $\mathcal{V}$ as an orbit in $o(1,4)^*$ or as T^+S^3 is symplectic. For this purpose it is convenient to make use of the following lemma:

Let ξ and η be vector fields on the differentiable manifold, M, and let $\hat{\xi}$ and $\hat{\eta}$ be the induced vector fields on T^*M. Let α be the fundamental one form on T^*M, let $\omega = d\alpha$ be the fundamental two form, let z be some point of T^*M, and let $x = \pi z$ be its base point in M. Then

$$\langle \hat{\xi}_z \wedge \hat{\eta}_z, \omega \rangle = -\langle [\hat{\xi}, \hat{\eta}]_z, \alpha \rangle = -\langle [\xi, \eta]_x, z \rangle.$$

Proof of Lemma. Since $\hat{\xi}$ is an infinitesimal automorphism of the cotangent bundle structure, we have $D_{\hat{\xi}}\alpha = 0$, where D denotes Lie derivative. Thus $0 = D_{\hat{\xi}}\alpha = \hat{\xi} \rfloor d\alpha + d(\hat{\xi} \rfloor \alpha)$ or

$$\hat{\xi} \rfloor \omega = -d\langle \xi, \alpha \rangle.$$

Thus

$$\langle \hat{\xi} \wedge \hat{\eta}, \omega \rangle = -\hat{\eta} \rfloor \hat{\xi} \rfloor \omega = -\hat{\eta} \rfloor d(\hat{\xi} \rfloor \alpha) = -D_{\hat{\eta}}(\hat{\xi} \rfloor \alpha) = [\hat{\xi}, \hat{\eta}] \rfloor \alpha$$

since $D_{\hat{\eta}}\alpha = 0$. Evaluating at z and using the definition $\langle \zeta_z, \alpha \rangle = \langle d\pi\zeta_z, z \rangle$ for any tangent vector ζ_z at z proves the lemma. Taking $z = p$ and $x = [e_0 + q]$, and using the formulas derived above and the lemma, proves that the identifications of $\mathcal{V}$ with T^+S^3 is a symplectic diffeomorphism.

Notice that the group $SO(1,4)$ not only preserves the symplectic structure on T^*S^3, it also preserves the cotangent fibration. On the other hand geodesic flow certainly does not preserve the cotangent fibration, and hence $SO(2)$ does not. Since $SO(1,4)$ is a maximal connected subgroup of $SO(2,4)$ we conclude that $SO(1,4)$ is the largest connected subgroup preserving the cotangent fibration.

§7. Homogeneous symplectic spaces.

Let the Lie group G act on the manifold X. Thus we are given a smooth map $G \times X \to X$, $(a,x) \to ax$ which is a group action. The differential of this map induces a map of $T_aG \times T_xX \to T_{ax}X$. In particular, taking $a = e$, the identity of G, then for any $\xi \in g = T_eG$, the Lie algebra of G, we obtain a vector field $\bar{\xi}$ on X where $\bar{\xi}_x$ is the image of $(\xi, 0)$ under the map $T_eG \times T_xX \to T_xX$. Thus $\bar{\xi}$ is the infinitesimal generator of the flow $x \to (\exp t\xi)x$.

As

$$\frac{d}{dt}\left\{\left[a^{-1}(\exp t\xi)a\right]x\right\} = (a^*\bar{\xi})(x)$$

it follows that

$$a^*\bar{\xi} = (Ada^{-1})\xi.$$

If we set $a = \exp s\eta$ and differentiate with respect to s we get

$$[\bar{\eta}, \bar{\xi}] = -[\eta, \xi].$$

(The reason for the confusing change of sign is that we are identifying the Lie algebra of G with the set of left invariant vector fields G, and hence they generate *right* translations. The opposite convention is sometimes used, for example by Souriau [9].)

If we set

$$\hat{\xi} = -\bar{\xi}$$

then the map $\xi \to \hat{\xi}$ gives a homomorphism of g into the Lie algebra of vector fields of X.

Now suppose that X is a symplectic manifold, and that G acts as a group of symplectic automorphisms. Then $\hat{\xi}$ satisfies

$$D_{\hat{\xi}}\omega = 0 \qquad \text{or} \qquad d(\hat{\xi} \lrcorner \omega) = 0.$$

We say that the action is *strongly symplectic if* $\hat{\xi} \lrcorner \omega = df_\xi$ for some function f_ξ. Notice that

$$\begin{aligned} [\hat{\xi}, \hat{\eta}] \lrcorner \omega &= D_{\hat{\xi}}(\hat{\eta} \lrcorner \omega) \\ &= \hat{\xi} \lrcorner d(\hat{\eta} \lrcorner \omega) + d(\hat{\xi} \lrcorner (\hat{\eta} \lrcorner \omega)) \\ &= d(\hat{\xi} \lrcorner \hat{\eta} \lrcorner \omega) \end{aligned}$$

so that $[\xi, \eta]^\wedge \lrcorner \omega$ is always of the form $-df$. Thus if $[g, g] = g$, then all symplectic actions are strongly symplectic. (This happens, for example, if g is semi-simple or if g is the semi-direct product of a semi-simple algebra k acting on a finite dimensional vector space V with no trivial component in the representation of k on V.) Suppose that the action is strongly symplectic. By choosing a basis of the Lie algebra and an f_{ξ_i} for each ξ_i in the basis, we can write

$$\hat{\xi} \lrcorner \omega = -d \sum a_i f_{\xi_i} \qquad \text{if } \xi = \sum a_i \xi_i.$$

Put another way, we can define a function $f: X \to g^*$ where

$$\langle \xi, f \rangle = \sum a_i f_{\xi_i}$$

and then

$$\hat{\xi} \lrcorner \omega = -d(\langle \xi, f \rangle).$$

Notice that f is determined only up to an additive constant (in g^*). Now for any $a \in G$ we have

$$a^*(\hat{\xi} \lrcorner \omega) = d\langle \xi, a^* f\rangle$$

since ξ is a constant in this equation. But $a^*\omega = \omega$ and $a^*\hat{\xi} = (Ada^{-1}\xi)^\wedge$ so setting $\eta = Ada^{-1}\xi$ we see that

$$\begin{aligned} \hat{\eta} \lrcorner \omega &= d\langle Ada\eta, a^* f\rangle \\ &= d\langle \eta, (Ada)^*(a^* f)\rangle. \end{aligned}$$

Here $(Ada)^* : g^* \to g^*$ is just the adjoint of Ada. In particular

$$(Ada)^*(a^* f) - f$$

is a constant on X. Replacing a by a^{-1} gives us the fact that

$$\rho_a f - f = z_a$$

is a constant where ρ_a is the representation

$$\rho_a f(x) = (Ad^{\#} a) f(a^{-1} x)$$

and $Ad^{\#}$ is the representation of G on g^* given by the contragredient to the adjoint representation. It is clear that $a \to z_a$ is a cocycle of G with values in g^*, i.e., satisfies the identity

$$z_{ab} = (Ad^{\#} a) z_b + z_a.$$

Replacing f by $f + c$ where $c \in g^*$ is a constant has the effect of changing z_a by adding the coboundary $(Ad^* a)c - c$, and thus the cohomology class is well defined. It represents the obstruction to choosing f: $X \to g^*$ to be a G map. If this obstruction vanishes then

$$\langle \eta, a^* f\rangle = \langle Ada^{-1}\eta, f\rangle$$

and setting $a = \exp t\xi$ and differentiating with respect to t gives

$$D_{\hat{\xi}}\langle \eta, f\rangle = \langle [\xi, \eta], f\rangle$$

or

$$D_{\hat{\xi}} f_\eta = f_{[\xi,\eta]}.$$

But $D_{\hat{\xi}} f_\eta = \hat{\xi} \lrcorner df_\eta$ is just the definition of the Poisson bracket, $\{f_\xi, f_\eta\}$. Thus in this case

$$\{f_\xi, f_\eta\} = f_{[\xi,\eta]}$$

i.e., the map $\xi \to f_\xi$ is a homomorphism of Lie algebras. An action of g on X together with such a lifting of the homomorphism $\xi \to \hat{\xi}$ to $\xi \to f_\xi$ (if it is possible) is called a *Hamiltonian* action of G on X. The symplectic actions are of basic importance in classical mechanics. (See for example Souriau [9] for an elegant discussion of many of the classical laws of physics from this point of view.) In particular, a symplectic action is called *elementary* if the action of G on X is transitive. (This is the classical analogue of the quantum notion of an elementary particle as an irreducible representation of G.) If G acts transitively on X and this action is Hamiltonian, then the equation

$$(Ad^{\#} a)f = f \circ a$$

shows that f maps X onto an orbit. Notice also that f is an *immersion*. Indeed since G acts transitively, the vector fields ξ span the tangent space at each point. If $df_x(\hat{\xi}) = \langle \xi, df \rangle_x = 0$ then $\langle \hat{\xi}, df_\eta \rangle_x = 0$ for all η, implying that $\hat{\xi}_x = 0$ since the $df_\eta = \hat{\eta} \rfloor \omega$ span the cotangent space. Thus X is a covering space of an orbit in g^*. This suggests that the orbits in g^* are symplectic manifolds which is indeed true, a fact due to Kirillov, Kostant and Souriau. We shall develop these facts from a more general point of view as developed by Chu [10].

Let G be a Lie group and $X = G/H$ a homogeneous space for G where H is a closed subgroup, and let $\pi\colon G \to G/H = X$ be the projection. If Ω is an invariant form on X then it is clear that $\sigma = \pi^* \Omega$ is a left invariant form on G which satisfies

(i) $\xi \rfloor \sigma = 0$ for all $\xi \in h$ where h is the Lie algebra of H;

(ii) σ is invariant under right multiplication by elements of H, and hence under Ad for elements of H.

Conversely, it is clear that any left invariant form on G satisfying (i) and (ii) arises from G/H. If Ω is a symplectic form then it is clear that a left invariant vector field will satisfy $\xi \rfloor \sigma = 0$ if and only if $\xi \in h$. Furthermore, since $d\sigma = 0$, the set of all vector fields satisfying $\xi \rfloor \sigma = 0$ forms an integrable subbundle of TG, and in particular, the left invariant ones form a subalgebra of the Lie algebra of G. Let us call it h_σ. We have thus recovered h. Let H_σ be the group generated by h_σ. Notice that for any $\xi \in h_\sigma$ we have $D_\xi \sigma = \xi \rfloor d\sigma + d(\xi \rfloor \sigma) = 0$ so that σ is invariant under H_σ. The only problem is that H_σ need not be closed. Let us say that σ is *regular* if H_σ is closed. Notice that if G/H is a symplectic homogeneous space, so that H is a closed subgroup, and if we construct σ as above then H_σ is just the connected component of the identity in H (and hence a closed subgroup of G). We have thus established:

PROPOSITION 7.1 (*Chu* [10]). *Each 2p-dimensional homogeneous symplectic space determines a left invariant regular closed two form of rank 2p on G. Conversely, a regular closed two form determines a homogeneous symplectic space. Up to covering, the space of all homogeneous symplectic manifolds for G is the same as the space of* *orbits of G acting on* $Z^2_{\text{reg}}(g)$, *where* $Z^2_{\text{reg}}(g)$ *denotes the set of regular two cocycles of* g.

Notice that if $\sigma = d\beta$ where β is a left invariant one form then σ is automatically regular. Indeed $D_\xi \beta = \xi \rfloor d\beta + d(\xi \rfloor \beta) = \xi \rfloor \sigma$ for any left invariant vector field ξ since $\xi \rfloor \beta$ is constant. Thus $\xi \in h_\sigma$ if and only if $D_\xi \beta = 0$. Now $H = \{a \mid (Ad^\# a)\beta = \beta\}$ is clearly a closed subgroup, and H_σ is the identity component. Thus, if $H^2(g) = \{0\}$, every cocycle is regular. We can also show that if $H^1(g) = \{0\}$ then every cocycle is regular. Indeed, to say that $H^1(g) = \{0\}$ is the same as saying that d: $g^* \to \wedge^2(g^*)$ is injective. But then $\xi \rfloor \sigma = 0$ is equivalent to $D_\xi \sigma = d(\xi \rfloor \sigma) = 0$ so h_σ is the Lie algebra of the isotropy group of σ. This argument was pointed out to us by Ofer Gabber. In any event, it is clear from the foregoing discussion that:

PROPOSITION 7.2 (*Kirillov-Kostant-Souriau*). *Each orbit,* $Ad^\#(G)\beta$ *for* $\beta \in g^*$ *is a symplectic manifold whose symplectic structure is induced from* $d\beta$.

We now assert:

PROPOSITION 7.3 (*Chu* [4]). *If G is a simply connected Lie group then every left invariant closed two form is regular.*

We sketch the proof. Let σ be a closed two form. We can think of σ as a one cocycle, f, from g to g^*, where $f(\xi) = \xi \rfloor \sigma$. Here f is a cocycle relative to the action, $ad^\#$, of g on g^*. Hence f defines an action of g as affine transformations on g^* via

$$\begin{aligned} \xi \cdot \theta &= (ad^\# \xi)\theta + f(\xi) \\ &= (ad^\# \xi)\theta + \xi \rfloor \sigma. \end{aligned}$$

Since G is simply connected this defines an affine action of G on g^*. It is clear that $\xi \in h_\sigma$ if and only if $\xi \cdot 0 = 0$. Thus H_σ is the identity component of the isotropy group of the origin and hence closed.

As the orbits in g^* represent (up to coverings) the "universal" elementary symplectic homogeneous spaces, it becomes important to analyze them. In this regard, for complex semi-simple groups see Kostant [11], for real semi-simple groups, see Rothschild [12], while for nilpotent groups see [20].

Let σ be a left invariant closed two form on G and suppose that the subalgebra h has minimum dimension among all subalgebras of the form h_σ. This implies that if σ_t is a curve of closed two forms with $\sigma_0 = \sigma$, then any $\xi \in h_\sigma$ can be extended to a curve ξ_t with $\xi_0 = \xi_0$ and $\xi_t \in h_{\sigma_t}$. (Indeed, choose a subspace m complementary to h_σ in the Lie algebra TG_e. Since dim h_σ is minimal, this implies that dim h_σ = dim $h_{\sigma'}$ for all σ' close to σ. Then projection along m defines an isomorphism of h_σ with $h_{\sigma'}$ for all σ' close to σ.). In particular, let θ be any left invariant one form and consider

$$\sigma_t = \sigma + td\theta.$$

We can write $\xi_t = \xi + t\xi' + O(t^2)$. Examining the coefficient of t in the equation $\xi_t \lrcorner \sigma_t = 0$ gives

$$\xi \lrcorner d\theta + \xi' \lrcorner \sigma = 0.$$

Let η be some other element of h_σ and take the interior product of this last equation with η. The term $\eta \lrcorner \xi' \lrcorner \sigma = -\xi' \lrcorner \eta \lrcorner \sigma = 0$ and we get

$$\eta \lrcorner \xi \lrcorner d\theta = 0.$$

Now, since η and θ are both left invariant, $\eta \lrcorner \theta$ is a constant and therefore $0 = D_\xi(\eta \lrcorner \theta) = D_\xi \eta \lrcorner \theta + \eta \lrcorner D_\xi \theta = [\xi, \eta] \lrcorner \theta + \eta \lrcorner \xi \lrcorner d\theta$, since $\xi \lrcorner \theta$ is also constant. Thus

$$[\xi, \eta] \lrcorner \theta = 0.$$

Since this holds for arbitrary θ we conclude that $[\xi, \eta] = 0$. We have thus proved

PROPOSITION 7.4. *Let σ be a left invariant closed two form such that h_σ has minimal dimension. Then h_σ is commutative. In particular, let X be a homogeneous symplectic manifold of G with maximal dimension. Then the connected component of the isotropy group of any point of X is commutative.*

For the case that $\sigma = d\theta$ is an exact two form this result was obtained by Duflo and Vergne [14]. (It is just a trivial observation to remark that their proof works just as well for the case of closed two forms.) For the case where G is a semi-simple group, the dual of the Lie algebra can be identified with the Lie algebra via the Killing form. In this case, to say that $h_{d\theta}$ has minimal dimension becomes the assertion that the centralizer of the corresponding element, θ, has minimal dimension, and Proposition 7.4 reduces to the classical assertion that for such regular elements the centralizer is abelian. For regular semi-simple elements the subalgebra $h_{d\theta}$ is a Cartan subalgebra.

For semi-simple subalgebras one has a conjugacy theorem for Cartan subalgebras, which, in the real case, can be formulated as asserting that if θ is generic, then $h_{d\theta'}$ is conjugate to $h_{d\theta}$ under the adjoint group if θ' is sufficiently close to θ. One can ask to what extent this remains true in the general case. It is not true for all Lie algebras as is shown by the following example: let $g = \mathbf{R} + V$ where V is the trivial Lie algebra (a vector space with trivial bracket) and $[r, v] = rv$ for $r \in \mathbf{R}$ and $v \in V$. It is easy to see that for any $\theta \in g^*$ which does not vanish on V the subalgebra $h_{d\theta}$ consists of the hyperplane in V defined by the equation

$\theta(v) = 0$, corresponding to two dimensional orbits in g^*. It is clear that no two such subalgebras are conjugate to one another if they are distinct. Let us call a θ in g^* *stable* if $h_{d\theta'}$ is conjugate to $h_{d\theta'}$ for all θ' close to θ.

PROPOSITION 7.5. *Suppose that $h_{d\theta}$ has minimal dimension and that $[g, h_{d\theta}] \cap h_{d\theta} = \{0\}$. Then θ is stable and conversely.*

PROOF. It is clear that for any θ' on the orbit of θ the algebras $h_{d\theta}$ and $h_{d\theta'}$ are conjugate. Thus we will be done if we can find a submanifold, W, transversal to the orbit through θ with the property that $h_{d\theta'} = h_{d\theta}$ for all $\theta' \in W$ (near θ). By the implicit function theorem we can reduce the problem to the corresponding infinitesimal problem: to show that every θ' can be written as $\theta_1 + \theta_2$ where $\theta_1 \in g \rfloor d\theta$ (the tangent space to the orbit) and $h_{d\theta} \rfloor d\theta_2 = 0$ (which, on account of the minimality of dim $h_{d\theta}$ is the same as saying that $h_{d(\theta+\theta_2)} = h_{d\theta}$ if θ_2 is sufficiently small). It therefore suffices to show that no vector in g can be annihilated by all such θ_1 and θ_2. Now to say that $\langle \xi, g \rfloor d\theta \rangle = 0$ is the same as saying that $\xi \rfloor d\theta = 0$, i.e. that $\xi \in h_{d\theta}$. To say that $\langle \xi, \theta_2 \rangle = 0$ for all θ_2 means that $\langle \xi, \theta_2 \rangle = 0$ for all θ_2 with the property that $\langle [g, h_{d\theta}], \theta_2 \rangle = 0$, i.e. that $\xi \in [g, h_{d\theta}]$. By hypothesis this implies that $\xi = 0$.

If θ is stable, then $h_{d\theta}$ must have the generic dimension, which is the minimal dimension. Suppose that there are some η in $h_{d\theta}$ with $0 \neq \sum [\eta, \zeta]$ in $h_{d\theta}$ for some ζ in g. Choose γ with $\langle \sum [\eta, \zeta], \gamma \rangle \neq 0$. If we apply the condition for the existence of a conjugacy of $h_{d(\theta+t\gamma)}$ with $h_{d\theta}$ and compare coefficients of t, it is easy to see that we must be able to solve the equations

$$\langle [[\xi, \eta], \zeta], \theta \rangle = \langle [\eta, \zeta], \gamma \rangle$$

for all η in h and ζ in g. Choosing $\sum [\eta, \zeta] \in h_{d\theta}$ and using Jacobi's identity on the left gives zero while the right side does not vanish, giving a contradiction.

Observe that Proposition 7.5 is not true if we replace the coboundary $d\theta$ by a cocycle, σ. Indeed, consider the trivial three dimensional algebra. Here every two form is a cocycle and, for non-zero σ, the subalgebra h_σ consists of the line $\xi \rfloor \sigma = 0$, and no distinct lines are conjugate since the adjoint group acts trivially. On the other hand, $[g, g] = 0$, so the condition $[g, h_\sigma] \cap h_\sigma = 0$ is certainly satisfied.

In order to understand this example it is useful to observe that for any Lie algebra g, we can form the central extension of g by $H^2(g)$ as follows: choose a basis $c_1, \ldots, c_k$ for $H^2(g)$ and cycles $z_1, \ldots, z_k$ representing the c's. Then define $[(v, x), (w, y)] = (z_1(x, y)c_1 + \cdots + z_k(x, y)c_k, [x, y])$ where v and w are in $H^2(g)$ and x and y are in g. This gives a Lie algebra structure to $H^2(g) + g$.

If $\theta \in (H^2(g) + g)^*$ is given by $\theta(v, x) = a_i$ where $v = \sum a_i c_i$ then it is clear that $d\theta = z_i$. In this way every cocycle of g can be regarded as a coboundary in the extended algebra. If σ is a cocycle of g corresponding to the coboundary $d\theta$ of the extended algebra, it is clear that $h_{d\theta} = H^2(g) + h_\sigma$. If σ is stable then so is θ and conversely. We must therefore require the stability criterion in the extended algebra.

We can extend the assertion and proof of Proposition 7.5 as follows: Suppose that ℓ is a Lie algebra, and g is an ideal in ℓ. Then any element of ℓ acts on g by Lie bracket and hence on g^* by the contragredient action. For $\xi \in \ell$ and $\theta \in g^*$, we let $\xi \rfloor d\theta$ denote the action of ξ on θ so that $(\xi \rfloor d\theta)(\eta) = -[\xi, \eta] \rfloor \theta$. Let L be a Lie group with Lie algebra ℓ, and suppose that L acts on g so that its infinitesimal action is the given action of bracketing by ℓ. We say that θ is L stable if $h_{d\theta'}$ is conjugate to $h_{d\theta}$ by an element of L, for all θ' close to θ. Then the condition for L stability is that

$$[g, h_{d\theta}] \cap (\ell \rfloor d\theta)^{\perp} = 0.$$

The proof is as before.

In particular, we can let L be the group of all automorphisms of g, in which case ℓ is the algebra of all derivations of g, and we get a condition for stability under all automorphisms. One can construct algebras for which no form θ is stable under the group of all automorphisms, cf. [15].

We would now like to classify the homogeneous symplectic manifolds for various interesting Lie groups. We will do this by reducing the problem to studying the behavior of closed two forms with respect to certain subgroups. In particular, we will make the following assumption about the Lie algebra, g, of G. We will assume that there are two subspaces, k and p, of g such that

$$g = k + p \qquad k \cap p = \{0\}$$

$$[k,k] \subset k \qquad \text{and} \qquad [k,p] \subset p.$$

Thus we are assuming that k is a subalgebra of g and that p is a supplementary subspace to k which is stable under the action of k. We do not make any further assumptions at the moment about p. Thus $[p,p]$ will have both a k and a p component which we denote by r and s respectively: for η and η' in p we have $[\eta, \eta'] = r(\eta, \eta') + s(\eta, \eta')$ where $r(\eta, \eta') \in k$ and $s(\eta, \eta') \in p$. Jacobi's identity implies some identities on r and s. It is easy to check that these are

$$\mathfrak{S} r(s(\eta, \eta'), \eta'') = 0$$

$$\mathfrak{S}\{s(s(\eta, \eta'), \eta'') + [r(\eta, \eta'), \eta'']\} = 0$$

where $\mathfrak{S}$ denotes cyclic sum. Also

$$[\xi, r(\eta, \eta')] = r(\xi \cdot \eta, \eta') + r(\eta, \xi \cdot \eta')$$

where $\xi \in k$ and η, $\eta' \in p$ and we have written $\xi \cdot \eta$ for $[\xi, \eta]$, thinking of k acting on p. We also have the equation

$$\xi \cdot s(\eta, \eta') = s(\xi \cdot \eta, \eta') + s(\eta, \xi \cdot \eta').$$

In addition we have the identity asserting that k acts as a Lie algebra of linear transformations on p and Jacobi's identity for k. Conversely, starting from any action of a Lie algebra k on a vector space p together with r and s satisfying the above identities it is clear that $g = k + p$ becomes a Lie algebra. Let us give some illustration of this situation:

A) $r = s = 0$. In this case p is a supplementary abelian ideal, and k acts as linear transformations on p. In other words, g is the semidirect product of k and p where k is a Lie algebra with a given linear representation of k on p. Any such linear representation of k gives rise to a Lie algebra, g, which is called the associated affine algebra.

B) $r = 0$. Here all that is assumed is that p is a supplementary ideal to k. An important illustration of this situation is the case of the Galilean group. Recall that the Galilean group can be regarded as the group of all five by five matrices of the form

$$\begin{bmatrix} A & v & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

where $A \in O(3)$, $v \in \mathbf{R}^3$, $x \in \mathbf{R}^3$ and $t \in \mathbf{R}$. Such a matrix carries the space time point (x_0, t_0) into the space time point $(Ax_0 + x + t_0 v, t + t_0)$. The corresponding Lie algebra consists of all matrices of the form

$$\begin{bmatrix} a & v & x \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$$

where $a \in o(3)$ and v, x, t as before. Here we can take $k \sim o(3)$ to consist of the subalgebra with $x = v = t = 0$ and p to be the seven dimensional subalgebra with $a = 0$. Denoting an element of p by (v, x, t) we see that $[(v,x,t),(v',x',t')] = s((v,x,t),(v',x',t')) = (0, t'v - tv', 0)$ and $\xi \cdot (v,x,t) = (\xi \cdot v, \xi \cdot x, 0)$ where $\xi \cdot v$ denotes the usual action of $\xi \in o(3)$ on $v \in \mathbf{R}^3$ and similarly for $\xi \cdot x$.

C) The case where g is semi-simple and k, p corresponds to a Cartan decomposition. Here $s = 0$.

D) The case where k is an ideal. Here the action of k on p is trivial. For example, in the case of the Heisenberg algebra we can take k to be the center. For this case p is a symplectic vector space, $k = \mathbf{R}$ acts trivially on p and r is the symplectic two form, while $s = 0$.

Let $f \in \wedge^2 g^*$ be a two form. Identifying $\wedge^2 g^*$ with $\wedge^2 k^* \oplus k^* \otimes p^* \oplus \wedge^2 p^*$ allows us to write $f = a + b + c$ so that

$$f(\xi + \eta, \xi' + \eta') = a(\xi, \xi') + b(\xi, \eta') - b(\xi', \eta) + c(\eta, \eta').$$

Now $df \in \wedge^3 g^*$ is given by $df(\chi, \chi', \chi'') = \mathfrak{S} f([\chi, \chi'], \chi'')$ where $\mathfrak{S}$ denotes cyclic sum. Writing $\chi = \xi + \eta$ etc. the equation $df = 0$ becomes

$$\mathfrak{S}\{a([\xi,\xi'] + r(\eta,\eta'),\xi'') + b([\xi,\xi'] + r(\eta,\eta'),\eta'') - b(\xi'',\xi\cdot\eta' - \xi'\cdot\eta + s(\eta,\eta')) + c(\xi\cdot\eta - \xi'\cdot\eta + s(\eta,\eta'),\eta'')\} = 0.$$

We now derive various identities for a, b, and c by considering special cases of this identity:

i) $\xi = \xi' = \xi'' = 0$. In this case the identity becomes

$$\mathfrak{S}\{b(r(\eta,\eta'),\eta'') + c(s(\eta,\eta'),\eta'')\} = 0. \qquad (*)$$

For the case of the affine algebra this identity is vacuous. If p is a subalgebra so that $r = 0$, only the identity involving c remains. For example, a direct computation in the case of the Galilean group shows that $(*)$ reduces to the condition $c((0,x,0),(0,x',0)) = 0$. For the case of the Cartan decomposition only the identity involving b remains. Similarly for the case of the Heisenberg algebra.

ii) $\xi = \xi' = 0$, $\eta'' = 0$. In this case the identity becomes

$$a(r(\eta,\eta'),\xi'') - b(\xi'',s(\eta,\eta')) + c(\xi''\cdot\eta,\eta') + c(\eta,\xi''\cdot\eta') = 0. \qquad (**)$$

For the case of the affine algebra both r and s vanish and this identity becomes

$$c(\xi\cdot\eta,\eta') + c(\eta,\xi\cdot\eta') = 0 \qquad (**)_{\mathrm{A}}$$

which asserts that the antisymmetric form c is invariant under the action of k. For example, in the case of the Poincaré algebra where $k = o(3,1)$ and $p = \mathbf{R}^4$ there is no invariant antisymmetric form so we conclude that $c = 0$.

In the case that we only assume that p is a subalgebra so that r vanishes the identity becomes

$$c(\xi\cdot\eta,\eta') + c(\eta,\xi\cdot\eta') = b(\xi,s(\eta,\eta')) \qquad (**)_{\mathrm{B}}$$

For example, in the case of the Galilean algebra, if we apply this identity to $\eta = (v,x,0)$ and $\eta' = (v',x',0)$ the right hand side vanishes and we conclude that c, when restricted to $(\mathbf{R}^3 + \mathbf{R}^3) \wedge (\mathbf{R}^3 + \mathbf{R}^3)$ is invariant under the action of $o(3)$, which acts diagonally on $\mathbf{R}^3 + \mathbf{R}^3$. There is obviously only one such invariant (up to scalar multiples) and it is given by

$$c((v,x,0),(v',x',0)) = m(\langle v,x'\rangle - \langle v',x\rangle)$$

where $\langle\ ,\ \rangle$ denotes the Euclidean scalar product. If we take $\eta = (0,x,0)$ and $\eta' = (0,0,t)$ the right side of $(**)_{\mathrm{B}}$ still vanishes. On the left the term $\xi\cdot\eta'$ vanishes and $\xi\cdot x$ is arbitrary. We conclude that

$$c((0,x,0),(0,0,t)) = 0.$$

Thus

$$c((v,x,t),(v',x',t')) = m(\langle v,x'\rangle - \langle v',x\rangle) + \langle \ell, t'v - tv'\rangle$$

for some $\ell \in \mathbf{R}^3$ where $(**)_B$ implies that

$$\langle \ell, \xi \cdot v\rangle = b(\xi,(0,v,0)).$$

In the case of a Cartan decomposition, or, more generally when $s = 0$ the identity $(**)$ becomes

$$c(\xi\cdot\eta,\eta') + c(\eta,\xi\cdot\eta') = a(\xi, r(\eta,\eta')). \qquad (**)_C$$

For the case where k is an ideal $(**)$ becomes

$$a(\xi, r(\eta,\eta')) + b(\xi, s(\eta,\eta')) = 0. \qquad (**)_D$$

iii) $\xi = 0$, $\eta' = \eta'' = 0$. In this case neither a nor c contributes and we obtain the identity

$$b([\xi',\xi''],\eta) + b(\xi'',\xi'\cdot\eta) - b(\xi',\xi''\cdot\eta) = 0. \qquad (***)$$

This identity says that the map from k to p^* sending $\xi \rightsquigarrow b(\xi,\cdot)$ is a cocycle. If k is semisimple, then Whitehead's lemma asserts that b must be a coboundary, i.e. that there exists a $\theta \in p^*$ such that

$$b(\xi,\eta) = \theta(\xi\cdot\eta). \qquad (***)_S$$

Suppose that instead of assuming that k is semi-simple we assume that k contains an element in its center which acts as the identity transformation on p. Taking ξ' to be this element and ξ'' to be an arbitrary ξ in $(***)$ we see that $(***)_S$ holds with $\theta(\eta) = b(\xi',\eta)$. Thus

if either k is semi-simple or k contains an element in its center acting as the identity transformation on p then $(***)_S$ holds.

For example, in the case of the Galilean algebra, we see that the bilinear form b is given by

$$b(\xi,(v,x,t)) = \langle \ell', \xi\cdot v\rangle + \langle \ell, \xi\cdot x\rangle$$

where ℓ' and ℓ are elements of $\mathbf{R}^3$.

iv) $\eta = \eta' = \eta'' = 0$. In this case we simply obtain the identity which asserts that a is a cocycle in $\wedge^2 k^*$. Again, if k is semi-simple we can conclude that a must be a coboundary. In the case of the Galilean algebra we have thus established that the most general cocycle can be written as

$$f((\xi,v,x,t),(\xi',v',x',t')) = \tau([\xi,\xi']) + \langle \ell', \xi v' - \xi' v\rangle$$
$$+ \langle \ell, \xi x' - \xi' x + t'v - tv'\rangle + m(\langle v,x'\rangle - \langle v',x\rangle),$$

where $\tau \in o(3)^*$. Now the sum of the first three terms can be written as $\theta([(\xi, v, x, t), (\xi', v', x', t')])$ where $\theta = (\tau, \ell', \ell, 0) \in g^*$, i.e., as a coboundary. On the other hand it is clear that the last term is definitely not a coboundary. We have thus recovered a result first proved by Bargmann.

If G is the Galilean group then $H^2(g)$ is one dimensional and, up to coboundaries, a cocycle can be written as

$$f((\xi, v, x, t), (\xi', v', x', t')) = m(\langle v, x'\rangle - \langle v', x\rangle).$$

We now turn to the problem of describing the action of G on the space of two-cycles in order to determine when two such cycles define equivalent symplectic structures. We begin with the case of the semi-direct product, i.e. the affine algebra. Every element of the simply connected group corresponding to g can be written as $m \exp \eta$ where $m \in K = \exp k$, and $\eta \in p$. Now K leaves both k and p invariant so that the action of K on $f = a + b + c$ does not mix the summands and the action on each summand is the appropriate exterior or tensor product of the contragredient representation. In case $f = d\theta$ for $\theta \in g^* = k^* + p^*$, $mf = dm\theta$ where K acts on g^* via the contragredient representation. We must therefore examine the action of $\exp \eta$. Now

$$[\eta, \eta'] = 0 \qquad \text{and} \qquad [\eta, \xi] = -\xi \cdot \eta.$$

Thus

$$Ad\,(\exp -\eta)(\xi' + \eta') = \exp(ad\, -\eta)(\xi' + \eta') = (\xi' + \xi' \cdot \eta + \eta').$$

Therefore

$$\begin{aligned}(\exp \eta) f(\xi' + \eta', \xi'' + \eta'') &= f(\exp ad\, -\eta(\xi' + \eta'), \exp ad\, -\eta(\xi'' + \eta''))\\ &= a(\xi', \xi'') + b(\xi', \xi'' \cdot \eta) - b(\xi'', \xi' \cdot \eta) + c(\xi' \cdot \eta, \xi'' \cdot \eta)\\ &= f(\xi' + \xi' \cdot \eta + \eta', \xi'' + \xi'' \cdot \eta + \eta'')\\ &\quad + b(\xi', \eta'') - b(\xi'', \eta') + c(\xi' \cdot \eta, \eta'') + c(\eta', \xi'' \cdot \eta) + c(\eta', \eta'').\end{aligned}$$

Now by (***), $b(\xi', \xi'' \cdot \eta) - b(\xi'', \xi' \cdot \eta) = b([\xi', \xi''], \eta)$ and, by $(**)_A$

$$c(\xi' \cdot \eta, \xi'' \cdot \eta) = -c(\eta, \xi' \cdot \xi'' \cdot \eta) = c(\xi' \cdot \xi'' \cdot \eta, \eta) = \tfrac{1}{2} c([\xi', \xi''] \cdot \eta, \eta).$$

We can thus write

$$(\exp \eta)(a + b + c) = (a + d(b_\eta + \tfrac{1}{2} c_{\eta\eta})) + (b + dc_\eta) + c$$

where b_η and $c_{\eta\eta} \in k^*$ are defined by

$$b_\eta(\xi) = b(\xi, \eta) \qquad \text{and} \qquad c_{\eta\eta}(\xi) = c(\xi \cdot \eta, \eta)$$

while $c_\eta \in p^*$ is defined by

$$c_\eta(\eta') = -c(\eta, \eta').$$

In the important special case where $(a + b) = d(\tau + \theta)$ is exact, where $\tau \in k^*$ and $\theta \in p^*$ we can write

$$(\exp \eta)(d(\tau + \theta) + c) = d((\tau + b_\eta + \tfrac{1}{2}c_{\eta\eta}) + (\theta + c_\eta)) + c.$$

We can therefore describe the situation as follows. The c component is invariant under the action of G. It is invariant under $\exp p$ by the above computation and it is invariant under the action of K by $(**)_A$. For a given choice of c we can move θ into $(h^{*-1})\theta + c_\eta$ where $h \in K$ and $\eta \in p$. This determines an action of $K \times p$ on p^*. Suppose that we have parametrized the orbits of this action and have, in fact, chosen a cross-section for these orbits. For a given orbit we have thus picked a fixed θ. This determines a subgroup of G, the isotropy group of θ. The corresponding algebra consists of those (ξ, η) for which $\xi\theta + c_\eta = 0$. The set of ξ which occur form a subalgebra of k which we denote by k_θ. Thus $\xi \in k_\theta$ if and only if there exists an $\eta_\xi \in p$ such that $\theta(\xi \cdot \eta) = c(\eta_\xi, \eta)$ for all $\eta \in p$. It is easy to check that the identity $(**)_A$ implies that the assignment $\xi \rightsquigarrow \eta_\xi$ is a cocycle of k_θ with values in p. If this cocycle is a coboundary (for instance if k_θ is semi-simple or contains the identity operator) then we can find an $\bar{\eta}$ such that $\xi\theta - c_{\xi\bar{\eta}} = \xi(\theta - c_{\bar{\eta}}) = 0$. Thus by changing our choice of θ within the orbit we can arrange that k_θ consist exactly of those ξ for which $\xi\theta = 0$. Notice that this equation is equivalent to the equation $\theta_\eta(\xi) = 0$ for all η. If we consider the action of $\exp \eta$ on the k^* component, it adds exactly $\theta_\eta + \frac{1}{2}c_{\eta\eta}$. If $c_{\eta\eta} = 0$, we see that the orbit of τ is just the complete inverse image of the orbit of $\rho_\theta(\tau)$ under K_θ where $\rho_\theta \colon k^* \to k_\theta^*$ is the projection dual to the injection of $k_\theta \to k$. In this case the cocycles are thus parametrized by c, θ ranging over a cross-section of the action of G on k^* determined by c, and χ ranging over a cross-section for the action of K_θ on k_θ^*.

For example, for the case of the Lie algebra of the Poincaré group we have already seen that $c = 0$. The orbits of G on p^* are thus the same as the orbits of $K = SO(3, 1)$ on p^* and consist of single sheeted hyperboloids $\theta^2 = m^2 > 0$, $\theta_0 > 0$ and $\theta^2 = m^2 > \theta$, $\theta_0 < 0$; the forward light cone $\theta^2 = 0$, $\theta_0 > 0$, the backward light cone $\theta^2 = 0$, $\theta_0 < 0$, the single sheeted hyperboloids $\theta^2 = -m^2 < 0$; and the origin. We thus choose cross-sections for these orbits as follows:

$$(m, 0, 0, 0) \qquad (-m, 0, 0, 0) \qquad (1, 1, 0, 0)$$
$$(-1, 1, 0, 0) \qquad (0, m, 0, 0) \qquad \text{and} \qquad (0, 0, 0, 0).$$

It is easy to see that the group $K_{(m,0,0,0)}$ is exactly $SO(3)$. Its orbits acting on the dual of its Lie algebra are spheres. If we call s the radius of these spheres, we see that a family of orbits is parametrized by the two real parameters $m > 0$ and

$s \geq 0$. Here m is the "mass" and s is the "spin". For "mass zero" i.e. for (1, 1, 0, 0) or (−1, 1, 0, 0) it is easy to see that the corresponding isotropy group is the Euclidean group, $E(2)$. The orbits of $E(2)$ in the dual of its Lie algebra are easily seen to be cylinders (of radius r, say) and points on the axis $r = 0$. If we let the real parameter s describe the points on this axis we see that the symplectic structures corresponding to (1, 1, 0, 0) are parametrized by $r > 0$ and, if $r = 0$ by an arbitrary real parameter, s. The case $r > 0$ does not arise in known physical systems; for $r = 0$ the parameter s is also called the "spin". The isotropy group of $(0, m, 0, 0)$ is $SL(2)$. Its orbits are again hyperboloids, forward and backward "light cone" and the origin. No particles with negative mass^2 ("tachyons") seem to occur.

Let us now do a slightly more complicated computation–determining the symplectic homogeneous spaces for the Galilean group. Here p is not abelian. However, it is easy to check that

$$Ad\,(\exp -(0,v,x,t))(\xi,w,y,s) = (\xi, w + \xi v, y + \xi(x + \tfrac{1}{2}tv) + tw - sv, s).$$

We have already written the form of the most general cocycle, f, of the Galilean algebra. Under the action of $\exp(\eta)$ it is easy to check that ℓ' is moved into $\ell' + mx$ and ℓ is changed into $\ell - mv$. Thus by suitable choice of x and v we can arrange that both ℓ and ℓ' vanish, provided that $m \neq 0$. Now $\vec{p} = -\ell$, being dual to the translation vector, x, has the character of linear momentum. By applying a pure "velocity transformation" $\exp(0, v, 0, 0)$, p is moved into $\vec{p} + mv$. Thus m is just the ratio between momentum and velocity, and hence corresponds to the usual notion of mass. Our choice of v amounts to making a change to a new frame of reference in which the center of gravity is at rest. The physical interpretation of $-\ell'/m$ is that it is the position of the center of mass in the frame in which it is at rest. By shifting the origin of the coordinate system we can arrange that this is the origin.) Once we have arranged that $\ell = \ell' = 0$, the only possibility left for η (in the case of non-zero m) is $\eta = (0, 0, t)$ and it is clear that $\exp(0, 0, -t)$ acts trivially. Thus we are left with the action of $SO(3)$ on $k^* = o(3)^*$. Again, the orbits are spheres, parametrized by, their radius, a non-negative parameter, s. Thus for $m \neq 0$ the homogeneous symplectic manifolds for the Galilean group are parametrized by m and $s \geq 0$, the "mass" and the spin. For $m = 0$, we cannot change ℓ while ℓ' is moved into $\ell' + t\ell$. On the other hand τ is moved into

$$\tau + \langle \ell', \cdot\, v\rangle + \langle \ell, \cdot\, (x + \tfrac{1}{2}tv)\rangle.$$

If we identify τ as a vector in three space this last expression can be written as

$$\tau + \ell' \times v + \ell \times (x + \tfrac{1}{2}tv)$$

where $\times$ denotes vector product. In this case it is more convenient to let G act by letting $SO(3)$ act first and then $\exp p$. By applying a suitable element of $SO(3)$ we can arrange that $\ell = (f, 0, 0)$ and then, if $f \neq 0$ that $\ell' = (0, b, c)$. If ℓ and ℓ'

are independent by suitable choice of v and x we can arrange that $\tau = 0$. If $f \neq 0$ and $b = c = 0$, then we can arrange that $\tau = (\pm sf, 0, 0)$ where $s \geq 0$ and $f > 0$. (This corresponds to the case of a particle of zero mass, travelling with infinite velocity. Here the condition that $b = c = 0$ amounts to the condition that the "disturbance is transverse" and the parameter f is the "inverse of the wave length", i.e. the "color". The parameter s is called the spin and the + or − is called the helicity. For details, see Souriau [9, p. 195].)

Let us give a procedure for interpreting, in terms of "particles" moving in space time, the meaning of the symplectic manifolds that we have descibed above for the Poincaré and Galilean groups. (We are indebted to Thomas Ungar for help in the ensuing discussion.) Let G be any Lie group and $M = G/L$ a homogeneous space for G, where L is some closed subgroup. (In the case at hand, G is either the Poincaré or Galilean group and M consists of space time.) Let S be some homogeneous symplectic space for G. We would like to find a homogeneous space, N, for G which is fibered over S (and so carries a presymplectic structure coming from S) and is also fibered over M (so that it makes sense to talk of the "position in M" of a point of N). Thus we wish to have the following double fibration:

$$\begin{array}{ccc} & N & \\ \swarrow & & \searrow \\ S & & M \end{array}$$

If $S = G/H$ and $M = G/L$, then the "smallest" N that will do will be of the form $G/H \cap aLa^{-1}$ for some $a \in G$, where a is chosen so that $H \cap aLa^{-1}$ has maximal dimension. The image in M of a typical fiber of N over S will look like the orbit through the point aL of N under the action of the group H. In particular the dimension of the image in M of a typical fiber will be equal to the dimension of $H/H \cap aLa^{-1}$.

For example, let us consider the case where G is the Poincaré group and $L = O(1,3)$ is the Lorentz group so that $M = G/L$ is just space time. Suppose we first consider a positive mass orbit of the Poincaré group as described above. A typical point on this orbit is (p, τ) where $p = (m, 0, 0, 0)$ and $\tau \in o(1,3)^*$. The isotropy algebra of p is a subalgebra $o(3)$ and we can consider τ as an element of $o(3)^*$. The isotropy group of (p, τ) then consists of all translations through vectors tp and all elements of $O(3)$ which preserve τ. Thus $H \sim \mathbf{R} \times O(2)$ if $\tau \neq 0$ and $H \sim \mathbf{R} \times O(3)$ if $\tau = 0$. If a_v is translation through the vector v then $a_v L a_v^{-1}$ consists of all transformations of Minkowski space which send the vector w into $Aw + v - Av$ where $A \in O(1,3) = L$. Thus $\dim(H/H \cap aLa^{-1}) = 1$ for any element of the Poincaré group, so we might as well take $aLa^{-1} = L$. Thus the image in space time of a typical fiber will look like the set $x + tp$. In the case of $s = 0$, we see that N is seven dimensional and the six dimensional manifold S consists of the space of "world lines" of particles of a given rest mass moving in space time. If $s \neq 0$, we can think of the particle as "spinning" (in the space like three space orthogonal to its world line) with total angular momentum ms. The "spin axis" can vary over a two sphere. Thus N is nine dimensional in this case and S is eight dimensional. The analysis for the non-zero mass symplectic homogeneous manifolds for the Galilean group is entirely analogous.

Let us examine the situation for the six dimensional mass zero orbits of the Poincaré group with non-zero spin. Here a typical point is of the form (u, κ), where u is the null vector $(1, 1, 0, 0)$ and $\kappa \in o(1,3)^*$ induces a non-zero point orbit in $e(2)^*$ where $e(2)$ is the Lie algebra of the subgroup $E(2)$ fixing the point u (and this subgroup is isomorphic to the two dimensional Euclidean group). The isotropy algebra h, in this case is four dimensional, and can be described as follows: Let us write (B, b) for the element of the Poincaré algebra whose linear component is B and whose translation component is b. Let $e_2 = (0,0,1,0)$ and $e_3 = (0,0,0,1)$; let $A_2 \in o(1,3)$ be defined by $A_2 e_2 = u$, $A_2(1,-1,0,0) = -2e_2$ and similarly $A_3 e_3 = u$ and $A_3(1,-1,0,0) = 2e_3$. Finally let B denote infinitesimal rotation in the e_2, e_3 plane, so that $Bu = B(1,-1,0,0) = 0$ and $Be_2 = e_3$, $Be_3 = -e_2$. Then h is spanned by

$$(0, u),\ (A_3, se_2),\ (-A_2, se_3),\ (B, 0)$$

where s is the "spin" of the mass zero particle and $h \cap o(1,3)$ is one dimensional. It is easy to check that this is the maximal dimension of intersection. So again, we take $L = O(1,3)$. This time the fibers are three dimensional. The image of a typical fiber is now a set of the form $x + u^{\perp}$ where $u^{\perp}$ is the three dimensional space of all vectors orthogonal to u. We can think of this as a plane in space moving with the speed of light in the direction determined by u. We leave the corresponding computation for the Galilean group to the reader.

Let us now compare the symplectic homogeneous spaces of the Galilean group with those of the Poincaré group. To do so, we wish to regard the Poincaré group as a "deformation" of the Galilean group, or as is more commonly stated in the physical literature, we wish to regard the Galilean group as a "contraction" of the Poincaré group as the speed of light goes to infinity. We first describe what this means. Suppose that we choose a definite splitting of space time into space and time and write a point in space time as a column vector with entries (t, x_1, x_2, x_3) which we shall write as $\binom{t}{x}$. We can write the most general element of the Poincaré algebra as the five by five matrix of the form

$$\begin{bmatrix} 0 & v_1 & v_2 & v_3 & t \\ v_1 & 0 & a_{12} & a_{13} & x_1 \\ v_2 & -a_{12} & 0 & a_{23} & x_2 \\ v_3 & -a_{13} & -a_{23} & 0 & x_3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which we shall denote more simply by

$$\begin{bmatrix} 0 & v^t & t \\ v & A & x \\ 0 & 0 & 0, \end{bmatrix}, \qquad A \in o(3).$$

This is the form of the Poincaré algebra relative to a space time splitting and a coordinate system in which the speed of light is one. To find the form of the Poincaré algebra relative to coordinates in which the speed of light is c, we must expand the space coordinates by a factor of c. This has the effect of conjugating the above matrices by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so as to obtain matrices of the form

$$\begin{bmatrix} 0 & c^{-1}v^t & t \\ cv & A & cx \\ 0 & 0 & 0 \end{bmatrix}.$$

For each positive value of c we obtain a different subalgebra of five by five matrices. They are all isomorphic (and indeed conjugate) to the Poincaré algebra. Within the framework of our distended coordinates, the vector cv gives a "velocity" (or "boost") transformation and the vector cx gives a spatial translation. If we are accustomed to dealing with velocities and displacements which are "small relative to the velocity of light", it makes sense to introduce $\bar{v} = cv$ and $\bar{x} = cx$ and so parametrize the elements of one such algebra (corresponding to a fixed c) as

$$\begin{bmatrix} 0 & c^{-2}\bar{v}^t & t \\ \bar{v} & A & \bar{x} \\ 0 & 0 & 0 \end{bmatrix}.$$

Let us now set $\epsilon = c^{-2}$ and drop the bars over the v's and x's. We thus obtain, for each $\epsilon > 0$ a map of the ten dimensional space spanned by the A's, v's, x's and t's into the space of five by five matrices. Explicitly, this map is

$$(A, v, x, t) \rightsquigarrow \begin{bmatrix} 0 & \epsilon v^t & t \\ v & A & x \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that for $\epsilon = 0$ the image is exactly the Galilean algebra. The limit $\epsilon \to 0$ is, of course, the same as $c \to \infty$. Let us denote the underlying ten dimensional vector space spanned by the A's, v's, x's and t's by g. Then for any $\epsilon \geqslant 0$ we have defined a linear map of g into a subalgebra of the Lie algebra of five by five matrices. This induces a Lie bracket, depending on ϵ on the space g, which we shall denote by $[\ ,\ ]_\epsilon$. Explicitly, if

$$\xi_1 = (A_1, v_1, x_1, t_1) \qquad \text{and} \qquad \xi_2 = (A_2, v_2, x_2, t_2)$$

then

$$[\xi_1, \xi_2]_\epsilon = ([A_1, A_2] + \epsilon(v_1 \otimes v_2^t - v_2 \otimes v_1^t), A_1 v_2 - A_2 v_1,$$
$$A_1 x_2 + t_2 v_1 - A_2 x_1 - t_1 v_2, \epsilon(\langle v_1, x_2\rangle - \langle v_2, x_1\rangle)).$$

Similarly, for each ϵ we get a map, $d_\epsilon\colon \wedge^k(g^*) \longrightarrow \wedge^{k+1}(g^*)$. In particular we get a varying space of 2-cocycles, $Z_\epsilon^2 \subset \wedge^2(g^*)$. Notice that in the case at hand, the dimension does not vary for all $\epsilon \geqslant 0$. Indeed, for $\epsilon > 0$ the algebras are all isomorphic to the Poincaré algebra, so we know that $\dim Z_\epsilon^2 = \dim g = 10$. For $\epsilon = 0$, we have already seen that $\dim H^2(g) = 1$ since the algebra is the Galilean algebra. On the other hand $[g, g]_0$ consists of all elements with t component equal to zero. Thus d_0 has a one dimensional kernel and hence a nine dimensional image. Let us introduce "coordinates" $(\tau, L, \vec{p}, E)$ dual to (A, v, x, t). (Here $\vec{p}$ is dual to x, and hence as we have already remarked is to be interpreted as linear momentum and E is dual to time translation and hence should be regarded as "energy".) Let $\nu \in \wedge^2(g^*)$ be the bilinear form given by

$$\nu(\xi_1, \xi_2) = \langle v_1, x_2\rangle - \langle v_2, x_1\rangle.$$

We have already seen that for $\epsilon = 0$ the bilinear form ν is a cocycle, and in fact, the cohomology class of ν generates $H_0^2(g)$, i.e., that the most general cohomology class is of the form $m[\nu]$ where m is the mass parameter for the Galilean group. We claim that ν is a cocycle for $\epsilon > 0$ as well (and hence of course a coboundary for positive values of ϵ). Indeed, let $\theta \in g^*$ be the element

$$\theta = (0, 0, 0, 1)$$

in terms of the coordinates that we are using. Thus $\theta(\xi) = t$ is just the t component of ξ. Then, using the above formula for $[\ ,\ ]_\epsilon$ we see that

$$d_\epsilon \theta(\xi_1, \xi_2) = \theta([\xi_1, \xi_2]_\epsilon) = \epsilon(\langle v_1, x_2\rangle - \langle v_2, x_1\rangle)$$

or

$$d_\epsilon \theta = \epsilon \nu.$$

Notice that for $\epsilon > 0$, θ is uniquely determined by this equation. Thus, for $\epsilon > 0$ the "mass cocycle" $m\nu$ is a coboundary, and indeed the coboundary of the unique element $(0,0,0,E)$ where $E = \epsilon^{-1}m$. If we substitute our definition of $\epsilon = c^{-2}$ we obtain Einstein's famous formula $E = mc^2$ relating mass and energy. The orbit through the point $(\tau, 0, 0, E)$, $\tau \in o(3)^*$ of the Poincaré group (corresponding to speed of light c) then goes over, as $c \to \infty$ to the symplectic homogenous space of the Galilean group (with no trivial cohomology class) with mass m (and spin $\|\tau\|$). The mass² of the orbit (in the Poincaré group) is m^2c^4. If $(\tau_1, L_1, \vec{p}_1, E_1)$ is some other point on this orbit, then

$$E_1^2 - c^2\vec{p}_1^{\,2} = m^2c^4$$

gives the relation between mass, energy and momentum.

We can compute the space of cocycles, and the corresponding symplectic manifolds for the Galilean group from a slightly different point of view. Let $SO(3) \times \mathbf{R}^3$ act on $\mathbf{R}^4$ by $(A,v)\cdot(x,t) = (Ax + tv, t)$. Here (x,t) is a vector in $\mathbf{R}^4$, with x a vector in $\mathbf{R}^3$ and t in $\mathbf{R}$. We can regard the Galilean group as the semi-direct product of $SO(3) \times \mathbf{R}^3$ with $\mathbf{R}^4$. Again we have a (k,p) decomposition but this time with $k = o(3) \oplus \mathbf{R}^3$ (semi-direct) and $p = \mathbf{R}^4$. It is easy to check that there are no invariant antisymmetric two forms on $\mathbf{R}^4$, so that $c = 0$. We can write b as

$$b = b_1(\xi, x) + b_2(\xi, t) + b_3(v, x) + b_4(v, t)$$

where ξ is in $o(3)$ and v in $\mathbf{R}^3$. Condition (***) implies that

$$b_3(\xi \cdot v, x) + b_4(\xi \cdot v, t) = b_1(\xi, tv) - b_3(v, \xi \cdot x) + b_2(\xi, t)$$

and

$$b_1([\xi,\xi'],x) + b_2([\xi,\xi'],t) = b_1(\xi, \xi' \cdot x) - b_1(\xi', \xi \cdot x).$$

The second equation implies that b_1 is a cocycle of $o(3)$ with values in $\mathbf{R}^{3*}$, and hence a coboundary, and that $b_2 = 0$. The first of these equations, with $t = 0$, implies that $b_3(v,x) = m\langle v, x\rangle$ while for $t \neq 0$ it implies that $b_4(\xi \quad v, t) = b_1(\xi, tv)$. Thus

$$b = \langle l, \xi \cdot x + tv\rangle + m\langle v, x\rangle.$$

The fact that a is a cocycle in $\wedge^2 k^*$ implies that it is a coboundary, $a = d(\tau + l')$ where $\tau \in o(3)^*$ and $l' \in \mathbf{R}^{3*}$. We thus recover the expression that we derived above for the most general cocycle for the Galilean algebra. The analysis of the operation of the Galilean group on the space of cocycles proceeds as before.

Let us now do a computation at the opposite extreme—the Heisenberg algebra. In this case k is the one dimensional center that we may identify with $\mathbf{R}$, and p is a symplectic vector space with r identified as its symplectic two form. The action of k on p is trivial and $s = 0$. Since k is one dimensional, $a = 0$ and conditions (**) and (***) are vacuous. Condition (*) can be interpreted as follows: let ω denote the symplectic form on p and let $\kappa \in p^*$ be defined by

$\kappa = b(1, \cdot)$. Then $(*)$ becomes $\omega \wedge \kappa = 0$. If $\dim p = 2$, this imposes no condition. If $\dim p > 2$, then this implies $\kappa = 0$. Indeed, for $\dim p > 2$ we can write $\omega = \theta \wedge \kappa + \omega'$ for some suitable $\theta \in p^*$ and where $\omega' \wedge \kappa \neq 0$. Any $c \in \wedge^2 p^*$ gives a cocycle. The element $\omega \in \wedge^2 p^*$ is a coboundary, $\omega = d\ell$ where ℓ is any element of g^* satisfying $\ell(1) = 1$, with 1 denoting the basis element of the center which we have identified with **R**. It is clear that the only coboundaries are the multiples of ω. Now

$$[(s, v), (t, w)] = (\omega(v, w), 0) \qquad \text{where } s, t \in k = \mathbf{R} \qquad \text{and} \qquad v, w \in p.$$

Therefore

$$Ad\,(\exp(s, v))(t, w) = (t + \omega(v, w), w).$$

Thus the group acts trivially on $\wedge^2 p^*$. For $\dim p = 2$ it maps a non-zero $b \in k^* \otimes p^*$ onto $b + c$ where c ranges all over $\wedge^2 p^*$. For $\dim p > 2$ the action of the group on the space of cocycles is completely trivial. Thus $H^2(g) \sim k^* \otimes p^*$ for $\dim p = 2$ while $H^2(g) \sim \wedge^2 p^* / \{\omega\}$ for $\dim p > 2$. The orbits of G acting on g^* via the contragredient representation are the hyperplanes $\ell(1,0) = \text{const}$ for $\ell(1,0) \neq 0$, and the points in the subspace $\ell(1,0) = 0$. Thus these orbits either have dimension equal to $\dim p$ or are zero dimensional. The symplectic manifolds corresponding to the non-vanishing cohomology classes for $\dim p = 2$ are all two dimensional while for $\dim p > 2$ the dimension corresponding to an element of $\wedge^2 p^*$ is equal to its rank. We now list the homogeneous symplectic manifolds for the low dimensional Lie algebras.

$n = 1$.

There is only one Lie algebra of dimension one, the trivial Lie algebra, the corresponding simply connected group is just the additive group of real numbers, which obviously does not act transitively on any symplectic manifold of positive dimension. Hence the only homogeneous symplectic manifold is a point. Nevertheless, even in this most trivial example there are a number of interesting lessons to be learned. The action of the adjoint group is trivial, and hence so is the co-adjoint action. Thus the orbits of G in g^* consist entirely of points. This is obviously the case for any commutative Lie algebra. Although the orbits are all distinct, they all correspond to the same symplectic manifold, because the operator d is trivial, and hence all orbits give the zero cocycle. From the point of view of "classical mechanics" these orbits are all the same. But from the "quantum" viewpoint, i.e. from the point of view of representation theory, they all correspond to different infinitesimal characters, to different representations. This is already true at the level of "prequantization", cf. Chapter V, where one is interested in classifying homogeneous Hermitian line bundles with connection. Another comment is in order, even at the level of "classical mechanics". While it is obvious that the real line cannot act transitively on any manifold whose dimension is greater than one, the action can be "ergodic" in any of the various senses, e.g. topologically transitive, or metrically transitive, or mixing, etc. Each

of these concepts corresponds to a different mathematical formulation of the intuitive notion of "irreducibility" for a mechanical system. The notion of transitivity, the one we are dealing with, is just the most simple of these concepts.
$n = 2$.

There are two Lie algebras of dimension two, the trivial Lie algebra and the Lie algebra with basis $\{\xi, \eta\}$ and bracket relations $[\xi, \eta] = \eta$. We shall call this second algebra the scale algebra. It corresponds to the group of symmetries in which one can change the origin of measurement (time translation) and the choice of units (scaling). For both algebras all two forms are cocycles since the algebra is two dimensional. However the operator $d\colon g^* \to \wedge^2 g^*$ behaves differently in each of the two cases.

The trivial algebra.

Here the operator d is trivial. Thus each element of $\wedge^2 g^*$ represents a different cohomology class and the action of G on $\wedge^2 g^*$ is trivial. Each element of $\wedge^2 g^*$ gives a distinct symplectic space. It is easy to see that the explicit realization of these spaces are given by $\omega = c\,dx \wedge dy$, $c \neq 0$, with $\xi \to \partial/\partial x$ and $\eta \to \partial/\partial y$. In addition, of course, there is the zero dimensional symplectic manifold corresponding to the orbits of g^* to which the remarks made above in the one dimensional case apply.

The scale algebra.

We have a $k + p$ decomposition with both k and p one dimensional. Hence $a = c = 0$, and, since d is non-trivial (or, by applying the general argument using the fact that k contains the identity), we know that every cocycle is a coboundary. Thus every b has the form $b(\xi, \cdot) = \ell([\xi, \cdot])$ for some $\ell \in p^*$. The group K acts on p and hence on p^* by multiplication by positive numbers, while $\exp p$ does not change $d\ell$. Hence there are three symplectic homogeneous spaces corresponding to the alternatives $\ell(\eta) > 0$, $\ell(\eta) = 0$, and $\ell(\eta) < 0$. Let (a, b) be the coordinates on g^* given by ξ and η, so that $\theta(\xi) = a$ and $\theta(\eta) = b$ if $\theta \in g^*$ has coordinates (a, b). Then a direct computation shows that

$$Ad^*_{\exp t\xi}(a, b) = (a, e^{-t}b) \qquad \text{and} \qquad Ad^*_{\exp t\eta}(a, b) = (a + tb, b).$$

For $b > 0$, say, we get a symplectic manifold, and

$$\hat{\xi} = b\frac{\partial}{\partial b}, \qquad \hat{\eta} = -b\frac{\partial}{\partial a}.$$

It is clear that the invariant two form ω must be given by

$$\omega = \kappa b^{-1}\, db \wedge da$$

for $\kappa \neq 0$. On the other hand, replacing (a, b) by (sa, sb), where s is an arbitrary constant, does not change $\hat{\xi}$ or $\hat{\eta}$, but replaces κ by $s\kappa$. This shows that the symplectic manifolds corresponding to $\ell(\eta) > 0$ and $\ell(\eta) < 0$ are equivalent, so that there is exactly one two dimensional homogeneous symplectic manifold for the scale algebra. There is, of course, also the zero dimensional manifold as well.

If we introduce the variables $x = a/b$, $y = b$, we can rewrite the two dimensional action as

$$(\exp t\xi)(x,y) = (e^t x, e^{-t} y) \qquad (\exp t\eta)(x,y) = (x + t, y).$$

In terms of these coordinates we have $\omega = dx \wedge dy$.

If we set $u = a$ and $v = -\log b$ we can describe the action as $\omega = du \wedge dv$, $\exp t\xi(u,v) = (u, v + t)$ and $\exp t\eta(u,v) = (u + te^{-v}, v)$.

$n = 3$.

The three dimensional Lie algebras over **R** are classified as follows (cf. Jacobson, *Lie algebras*, pp. 11–13):

(i) the trivial algebra,

(ii) the Heisenberg algebra,

(iii) the direct sum $\mathbf{R} + h$ where h is the two dimensional scale algebra.

$(\text{iv})_A$ the affine algebra $k + p$ where k is one dimensional and p is two dimensional. Here, a basis element of k acts on p via the linear transformation A, which is non-singular. Here A is determined only up to conjugacy on p and multiplication by any non-zero real number (since the basis of k was chosen arbitrarily). We will distinguish several possibilities, according to whether the trace of A is or is not zero. If $\operatorname{tr} A = 0$, and if A has real eigenvalues then we may arrange that A is diagonal with eigenvalues ± 1. Thus A is the matrix which infinitesimally preserves the indefinite metric xy on the (x,y) plane, and g is the algebra of (infinitesimal) motions for this metric. We list this algebra as:

(iv) $e(1,1)$. If $\operatorname{tr} A = 0$ and A has comlex eigenvalues, then the eigenvalues must be purely imaginary, and we can arrange that they are $\pm i$. Thus A is an infinitesimal rotation for the Euclidean metric in the plane and we are in the case,

(v) $e(2)$, the Lie algebra of the group of Euclidean motions in the plane,

(vi) the case of the affine algebra $\text{iv})_A$ where $trA \neq 0$,

(vii) the orthogonal algebra $o(3)$, and

(viii) $sl(2, \mathbf{R})$.

For the case of the trivial algebra, every element of $\wedge^2 g^*$ is a cocycle and no element is a coboundary; each non-zero element has a one dimensional kernel which will act trivially on the corresponding symplectic manifold, which is a homogeneous symplectic manifold for the quotient two dimensional trivial algebra. We have already discussed the Heisenberg algebra. For the case (iii) we can apply the k, p decomposition with $k = \mathbf{R}$ and with $p = h$, both ideals. Then $a = 0$ and c is a cocycle for the scale algebra, while condition $(**)_D$ implies that $b(k, [h,h]) = 0$. The space of possible b's is thus one dimensional and none of them is a coboundary, and thus $H^2(g)$ is one dimensional. Let x, y, and z be a basis of g with the bracket relations

$$[x,y] = y, \qquad [x,z] = [y,z] = 0,$$

and let

$$A_t = \exp tx, \qquad B_t = \exp ty, \qquad \text{and} \qquad C_t = \exp tz$$

be the corresponding one parameter subgroups. Then it follows from the above computation that the space of all two dimensional symplectic manifolds is parametrized by the constant $m = b(z, x)$. The actual manifolds are all $\mathbf{R}^2$ with coordinates (u, v) and $\omega = du\,dv$ and actions given by

$$A_t(u,v) = (u, v + t) \qquad tB_t(u,v) = (u + te^v, v) \quad \text{and} \quad C_t(u,v) = (u + mt, v).$$

We now turn to the remaining cases:

(iv) the algebra $e(1,1)$. Here we can choose a basis with

$$[z, x] = x, \qquad [z, y] = -y \qquad \text{and} \qquad [x, y] = 0.$$

Here k is spanned by z and p is spanned by x and y. Any c in $\wedge^2 p^*$ is a cocycle and is not a coboundary if $c \neq 0$. The operator $d\colon g^* \to \wedge^2 g^*$ has a one dimensional kernel and hence the space of coboundaries is two dimensional. Thus $dg^* = k^* \otimes p^*$, i.e. all b's are coboundaries. If $c \neq 0$, the map of $p \to k^* \otimes p^*$ sending $v \rightsquigarrow c(\cdot\, v, \cdot)$ is non-singular and hence surjective. Thus if $c \neq 0$, we can eliminate b by the action of G. There are thus two families of two dimensional symplectic manifolds, one parametrized by non-zero $c \in \wedge^2 p^*$ and the other parametrized by a cross-section for the orbits of C_t acting on p^*. The first family all consist of $\mathbf{R}^2$ with coordinates (u, v) and with varying forms $\omega = m\,du\,dv$ (where $m = c(x, y)$) and action

$$A_t(u,v) = (u + t, v) \qquad B_t(u,v) = (u, v + t) \qquad \text{and} \qquad C_t(u,v) = (e^t u, e^{-t} v).$$

To describe the orbits in p^* observe that x and y can be thought of as functions on g^* and hence on p^*, and the orbits of C_t are the various components of the hyperbolas $xy = \kappa$ for different values of the constant, κ. The actual orbits of G in g^* are given by the same equations, and are, in fact, just cylinders over these curves, with generators in the k^* direction. Again the orbits can all be identified with $\mathbf{R}^2$, with coordinates (u, v) and $\omega = du\,dv$. For the orbits on which $x \neq 0$ we can use the vectors $(1, \kappa)$ as cross-sections to the orbits; the corresponding actions are given by

$$A_t(u,v) = (u + te^{-v}, v), \qquad B_t(u,v) = (u - \kappa e^{+v}, v), \qquad C_t(u,v) = (u, v + t).$$

The case $\kappa = \infty$ is obtained in the limit as A_t acting as the identity and $B_t(u,v) = (u \pm te^v, v)$ and $C_t(u,v) = (u, v + t)$. It is interesting to give some interpretation to the parameters m and κ. Notice that the algebra $e(1,1)$ contains two copies of the scale algebra, namely z, x and z, y, with the group C_t multiplying x by e^t and multiplying y by e^{-t}. Now there are two situations where making a change in scale of one variable induces the inverse change of scale of a second variable, if the variables are dual to one another (i.e. represent coordinates in dual one dimensional vector spaces) or if the variables are inverse to one another. The first family of orbits corresponds to the duality situation, with the parameter m giving the duality between u and b. The second family of orbits corresponds to the situation where the scale algebra is acting on variables r and s related by $rs = \kappa$.

(v) The situation for the Euclidean algebra $e(2)$ is quite similar to that for $e(1,1)$. The cohomology is one dimensional, each non-zero element of $\wedge^2 p^*$ corresponding to a non-zero cohomology class and giving rise to a symplectic manifold with x and y acting as constant vector fields. The remaining symplectic manifolds are given by orbits in g^* which are the cylinders $x^2 + y^2 = r^2$, for r positive, together with the zero dimensional orbits on the z-axis.

(vi) For the affine algebra with $\operatorname{tr} A \neq 0$, there is no non-zero invariant c in $\wedge^2 p^*$, and thus the space of cocycles is two dimensional, the cohomology vanishes. The orbits in g^* are seen to be cylinders over the orbits of $\exp tz$ acting on p^*, and these provide all the two dimensional symplectic manifolds. The z-axis again splits into zero dimensional orbits.

(vii) The orthogonal algebra is semi-simple, so its cohomology vanishes. The orbits in g^* are given by the spheres $x^2 + y^2 + z^2 = r^2$, where x, y, and z are the usual basis of $o(3)$, with the bracket relations

$$[x,y] = z, \qquad [z,x] = y, \qquad [z,y] = -x.$$

(viii) The algebra $sl(2)$ is also semi-simple, so its cohomology also vanishes. We may choose a basis of, and z with the bracket relations

$$[x,y] = z, \qquad [z,x] = x, \qquad \text{and} \qquad [z,y] = -y.$$

Then $xy + z^2/2$ is invariant under the action of g, and thus, when considered as a function on g^* defines two dimensional surfaces which are invariant under G. The connected components of these level surfaces are clearly the orbits of G; they are the single sheeted hyperboloids, the double sheeted hyperboloids, and the two components of the light cone.

We now study the behavior of homogeneous symplectic manifolds under deformation of the Lie algebra structure. As an illustration of what can happen let us consider the deformation of $sl(2)$ into $e(1,1)$. We consider a three dimensional vector space with basis x, y, and z, and with bracket relations

$$[z,x] = x, \qquad [z,y] = -y, \qquad [x,y] = \epsilon z.$$

For $\epsilon \neq 0$, this algebra is isomorphic to $sl(2)$, while for $\epsilon = 0$ the algebra is $e(1,1)$. For all values of ϵ the function $xy + \epsilon z^2/2$ is invariant. The double sheeted hyperboloids, corresponding to positive values of this function for $\epsilon < 0$ (and to negative values for $\epsilon > 0$) clearly deform into the cylinders $xy = c$ for $\epsilon = 0$. It is interesting to examine the behavior of the single sheeted hyperboloids. They provide both the other cylinders and also the symplectic manifolds of

$e(1,1)$ corresponding to non-vanishing cohomology classes of $e(1,1)$: (Recall that $H^2(e(1,1)) = \mathbf{R}$ while $H^2(sl(2)) = 0$). Indeed, for a *fixed* value of $xy + \epsilon z^2/2$ the points near $x = 0$ of the hyperboloid (or near $y = 0$) clearly move off to infinity at the rate $\epsilon^{-1/2}$ and the hyperboloid splits into two cylinders. As to the orbits with non-vanishing cohomology, observe that the cocycles are of the form $hx^* \wedge y^*$, and, for all ϵ, we have

$$dz^* = \epsilon x^* \wedge y^*.$$

For $\epsilon = 0$ we know that $hx^* \wedge y^*$ is not a coboundary (for non-vanishing h) while for $\epsilon \neq 0$ the above equation shows that $hx^* \wedge y^* = d(h\epsilon^{-1}z^*)$. This suggests looking at the orbit through the point with $x = 0, y = 0$ and $z = h\epsilon^{-1}$, i.e. the orbit $xy + \epsilon z^2/2 = m_\epsilon$ where $m_\epsilon = (2\epsilon)^{-1}h^2$. A direct computation shows that if we consider a bounded region of x and y, the action on this portion of the orbit tends to the desired limiting action for $e(1,1)$.

§8. Multisymplectic structures and the calculus of variations.

The principal motivation that we have discussed for the study of symplectic manifolds has been via asymptotics. There is another route which historically led to symplectic manifolds, and that is classical mechanics, and in particular, its relation to variational principles, cf. Sternberg [2, Chapter III, §7 and Chapter IV], Loomis-Sternberg [13, Chapter XIII] and Souriau [9]. For systems with "finitely many degrees of freedom", i.e. for variational problems with one independent parameter—curves on a finite dimensional manifold—the "space of extremals" forms a finite dimensional symplectic manifold, cf., in this connection, Hermann [20] and [21], Garcia [22] and Dedecker [23].

In this section we sketch the geometry involved for variational problems in one or several independent variables. For the case of several independent variables our results will be rather formal. They would acquire more content in special instances if appropriate existence theorems in the theory of elliptic or of hyperbolic partial differential equations are introduced. However, we shall not go into these kinds of questions. The treatment here follows [17], and the "symplectic structure" is based also on [18]. We will restrict attention to the case of Lagrangians involving at most the first order derivatives. Let $Y \to X$ be a fibered manifold, i.e. we are given a differentiable map, π, from Y to X, such that near every point of Y we can introduce coordinates in which the map π is just projection onto a factor. Thus we can introduce local coordinates on Y of the form $(x,y) = (x^1, \ldots, x^n, y^1, \ldots, y^f)$ where n is the dimension of X and $n + f$ is the dimension of Y. A section of Y over X is a map $s\colon X \to Y$ satisfying $\pi \circ s = \mathrm{id}$. Thus s assigns, to each $x \in X$, a point, $s(x) \in \pi^{-1}(x)$. (The set $\pi^{-1}(x)$ is called the fiber over x. It is automatically a differentiable submanifold of Y.) Locally, a section is given by f functions $y^I = y^I(x^1, \ldots, x^n)$ where

$\ell = 1, \ldots, f$. Two sections, s_1 and s_2, are said to agree to first order at some point x_0 if $s_1(x_0) = s_2(x_0)$ and the functions $y_1^\ell(x)$ and $y_2^\ell(x)$ have the same first derivatives at x_0. (It is easy to check that this condition does not depend on the choice of coordinates of the form (x,y).) To agree to first order at a point x_0 is an equivalence relation. The equivalence class of a section s at x_0 is called the one jet of s at x_0 and is denoted by $j_1(s)(x_0)$. Since we will not be concerned in the main, with jets of higher order, we shall drop the subscript 1, and simply write $js(x_0)$. The jet, $js(x_0)$ is determined, locally, by the coordinates $(x,y,(y)) = (x^i, y^\ell, y_i^\ell)$, $i = 1, \ldots, n$, $\ell = 1, \ldots, f$, where the x^i are the coordinates of x_0, the y^ℓ are the coordinates of $s(x_0)$ and the y_i^ℓ the coordinates of the first partial derivatives of s evaluated at x_0. Thus the set of all jets at all possible points of X forms a manifold, which we denote by JY. We have the projections:

$$\pi_Y : JY \to Y \qquad \pi_Y(x,y,(y)) = (x,y)$$

and

$$\pi_X : JY \to X \qquad \pi_X(x,y,(y)) = x$$

so that

$$\pi_X = \pi \circ \pi_Y.$$

Thus JY is a fibered manifold over Y and is also a fibered manifold over X. If s is a section of Y, then s determines a section, js, of JY over X, where js assigns, to each point of X, the jet of s at that point. In terms of local coordinates, if s is given by the functions $y^\ell(x) = s^\ell(x)$, then js is given by $y^\ell = s^\ell(x)$, $y_i^\ell = (\partial s^\ell / \partial x^i)(x)$. Not every section of JY will be of the form js. Indeed, if u is a section of JY then u, in local coordinates gives y^ℓ and y_i^ℓ as functions of x, and they must be related by the equations

$$dy^\ell(x) - y_i^\ell(x) dx^i = 0 \qquad \text{(summation convention)}.$$

Put another way, let ω be the linear differential form which assigns tangent vectors to Y to tangent vectors to JY according to the formula

$$\omega = (\partial / \partial y^\ell) \otimes [dy^\ell - y_i^\ell \, dx^i]. \tag{8.1}$$

Then a section u, of JY over X is of the form $u = js$ if and only if

$$u^* \omega = 0. \tag{8.2}$$

Notice that we have given a definition of ω in terms of local coordinates. Actually, ω has an invariant definition: let ξ be a tangent vector to JY at a point z, where $z = js(x_0)$, with $x_0 \in X$. Then $d\pi_Y \xi$ is a tangent vector to Y at $y_0 = s(x_0)$ and $d\pi_X \xi$ is a tangent vector to X at x_0. If η is any tangent to X at x_0

then $ds_{x_0}\eta$ is a tangent vector to Y at y_0 which depends only on $z = js(x_0)$. Then we claim that

$$\langle \xi, \omega \rangle = d\pi_Y \xi - ds(d\pi_X \xi). \tag{8.3}$$

Indeed, in terms of the local coordinates, we can write

$$\xi = a^i(\partial/\partial x^i) + b^f(\partial/\partial y^f) + c_i^f(\partial/\partial y_i^f),$$

so that

$$d\pi_X \xi = a^i(\partial/\partial x^i),$$

$$d\pi_Y \xi = a^i(\partial/\partial x^i) + b^f(\partial/\partial y^f),$$

and

$$ds(d\pi_X \xi) = a^i(\partial/\partial x^i) + y_i^f a^i(\partial/\partial y^f),$$

which establishes the formula.

Notice that if ξ is a *vertical* tangent vector, i.e.

$$\text{if } d\pi_X \xi = 0 \text{ then } \langle \xi, \omega \rangle = d\pi_Y \xi. \tag{8.4}$$

Let L be a real valued function on JY, and let (vol) be an n-form on X. The basic problem of the calculus of variations is to find the extremals for integrals of the form

$$I_A[s] = \int_A L(js)(\text{vol}), \tag{8.5}$$

where A is some bounded region of X. Here s is allowed to vary over some class of sections of X. The usual problem (the fixed boundary problem) is the situation where A is a region with smooth boundary, ∂A, and s is restricted to take on assigned values on the boundary.

The main point of the *Hamilton-Cartan formalism* is to replace the integral (8.5) by an integral of the form $\int_A u^* \Theta$ where Θ is a suitable n-form defined on JY and u is a section of JY over X. This integral satisfies

$$\int_A u^* \Theta = \int_A L(js)(\text{vol})$$

if $u = js$. The n-form Θ has the property that an extremal for $\int_A u^* \Theta$, where u is allowed to vary over *all* sections of JY, is automatically of the form $u = js$, if L is "regular".

We now describe the construction of the form Θ. We begin by defining a form, θ, which maps $TJY \to TX$, i.e. which assigns, to each tangent vector, ξ, to JY at z a tangent vector, $\langle \xi, \theta \rangle$, to X at πz. We first give the formula in terms of local coordinates: Define the functions p_ℓ^i on JY by

$$p_\ell^i = \partial L / \partial y_i^\ell$$

and set

$$\theta = (\partial / \partial x^i) \otimes \left[\frac{1}{n} L dx^i + p_\ell^i (dy^\ell - y_j^\ell dx^j) \right]. \tag{8.6}$$

To give an invariant definition of θ we observe that L defines a bundle map, σ, from $JY \to Y$ to the vector bundle Hom $(VY, TX) \to Y$, where VY denotes the bundle of vertical tangent vectors to Y. The map σ (which is the Legendre transformation) is defined as $\sigma(z) = d_v L$ where d_v means computing the differential of L with respect to the fiber of JY over Y. (This makes invariant sense because JY is an "affine bundle" over Y whose associated vector bundle is Hom (TX, VY): Given two sections with $s_1(x_0) = s_2(x_0) = y_0$ the map $ds_{1*} - ds_{2*} \in$ Hom (TX_{x_0}, VY_{y_0}) depends only on $js_1(x_0)$ and $js_2(x_0)$.) Then, for $\xi \in TJY_z$,

$$\langle \xi, \theta \rangle = \left(\frac{1}{n} \right) L(z) d\pi_X \xi + \sigma(z) \langle \xi, \omega \rangle \tag{8.7}$$

where $\langle \xi, \omega \rangle \in VY$ so $\sigma(z)\langle \xi, \omega \rangle \in TX$. It follows from the definitions that if s is any section of Y then

$$\begin{aligned} (js)^* \theta &= \left(\frac{1}{n} \right) L(js) \text{ id} \\ &= \left(\frac{1}{n} \right) L(js) \partial / \partial x^i \otimes dx^i \end{aligned} \tag{8.8}$$

in local coordinates.

The form Θ is defined, in local coordinates as

$$\begin{aligned} \Theta = &(L - p_\ell^i y_i^\ell) dx^1 \wedge \cdots \wedge dx^n \\ &+ \sum_{\ell, i} (-1)^{i+1} p_\ell^i dy^\ell \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \end{aligned} \tag{8.9}$$

where we have chosen our coordinates so that (vol) $= dx^1 \wedge \cdots \wedge dx^n$.

Invariantly, the definition of Θ is

$$\Theta = \theta \mathbin{\overline{\wedge}} \pi^* (\text{vol}) \tag{8.10}$$

where, the operation, $\overline{\wedge}$, pairs a TX valued p-form on JY with a q-form on X to get a $p + q - 1$ form on JY according to the rule

$$(\alpha \otimes \eta) \mathbin{\overline{\wedge}} \tau = \alpha \wedge (\eta \lrcorner \tau)$$

if η is a vector field on X and α a p-form on JY. (For the details we refer the reader to [14].)

It follows from (8.10) that for any section, u, of JY over X we have

$$u^*\Theta = u^*\theta \,\overline{\wedge}\, \text{vol}$$

and therefore

$$u^*\Theta = L(u)(\text{vol}) + \sigma(u) \cdot u^*\omega. \tag{8.11}$$

In particular, if $u = js$, so that $u^*\omega = 0$, we get

$$(js)^*\Theta = L(js)(\text{vol}). \tag{8.12}$$

Suppose that u_t is some one parameter family of sections of JY, $u_0 = u$. Let ξ denote the vector field along u giving the tangent to the deformation, so that $\xi(x) \in TJY_{u(x)}$ is the tangent vector to the curve $t \rightsquigarrow u_t(x)$. Then the basic formula of the differential calculus asserts that

$$\frac{d}{dt} u_t^*\Theta \,\Big|_{t=0} = u^*(\xi \lrcorner d\Theta) + du^*(\xi \lrcorner \Theta).$$

If u is to be an extremal for the integral $\int_A u_t^*\Theta$ among all u_t which satisfy the condition $\pi_Y u_t = \pi_Y u$ on ∂A then we obtain the "Euler equations"

$$u^*(\xi \lrcorner \Omega) = 0 \tag{8.13}$$

where $\Omega = d\Theta$ and ξ is allowed to be any vector field on JY. A computation (which can either be done in local coordinates or invariantly, as in [17, pp. 219–220]) shows that if η is any vector field on JY satisfying $d\pi_Y \eta = 0$ then

$$u^*(\eta \lrcorner \Omega) = \text{tr}\,[(u^* D_\eta \sigma(L)) \circ u^*\omega]\,\text{vol} \tag{8.14}$$

where $u^*\omega \in \text{Hom}\,(TX, VY)$ and $u^* D_\eta \sigma \in \text{Hom}\,(VY, TX)$, so that the trace makes sense. If $u = js$ then $u^*\omega = 0$ so that $u^*(\eta \lrcorner \Omega) = 0$ for all η satisfying $d\pi_Y \eta = 0$. If s_t is a one parameter family of sections of Y with $s_0 = s$ and $s_{t/\partial A} = s_{/\partial A}$, then js_t is a one parameter family of sections of JY where the tangent vector, η, satisfies $d\pi_Y \eta = \xi$ where ξ is the tangent field along s. Then if

$$\frac{d}{dt} I_A(s_t) = 0$$

we conclude that

$$\int u^*(\eta \lrcorner \Omega) = 0$$

for all such η which implies that

$$u^*(\eta \lrcorner \Omega) = 0$$

for all such η. This implies that $u^*(\eta \lrcorner \Omega) = 0$ for all η, satisfying $d\pi_X \eta = 0$. But since $u^* \Omega = 0$ (as Ω is an $n + 1$ form and X is only n dimensional) we conclude that $u^*(\eta \lrcorner \Omega) = 0$ for all η. We thus see that *if s is an extremal for I_A then $u = js$ satisfies* (8.13).

The same argument shows that *if $u = js$* and u satisfies (8.13) then u is an extremal. But we can say a lot more. Suppose we merely assume that u satisfies (8.13). By (8.14) this implies that $u^* \omega$, a section of Hom (TX, VY) is perpendicular to the sub-bundle of Hom (VY, TX) spanned by all $D_\eta \sigma$. If the $D_\eta \sigma$ span all of Hom $(VY, TX)_z$ then condition (8.13) automatically implies that $u^* \omega = 0$, i.e. that $u = js$. The condition that $D_\eta \sigma$ span all of Hom $(VY, TX)_z$ is known as the "regularity condition" on the Lagrangian L. In terms of local coordinates it says that the Hessian

$$\partial^2 L / \partial y_i^\ell \, \partial y_j^k$$

be a non-degenerate fn by fn matrix. Thus, for regular Lagrangians, the *equation*

$$u^*(\eta \lrcorner \Omega) = 0$$

on all η is equivalent to the pair of conditions

$$u = js \qquad s \text{ is an extremal of } I.$$

Let ξ be a vector field on JY. The condition that ξ (infinitesimally) preserve Ω is

$$0 = D_\xi \Omega = d(\xi \lrcorner \Omega).$$

Locally, this is equivalent to the stronger condition

$$\xi \lrcorner \Omega = -d\tau. \tag{8.15}$$

Let us call a vector field satisfying (8.15) *Hamiltonian*. Notice that if u satisfies (8.13) then

$$du^*(\tau) = 0,$$

i.e. the form $u^*(\tau)$ is closed. We will want to think of τ as if it defines a functional on extremals by the "formula"

$$\tau(u) = \int_C u^* \tau,$$

where C is a suitable $n - 1$ dimensional submanifold of X. If, for instance, $X = M \times \mathbf{R}$ and τ had "compact support" in the M-variables, then we would choose C to be a "space like surface" $\{(m, t(m))\}$ and the value of the integral would not depend on the particular choice of the space like surface, C, i.e. on the choice of the function, t. Moreover, modifying τ by adding an exact form, $d\nu$, (of compact support in M) would not change the value of the integral. With this in mind, we define the "algebra of currents" to consist of all $n-1$ forms τ for which there exists a ξ such that (8.15) holds. We define the "algebra of charges" to consist of equivalence classes $[\tau]$ where τ satisfies (8.15) and where

$$[\tau] = [\tau'] \qquad \text{if } \tau' = \tau + d\nu, \qquad \text{for some } n-2\text{-form } \nu.$$

We define the Poisson bracket

$$\{\tau_1, \tau_2\} = \xi_1 \lrcorner d\tau_2$$

where $\xi_i \lrcorner \Omega = -d\tau_i$. Observe that this does not depend on the particular choice of ξ_i. Indeed

$$\xi_1 \lrcorner d\tau_2 = -\xi_1 \lrcorner \xi_2 \lrcorner \Omega = \xi_1 \lrcorner \xi_2 \lrcorner \Omega = -\xi_2 \lrcorner d\tau_1$$

which shows that if $\xi_1 \lrcorner \Omega = 0$ then $\xi_1 \lrcorner d\tau_2 = 0$ and so $\{\tau_1, \tau_2\}$ is independent of which ξ_1 we choose to satisfy $\xi_1 \lrcorner \Omega = -d\tau_1$. Also we see that $\{\tau_1, \tau_2\}$ is antisymmetric in τ_1 and τ_2. Also observe that

$$\xi_1 \lrcorner d(d\nu) = 0$$

so the Poisson bracket induces an operation:

$$\{[\tau_1], [\tau_2]\} = [\{\tau_1, \tau_2\}]$$

which we also call the Poisson bracket. Notice that

$$D_1(\xi_2 \lrcorner \Omega) = [\xi_1, \xi_2] \lrcorner \Omega$$

and

$$\begin{aligned} D_{\xi_1}(\xi_2 \lrcorner \Omega) &= D_{\xi_1} d\tau_2 = d(D_{\xi_1}\tau_2) \\ &= d(\xi_1 \lrcorner d\tau_2) + d(d(\xi_1 \lrcorner \tau_2)) = d(\xi_1 \lrcorner d\tau_2) \end{aligned}$$

which shows that

$$[\xi_1, \xi_2] \lrcorner \Omega = -d\{\tau_1, \tau_2\}.$$

Also, we see that

$$\begin{aligned} \{\tau_1, \{\tau_2, \tau_3\}\} &= [\xi_2, \xi_3] \lrcorner d\tau_1 \\ &= -D_{\xi_2}(\xi_3 \lrcorner d\tau_1) + \xi_3 \lrcorner D_{\xi_2} d\tau_1 \\ &= -D_{\xi_2}(\xi_1 \lrcorner d\tau_3) + \{\{\tau_1, \tau_2\}, \tau_3\} \\ &= \xi_2 \lrcorner d(\xi_1 \lrcorner d\tau_3) + d(\xi_2 \lrcorner \xi_1 \lrcorner \tau_3) + \{\{\tau_1, \tau_2\}, \tau_3\} \\ &= \{\{\tau_1, \tau_2\}, \tau_3\} + \{\tau_2, \{\tau_1, \tau_3\}\} + d(\xi_1 \lrcorner \xi_2 \lrcorner \xi_3 \lrcorner \Omega). \end{aligned}$$

Thus the algebra of currents need not satisfy Jacobi's identity but the algebra of charges does. Of course, if dim $X = 1$, the last term vanishes.

Let φ_t be a one parameter family of automorphisms of JY which satisfies

$$\varphi_t^* \Theta = \Theta + d\alpha_t .$$

(For example φ_t might arise from a one parameter family of automorphisms of Y which preserves L.) If η is the corresponding vector field, then

$$D_\eta \Theta = d\dot{\alpha}_0 \qquad \eta \lrcorner \Omega = d(\dot{\alpha}_0 - \eta \lrcorner \Theta)$$

so that η is Hamiltonian with $\tau = \dot{\alpha}_0 - \eta \lrcorner \Theta$. Thus φ_t gives rise to the "conserved current" $u^* \tau$ for any extremal, u. This is the content of "Noether's theorem".

For the case where $n = \dim X = 1$, the condition on τ (which is now a form of degree zero, i.e. a function) that is imposed by equation (8.15) is that τ be constant along extremals. Indeed, in this case, the extremals are the integral curves to the line element field spanned by η satisfying $\eta \lrcorner \Omega = 0$, and the condition $\langle \eta, d\tau \rangle = 0$ is equivalent to (8.15) since the two form Ω has rank $2f$ and JY has dimension $2f + 1$. Thus, for the case $n = 1$, the "algebra of currents" is an honest Lie algebra, can be identified with the "algebra of charges", and consists of *all* smooth functions on extremals.

For $n > 1$, the condition (8.15) is much more restrictive. The "functions on extremals" corresponding to $[\tau]$ where τ satisfies (8.15) will, in general, constitute only a small subspace of the space of functions on extremals. In fact, for many interesting Lagrangians, the space of such $[\tau]$ will be finite dimensional. If the Lagrangian is "quadratic", that is can be expressed as a quadratic function of the coordinates of JY when suitable coordinates are chosen on X and Y, then there is an infinite dimensional space of $[\tau]$ which are "sufficient" in a sense that we will not specify here.

The problem then arises as to how to introduce a reasonable class of functions on extremals. One possible method, suggested by [18], is to consider an "infinitesimal version" of the construction of $[\tau]$. This involves considering the so called "second variation" and Jacobi fields. Roughly speaking, the situation is as follows: let u_t be a one parameter family of extremals, and let ξ be the vector field along u tangent to u_t at $t = 0$. Then ξ satisfies the Jacobi equation

$$u^*(\eta \lrcorner d(\xi \lrcorner \Omega)) = 0, \qquad \text{for all vector fields } \eta \text{ along } u.$$

The set of ξ satisfying the above equation can be described as the extremals of a quadratic Lagrangian defined on the vector bundle of all vector fields along u, and this quadratic Lagrangian is the second variation. We refer to [17] for details, especially pp. 255–262. We can think of the set of ξ satisfying the above equation as the "tangent space" to the set of extremals at u. We can then define a bilinear antisymmetric form Ξ from this "tangent space" to $\wedge^{n-1} X$, sending ξ_1, ξ_2 into

$u^*(\xi_1 \lrcorner \xi_2 \lrcorner \Omega)$. One checks that this form actually is closed for any ξ_1, ξ_2 in the "tangent space" to u, and hence if ξ_1 or ξ_2 had suitable "compact support in the space like direction" would define an antisymmetric two form on the "tangent space" by integration over some space like surface. This two form would then be the candidate for the symplectic form on the "manifold of all extremals". With this symplectic form one then considers those "functions on extremals" f, which satisfy $df_u = \int \xi \lrcorner \Xi$, the integral being taken over the space like surface. We refer the reader to [18] for details.

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