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A GENERALIZATION OF THE HOPF INVARIANT[1]

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Little progress has been made in the study of homotopy groups of spheres since the appearance of Freudenthal's paper [9] on the subject in 1937. Perhaps one reason for this lack of progress has been the fact that there were very few methods for determining whether or not a mapping of one sphere on another is essential or not; in other language, whether the n^{th} homotopy group $\pi_n(S^r)$ is different from zero. One such method is furnished by the Brouwer degree of a mapping of S^r on itself [1]; another by the Hopf invariant of a mapping of S^{2r-1} on S^r [10, 11]. These methods are, however, limited in scope, applying as they do only to mappings of S^n on S^r with $n = r$ or $n = 2r - 1$. In the intermediate cases, $r < n < 2r - 1$, Freudenthal's results reduce the problem of the determination of $\pi_n(S^r)$ for $n < 2r - 1$ to that of calculating $\pi_{2k-1}(S^k)$ with $k = n - r + 1$. But almost nothing was known about $\pi_n(S^r)$ for $n > 2r - 1$.

The present paper attempts a step in this direction by defining a homomorphism $H:\pi_n(S^r) \to \pi_n(S^{2r-1})$ for each $n < 3r - 3$. This homomorphism is a generalization of the Hopf invariant in the following sense. The Hopf invariant can be described as a homomorphism of $\pi_{2r-1}(S^r)$ into the additive group of integers; the group $\pi_{2r-1}(S^{2r-1})$ is known to be isomorphic with the group of integers, and under a suitably chosen such isomorphism H is identical with this homomorphism in case $n = 2r - 1$.

The Hopf invariant is closely connected with the suspension homomorphism $E_n : \pi_n(S^r) \to \pi_{n+1}(S^{r+1})$ [9]. In fact, one of Freudenthal's theorems states that the image of $E_{2r-2} : \pi_{2r-2}(S^{r-1}) \to \pi_{2r-1}(S^r)$ is the subgroup of those elements of $\pi_{2r-1}(S^r)$ whose Hopf invariant is zero; i.e., the image of E_{2r-2} is the kernel of H. One criterion for the usefulness of a generalization of the Hopf invariant would seem to be the truth of the theorem that the image of E_n is always equal to the kernel of H. This we have not been able to prove or disprove. It is immediately evident from the definition of H that the image of E_n is contained in the kernel of H; but the opposite inclusion seems to be much more difficult.

Another of Freudenthal's theorems deals with the *kernel* of E_n. Again we are able to obtain only a partial generalization of Freudenthal's results in the general case. However, in the special case of $E_{2r-1} : \pi_{2r-1}(S^r) \to \pi_{2r}(S^{r+1})$ we are able to prove a stronger result than that obtained by Freudenthal. In fact, we prove that the kernel of E_{2r-1} is a *cyclic* group, and exhibit a generator of this group. In case r is odd, the kernel of E_{2r-1} is infinite cyclic; if r is even, it is either zero or cyclic of order two, according as $\pi_{2r+1}(S^{r+1})$ contains an element with Hopf invariant 1 or not.

The problem of the existence of elements of $\pi_{2r-1}(S^r)$ with given Hopf invariant was proposed by Hopf [11], who proved that if r is odd, then every

[1] Presented to the American Mathematical Society, April 26, 1946, and September 9, 1948. Some of the results of this paper were announced in [20].

element of $\pi_{2r-1}(S^r)$ has Hopf invariant zero; while if r is even, there exist elements with Hopf invariant 2 and therefore with any even integer as Hopf invariant; and that there exist elements with invariant 1 if $r = 2, 4$, or 8. We shall prove here the first result in the negative direction for r even; if $r \equiv 2 \pmod 4$ and $r > 2$, no map with odd Hopf invariant exists. As a consequence S^{4k+1} cannot admit a continuous multiplication with two-sided identity if $k > 0$.

Sections 1–3 of the paper are preliminary in nature. Section 1 contains a list of notations used throughout the paper. Section 2 is devoted to a discussion of homotopy groups, and Section 3 describes the properties of some of the many operations involving homotopy groups which have appeared either explicitly or implicitly in the literature.

The original definition of the Hopf invariant can be described roughly by the statement that the Hopf invariant measures the extent to which the counter-image of two points (say the north and south poles) are linked. In the case $n = 2r - 1$, this linking can be described combinatorially by means of a linking number; but if $n > 2r - 1$, the necessary combinatorial machinery is lacking and we are forced to use a different approach. The counter-images of the north and south poles are "separated" by collapsing an equator $S^{r-1} \subset S^r$ to a point to obtain a space which may be regarded as the union $S^r \vee S^r$ of two tangent spheres. Thus each map of S^n into S^r determines a map of S^n into $S^r \vee S^r$. In Section 4 we prove first a general theorem on the homotopy groups of the union of two spaces A and B which have just one point in common, and then proceed to obtain more precise results when A and B are specialized to be spheres. A useful tool here is a theorem of J. H. C. Whitehead [22] on the homotopy groups of a space obtained from another by adjoining an open cell. We obtain in passing a theorem on $\pi_n(S^p \vee S^q)$ which generalizes one due to J. H. C. Whitehead [21]. In Section 5 we apply the results of the preceding section to define and prove a series of properties of the Hopf invariant.

In Sections 6 and 7 we investigate the kernel of the suspension homomorphism. The principal tool here is a generalization (similar to the previous generalization of the Hopf invariant) of a pair of numerical invariants associated by Freudenthal with each nullhomotopy of the suspension of a given map. A formula is proved which relates these generalized Freudenthal invariants with the Hopf invariant of the given map. In this section we also prove the strengthened form of the Freudenthal theorem mentioned earlier.

Section 8 is devoted to the construction of some essential mappings of spheres on spheres. The generalized Hopf invariant is used to prove that certain elements of $\pi_n(S^r)$ are different from zero; the generalized Freudenthal invariants are also used to prove that the suspensions of some of these elements are non-zero. We prove in fact that $\pi_n(S^r) \neq 0$ for $(n, k) = (14, 7)$, $(14, 4)$, $(8k, 4k)$, $(8k + 1, 4k + 1)$, $(16k + 2, 8k)$, $(16k + 3, 8k + 1)$. Section 9 is devoted to the proof of the afore-mentioned theorem on the non-existence of maps with Hopf invariant 1. Section 10 contains some concluding remarks and a few conjectures.

1. Preliminaries

This section contains a list of notations to be used throughout the paper. Many of the spaces we consider will be subspaces of Cartesian n-space C^n for some integer n; it will be convenient to consider all the spaces C^n as subspaces of Cartesian space of infinitely many dimensions. Thus we define C^n to be the set of all infinite sequences $x = (x_1, \cdots, x_n, \cdots)$ of real numbers such that $x_i = 0$ for $i > n$, and set $C = \bigcup_{n=1}^{\infty} C^n$. The space C is metrized by the distance function

$$||x - y|| = (\textstyle\sum_{i=1}^{\infty}(x_i - y_i)^2)^{\frac{1}{2}} \qquad (x, y \in C). \tag{1.1}$$

It will facilitate the writing of formulas to ignore the distinction between a finite and an infinite sequence; thus the symbols $(x_1, \cdots, x_n)$ and $(x_1, \cdots, x_n, 0, \cdots)$ will denote the same point $x \in C^n$.

The *unit n-cube* E^n is the subset of C^n defined by

$$E^n = \{x \in C^n \mid -1 \leqq x_i \leqq 1 \text{ for } i = 1, \cdots, n\}. \tag{1.2}$$

We distinguish two subsets $\dot{E}^n$ and J^{n-1} of E^n:

$$\begin{cases} \dot{E}^n = \{x \in E^n \mid \prod_{i=1}^{n} (1 - x_i^2) = 0\}, \\ J^{n-1} = \{x \in E^n \mid (1 + x_n) \prod_{i=1}^{n-1} (1 - x_i^2) = 0\}; \end{cases} \tag{1.3}$$

$\dot{E}^n$ is the boundary of E^n, while J^{n-1} is an (n $-$ 1)-cell contained in $\dot{E}^n$.

The *unit n-sphere* S^n is defined by

$$S^n = \{x \in C^{n+1} \mid \textstyle\sum_{i=1}^{n+1} x_i^2 = 1\}. \tag{1.4}$$

We denote by y_* the point $(1, 0, \cdots, 0, \cdots)$ of C and observe that $y_* \in S^n$ for every $n \geqq 0$. We also define

$$\begin{cases} E_+^n = \{x \in S^n \mid x_{n+1} \geqq 0\}, \\ E_-^n = \{x \in S^n \mid x_{n+1} \leqq 0\}, \\ S_0^{n-1} = \{x \in S^n \mid x_2 = 0\}, \\ K_+^n = \{x \in S^n \mid x_2 \geqq 0\}, \\ K_-^n = \{x \in S^n \mid x_2 \leqq 0\}, \\ z^n = (1, 1, \cdots, 1) \in E^n. \end{cases} \tag{1.5}$$

We next define a mapping d_n: $S^n \times E^1 \rightarrow S^{n+1}$ as follows.[2] If $x \in S^n$, $0 \leqq t \leqq 1$, then $d_n(x, t)$ is the point of E_+^{n+1} whose "vertical" projection into the equatorial plane $x_{n+2} = 0$ is the point which separates the line segment from x to y_* in

[2] If X, Y are topological spaces, a *mapping* $f: X \rightarrow Y$ is a continuous function f on X to Y. If $A \subset X$, $B \subset Y$, a mapping $f: (X, A) \rightarrow (Y, B)$ is a mapping $f: X \rightarrow Y$ such that $f(A) \subset B$. If $f: (X, A) \rightarrow (Y, B)$ and $g: (Y, B) \rightarrow (Z, C)$ are mappings, $g \circ f$ denotes the composite mapping of (X, A) into (Z, C).

the ratio $t: 1 - t$; while if $x \in S^n$, $-1 \leqq t \leqq 0$, then $d_n(x, t)$ is the mirror image with respect to the equatorial plane of the point $d_n(x, -t)$. Thus

$$(1.6)\quad d_n(x, t) = \begin{cases} (t + (1-t)x_1, (1-t)x_2, \cdots, (1-t)x_{n+1}, \\ \qquad (2t(1-t)(1-x_1))^{1/2}) \quad (0 \leqq t \leqq 1), \\ (-t + (1+t)x_1, (1+t)x_2, \cdots, (1+t)x_{n+1}, \\ \qquad -(-2t(1+t)(1-x_1))^{1/2}) \quad (-1 \leqq t \leqq 0). \end{cases}$$

The following properties of the map d_n are easily verified:

(1.7) d_n maps $(S^n - y_*) \times [0, 1)$ topologically on $E_+^{n+1} - y_*$;

(1.8) d_n maps $(S^n - y_*) \times (-1, 0]$ topologically on $E_-^{n+1} - y_*$;

(1.9) d_n maps $(S^n \times 1) \cup (S^n \times (-1)) \cup (y_* \times E^1)$ into y_*;

$$(1.10)\quad d_n(x, 0) = x \text{ for } x \in S^n.$$

For $n \geqq 2$, let $\rho_n: S^n \to S^n$ be the reflection of S^n about the $(n-1)$-dimensional plane $x_2 = x_3 = 0$;

$$(1.11)\quad \rho_n(x_1, \cdots, x_{n+1}) = (x_1, -x_2, -x_3, x_4, \cdots, x_{n+1}), \qquad (x \in S^n).$$

Let further $\tau_n: S^n \to S^n$ be the rotation through 90° about the great $(n-2)$-sphere $S_0^{n-1} \cap S^{n-1}$, so that

$$(1.12)\quad \tau_n(x_1, \cdots, x_{n+1}) = (x_1, -x_{n+1}, x_3, \cdots, x_n, x_2).$$

Note that

$$(1.13)\quad \begin{aligned} \tau_n(E_+^n) &= K_-^n, \quad \tau_n(E_-^n) = K_+^n, \\ \tau_n(K_+^n) &= E_+^n, \quad \tau_n(K_-^n) = E_-^n, \end{aligned}$$

and that τ_n is homotopic to the identity map of S^n on itself.

For $n \geqq 1$, let $\theta_n: (E^n, \dot{E}^n) \to (E^n, \dot{E}^n)$ be the reflection of E^n about the $(n-1)$-dimensional plane $x_1 = 0$;

$$(1.14)\quad \theta_n(x_1, \cdots, x_n) = (-x_1, x_2, \cdots, x_n).$$

We next define (for $n \geqq 1$) a mapping $\varphi_n': S^n \to S^n$ by the formula

$$(1.15)\quad \varphi_n'(x_1, \cdots, x_{n+1}) = \begin{cases} (2x_1 - 1, 2^{1/2}(x_2^2 - (1-x_1)^2)^{1/2}, \\ \qquad 2^{1/2}x_3, \cdots, 2^{1/2}x_{n+1}) \quad (1 - x_1 \leqq x_2 \leqq 1), \\ \left(1 - 2\dfrac{x_2^2}{1-x_1}, -2^{1/2}\dfrac{x_2}{1-x_1}((1-x_1)^2 - x_2^2)^{1/2},\right. \\ \left. 2^{1/2}\dfrac{x_2x_3}{1-x_1}, \cdots, 2^{1/2}\dfrac{x_2x_{n+1}}{1-x_1}\right) \quad (0 \leqq x_2 \leqq 1 - x_1), \\ y_* \qquad (-1 \leqq x_2 \leqq 0). \end{cases}$$

and note that φ_n' maps $K_+^n - S_0^{n-1}$ topologically on $S^n - y_*$, while $\varphi_n'(K_-^n) = y_*$. Moreover φ_n' is homotopic to the identity map of S^n onto itself. It can be verified by direct computation that

$$\varphi_{n+1}'(d_n(x, t)) = d_n(\varphi_n'(x), t). \tag{1.16}$$

We also define a mapping $\varphi_n'': S^n \to S^n$ by the formula

$$\varphi_n''(x) = \varphi_n'(\rho_n(x)) \qquad (x \in S^n). \tag{1.17}$$

Since ρ_n and φ_n' are homotopic to the identity, so is φ_n''.

We denote by $S^p \vee S^q$ the subspace $(S^p \times y_*) \cup (y_* \times S^q)$ of the product space $S^p \times S^q$, and define a mapping $\varphi_n : S^n \to S^n \vee S^n$ by the formula

$$\varphi_n(x) = (\varphi_n'(x), \varphi_n''(x)) \qquad (x \in S^n). \tag{1.18}$$

The map φ_n has the following properties:

(1.19) φ_n maps $K_+^n - S_0^{n-1}$ topologically on $(S^n - y_*) \times y_*$;

(1.20) φ_n maps $K_-^n - S_0^{n-1}$ topologically on $y_* \times (S^n - y_*)$;

$$\varphi_n(S_0^{n-1}) = y_* \times y_* . \tag{1.21}$$

Let $\bar{\delta}_n: (S^n \times S^n \times E^1, (S^n \vee S^n) \times E^1) \to (S^{n+1} \times S^{n+1}, S^{n+1} \vee S^{n+1})$ be the mapping defined by

$$\bar{\delta}_n(y, y', t) = (d_n(y, t), d_n(y', t)) \qquad (y, y' \in S^n, t \in E^1), \tag{1.22}$$

and define $\delta_n: (S^n \vee S^n) \times E^1 \to S^{n+1} \vee S^{n+1}$ by

$$\delta_n = \bar{\delta}_n \mid (S^n \vee S^n) \times E^1. \tag{1.23}$$

Since $\varphi_{n+1}'(d_n(x, t)) = d_n(\varphi_n'(x), t)$ and

$$\begin{aligned} \varphi_{n+1}''(d_n(x, t)) &= \varphi_{n+1}'(\rho_{n+1}(d_n(x, t))) \\ &= \varphi_{n+1}'(d_n(\rho_n(x), t)) \\ &= d_n(\varphi_n'(\rho_n(x)), t) \\ &= d_n(\varphi_n''(x), t) \end{aligned} \tag{1.24}$$

we have

$$\varphi_{n+1}(d_n(x, t)) = \delta_n(\varphi_n(x), t). \tag{1.25}$$

Let $\sigma_n: (S^n \times S^n, S^n \vee S^n) \to (S^n \times S^n, S^n \vee S^n)$ be the mapping such that

$$\sigma_n(x, x') = (x', x) \qquad (x, x' \in S^n), \tag{1.26}$$

and observe that

$$\varphi_n(\rho_n(y)) = \sigma_n(\varphi_n(y)) \qquad (y \in S^n). \tag{1.27}$$

Define also $\sigma_n': S^n \vee S^n \to S^n \vee S^n$ by

$$\sigma_n' = \sigma_n \mid S^n \vee S^n. \tag{1.28}$$

We next define a mapping ψ_n: $(E^n, \dot{E}_n) \rightarrow (S^n, y_*)$.
For $n = 1$, ψ_n is defined by

$$(1.29)\quad \psi_1(t) = d_0(-1, t) = \begin{cases} (2t - 1, 2(t(1 - t))^{\frac{1}{2}}) & (0 \leqq t \geqq 1), \\ (-2t - 1, -2(-t(1 + t))^{\frac{1}{2}}) & (-1 \leqq t \leqq 0). \end{cases}$$

The definition of ψ_n is completed inductively by setting

$$(1.30)\quad \psi_{n+1}(x_1, \cdots, x_{n+1}) = d_n(\psi_n(x_1, \cdots, x_n), x_{n+1}) \qquad (x \in E^{n+1}).$$

It is easy to see that ψ_n is topological on $E^n - \dot{E}^n$, while $\psi_n(\dot{E}^n) = y_*$.

The maps ψ_n are used to define mappings $\bar{\psi}'_{p,q}$: $(E^p \times E^q, (E^p \times \dot{E}^q) \cup (\dot{E}^p \times E^q)) \rightarrow (S^p \times S^q, S^p \vee S^q)$;

$$(1.31)\quad \bar{\psi}'_{p,q}(x, y) = (\psi_p(x), \psi_q(y)) \qquad (x \in E^p, y \in E^q).$$

The cube E^{p+q} has a natural representation as the product space of E^p and E^q; this representation is realized by the mapping $\eta_{p,q}$: $E^p \times E^q \rightarrow E^{p+q}$ defined by

$$(1.32)\quad \eta_{p,q}(x, y) = (x_1, \cdots, x_p, y_1, \cdots, y_q) \qquad (x \in E^p, y \in E^q).$$

The map $\eta_{p,q}$ is a homeomorphism which maps $E^p \times \dot{E}^q \cup \dot{E}^p \times E^q = (E^p \times E^q)^\cdot$ onto $\dot{E}^{p+q}$. We now set

$$(1.33)\quad \bar{\psi}_{p,q} = \bar{\psi}'_{p,q} \circ \overset{-1}{\eta}_{p,q}$$

so that $\bar{\psi}_{p,q}$: $(E^{p+q}, \dot{E}^{p+q}) \rightarrow (S^p \times S^q, S^p \vee S^q)$, and further define $\psi_{p,q} = \bar{\psi}_{p,q} \mid \dot{E}^{p+q}$, and $\psi'_{p,q} = \bar{\psi}'_{p,q} \mid (E^p \times E^q)^\cdot$, so that $\psi_{p,q}$: $\dot{E}^{p+q} \rightarrow S^p \vee S^q$ and $\psi'_{p,q}$: $(E^p \times E^q)^\cdot \rightarrow S^p \vee S^q$.

2. Homotopy groups

We list here for reference the definitions and properties of homotopy groups needed in the sequel.[3] There are in current use two different (but equivalent) definitions of homotopy groups. In the first of these definitions the cube E^n is used as antecedent space; in the second the sphere S^n is used instead of E^n. Moreover it is frequently convenient to replace E^n (or S^n) by a homeomorphic copy. In order to make consistent use of such homeomorphs of E^n (or S^n), questions of orientation must be settled. We begin by selecting orientations of E^n and S^n.

We recall that an orientation of E^n is simply a generator of the (infinite cyclic) integral relative homology group $H_n(E^n, \dot{E}^n)$; while an orientation of S^n is a generator of the (likewise infinite cyclic) integral homology group $H_n(S^n)$ (or preferably here but equivalently, a generator of $H_n(S^n, y_*)$). We first orient E^1 by considering E^1 as the ordered 1-simplex whose first vertex is -1 and whose last vertex is $+1$; the identity map of E^1 into itself is a singular 1-simplex (in the sense of Eilenberg [6]), which is a 1-cycle modulo $\dot{E}^1$; the homology class of

[3] For an exposition of the elements of homotopy theory see [8; 12].

this 1-cycle is an orientation ω_1 of E^1. We now suppose that orientations ω'_{n-1} of S^{n-1} and ω_n of E^n have been selected and proceed to define inductively first ω'_n and then ω_{n+1}. The map ψ_n: $(E^n, \dot{E}^n) \to (S^n, y_*)$ induces a homomorphism ψ_n^*: $H_n(E^n, \dot{E}^n) \to H_n(S^n, y_*)$; we set $\omega'_n = \psi_n^*(\omega_n)$. Next let r_n be the radial projection (from the origin) of S^n on $\dot{E}^{n+1}$ (i.e., if $x \in S^n$, $r_n(x)$ is the point in which the half-line beginning at the origin and containing x intersects $\dot{E}^{n+1}$). The map r_n induces an isomorphism r_n^*: $H_n(S^n, y_*) \to H_n(\dot{E}^{n+1}, y_*)$ and therefore $r_n^*(\omega'_n)$ is an orientation ω''_n of E^{n-1}. The boundary operator ∂_{n+1}^* maps $H_{n+1}(E^{n+1}, \dot{E}^{n+1})$ isomorphically onto $H_n(\dot{E}^{n+1}, y_*)$; we set $\omega_{n+1} = \partial_{n+1}^{*-1}(\omega''_n)$. The choice of orientations is indicated by the diagram

$$(2.1) \quad \begin{aligned} &H_1(E^1, \dot{E}^1) \to \cdots \to H_n(E^n, \dot{E}^n) \xrightarrow{\psi_n^*} H_n(S^n, y_*) \\ &\qquad \xrightarrow{r_n^*} H_n(\dot{E}^{n+1}, y_*) \xrightarrow{\partial_{n+1}^{*-1}} H_{n+1}(E^{n+1}, \dot{E}^{n+1}) \to \cdots . \end{aligned}$$

Now let X be a topological space, A a subspace of X, and x_* a point of A. Let $F^n(X, A, x_*)$ be the set of all mappings of $(E^n, \dot{E}^n, J^{n-1})$ into (X, A, x_*). If f, $g \in F^n(X, A, x_*)$ we say that f is homotopic to g if and only if there is a mapping F: $(E^n \times E^1, \dot{E}^n \times E^1, J^{n-1} \times E^1) \to (X, A, x_*)$ such that $F(x, -1) = f(x)$ and $F(x, 1) = g(x)$ for $x \in E^n$. Homotopy is an equivalence relation; the set of all homotopy classes of elements of $F^n(X, A, x_*)$ is denoted by $\pi_n(X, A, x_*)$. In order to introduce a group operation in $\pi_n(X, A, x_*)$ for $n \geqq 2$ and for $n = 1$ if $A = x_*$ we first define an operation in $F^n(X, A, x_*)$: if $f, g \in F^n(X, A, x_*)$ we set

$$(2.2) \quad (f + g)(x_1, \cdots, x_n) = \begin{cases} f(2x_1 + 1, x_2, \cdots, x_n) & (-1 \leqq x_1 \leqq 0), \\ g(2x_1 - 1, x_2, \cdots, x_n) & (0 \leqq x_1 \leqq 1). \end{cases}$$

If $n > 1$ or if $n = 1$ and $A = x_*$, $f + g$ is again an element of $F^n(X, A, x_*)$ and its homotopy class depends only on the homotopy classes of f and g and in these cases the operation $+$ induces an operation (also denoted by $+$) in $\pi_n(X, A, x_*)$. If x'_* is a point which belongs to the same path-component of A as x_*, the groups $\pi_n(X, A, x_*)$ and $\pi_n(X, A, x'_*)$ are isomorphic; this isomorphism is in general not uniquely defined but depends on a homotopy class of paths joining x_* to x'_* in A. In most of the cases in which homotopy groups will be used in this paper the pair (X, A) will be n-simple; i.e., X and A are pathwise connected and the above isomorphism is independent of the path. In the notation for homotopy groups we shall frequently suppress the base point x_*, and also the subspace A if $A = x_*$.

If (X, A, x_*) and (X', A', x'_*) are triples and f: $(X, A, x_*, \to (X', A', x'_*)$ is mapping, then f induces a homomorphism f: $\pi_n(X, A, x_*) \to \pi_n(X', A', x'_*)$ which depends only on the homotopy class of f. If g: $(E^n, \dot{E}^n, J^{n-1}) \to (X, A, x_*)$ is an element of the homotopy class $\alpha \in \pi_n(X, A, x_*)$, then the map $f \circ g$: $(E^n, \dot{E}^n, J^{n-1}) \to (X', A', x'_*)$ is an element of $f(\alpha)$.

If $f \in F^n(X, A, x_*)$ belongs to the homotopy class $\alpha \in \pi_n(X, A, x_*)$, then the homotopy class of the mapping ∂f: $(E^{n-1}, \dot{E}^{n-1}) \to (A, x_*)$ defined by

$$(2.3) \quad \partial f(x_1, \cdots, x_{n-1}) = f(x_1, \cdots, x_{n-1}, 1)$$

depends only on α and is denoted by $\partial(\alpha)$. The function $\partial:\pi_n(X, A, x_*) \to \pi_{n-1}(A, x_*, x_*)$ is a homomorphism, called the *boundary operator.*

The homotopy groups of a space X, a subspace A, and the relative homotopy groups of (X, A) are connected by a sequence of homomorphisms:

$$(2.4) \qquad \cdots \to \pi_n(A) \xrightarrow{i} \pi_n(x) \xrightarrow{j} \pi_n(x, A) \xrightarrow{\partial} \pi_{n-1}(A) \to \cdots \to \pi_1(x).$$

Here $i:(A, x_*, x_*) \to (X, x_*, x_*)$ and $j:(X, x_*, x_*) \to (X, A, x_*)$ are the appropriate identity maps. This sequence (which is referred to as the *homotopy sequence* of the triple (X, A, x_*) is *exact* in the sense that the kernel of each homomorphism is the sequence is the image of the preceding homomorphism.

The group $\pi_n(X, A)$ is abelian if $n \geqq 3$, and $\pi_2(X, A)$ is abelian if $\pi_1(A) = 0$; thus $\pi_2(X)$ is abelian.

An *oriented n-cell* is a quadruple $(E, \dot{E}, x, \omega)$, where E is a topological space homeomorphic with E^n under a homeomorphism which maps $(E^n, \dot{E}^n)$ onto $(E, \dot{E})$, x is a point of $\dot{E}$, and ω is a generator of $H_n(E, \dot{E})$. Let $(E, \dot{E}, x, \omega)$ be an oriented n-cell and let $f:(E, \dot{E}, x) \to (X, A, x_*)$ be a mapping. Choose a mapping $h:(E^n, \dot{E}^n) \to (E, \dot{E})$ (not necessarily a homeomorphism) such that $x \in h(J^{n-1}) \subset f^{-1}(x_*)$ and $h^*(\omega_n) = \omega$. Then $f \circ h \in F^n(X, A, x_*)$ and the homotopy class α of $f \circ h$ depends only on the homotopy class of f and not on the particular representative chosen, nor on the choice of h subject to the above conditions. We shall say that f is a *representative* of α. It is easy to see that if $(E, \dot{E}, x, \omega)$ is an oriented n-cell and $\alpha \in \pi_n(X, A, x_*)$ there is one and only one homotopy class of mappings: $(E, \dot{E}, x) \to (X, A, x_*)$ each of whose elements represents α. Thus $\pi_n(X, A)$ could have been described as the set of homotopy classes of mappings of $(E, \dot{E}, x)$ into (X, A, x_*); the only role played by the orientation ω is to allow us to compare mappings of different spaces E into X.

An *oriented n-sphere* is a triple (S, x, ω), where S is a topological space homeomorphic with S^n, $x \in S$, and ω is a generator of $H_n(S^n, x)$. Let (S, x, ω) be an oriented n-sphere and let $f:(S, x) \to (X, x_*)$ be a mapping. Choose a mapping $h:(E^n, \dot{E}^n) \to (S, x)$ such that $h^*(\omega_n) = \omega$. Then $f \circ h$ is a mapping: $(E^n, \dot{E}^n) \to (X, x_*)$ whose homotopy class α does not depend on the choice of f in its homotopy class nor on the choice of h subject to the above condition. As above, we shall say that f *represents* α. Again, if (S, x, ω) is an oriented n-sphere, the elements of $\pi_n(X)$ are in 1:1 correspondence with the homotopy classes of mappings: $(S, x) \to (X, x_*)$.

We list some standard oriented cells and spheres for future reference. Hereafter, whenever reference is made to the element of a homotopy group represented by a map of one of the cells listed here, it will be understood that the cell or sphere is oriented by the orientation given here.

Cells: $(E^n, \dot{E}^n, x, \omega_n)$, where x is any point of J^{n-1};
$(E^p \times E^q, (E^p \times E^q)^{\cdot}, x, \overset{*-1}{\eta}_{p,q}(\omega_{p+q}))$, where x is any point of $(E^p \times E^q)^{\cdot}$.

Spheres: (S^n, y_*, ω_n');
$(\dot{E}^{n+1}, x, \omega_n'')$ where x is any point of $\dot{E}^{n+1}$;
$((E^p \times E^q)^{\cdot}\ x, \omega)$ where x is any point of $(E^p \times E^q)^{\cdot}$ and $\omega = \overset{*}{\partial}_{p+q}(\overset{*-1}{\eta}_{p,q}(\omega_{p+q}))$.

It will also be convenient to allow a certain latitude in the choice of the image space similar to that we have already allowed in the antecedent. However, we shall do this only in the case of cells and spheres.

Let $(E, \dot{E}, x, \omega)$ and $(E', \dot{E}', x', \omega')$ be oriented n-cells. A map h: $(E, \dot{E}) \to (E', \dot{E}')$ is called *admissible* in case $h_n^*(\omega) = \omega'$. Similarly, if (S, x, ω) and (S', x', ω') are oriented n-spheres, an *admissible* map $h: S \to S'$ is any one such that $h_n^*(\omega) = \omega'$.

Let (S, x, ω) be an oriented r-sphere, and h: $S \to S^r$ an admissible map. Then $\boldsymbol{h}$ is an isomorphism of $\pi_n(S)$ onto $\pi_n(S^r)$; since any two admissible maps are homotopic, they induce the same homomorphism. Thus each element $\alpha \in \pi_n(S)$ determines a unique element $\boldsymbol{h}(\alpha) \in \pi_n(S^r)$ and we shall say that any representative of α is also a representative of $\boldsymbol{h}(\alpha)$. Moreover, if (S, x, ω) and (S', x', ω') are oriented spheres of dimensions r, r' respectively, if $h: S \to S^r$ and $h: S' \to S^{r'}$ are admissible maps, and if $f: \pi_n(S) \to \pi_{n'}(S')$, $g: \pi_n(S^r) \to \pi_{n'}(S^{r'})$ are homomorphisms such that $\boldsymbol{h}' \circ f = g \circ \boldsymbol{h}$, we shall say that f and g are *equivalent*. The term *equivalent* will be used in an analogous sense to refer to other operations involving homotopy groups of spheres.

Let $(E, \dot{E}, x, \omega)$ be an oriented $(n+1)$-cell. Just as above, all admissible maps of $(E, \dot{E})$ into $(E^{n+1}, \dot{E}^{n+1})$ induce the same isomorphism $\boldsymbol{h}$ of $\pi_{m+1}(E, \dot{E})$ onto $\pi_{m+1}(E^{n+1}, \dot{E}^{n+1})$, and any representative of an element $\alpha \epsilon \pi_{m+1}(E, E)$ will be said to be a representative of $\boldsymbol{h}(\alpha)$. The notion of equivalence can then be extended to compare operations involving relative homotopy groups of oriented cells modulo their boundaries.

If $(E, \dot{E}, x, \omega)$ is an oriented $(n + 1)$-cell, then $(\dot{E}, x, \partial_{n+1}^*(\omega))$ is an oriented n-sphere, which we shall refer to as the *boundary* of the oriented cell $(E, \dot{E}, x, \omega)$. The homotopy groups of E all vanish; hence by the exactness of the homotopy sequence of $(E, \dot{E})$, the homomorphism $\boldsymbol{\partial}: \pi_{m+1}(E, \dot{E}) \to \pi_m(\dot{E})$ is an isomorphism onto. Then if $h: \dot{E} \to S^n$ is an admissible mapping, the homomorphism $\boldsymbol{h} \circ \boldsymbol{\partial}$: $\pi_{m+1}(E, \dot{E}) \to \pi_m(S^n)$ is an isomorphism onto which is independent of the choice of the admissible mapping h. This being the case, we may use the term *equivalent* in a still wider sense to compare operations involving the homotopy groups of $(E, \dot{E})$ with analogous operations involving homotopy groups of spheres.

We now consider the effect on the homotopy group $\pi_n(S^r)$ of a reversal of orientation in S^n or S^r. The group $\pi_p(S^p)$ is an infinite cyclic group generated by the homotopy class ι_p of the identity map. Let $h_p: S^p \to S^p$ be a mapping representing $-\iota_p$. If $f: S^n \to S^r$ is a mapping representing an element $\alpha \epsilon \pi_n(S^r)$, then $f \circ h_n$ is a representative of $f(-\iota_n) = -f(\iota_n) = -\alpha$, since f is a homomorphism. Thus reversal of orientation in S^n induces a change of sign in $\pi_n(S^r)$ for each r. On the other hand, the correspondence $f \to h_r \circ f$ induces the endomorphism $U_{n,r} = \boldsymbol{h}_r$ of $\pi_n(S^r)$; since we may choose h_r to be the reflection about an $(r - 1)$-dimensional great sphere in S^r, so that $h_r \circ h_r$ is the identity, it follows that $U_{n,r} \circ U_{n,r}$ is the identity. Hence $U_{n,r}$ is an automorphism of period 2. In general $U_{n,r}$ is not merely a change in sign; for example $U_{3,2}$ is the identity automorphism of $\pi_3(S^2)$.

We now list without proof some lemmas which will be useful in handling homotopy groups.

LEMMA 2.6. *Let* $f, g: (S^r, y_*) \to (X, x_*)$, *and let* $f \vee g: (S^r \vee S^r, y_* \times y_*) \to (X, x_*)$ *be the mapping defined by*

$$(2.7)\qquad \begin{cases} (f \vee g)(y, y_*) = f(y) & (y \in S^r), \\ (f \vee g)(y_*, y) = g(y) & (y \in S^r). \end{cases}$$

Suppose that f, g *represent* $\alpha, \beta \in \pi_r(X)$ *respectively. Then* $(f \vee g) \circ \varphi_r : (S^r, y_*) \to (X, x_*)$ *represents* $\beta + \alpha$.

LEMMA 2.8. *Let* $f, g: (S^n, y_*) \to (X, x_*)$ *be mappings representing* $\alpha, \beta \in \pi_n(X)$ *respectively. Suppose that*

$$(2.9)\qquad f(x_1, \cdots, x_{n+1}) = g(x_1, \cdots, x_n, -x_{n+1}) \qquad \text{for } x \in E^n_+ .$$

Let h *be the mapping*: $(S^n, y_*) \to (X, x_*)$ *such that*

$$(2.10)\qquad h(x) = \begin{cases} f(x) & (x \in E^n_-), \\ g(x) & (x \in E^n_+). \end{cases}$$

Then h *represents* $\alpha + \beta$.

LEMMA 2.11. *Let* f, g *be maps*: $(E^n, \dot{E}^n) \to (X, A)$ *such that*

$$(2.12)\qquad f(1, x_2, \cdots, x_n) = g(-1, x_2, \cdots, x_n) \qquad ((x_2, \cdots, x_n) \in E^{n-1}),$$

and suppose that (X, A) *is* n*-simple, and that* f, g *represent* $\alpha, \beta, \in \pi_n(X, A)$ *respectively. Let* $h: (E^n, \dot{E}^n) \to (X, A)$ *be the map defined by*

$$(2.13)\qquad h(x_1, \cdots, x_n) = \begin{cases} f(2x_1 + 1, x_2, \cdots, x_n) & (-1 \leqq x_1 \leqq 0), \\ g(2x_1 - 1, x_2, \cdots, x_n) & (0 \leqq x_1 \leqq 1). \end{cases}$$

Then $h: (E^n, \dot{E}^n) \to (X, A)$ *represents* $\alpha + \beta$.

LEMMA 2.14. *Let* $(E, \dot{E}, x, \omega)$ *by an oriented* n*-cell, and let* $f: (E, \dot{E}, x) \to (X, A, x_*)$ *represent* $\alpha \in \pi_n(X, A, x_*)$. *Then if* $(\dot{E}, x, \omega')$ *is the boundary of* $(E, \dot{E}, x, \omega)$, *the map* $f \mid (\dot{E}, x): (\dot{E}, x) \to (A, x_*)$ *represents* $\partial(\alpha)$.

3. Some operations in homotopy groups

We consider here some of the operations involving homotopy groups which will be needed later.

I. *The product* [21]. This operation associates with each pair

$$(\alpha, \beta)(\alpha \in \pi_p(X), \beta \in \pi_q(X))$$

an element $[\alpha, \beta] \in \pi_{p+q-1}(X)$. To define $[\alpha, \beta]$, let $f: (S^p, y_*) \to (X, x_*)$ and $g: (S^q, y_*) \to (X, x_*)$ be representatives of α, β respectively. Define a map $f \vee g: (S^p \vee S^q, y_* \times y_*) \to (X, x_*)$ by the formula

$$(3.1)\qquad \begin{aligned} (f \vee g)(y, y_*) &= f(y) & (y \in S^p), \\ (f \vee g)(y_*, y) &= g(y) & (y \in S^q) \end{aligned}$$

and let $[f, g] = (f \vee g) \circ \psi_{p,q}$. Then $[f, g]$ is a mapping of $(\dot{E}^{p+q}, z^{p+q})$ into (X, x_*) which represents an element $[\alpha, \beta] \in \pi_{p+q-1}(X)$ depending only on α and β. The following properties of the product operation are known [21, 18]:

(3.2) $$[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta] \qquad (p > 1);$$

(3.3) $$[\alpha, \beta_1 + \beta_2] = [\alpha, \beta_1] + [\alpha, \beta_2] \qquad (q > 1);$$

(3.4) $$[\beta, \alpha] = (-1)^{pq}[\alpha, \beta].$$

(3.5) *Let f, g be mappings: $(E^p \times E^q)^{\cdot} \to X$ representing elements $\alpha, \beta \in \pi_{p+q-1}(X)$, respectively, and such that*

(3.6) $$f(\dot{E}^p \times E^q) = g(E^p \times \dot{E}^q) = x_*.$$

Let f' be the mapping: $(E^p, \dot{E}^p) \to (X, x_*)$ defined by

(3.7) $$f'(x) = f(x, z^q),$$

and let g' be the mapping: $(E^q, \dot{E}^q) \to (X, x_*)$ defined by

(3.8) $$g'(x) = g(z^p, x),$$

and let $\alpha' \in \pi_p(X)$, $\beta' \in \pi_q(X)$ be the elements represented by f', g' respectively. Define a mapping $h: (E^p \times E^q)^{\cdot} \to X$ by

(3.9) $$h(x, y) = \begin{cases} f(x, y) & ((x, y) \in E^p \times \dot{E}^q), \\ g(x, y) & ((x, y) \in \dot{E}^p \times E^q); \end{cases}$$

then the element of $\pi_{p+q-1}(X)$ represented by h is $\alpha + \beta + [\alpha', \beta']$.

II. *Composition* [9]. This operation associates with each pair (α, β) with $\alpha \in \pi_n(S^r)$, $\beta \in \pi_r(X)$ an element $\beta \circ \alpha \in \pi_n(X)$. If $f: (S^n, y_*) \to (S^r, y_*)$ and $g: (S^r, y_*) \to (X, x_*)$ are representatives of α, β respectively, then the map $g \circ f: (S^n, y_*) \to (X, x_*)$ is a representative of $\beta \circ \alpha$. The composition operation has the following properties:

(3.10) $\beta \circ \alpha = g(\alpha)$,

(3.11) $\beta \circ (\alpha_1 + \alpha_2) = \beta \circ \alpha_1 + \beta \circ \alpha_2 \qquad (\alpha_1, \alpha_2 \in \pi_n(S^r), \beta \in \pi_r(X))$,

(3.12) $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha \qquad (\alpha \in \pi_n(S^r), \beta \in \pi_r(S^s), \gamma \in \pi_s(X))$,

(3.13) $\gamma \circ [\alpha, \beta] = [\gamma \circ \alpha, \gamma \circ \beta] \qquad (\alpha \in \pi_p(S^r), \beta \in \pi_q(S^r), \gamma \in \pi_r(X))$.

On the other hand, the right distributive law analogous to (3.11) is not in general true.

III. *The join.* If A and B are topological spaces, their join $A * B$ is the space obtained from $A \times B \times E^1$ by identifying each set of the form $a \times B \times 1$ with $a \in A$ and each set of the form $A \times b \times (-1)$ with $b \in B$. Under the above identification the set $a \times b \times E^1$ is mapped homeomorphically onto a subset $\overline{ab}$ of $A * B$, called the *line segment from a to b*. Thus A and B may be regarded as subsets of $A * B$, and each point of $(A * B) - (A \cup B)$ lies on a unique line segment joining a point of A to a point of B.

If A, B, A', B' are topological spaces and $f:A \to A'$, $g:B \to B'$ are mappings, there is a mapping $f * g:A * B \to A' * B'$, called the *join* of f and g, such that

(3.14) $$(f * g) \mid A = f;$$

(3.15) $$(f * g) \mid B = g;$$

(3.16) $f * g$ maps each line segment $\overline{ab}$ linearly onto the line segment $\overline{f(a)g(b)}$.

The map $f * g$ is induced by the map $f \otimes g:A \times B \times E^1 \to A' \times B' \times E^1$ such that

(3.17) $$(f \otimes g)(x, y, t) = (f(x), g(y), t).$$

If A, B are spheres of dimensions p, q respectively, then $A * B$ is a sphere of dimension $p + q + 1$. For if $(x, y, t) \in S^p \times S^q \times E^1$, and

(3.18) $$\lambda = (1 - t)[2(1 + t^2)]^{-\frac{1}{2}}, \qquad \mu = (1 + t)[2(1 + t^2)]^{-\frac{1}{2}},$$

(3.19) $$\chi(x, y, t) = \eta_{p+1,q+1}(\lambda x, \mu y),$$

then the mapping $\chi:S^p \times S^q \times E^1 \to S^{p+q+1}$ induces a homeomorphism $\bar{\chi}$ of $S^p * S^q$ onto S^{p+q+1}. We orient $S^p * S^q$ by the requirement that $\bar{\chi}$ preserves orientation.

If A, B are topological spaces, $a_0 \in A$, $b_0 \in B$, and if $f:(S^p, y_*) \to (A, a_0)$, $g:(S^q, y_*) \to (B, b_0)$ are mappings, then $f * g$ maps $(S^p * S^q, y_*)$ into $(A * B, a_0)$, and the homotopy class of $f * g$ depends only on those of f and g. The correspondence $(f, g) \to f * g$ therefore induces an operation associating with $\alpha \in \pi_p(A)$, $\beta \in \pi_q(B)$ an element $\alpha * \beta \in \pi_{p+q+1}(A * B)$.

(3.20) *The join operation is bilinear*; i.e.,

(3.21) $$\alpha * (\beta_1 + \beta_2) = (\alpha * \beta_1) + (\alpha * \beta_2) \qquad (q > 0);$$

(3.22) $$(\alpha_1 + \alpha_2) * \beta = (\alpha_1 * \beta) + (\alpha_2 * \beta) \qquad (p > 0).$$

To prove (3.21), let $f:(S^p, y_*) \to (A, a_0)$, and $g_1, g_2 : (S^q, y_*) \to (B, b_0)$ be mappings such that f represents and g_1 represents β_i for $i = 1, 2$, and such that $g_2(E^q_+) = g_1(E^q_-) = y_*$. Then the map g defined by

(3.23) $$g(y) = \begin{cases} g_1(y) & (y \in E^q_+) \\ g_2(y) & (y \in E^q_-) \end{cases}$$

is by (2.8) a representative of $\beta_1 + \beta_2$. From (3.17) it follows that if $y = (y_1, \cdots, y_{q+1}) \in E^q_-$, then $y' = (y_1, \cdots, y_q, -y_{q+1}) \in E^q_+$, and

(3.24) $$(f \otimes g_1)(x, y, t) = (f(x), y_*, t) = (f \otimes g_2)(x, y', t).$$

Letting $h = (f * g) \circ \bar{\chi}^{-1}$, $h_i = (f * g_i) \circ \bar{\chi}^{-1}$, we see that h represents $\alpha * \beta$, h_i represents $\alpha * \beta_i$, and that the hypotheses of (2.8) are satisfied by the triple (h_1, h_2, h). Hence (3.21) follows from (2.8). The proof of (3.22) is similar.

We also need the following properties of the join operation:

(3.25) *If* $\alpha \in \pi_p(S^r)$, $\beta \in \pi_q(S^s)$, $\alpha' \in \pi_r(A)$, $\beta' \in \pi_s(B)$, *then*

$$(\alpha' * \beta') \circ (\alpha \bar{*} \beta) = (\alpha' \circ \alpha) \bar{*} (\beta' \circ \beta). \tag{3.26}$$

(3.27) *If* $\alpha \in \pi_p(S^r)$, $\beta \in \pi_q(S^s)$, *then*

$$U_{p+q+1,r+s+1}(\alpha * \beta) = U_{p,r}(\alpha) * \beta = \alpha * U_{q,s}(\beta); \tag{3.28}$$

$$\beta * \alpha = \begin{cases} (-1)^{(p+1)(q+1)}(\alpha * \beta) \text{ if } r \text{ or } s \text{ is odd}; \\ (-1)^{(p+1)(q+1)} U_{p,r}(\alpha) * \beta \text{ if } r \text{ and } s \text{ are even}. \end{cases} \tag{3.29}$$

To prove (3.25), let $f: S^p \to S^r$, $f': S^r \to A$, $g: S^q \to S^s$, $g': S^s \to B$ be mappings representing α, α', β, β' respectively. It follows from (3.17) that

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g). \tag{3.30}$$

and therefore the corresponding relation with $\otimes$ replaced by $*$ holds.

To prove (3.28), note that $\iota_{r+s+1} = \iota_r \circ \iota_s$, and therefore $-\iota_{r+s+1} = -(\iota_r * \iota_s) = (-\iota_r) * \iota_s = \iota_r * (-\iota_s)$ by (3.20). These facts, together with (3.25) and the definition of $U_{m,n}$ (§2), imply (3.28).

To prove (3.29), define a map $\zeta'_{p,q}: S^{p+q+1} \to S^{p+q+1}$ by

$$\zeta'_{p,q}(\eta_{p+1,q+1}(x, y)) = \eta_{q+1,p+1}(y, x) \tag{3.31}$$

for $(x, y) \in \overset{-1}{\eta}_{p+1,q+1}(S^{p+q+1})$. The map $\zeta'_{p,q}$ is the restriction to S^{p+q+1} of a linear map of C^{p+q+2} into itself of determinant $(-1)^{(p+1)(q+1)}$ and therefore $\zeta'_{p,q}$ represents $(-1)^{(p+1)(q+1)}\iota_{p+q+1}$. Let $\zeta_{p,q}: S^q \times S^q \times E^1 \to S^p \times S^q \times E^1$ be the map such that

$$\zeta_{p,q}(x, y, t) = (y, x, -t). \tag{3.32}$$

From (3.18) and (3.19) it follows that

$$\chi \circ \zeta_{p,q} = \zeta'_{p,q} \circ \chi \tag{3.33}$$

and therefore $\zeta'_{p,q}$ is the map of S^{p+q+1} into itself induced by $\zeta_{p,q}$. Next note that

$$(g \otimes f) \circ \zeta_{p,q} = \zeta_{r,s} \circ (f \otimes g); \tag{3.34}$$

for if $(x, y, t) \in S^p \times S^q \times E^1$, we have

$$\begin{aligned} (g \otimes f)(\zeta_{p-q}(x, y, t)) &= (g \otimes f)(y, x, -t) = (g(y), f(x), -t) \\ &= \zeta_{r,s}(f(x), g(y), -t) \\ &= \zeta_{r,s}((f \otimes g)(x, y, t)) \end{aligned} \tag{3.35}$$

by (3.32) and (3.17). It follows easily that

$$(g \bar{*} f) \circ \zeta'_{p,q} = \zeta'_{r,s} \circ (f \bar{*} g). \tag{3.36}$$

(3.29) now follows from (3.36), (3.28), and (3.11).

Next let $f:(E^{p+1}, \dot{E}^{p+1}) \to (E^{r+1}, \dot{E}^{r+1})$ and $g:(E^{q+1}, \dot{E}^{q+1}) \to (E^{s+1}, \dot{E}^{s+1})$ be mappings representing $\alpha \in \pi_{p+1}(E^{r+1}, \dot{E}^{r+1})$, $\beta \in \pi_{q+1}(E^{s+1}, \dot{E}^{s+1})$, respectively. Define a map $f \bigstar g:(E^{p+1} \times E^{q+1}, (E^{p+1} \times E^{q+1})^{\cdot}) \to (E^{r+1} \times E^{s+1}, (^{\cdot}E^{r+1} \times E^{s+1})^{\cdot})$ by

$$(3.37) \qquad (f \bigstar g)(x, y) = (f(x), g(y)).$$

The element $\alpha \bigstar \beta$ of $\pi_{p+q+2}(E^{r+1} \times E^{s+1}, (E^{r+1} \times E^{s+1})^{\cdot})$ represented by $f \bigstar g$ depends only on α and β. It is easy to see that

(3.38) *The operation $\bigstar$ is equivalent to the join operation* $*$.

IV. *The Hopf homomorphism* [10, 11]. This is a homomorphism H_0 of $\pi_{2n-1}(S^n)$ into the additive group of integers. Let $\alpha \in \pi_{2n-1}(S^n)$ and let $f:(S^{2n-1}, y_*) \to (S^n, y_*)$ be a representative of α which is a simplicial map of a subdivision K of S^{2n-1} into a subdivision L of S^n such that y_* is an interior point of some n-simplex of L. Let $P = f^{-1}(y_*)$. Let K' be the first barycentric subdivision of K, the new vertices being chosen whenever possible to be points of P; and let $\tilde{K}$ be the dual subdivision associated with K'. Then P is a subcomplex of $\tilde{K}$. Let $\mathfrak{D}$ be the duality operator [13] associated with the orientation ω'_{2n-1} of S^{2n-1}; $\mathfrak{D}$ associates with each oriented r-simplex σ_r of K an oriented $(2n - 1 - r)$-cell $\mathfrak{D}\sigma_r$ of K. Now let τ_n be the oriented n-simplex of L which contains y_* and is oriented concordantly with ω'_n ; and let z^n be the integral n-cocycle of L such that

$$(3.39) \qquad \begin{aligned} z^n(\tau_n) &= 1, \\ z^n(\tau'_n) &= 0 \text{ if } \tau'_n \neq \pm \tau_n . \end{aligned}$$

The image $f'z^n$ of z^n under the cochain mapping f' defined by f is an n-cocycle of K, and $\mathfrak{D}(f'z^n)$ is an $(n - 1)$-cycle of P. Let c_n be an n-dimensional chain of $\tilde{K}$ whose boundary is $\mathfrak{D}(f'z^n)$. Then c_n is a relative n-cycle of S^{2n-1} modulo P; the image under the homomorphism $f^*:H_n(S^{2n-1}, P) \to H_n(S^n, y_*)$ induced by f of the homology class of c_n is a certain multiple $H_0(\alpha)$ of ω'_n . The integer $H_0(\alpha)$ is by definition the Hopf invariant of α. Hopf has proved that $H_0(\alpha)$ does not depend on the choice of f in its homotopy class and that the mapping $\alpha \to H_0(\alpha)$ is a homomorphism (which we shall refer to as the *Hopf homomorphism*) of $\pi_{2n-1}(S^n)$ into the group of integers having the following additional properties:

(3.40) *if* $\alpha \in \pi_{2n-1}(S^n)$ *and* $\beta = k\iota_{2n-1} \in \pi_{2n-1}(S^{2n-1})$, *then* $H_0(\alpha \circ \beta) = k \cdot H_0(\alpha)$;

(3.41) *if* $\alpha \in \pi_{2n-1}(S^n)$ *and* $\beta = k\iota_n \in \pi_n(S^n)$, *then* $H_0(\beta \circ \alpha) = k^2 \cdot H_0(\alpha)$;

(3.42) *if n is odd, then* $H_0(\alpha) = 0$ *for every* $\alpha \in \pi_{2n-1}(S^n)$;

(3.43) *if n is even, there exists* $\alpha \in \pi_{2n-1}(S^n)$ *such that* $H_0(\alpha) = 2$;

(3.44) *if* $n = 2, 4,$ *or* 8, *there exists* $\alpha \in \pi_{2n-1}(S^n)$ *such that* $H_0(\alpha) = 1$.

It is likewise easy to see that

(3.45) *if n is even,* $H_0([\iota_n , \iota_n]) = \pm 2$.

V. *The suspension* [9]. This operation is a special case of the join. Specifically, if $\alpha \in \pi_p(S^r)$, let $E(\alpha) = \alpha * \iota_0$. Thus $E(\alpha) \in \pi_{p+1}(S^{r+1})$, and E is a homomorphism of $\pi_p(S^r)$ into $\pi_{p+1}(S^{r+1})$, called the *suspension homomorphism.* Freudenthal [9] has proved the following properties of E:

(3.46) *E is onto if $p \leqq 2r - 1$;*

(3.47) *E is an isomorphism if $p < 2r - 1$;*

(3.48) *if $p = 2r$, the image of E is the subgroup of $\pi_{2r+1}(S^{r+1})$ consisting of those elements of Hopf invariant zero.*

The following is a slightly strengthened form of one of Freudenthal's theorems:

(3.49) *if $p = 2r - 1$, the kernel of E is the cyclic subgroup of $\pi_{2r-1}(S^r)$ generated by $[\iota_r, \iota_r]$; if r is even, $[\iota_r, \iota_r]$ has Hopf invariant ± 2 and therefore has infinite order; if r is odd, $2[\iota_r, \iota_r] = 0$, and $[\iota_r, \iota_r] = 0$ if and only if there is an element of $\pi_{2r+1}(S^{r+1})$ with Hopf invariant 1.*

The proof of (3.49) will be given in §7.

If $f:(S^p, y_*) \to (S^r, y_*)$, we denoted by $E_0 f$ the mapping $f * i_0$, where i_0 is the identity map of S^0 on itself, so that if f represents $\alpha \in \pi_p(S^r)$, then $E_0 f$ represents $E(\alpha)$. The mapping $E_0 f$ has the following properties:

$$(E_0 f)(E_+^{p+1}) \subset E_+^{r+1}; \tag{3.50}$$

$$(E_0 f)(E_-^{p+1}) \subset E_-^{r+1}; \tag{3.51}$$

$$E_0 f \mid S^p = f. \tag{3.52}$$

It is clear that any map of S^{p+1} into S^{r+1} having properties (3.50)–(3.52) is a representative of $E(\alpha)$.

Suppose that f: $(E^p, \dot{E}^p) \to (S^r, y_*)$ represents $\alpha \in \pi_p(S^r)$. Then the map $E_1 f:(E^{p+1}, \dot{E}^{p+1}) \to (S^{r+1}, y_*)$ defined by

$$E_1 f(x_1, \cdots, x_{p+1}) = d_r(f(x_1, \cdots, x_p), x_{p+1}) \qquad (x \in E^{p+1}) \tag{3.53}$$

is a representative of $E(\alpha)$. Since ψ_p is topological on $E^p - \dot{E}^p$, while $\psi_p(\dot{E}^p) = f(\dot{E}^p) = y_*$, there is a unique map $f':(S^p, y_*) \to (S^r, y_*)$ such that $f = f' \circ \psi_p$. Similarly, there is a unique map $g':(S^{p+1}, y_*) \to (S^{r+1}, y_*)$ such that $E_1 f = g' \circ \psi_{p+1}$. Now $E_1 f$ maps the half-cube $\{x_{p+1} \geqq 0\}$ into E_+^{r+1} and the half-cube $\{x_{p+1} \leqq 0\}$ into E_-^{r+1}; since ψ_{p+1} maps $\{x_{p+1} \geqq 0\}$ into E_+^{p+1} and $\{x_{p+1} \leqq 0\}$ into E_-^{p+1}, it follows that g' maps E_+^{p+1} into E_+^{r+1} and E_-^{p+1} into E_+^{r+1}. Since

$$E_1 f \mid E^p = f$$

and $\psi_{p+1} \mid E^p = \psi_p$, it follows that $g' \mid S^p = f'$. Therefore, since f' represents α, g' is a representative of $E(\alpha)$ and hence $E_1 f$ likewise represents $E(\alpha)$.

Let f: $(E^{p+1}, \dot{E}^{p+1}) \to (E^{k+1}, \dot{E}^{k+1})$ be a representative of $\alpha \in \pi_{p+1}(E^{k+1}, \dot{E}^{k+1})$. Let $E_0' f$: $(E^{p+2}, \dot{E}^{p+2}) \to (E^{k+2}, \dot{E}^{k+2})$ be the mapping such that

$$E_0' f(x_1, \cdots, x_{p+2}) = \eta_{k+1,1}(f(x_1, \cdots, x_{p+1}), x_{p+2}). \tag{3.54}$$

Clearly f homotopic to g implies $E_0'f$ is homotopic to $E_0'g$ and therefore the correspondence $f \to E_0'f$ induces a mapping $E': \pi_{p+1}(E^{k+1}, \dot{E}^{k+1}) \to \pi_{p+2}(E^{k+2}, \dot{E}^{k+2})$. Moreover, it is easy to see that E' is a homomorphism equivalent to E.

Let $f: (E^{p+1}, \dot{E}^{p+1}) \to (E^{r+1}, \dot{E}^{r+1})$ be a mapping representing

$$\alpha \in \pi_{p+1}(E^{r+1}, \dot{E}^{r+1}),$$

and let α' be the element of $\pi_p(S^n)$ represented by $f \mid \dot{E}^{p+1}$. Let

$$f': (S^{p+1}, y_*) \to (S^{r+1}, y_*)$$

be the unique mapping such that $f' \circ \psi_{p+1} = \psi_{r+1} \circ f$. Then [2, §10(c)]:

(3.55) *f' is a representative of $E(\alpha') \in \pi_{p+1}(S^{r+1})$.*

If further $g: (E^{r+1}, \dot{E}^{r+1}) \to (X, x_*)$ represents $\beta \in \pi_{r+1}(X)$, then

(3.56) *the map $g \circ f: (E^{p+1}, \dot{E}^{p+1}) \to (X, x_*)$ represents $\beta \circ E(\alpha') \in \pi_{p+1}(X)$.*

For if $g': (S^{r+1}, y_*) \to (X, x_*)$ is such that $g' \circ \psi_{r+1} = g$, then

$$g' \circ f' \circ \psi_{p+1} = g' \circ \psi_{r+1} \circ f = g \circ f; \tag{3.57}$$

since $g' \circ f'$ represents $\beta \circ E(\alpha')$, so does $g \circ f$.

We shall also need the following result:

(3.58) *if $\alpha \in \pi_r(X)$, $\beta \in \pi_s(X)$, $\alpha' \in \pi_{p-1}(S^{r-1})$, $\beta' \in \pi_{q-1}(S^{s-1})$, then*

$$[\alpha, \beta] \circ (\alpha' * \beta') = [\alpha \circ E(\alpha'), \beta \circ E(\beta')]. \tag{3.59}$$

For let

$$\begin{cases} f: (E^r, \dot{E}^r) \to (X, x_*), \\ g: (E^s, \dot{E}^s) \to (X, x_*), \\ f': (E^p, \dot{E}^p) \to (E^r, \dot{E}^r), \\ g': (E^q, \dot{E}^q) \to (E^s, \dot{E}^s) \end{cases} \tag{3.60}$$

be mappings such that f represents α, g represents β, $f' \mid \dot{E}^p$ represents α', and $g' \mid \dot{E}^q$ represents β'. Let h' be the mapping $(f' \bigstar g') \mid (E^p \times E^q)^{\cdot}$, and let h: $(E^r \times E^s)^{\cdot} \to X$ be the mapping such that

$$h(x, y) = \begin{cases} f(x) & ((x, y) \in E^r \times \dot{E}^s), \\ g(y) & ((x, y) \in \dot{E}^r \times E^s). \end{cases} \tag{3.61}$$

Then h' represents $\alpha' * \beta'$, while h represents $[\alpha, \beta]$. Now h' maps $E^p \times \dot{E}^q$ into $E^r \times \dot{E}^s$ and $\dot{E}^p \times E^q$ into $\dot{E}^r \times E^s$. Hence, if $(x, y) \in E^p \times \dot{E}^q$,

$$h(h'(x, y)) = h(f'(x), g'(y)) = f(f'(x)); \tag{3.62}$$

and if $(x, y) \in \dot{E}^p \times E^q$

$$h(h'(x, y)) = h(f'(x), g'(y)) = g(g'(y)). \tag{3.63}$$

Thus $h \circ h'$ represents the product $[\alpha'', \beta'']$ of the elements of $\pi_p(X)$, $b_q(X)$ represented by $f \circ f'$ and $g \circ g'$, respectively. According to (3.56), $\alpha'' = \alpha \circ E(\alpha')$ and $\beta'' = \beta \circ E(\beta')$; since $h \circ h'$ represents $[\alpha, \beta] \circ (\alpha' * \beta')$, (3.58) holds.

The following result has been proved by J. H. C. Whitehead [22, Theorem 9]:

(3.64) *if* $\alpha \in \pi_{n-1}(S^{r-1})$, *and* $\beta_1, \beta_2 \in \pi_r(x)$, *then*

$$(3.65) \qquad (\beta_1 + \beta_2) \circ E(\alpha) = \beta_1 \circ E(\alpha) + \beta_2 \circ E(\alpha).$$

The author has proved [18, Theorem 3.11]:

(3.66) *if* $\alpha \in \pi_p(S^r)$, $\beta \in \pi_q(S^r)$, *then* $E([\alpha, \beta]) = 0$.

Freudenthal [9] has pointed out that

(3.67) *If* $\alpha \in \pi_p(S^r)$, $\beta \in \pi_r(S^s)$, *then* $E(\beta \circ \alpha) = E(\beta) \circ E(\alpha)$.

(3.68) If $\alpha \in \pi_p(S^r)$, then $E((-\iota_r) \circ \alpha) = -E(\alpha)$.

VI. *The Hopf construction* [11]. This construction associates with each homotopy class of mappings of $\dot{E}^{p+1} \times \dot{E}^{q+1}$ into S^r an element of $\pi_{p+q+1}(S^{r+1})$. If $f: \dot{E}^{p+1} \times \dot{E}^{q+1} \to S^r$ is a mapping, let $Gf: (E^{p+1} \times E^{q+1})^{\cdot} \to S^{r+1}$ be the mapping defined by

$$(3.69) \quad Gf(x, y) = \begin{cases} d_r\left(f\left(\frac{x}{|x|}\right), y\right), 1 - |x|\Big) & ((x, y) \in E^{p+1} \times \dot{E}^{q+1}, x \neq 0), \\ d_r\left(f\left(x, \frac{y}{|y|}\right), |y| - 1\right) & ((x, y) \in \dot{E}^{p+1} \times E^{q+1}, y \neq 0), \\ y_* & (x = 0 \text{ or } y = 0). \end{cases}$$

(Here, for $x \in E^{p+1}$, $|x| = \max(|x_1|, \ldots, |x_{p+1}|)$; thus $|x| = 0$ if and only if $x = 0$ and $|x| = 1$ if and only if $x \in \dot{E}^{p+1}$.) It is easily verified that Gf is a mapping and that the homotopy class of Gf depends only on that of f. Observe that $Gf(x, y) = f(x, y)$ for $(x, y) \in \dot{E}^{p+1} \times \dot{E}^{q+1}$ and that

$$Gf(E^{p+1} \times \dot{E}^{q+1}) \subset E_+^{r+1}, \qquad Gf(\dot{E}^{p+1} \times E^{q+1}) \subset E_-^{r+1}.$$

Clearly any map $F: (E^{p+1} \times E^{q+1})^{\cdot} \to S^{r+1}$ having these properties is homotopic to Gf.

The map $f: (\dot{E}^{p+1} \times \dot{E}^{q+1}) \to S^r$ is said to have type (α, β) with $\alpha \in \pi_p(S^r)$, $\beta \in \pi_q(S^r)$ if and only if $f | \dot{E}^{p+1} \times y$ represents α and $f | x \times \dot{E}^{q+1}$ represents β for some (and therefore for all) $x \in \dot{E}^{p+1}$, $y \in \dot{E}^{q+1}$. (The spheres $\dot{E}^{p+1} \times y$ and $x \times \dot{E}^{q+1}$ are oriented by the requirement that the mappings $x' \to (x', y)$ and $y' \to (x, y')$ shall preserve orientation.) Hopf has proved [11]:

(3.70) *if* $f: (\dot{E}^{r+1} \times \dot{E}^{r+1}) \to S^r$ *has type* $(p\iota_r, q\iota_r)$, *then the Hopf invariant of* Gf *is* $\pm pq$ (*the sign depending only on* r).

We shall also need the following results due to Eilenberg [5]:

(3.71) *if* $\alpha \in \pi_{2r+1}(S^{r+1})$, *and* $H_0(\alpha) = kl$ *where* k *and* l *are integers, then there is a mapping* $f: (\dot{E}^{r+1} \times \dot{E}^{r+1}) \to S^r$ *of type* $(k\iota_r, l\iota_r)$ *such that* Gf *is a representative of* $\pm\alpha$.

It is also known [18] that

(3.72) *if* $\alpha \in \pi_p(S^r)$, $\beta \in \pi_q(S^r)$, *then* $[\alpha, \beta] = 0$ *if and only if there is a map* $f: S^p \times S^q \to S^r$ *of type* (α, β).

4. Homotopy groups of the union of two spaces with one point in common.

Let A, B be topological spaces, $a_0 \in A$, $b_0 \in B$. Denote by $A \vee B$ the subset $A \times b_0 \cup a_0 \times B$ of the product space $A \times B$. The space $A \vee B$ may be regarded as the space obtained from the disjoint union of A and B by identifying the points a_0 and b_0. We investigate the higher homotopy groups of $A \vee B$, the fundamental group of $A \vee B$ being known [15, §52].

Let p_1, p_2 be the projections of $A \times B$ on A and B respectively:

$$p_1(a, b) = a, \qquad p_2(a, b) = b \qquad ((a, b) \in A \times B). \tag{4.1}$$

Let μ_1, μ_2 be the injections of A, B into $A \vee B$:

$$\begin{aligned} \mu_1(a) &= (a, b_0) && (a \in A), \\ \mu_2(b) &= (a_0, b) && (b \in B). \end{aligned} \tag{4.2}$$

Let j be the injection of $A \vee B$ into $A \times B$, k the injection of $(A \times B, a_0 \times b_0)$ into $(A \times B, A \vee B)$.

Consider the homomorphism $\mathfrak{p}$ of $\pi_n(A \times B)$ into the direct sum $\pi_n(A) \oplus \pi_n(B)$;

$$\mathfrak{p}(\alpha) = (\mathfrak{p}_1(\alpha), \mathfrak{p}_2(\alpha)) \qquad (\alpha \in \pi_n(A \times B)). \tag{4.3}$$

It is known that $\mathfrak{p}$ is an isomorphism onto.

Define a homomorphism $\lambda: \pi_n(A \times B) \to \pi_n(A \vee B)$ by the formula:

$$\lambda(\alpha) = \mathfrak{u}_1(\mathfrak{p}_1(\alpha)) + \mathfrak{u}_2(\mathfrak{p}_2(\alpha)) \qquad (\alpha \in \pi_n(A \times B)). \tag{4.4}$$

Then $\mathfrak{j} \circ \lambda: \pi_n(A \times B) \to \pi_n(A \times B)$ is the identity.

It is sufficient to prove that $\mathfrak{p}_1 \circ \mathfrak{j} \circ \lambda = \mathfrak{p}_1$ and $\mathfrak{p}_2 \circ \mathfrak{j} \circ \lambda = \mathfrak{p}_2$ because $\mathfrak{p}$ is an isomorphism. But

$$\begin{cases} p_1 \circ j \circ \mu_1 \circ p_1(a, b) = p_1 \circ j \circ \mu_1(a) = p_1 \circ j(a, b_0) = p_1(a, b_0) = a, \\ p_1 \circ j \circ \mu_2 \circ p_2(a, b) = p_1 \circ j \circ \mu_2(b) = p_1 \circ j(a_0, b) = p_1(a_0, b) = a_0, \end{cases} \tag{4.5}$$

and hence

$$\begin{cases} \mathfrak{p}_1 \circ \mathfrak{j} \circ \mathfrak{u}_1 \circ \mathfrak{p}_1(\alpha) = \mathfrak{p}_1(\alpha) \\ \mathfrak{p}_1 \circ \mathfrak{j} \circ \mathfrak{u}_2 \circ \mathfrak{p}_2(\alpha) = 0 \end{cases} \qquad (\alpha \in \pi_n(A \times B)). \tag{4.6}$$

Hence $\mathfrak{p}_1 \circ \mathfrak{j} \circ \lambda = \mathfrak{p}_1$. The proof that $\mathfrak{p}_2 \circ \mathfrak{j} \circ \lambda = \mathfrak{p}_2$ is similar.

Since $j \circ \lambda$ is the identity, it is clear that λ is an isomorphism into and that j is onto, and that $\pi_n(A \vee B)$ is the direct sum of the image of λ and the kernel of j. If $\alpha \in \pi_n(A \vee B)$, then $\alpha = \beta + \gamma$, where $\beta = \lambda(j(\alpha))$, $\gamma = \alpha - \lambda(j(\alpha))$, where $\beta \in$ Image λ and $j(\gamma) = j(\alpha) - j(\lambda(j(\alpha))) = j(\alpha) - j(\alpha) = 0$ since $j \circ \lambda =$ identity. The image of λ is mapped isomorphically by j onto $\pi_n(A \times B)$.

$$(4.7)\qquad \begin{aligned} \cdots \pi_{n+1}(A \vee B) \xrightarrow{j} \pi_{n+1}(A \times B) \xrightarrow{k} \pi_{n+1}(A \times B, A \vee B) \\ \xrightarrow{\partial} \pi_n(A \vee B) \xrightarrow{j} \pi_n(A \times B) \rightarrow \cdots \end{aligned}$$

Since j is onto, the exactness of the homotopy sequence of $(A \times B, A \vee B)$ implies that k maps $\pi_{n+1}(A \times B)$ into zero and therefore ∂ is an isomorphism into. Since the kernel of j is the image of ∂, we have shown that $\pi_n(A \vee B)$ is the direct sum of the two groups Image $\partial \approx \pi_{n+1}(A \times B, A \vee B)$ and Image $\lambda \approx \pi_n(A \times B)$. We have proved:

THEOREM 4.8. *If* $n > 1$, $\pi_n(A \vee B) \approx \pi_n(A) \oplus \pi_n(B) \oplus \pi_{n+1}(A \times B), A \vee B)$.

In the direct sum decomposition just established there are projections

$$(4.9)\qquad \begin{cases} P_1 = p_1 \circ j : \pi_n(A \vee B) \rightarrow \pi_n(A), \\ P_2 = p_2 \circ j : \pi_n(A \vee B) \rightarrow \pi_n(B), \\ Q : \pi_n(A \vee B) \rightarrow \pi_{n+1}(A \times B, A \vee B) \end{cases}$$

where $Q(\alpha) = \partial^{-1}(\alpha - \lambda(j(\alpha)))$. For later purposes it will be convenient to construct here an operation which associates with each mapping $f:(E^n, \dot{E}^n) \rightarrow (A \vee B, a_0 \times b_0)$ a mapping $Qf:(E^{n+1}, \dot{E}^{n+1}) \rightarrow (A \times B, A \vee B)$ having the property that if f represents $\alpha \in \pi_n(A \vee B)$, then Qf is a representative of $Q(\alpha)$.

First define two maps η', η'' of E^{n+1} into E^n by

$$(4.10)\qquad \eta'(x_1, \cdots, x_n, t) = \begin{cases} (-\frac{3}{2}(1+t)x_1 - \frac{1}{2}(1+3t), x_2, \cdots, x_n) & (-1 \leqq x_1 \leqq -\frac{1}{3}), \\ \frac{3}{2}(1+t)x_1 + \frac{1}{2}(1-t), x_2, \cdots, x_n) & (-\frac{1}{3} \leqq x_1 \leqq \frac{1}{3}), \\ (1, x_2, \cdots, x_n) & (\frac{1}{3} \leqq x_1 \leqq 1); \end{cases}$$

$$(4.11)\qquad \eta''(x_1, \cdots, x_n, t) = \begin{cases} (-1, x_2, \cdots, x_n) & (-1 \leqq x_1 \leqq -\frac{1}{3}), \\ (\frac{3}{2}(1+t)x_1 - \frac{1}{2}(1-t), x_2, \cdots, x_n) & (-\frac{1}{3} \leqq x_1 \leqq \frac{1}{3}), \\ (-\frac{3}{2}(1+t)x_1 + \frac{1}{2}(1+3t), x_2, \cdots x_n) & (\frac{1}{3} \leqq x_1 \leqq 1). \end{cases}$$

Now if $f:(E^n, \dot{E}^n) \to (A \vee B, a_0 \times b_0)$ is a mapping representing $\alpha \in \pi_n(A \vee B)$, define a mapping Qf by

$$Qf(x) = (p_1(f(\eta'(x))), p_2(f(\eta''(x)))) \qquad (x \in E^{n+1}) \tag{4.12}$$

Note that Qf is a mapping of $(E^{n+1}, \dot{E}^{n+1}, J^n)$ into $(A \times B, A \vee B, a_0 \times b_0)$,. and that the map $\partial(Qf) = Q_1 f$ is given by

$$Q_1 f(x_1, \cdots, x_n) = \begin{cases} (p_1(f(-3x_1 - 2, x_2, \cdots, x_n)), b_0) & (-1 \leqq x_1 \leqq -\frac{1}{3}), \\ (p_1(f(3x_1, x_2, \cdots, x_n)), p_2(f(3x_1, x_2, \cdots, x_n))) & (-\frac{1}{3} \leqq x_1 \leqq \frac{1}{3}), \\ (a_0, p_2(f(-3x_1 + 2, x_2, \cdots, x_n))) & (\frac{1}{3} \leqq x_1 \leqq 1). \end{cases} \tag{4.13}$$

The map $Q_1 f$ is clearly a representative of the element

$$\mathfrak{i}_1(\mathfrak{p}_1(j(-\alpha))) + \alpha + \mathfrak{i}_2(\mathfrak{p}_2(j(-\alpha))) = \alpha - \lambda(j(\alpha)),$$

and therefore Qf is a representative of $\partial^{-1}(\alpha - \lambda(j(\alpha))) = Q(\alpha)$, as desired.

If A and B are spheres, the structure of the group $\pi_{n+1}(A \times B, A \vee B)$ can be investigated further. We suppose $A = S^p, B = S^q, a_0 = b_0 = y_*$. The mapping $\bar{\psi}_{p,q}:(E^{p+q}, \dot{E}^{p+q}) \to (S^p \times S^q, S^p \vee S^q)$ is a topological mapping of $E^{p+q} - \dot{E}^{p+q}$ on $(S^p \times S^q) - (S^p \vee S^q)$. We now apply a theorem which differs from one of J. H. C. Whitehead [22, Theorem 8] only in the fact that it is stated in the language of relative homotopy groups. The theorem deals with the following situation. Suppose that X is a pathwise connected topological space such that $\pi_i(X) = 0$ for $i < r$; and suppose that X^* is a topological space containing X and that ψ is a mapping of $(E^m, \dot{E}^m)$ into (X^*, X) such that ψ maps $E^m - \dot{E}^m$ topologically on $X^* - X$. Then

THEOREM 4.14. *The homomorphism* $\psi:\pi_n(E^m, \dot{E}^m) \to \pi_n(X^*, X)$ *is onto if* $n < \min(m + r - 1, 2m - 1)$; ψ *is an isomorphism if* $n < \min(m + r - 2, 2m - 2)$.

The proof is contained in [22].

We proceed to apply Theorem 4.14 to the case $X^* = S^p \times S^q$, $X = S^p \vee S^q$, $\psi = \bar{\psi}_{p,q}$, $r = \min(p, q)$, $m = p + q$, to conclude that

THEOREM 4.15. $\bar{\psi}_{p,q}$ *maps* $\pi_{n+1}(E^{p+q}, \dot{E}^{p+q})$ *isomorphically onto*

$$\pi_{n+1}(S^p \times S^q, S^p \vee S^q) \textit{ if } n < p + q + \min(p, q) - 3.$$

Now consider the diagram

$$\begin{array}{ccc} \pi_{n+1}(E^{p+q}, \dot{E}^{p+q}) & \xrightarrow{\partial} & \pi_n(\dot{E}^{p+q}) \\ \bar{\psi}_{p,q} \downarrow & & \downarrow \psi_{p,q} \\ \pi_{n+1}(S^p \times S^q, S^p \vee S^q) & \xrightarrow{\partial'} & \pi_n(S^p \vee S^q). \end{array} \tag{4.16}$$

Since the commutativity relation $\psi_{p,q} \circ \partial = \partial' \circ \bar{\psi}_{p,q}$ holds, and since ∂ and $\bar{\psi}_{p,q}$ are isomorphisms onto, it follows that Image $\psi_{p,q}$ = Image ∂' and that $\psi_{p,q}$ is an isomorphism into. Hence Image $\partial' \approx \pi_n(\dot{E}^{p+q}) \approx \pi_n(S^{p+q-1})$ and therefore

THEOREM 4.17. *$\pi_n(S^p \vee S^q) \approx \pi_n(S^p) \oplus \pi_n(S^q) \oplus \pi_n(S^{p+q-1})$ if $n < p + q + \min(p, q) - 3$.*

Theorem 4.17 was proved by J. H. C. Whitehead in the case $n = p + q - 1$ [21, Theorem 2].

If $n < p + q + \min(p, q) - 3$, we define $\bar{Q}$ to be the projection of $\pi_n(S^p \vee S^q)$ on its direct summand $\pi_n(S^{p+q-1})$; $\bar{Q} = \mathbf{h} \circ \partial \circ \bar{\psi}_{p,q}^{-1} \circ Q$, where h is an admissible map of $\dot{E}^{p+q}$ on S^{p+q-1}.

It is clear from the definition of product that the map $\psi_{p,q}:\dot{E}^{p+q} \to S^p \vee S^q$ is a representative of the element $[\mu_1(\iota_p), \mu_2(\iota_q)] \in \pi_{p+q-1}(S^p \vee S^q)$. Now suppose that $\alpha \in \pi_n(S^p), \beta \in \pi_n(S^q), \alpha' \in \pi_r(S^p), \beta' \in \pi_{n-r+1}(S^q)$ and that $r \geqq p, n - r + 1 \geqq q$, and $n < p + q + \min(p, q) - 3$; and suppose further without loss of generality that $p \leqq q$. Then an easy calculation shows that $r < 2p - 2$ and $n - r + 1 < 2q - 2$ and therefore there are unique elements $\alpha'' \in \pi_{r-1}(S^{p-1})$, $\beta'' \in \pi_{n-r}(S^{q-1})$ such that $E(\alpha'') = \alpha'$ and $E(\beta'') = \beta'$.

THEOREM 4.18. *$\bar{Q}(\mu_1(\alpha) + \mu_2(\beta) + [\mu_1(\alpha'), \mu_2(\beta')]) = \alpha'' * \beta''$.*

For

$$
\begin{aligned}
(4.19) \qquad [\mu_1(\alpha'), \mu_2(\beta')] &= [\mu_1(\iota_p \circ E(\alpha'')), \mu_2(\iota_q \circ E(\beta''))] \\
&= [\mu_1(\iota_p) \circ E(\alpha''), \mu_2(\iota_q) \circ E(\beta'')] \\
&= [\mu_1(\iota_p), \mu_2(\iota_q)] \circ (\alpha'' * \beta'') = \psi_{p,q}(\alpha'' * \beta'').
\end{aligned}
$$

Since $\mu_1(\alpha)$ belongs to the direct summand $\mu_1(\pi_n(S^p))$, $\mu_2(\beta)$ belongs to the direct summand $\mu_2(\pi_n(S^q))$, and $[\mu_1(\alpha'), \mu_2(\beta')]$ belongs to the direct summand Image $\psi_{p,q}$, the truth of the theorem follows.

We now consider the effect on $\pi_n(S^r \vee S^r)$ of the mapping σ_r' defined in (1.28). First note that if $\bar{\sigma}_r:(E^r \times E^r, (E^r \times E^r)^{\cdot}) \to (E^r \times E^r, (E^r \times E^r)^{\cdot})$ is the map defined by

$$(4.20) \qquad \bar{\sigma}_r(x, y) = (y, x),$$

then $\bar{\sigma}_r$ is the restriction to $E^r \times E^r$ of a linear map of $C^r \times C^r$ into itself of determinant $(-1)^r$. Hence $\bar{\sigma}_r$ represents the element $(-1)^r\iota_{2r-1}$ of $\pi_{2r-1}(S^{2r-1})$. Moreover, from (1.26), (1.31), and (4.20), we see that

$$(4.21) \qquad \bar{\psi}'_{r,r} \circ \bar{\sigma}_r = \sigma_r \circ \bar{\psi}'_{r,r}.$$

Now if $f:(E^{n+1}, \dot{E}^{n+1}) \to (E^r \times E^r, (E^r \times E^r)^{\cdot})$ is a representative of $\gamma \in \pi_{n+1}(E^r \times E^r, (E^r \times E^r)^{\cdot})$ and if $n < 3r - 3$, then $\sigma_r \circ \bar{\psi}'_{r,r} \circ f = \bar{\psi}'_{r,r} \circ \bar{\sigma}_r \circ f$ represents $\psi'_{r,r}(((-1)^r\iota_r) \circ \gamma) = (-1)^r\psi'_{r,r}(\gamma)$. Hence $\sigma_r(\alpha) = (-1)^r\alpha$ if $\alpha \in \pi_{n+1}(S^r \times S^r, S^r \vee S^r)$. Since $p_1 \circ j \circ \sigma_r' = p_2 \circ j$ and $p_2 \circ j \circ \sigma_r' = p_1 \circ j$, we see that σ_r' interchanges the first two direct summands of $\pi_n(S^r \vee S^r)$. Hence

THEOREM 4.22. *If $\alpha \in \pi_n(S^r)$, and if $n < 3r - 3$, then*

$$
\begin{aligned}
(4.23) \qquad P_1(\sigma_r'(\alpha)) &= P_2(\alpha) \\
P_2(\sigma_r'(\alpha)) &= P_1(\alpha) \\
Q(\sigma_r'(\alpha)) &= (-1)^r Q(\alpha).
\end{aligned}
$$

5. The generalized Hopf invariant

The map $\varphi_r: S^r \to S^r \vee S^r$ induces a homomorphism $\boldsymbol{\varphi}_r: \pi_n(S^r) \to \pi_n(S^r \vee S^r)$; we denote by H' the composite homomorphism $Q \circ \boldsymbol{\varphi}_r: \pi_n(S^r) \to \pi_{n+1}(S^r \times S^r, S^r \vee S^r)$. For $n < 3r - 3$ we set $H = \bar{Q} \circ \boldsymbol{\varphi}_r: \pi_n(S^r) \to \pi_n(S^{2r-1})$. The homomorphism H is referred to as the *generalized Hopf homomorphism*, and if $\alpha \in \pi_n(S^r)$, the element $H(\alpha)$ is referred to as the *generalized Hopf invariant* of α. This section is devoted to the proofs of some properties of H.

THEOREM 5.1. *Let $f: \dot{E}^p \times \dot{E}^{n-p+1} \to S^{r-1}$ be a mapping of type (α, β) with $\alpha \in \pi_{p-1}(S^{r-1})$, $\beta \in \pi_{n-p}(S^{r-1})$, and let γ be the element of $\pi_n(S^r)$ represented by Gf. Then $H(\gamma) = (-1)^r(\alpha * \beta)$.*

Let $F = \tau_r \circ Gf: (E^p \times E^{n-p+1})^{\cdot} \to S^r$; since τ_r is homotopic to the identity, F is a representative of γ. Now $\varphi_r \circ F = \varphi_r \circ \tau_r \circ Gf$ maps $E^p \times \dot{E}^{n-p+1}$ into $\varphi_r(\tau_r(E^r_+)) = \varphi_r(K^r_-) = y_* \times S^r$, and similarly $\varphi_r \circ F$ maps $\dot{E}^p \times E^{n-p+1}$ on $S^r \times y_*$, while $\varphi_r \circ F(\dot{E}^p \times \dot{E}^{n-p+1}) = y_* \times y_*$. Let F_1, F_2 be the maps of $(E^p \times E^{n-p+1})^{\cdot}$ into $S^r \vee S^r$ such that

$$(5.2)\qquad \begin{cases} F_1(x, y) = \begin{cases} \varphi_r(F(x, y)) & ((x, y) \in E^p \times \dot{E}^{n-p+1}) \\ y_* \times y_* & ((x, y) \in \dot{E}^p \times E^{n-p+1}) \end{cases} \\ F_2(x, y) = \begin{cases} y_* \times y_* & ((x, y) \in E^p \times \dot{E}^{n-p+1}) \\ \varphi_r(F(x, y)) & ((x, y) \in \dot{E}^p \times E^{n-p+1}). \end{cases} \end{cases}$$

Now let $g_1: (E^p, \dot{E}^p) \to (S^r \vee S^r, y_* \times y_*)$, $g_2: (E^{n-p+1}, \dot{E}^{n-p+1}) \to (S^r \vee S^r, y_* \times y_*)$ be the maps such that

$$(5.3)\qquad \begin{cases} g_1(x) = F_1(x, z^{n-p+1}) & (x \in E^p), \\ g_2(y) = F_2(z^p, y) & (y \in E^{n-p+1}). \end{cases}$$

The map g_1 can be decomposed into the composite of three mappings $c \circ b \circ a$, where

$$(5.4)\qquad (E^p, \dot{E}^p) \xrightarrow{a} (E^r_+, S^{r-1}) \xrightarrow{b} (K^r_-, S^{r-1}_0) \xrightarrow{c} (y_* \times S^r, y_* \times y_*)$$

a is a map of $(E^p, \dot{E}^p)$ into (E^r_+, S^{r-1}) such that $a \mid \dot{E}^p$ represents α, $b = \tau_r \mid E^r_+$, and $c = \varphi_r \mid K^r_-$. We orient the cells E^r_+ and K^r_- coherently with the orientation ω'_r of S^r, and orient S^{r-1} and S^{r-1}_0 by the requirement that they shall be the boundaries of the oriented cells (E^r_+, S^{r-1}) and (K^r_-, S^{r-1}_0) respectively. It is easy to see that the orientation of S^{r-1} so defined is ω'_{r-1}, and that the map $\tau_r \mid S^{r-1}: S^{r-1} \to S^{r-1}_0$ reverses orientation. If $y_* \times S^r$ is oriented so that the map $y \to (y_*, y)$ of S^r into $y_* \times S^r$ preserves orientation, it follows from the fact that φ''_r is homotopic to the identity that c preserves orientation. Hence

$$(5.5)\qquad \begin{cases} a \mid \dot{E}^p: \dot{E}^p \to S^{r-1} \text{ represents } \alpha, \\ b \mid S^{r-1}: S^{r-1} \to S^{r-1}_0 \text{ represents } -\iota_{r-1}, \\ c: (K^r_-, S^{r-1}_0) \to (y_* \times S^r, y_* \times y_*) \text{ represents } \iota''_r, \end{cases}$$

and therefore

$$b \circ a \mid \dot{E}^p : \dot{E}^p \to S_0^{r-1} \text{ represents } (-\iota_{r-1}) \circ \alpha. \tag{5.6}$$

Thus $g_1 = c \circ b \circ a$ represents

$$\iota_r'' \circ E((-\iota_{r-1}) \circ \alpha) = \mathfrak{u}_2(E((-\iota_{r-1}) \circ \alpha)) = -\mathfrak{u}_2(E(\alpha)) \tag{5.7}$$

by (3.68). By a similar argument, g_2 represents $-\mathfrak{u}_1(E(\beta))$. Since

$$\varphi_r(F(x, y)) = \begin{cases} F_1(x, y) & ((x, y) \in E^p \times \dot{E}^{n-p+1}) \\ F_2(x, y) & ((x, y) \in \dot{E}^p \times E^{n-p+1}) \end{cases} \tag{5.8}$$

and since F_1 represents $\mathfrak{u}_1(\gamma)$, F_2 represents $\mathfrak{u}_2(\gamma)$, it follows from (3.5) that $\varphi_r \circ F$ represents

$$\begin{aligned} &\mathfrak{u}_1(\gamma) + \mathfrak{u}_2(\gamma) + [-\mathfrak{u}_2(E(\alpha)), -\mathfrak{u}_1(E(\beta))] \\ &\qquad = \mathfrak{u}_1(\gamma) + \mathfrak{u}_2(\gamma) + (-1)^{p(n-p+1)}[\mathfrak{u}_1(E(\beta)), \mathfrak{u}_2(E(\alpha))] \end{aligned} \tag{5.9}$$

and by Theorem 4.18 and (3.29)

$$\begin{aligned} H(\gamma) &= \bar{Q}(\varphi_r(\gamma)) = (-1)^{p(n-p+1)}(\beta * \alpha) \\ &= (-1)^{p(n-p+1)}(-1)^{p(n-p+1)}((((-1)^r\iota_r) \circ \alpha) * \beta) \\ &= (((-1)^r\iota_r) \circ \alpha) * \beta. \end{aligned} \tag{5.10}$$

Now is $n - p < r - 1$, then β is zero and we have proved $H(\gamma) = 0$ as desired. If $n - p + 1 \geqq r$, then $p \leqq n - r + 1 < 3r - 3 - r + 1 = 2r - 2$, so that $p - 1 < 2r - 3 = 2(r - 1) - 1$, and therefore $\alpha = E(\alpha')$ for some $\alpha' \in \pi_{p-2}(S^{r-2})$. Hence $(-\iota_r) \circ \alpha = -\alpha$, and therefore $((-1)^r\iota_r) \circ \alpha = (-1)^r\alpha$. Because the join is distributive, $H(\gamma) = ((-1)^r\alpha) * \beta = (-1)^r(\alpha * \beta)$ as desired.

Let $\alpha \in \pi_{2n-1}(S^n)$. By (3.71) there is a map $f:(\dot{E}^n \times \dot{E}^n) \to S^{n-1}$ of type $(\iota_{n-1}, H_0(\alpha)\iota_{n-1})$ such that Gf represents $\pm\alpha$. It follows that $H(\alpha) = \pm(\iota_{n-1} * H_0(\alpha)\iota_{n-1}) = \pm H_0(\alpha)(\iota_{n-1} * \iota_{n-1}) = \pm H_0(\alpha)\iota_{2n-1}$, and therefore $H(\alpha) = \pm H_0(\alpha)\iota_{2n-1}$. Thus H is a generalization of the Hopf homomorphism.

THEOREM 5.11. *If $\alpha \in \pi_{n-1}(S^{r-1})$ and $n < 3r - 3$, then $H(E(\alpha)) = 0$.*

For if $f:(E^{n-1}, \dot{E}^{n-1}) \to (S^{r-1}, y_*)$ is a representative of α, then $\tau_r \circ E_1 f$ is a representative of $E(\alpha)$. Now $\varphi_r \circ \tau_r \circ E_1 f$ maps the half-cube $\{x_n \geqq 0\}$ into $y_* \times S^r$ and maps the half-cube $\{x_n \leqq 0\}$ into $S^r \times y_*$. Hence $\varphi_r(E(\alpha))$ is the sum of an element of $\mathfrak{u}_1(\pi_n(S^r))$ and an element of $\mathfrak{u}_2(\pi_n(S^r))$, and it follows from Theorem 4.18 that $H(E(\alpha)) = 0$.

Let R_k be the group of rotations of S^k. Any map $f:S^p \to R_{k-1}$ determines a map $f:S^p \times S^{k-1} \to S^{k-1}$ by the formula

$$f(x, y) = (f(x))(y) \qquad (x \in S^p, y \in S^{k-1}). \tag{5.12}$$

The map f determines a mapping g of S^{p+k} into S^k by Hopf's construction, and the correspondence $f \to g$ induces a homomorphism $J:\pi_p(R_{k-1}) \to \pi_{p+k}(S^k)$. Let $\kappa:R_{k-1} \to S^{k-1}$ be the map such that

$$\kappa(r) = r(y_*) \qquad (r \in R_{k-1}). \tag{5.13}$$

If $\alpha \in \pi_p(R_{k-1})$, then f has type $(\kappa(\alpha), \iota_{k-1})$, and therefore $H(J(\alpha)) = (-1)^k(\kappa(\alpha) * \iota_{k-1})$ provided that $p + k < 3k - 3$ and therefore $p < 2k - 3$. But $\kappa(\alpha) * \iota_{k-1} = E(E(\cdots E(\kappa(\alpha))) \cdots) = E^k(\kappa(\alpha))$, the k-fold suspension of $\kappa(\alpha)$, and since $p < 2k - 3 = 2(k - 1) - 1)$, E^k is an isomorphism. Hence

COROLLARY 5.14. *If* $\alpha \in \pi_p(R_{k-1})$ *and* $\kappa(\alpha) \neq 0$, *then* $J(\alpha) \neq 0$.

This affords a method for constructing mappings with non-zero generalized Hopf invariant. Some such mappings will be constructed in Section 7.

THEOREM 5.15. *If* $\alpha \in \pi_n(S^r)$, *and* $\beta_1, \beta_2 \in \pi_r(X)$, *and if* $n < 3r - 3$, *then*

$$(5.16) \qquad (\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha + [\beta_1, \beta_2] \circ H(\alpha).$$

The theorem is true by definition of H in the special case $X = S^r \vee S^r$, $\beta_1 = \iota_r' = \mathfrak{y}_1(\iota_r)$, $\beta_2 = \iota_r'' = \mathfrak{y}_2(\iota_r)$; for $\varphi_r : S^r \to S^r \vee S^r$ is a representative of $\beta_1 + \beta_2$, so that $(\beta_1 + \beta_2) \circ \alpha = \boldsymbol{\varphi}_r(\alpha)$. By definition of $H(\alpha), \boldsymbol{\varphi}_r(\alpha) = \mathfrak{y}_1(\boldsymbol{p}_1(\boldsymbol{\varphi}_r(\alpha))) + \mathfrak{y}_2(\boldsymbol{p}_2(\boldsymbol{\varphi}_r(\alpha))) + \boldsymbol{\psi}_{r,r}(\boldsymbol{h}(H(\alpha)))$ where h is an admissible map of S^{2r-1} into $\dot{E}^{2r}$. Now $p_1 \circ \varphi_r = \varphi_r'$ and $p_2 \circ \varphi_r = \varphi_r''$; since φ_r' and φ_r'' are homotopic to the identity map of S^r on itself, we have

$$(5.17) \qquad \begin{cases} \mathfrak{y}_1(\boldsymbol{p}_1(\boldsymbol{\varphi}_r(\alpha))) = \mathfrak{y}_1(\alpha) = \beta_1 \circ \alpha, \\ \mathfrak{y}_2(\boldsymbol{p}_2(\boldsymbol{\varphi}_r(\alpha))) = \mathfrak{y}_2(\alpha) = \beta_2 \circ \alpha, \\ \boldsymbol{\psi}_{r,r}(\boldsymbol{h}(H(\alpha))) = [\beta_1, \beta_2] \circ H(\alpha). \end{cases}$$

This proves the formula in the special case.

Now let $g_i : S^r \to X$ be a representative of β_i for $i = 1, 2$; then $(g_i \vee g_2) \circ \varphi_r$ is a representative of $\beta_1 + \beta_2$. Letting $g = g_1 \vee g_2$, we have

$$(5.18) \qquad \begin{aligned} (\beta_1 + \beta_2) \circ \alpha = \boldsymbol{g}(\boldsymbol{\varphi}_r(\alpha)) &= \boldsymbol{g}(\iota_r' \circ \alpha + \iota_r'' \circ \alpha + [\iota_r', \iota_r''] \circ H(\alpha)) \\ &= \boldsymbol{g}(\iota_r' \circ \alpha) + \boldsymbol{g}(\iota_r'' \circ \alpha) + \boldsymbol{g}([\iota_r', \iota_r''] \circ H(\alpha)). \end{aligned}$$

If $f : S^n \to S^r$ is a representative of α, then $\iota_r' \circ \alpha$ is represented by $\mu_1 \circ f$, and therefore $\boldsymbol{g}(\iota_r' \circ \alpha)$ is represented by $g \circ \mu_1 \circ f = g_1 \circ f$; hence $\boldsymbol{g}(\iota_r' \circ \alpha) = \beta_1 \circ \alpha$. Similarly $\boldsymbol{g}(\iota_r'' \circ \alpha) = \beta_2 \circ \alpha$. Let $f' : S^n \to \dot{E}^{2r}$ be a map representing $H(\alpha)$; then $[\iota_r', \iota_r''] \circ H(\alpha)$ is represented by $\psi_{r,r} \circ f'$ and therefore $\boldsymbol{g}([\iota_r', \iota_r''] \circ H(\alpha))$ is represented by $g \circ \psi_{r,r} \circ f'$. Since $g \circ \psi_{r,r}$ represents $[\beta_1, \beta_2]$, $g \circ \psi_{r,r} \circ f'$ represents $[\beta_1, \beta_2] \circ H(\alpha)$ and the proof is complete.

THEOREM 5.19. *If* $\alpha \in \pi_n(S^r)$, $\beta \in \pi_r(S^s)$, *and* $n < 3s - 3$, $r - 3s < 3$, $n < 3r - 3$, *and if* $H(\alpha) = 0$, *then*

$$(5.20) \qquad H(\beta \circ \alpha) = H(\beta) \circ \alpha.$$

For

$$(5.21) \qquad \begin{aligned} \boldsymbol{\varphi}_s(\beta \circ \alpha) &= (\iota_s' + \iota_s'') \circ (\beta \circ \alpha) \\ &= ((\iota_s' + \iota_s'') \circ \beta) \circ \alpha \\ &= (\iota_s' \circ \beta + \iota_s'' \circ \beta + [\iota_s', \iota_s''] \circ H(\beta)) \circ \alpha \end{aligned}$$

by Theorem 5.15. Again applying Theorem 5.15, and recalling that $H(\alpha) = 0$,

$$(5.22) \qquad \varphi_s(\beta \circ \alpha) = (\iota_s' \circ \beta) \circ \alpha + (\iota_s'' \circ \beta) \circ \alpha + ([\iota_s', \iota_s''] \circ H(\beta)) \circ \alpha.$$

But on the other hand

$$(5.23) \qquad \begin{aligned} \varphi_s(\beta \circ \alpha) &= (\iota_s' + \iota_s'') \circ (\beta \circ \alpha) \\ &= \iota_s' \circ (\beta \circ \alpha) + \iota_s'' \circ (\beta \circ \alpha) + [\iota_s', \iota_s''] \circ H(\beta \circ \alpha); \end{aligned}$$

by the associative law for composition and because of the fact that left composition with $[\iota_s', \iota_s'']$ maps $\pi_n(S^{2s-1})$ isomorphically, we conclude that $H(\beta) \circ \alpha = H(\beta \circ \alpha)$ as desired.

THEOREM 5.24. *If $\alpha \in \pi_n(S^r)$, $\beta \in \pi_{r-1}(S^{s-1})$, and $n < 3s - 3 \leqq 3r - 3$, then*

$$(5.25) \qquad H(E(\beta) \circ \alpha) = (\beta * \beta) \circ H(\alpha).$$

For

$$(5.26) \qquad \begin{aligned} \varphi_s(E(\beta) \circ \alpha) &= ((\iota_s' + \iota_s'') \circ E(\beta)) \circ \alpha \\ &= ((\iota_s' \circ E(\beta)) + (\iota_s'' \circ E(\beta))) \circ \alpha \end{aligned}$$

by Theorem 5.15, recalling that $H(E(\beta)) = 0$. Hence, applying Theorem 5.15 again

$$(5.27) \qquad \varphi_s(E(\beta) \circ \alpha) = \iota_s' \circ E(\beta) \circ \alpha + \iota_s'' \circ E(\beta) \circ \alpha + [\iota_s' \circ E(\beta), \iota_s'' \circ E(\beta)] \circ H(\alpha).$$

But

$$(5.28) \qquad [\iota_s' \circ E(\beta), \iota_s'' \circ E(\beta)] = [\iota_s', \iota_s''] \circ (\beta * \beta)$$

by (3.59), and therefore

$$(5.29) \qquad \varphi_s(E(\beta) \circ \alpha) = \iota_s' \circ E(\beta) \circ \alpha + \iota_s'' \circ E(\beta) \circ \alpha + [\iota_s', \iota_s''] \circ (\beta * \beta) \circ H(\alpha).$$

As before, we conclude that

$$(5.30) \qquad H(E(\beta) \circ \alpha) = (\beta * \beta) \circ H(\alpha),$$

as desired.

THEOREM 5.31. *If $\alpha \in \pi_{p-1}(S^{r-1})$, $\beta \in \pi_{q-1}(S^{r-1})$, and $p + q < 3r - 2$, then*

$$(5.32) \qquad H([E(\alpha), E(\beta)]) = \begin{cases} 0 & (r \text{ odd}), \\ 2(\alpha * \beta) & (r \text{ even}). \end{cases}$$

We first note that $H(\alpha * \beta) = 0$ since $H(\alpha * \beta) \in \pi_{p+q-1}(S^{4r-3})$ and $p + q - 1 < 3r - 3 < 4r - 3$.

Now $[E(\alpha), E(\beta)] = [\iota_r \circ E(\alpha), \iota_r \circ E(\beta)] = [\iota_r, \iota_r] \circ (\alpha * \beta)$, by (3.58), and by Theorem 5.19

$$(5.33) \qquad \begin{aligned} H([E(\alpha), E(\beta)]) &= H([\iota_r, \iota_r] \circ (\alpha * \beta)) \\ &= H([\iota_r, \iota_r]) \circ (\alpha * \beta). \end{aligned}$$

It remains to compute $H([\iota_r, \iota_r])$.

Now

$$(5.34)\qquad \begin{aligned}\varphi_r([\iota_r, \iota_r]) &= (\iota_r' + \iota_r'')\circ[\iota_r, \iota_r]\\ &= [\iota_r' + \iota_r'', \iota_r' + \iota_r'']\\ &= [\iota_r', \iota_r'] + [\iota_r', \iota_r''] + [\iota_r'', \iota_r'] + [\iota_r'', \iota_r''],\end{aligned}$$

by (3.13), (3.2) and (3.3). But

$$(5.35)\qquad \begin{cases}\iota_r'\circ[\iota_r, \iota_r] = [\iota_r', \iota_r'],\\ \iota_r''\circ[\iota_r, \iota_r] = [\iota_r'', \iota_r''],\end{cases}$$

and therefore

$$(5.36)\qquad \begin{aligned}[\iota_r', \iota_r'']\circ H([\iota_r, \iota_r]) &= \varphi_r([\iota_r, \iota_r]) - \iota_r'\circ[\iota_r, \iota_r] - \iota_r''\circ[\iota_r, \iota_r]\\ &= [\iota_r', \iota_r''] + [\iota_r'', \iota_r']\\ &= [\iota_r', \iota_r''] + (-1)^r[\iota_r', \iota_r'']\end{aligned}$$

by (3.4) and Theorem 5.15.

If r is odd, $[\iota_r', \iota_r'']\circ H([\iota_r, \iota_r]) = 0$ and therefore $H([\iota_r, \iota_r]) = 0$, and $H([E(\alpha), E(\beta)]) = 0\circ(\alpha * \beta) = 0$. If r is even

$$(5.37)\qquad [\iota_r', \iota_r'']\circ H([\iota_r, \iota_r]) = 2[\iota_r', \iota_r''] = [\iota_r', \iota_r'']\circ(2\iota_{2r-1}),$$

and therefore $H([\iota_r, \iota_r]) = 2\iota_{2r-1}$. Hence for r even,

$$(5.38)\qquad \begin{aligned}H([E(\alpha), E(\beta)]) &= (2\iota_{2r-1})\circ(\alpha*\beta)\\ &= 2(\alpha*\beta)\end{aligned}$$

since $H(\alpha*\beta) = 0$ and therefore the right distributive law holds.

For $r = 2, 4, 8$, Hopf has constructed fibre maps $h_r: S^{2r-1} \to S^r$ which represent elements of $\pi_{2r-1}(S^r)$ with generalized Hopf invariant ι_{2r-1}. For these values of r we have the isomorphism [2, Theorem 15]:

$$(5.39)\qquad \pi_n(S^r) \approx \pi_n(S^{2r-1}) \oplus \pi_{n-1}(S^{r-1}).$$

More precisely, $E: \pi_{n-1}(S^{r-1}) \to \pi_n(S^r)$ and $\mathbf{h}_r: \pi_n(S^{2r-1}) \to \pi_n(S^r)$ are isomorphisms into, and $\pi_n(S^r)$ is the direct sum of the subgroups Image E and Image $\mathbf{h}_r$. This direct sum decomposition defines a "projection" $H_1: \pi_n(S^r) \to \pi_n(S^{2r-1})$; if $\beta \in \pi_n(S^{2r-1})$, $\alpha \in \pi_n(S^r)$, then $\beta = H_1(\alpha)$ if and only if there is an element $\gamma \in \pi_{n-1}(S^{r-1})$ such that $\alpha = \mathbf{h}_r(\beta) + E(\gamma)$.

THEOREM 5.40. *The homomorphism H_1 and H are identical if $n < 3r-3$.*

We have $H(\alpha) = H(\mathbf{h}_r(\beta)) + H(E(\gamma))$. Now $H(E(\gamma)) = 0$, and therefore $H(\alpha) = H(\mathbf{h}_r(\beta))$. Let α_r be the element of $\pi_{2r-1}(S^r)$ represented by h_r, so that $H(\alpha_r) = \iota_{2r-1}$. Since $\beta \in \pi_n(S^{2r-1})$, and $n < 3r - 3 < 4r - 3 = 2(2r - 1) - 1$, we have $H(\beta) = 0$, and therefore by Theorem 5.19,

$$(5.41)\qquad \begin{aligned}H(\alpha) &= H(\mathbf{h}_r(\beta)) = H(\alpha_r\circ\beta)\\ &= H(\alpha_r)\circ\beta\\ &= \iota_{2r-1}\circ\beta = \beta = H_1(\alpha).\end{aligned}$$

THEOREM 5.42. *If r is odd and $n < 3r - 3$, then $2H(\alpha) = 0$ for every $\alpha \in \pi_n(S^r)$.*[4]

Let $\alpha \in \pi_n(S^r)$, and let $f: S^n \to S^r$ be a mapping representing α. The map $\rho_r: S^r \to S^r$ defined in (1.11) is homotopic to the identity, and therefore $\rho_r \circ f$ represents α. But $\varphi_r \circ \rho_r \circ f = \sigma'_r \circ \varphi_r \circ f$ by (1.27), and therefore

$$\varphi_r(\alpha) = \sigma'_r(\varphi_r(\alpha)). \tag{5.43}$$

By Theorem (4.22), we have

$$Q(\varphi_r(\alpha)) = Q(\sigma'_r(\varphi_r(\alpha))) = (-1)^r Q(\varphi_r(\alpha)), \tag{5.44}$$

and therefore $2H(\alpha) = 2Q(\varphi_r(\alpha)) = 0$ if r is odd.

6. Construction of the Freudenthal invariants

In [9], Freudenthal defined, for each nullhomotopy h of the suspension of a map f of S^{2r-1} into S^r, a pair of integers c', c'', related by the formulas

$$\begin{cases} c' - c'' = \mathrm{H}_0(\alpha), \\ c' = (-1)^{r+1} c'', \end{cases} \tag{6.1}$$

where α is the element of $\pi_n(S^r)$ represented by f. In the present section we shall give a generalization of the Freudenthal invariants c', c'' by associating with each nullhomotopy h of the suspension of a map $f: S^n \to S^r$, a pair of elements of $\pi_{n+2}(S^{2r+1})$, which we shall refer to as the *generalized Freudenthal invariants.* (The sense in which these are invariants will be explained below.) The proof of the properties of the generalized Freudenthal invariants analogous to (6.1) will be given in §7.

The construction can be described briefly as follows. Suppose that $h: (E^{n+2}, J^{n+1}) \to (S^{r+1}, y_*)$ is a nullhomotopy of the suspension $E_1 f$ of a map $f: (E^n, \dot{E}^n) \to (S^r, y_*)$. We have already (4.13) associated with each map $g: (E^{n+1}, \dot{E}^{n+1}) \to (S^{r+1} \vee S^{r+1}, y_* \times y_*)$ representing $\alpha \in \pi_{n+1}(S^{r+1} \vee S^{r+1})$, a map $Qg: (E^{n+2}, \dot{E}^{n+2}) \to (S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$ whose homotopy class is $Q(\alpha)$. Now if $g = \varphi_{r+1} \circ (E_1 f)$, then Qg is nullhomotopic by Theorem 5.11. We first construct two nullhomotopies B', B'' (depending only on f) of Qg. On the other hand, the given nullhomotopy h of $E_1 f$ defines a nullhomotopy of g and therefore a nullhomotopy of Qg (since Q carries the constant map: $E^{n+2} \to y_* \times y_*$ into the constant map: $E^{n+2} \to y_* \times y_*$). The second nullhomotopy depends on h, but its initial value coincides with the initial values of the nullhomotopies B', B''; thus it fits with each of B', B'' to produce mappings K', K'' of $(E^{n+3}, \dot{E}^{n+3})$ into $(S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$. These mappings determine the desired generalized Freudenthal invariants by means of the isomorphism $\bar{\Psi}_{r+1,r+1} \circ \partial^{-1}$ of §4.

It will be convenient to formulate the association just described as a pair of homomorphisms. For this purpose we first introduce the function space F_{r+1}^{n+2}

[4] This theorem was pointed out to the author by J. H. C. Whitehead, to whom the proof given here is due.

whose elements are the nullhomotopies of suspensions of mappings of $(E^n, \dot{E}^n)$ into (S^r, y_*). Under a suitable equivalence relation the homotopy classes of such nullhomotopies form a group, and the generalized Freudenthal invariants can be best described as homomorphisms of this group into $\pi_{n+2}(S^{2r+1})$.

Let $f:(E^{n-2}, \dot{E}^{n-2}) \rightarrow (S^{r-1}, y_*)$ with $n > 2$, $r > 1$. A nullhomotopy of $E_1 f$ is a mapping $h:(E^n, J^{n-1}) \rightarrow (S^r, y_*)$ such that

$$h(x_1, \cdots, x_{n-1}, 1) = d_{r-1}(f(x_1, \cdots, x_{n-2}), x_{n-1}). \tag{6.2}$$

The map f is determined by h, because $d_{r-1} \mid S^{r-1} \times 0$ is the identity; thus (6.2) can be phrased

$$h(x_1, \cdots, x_{n-1}, 1) = d_{r-1}(h(x_1, \cdots, x_{n-2}, 0, 1), x_{n-1}). \tag{6.3}$$

We define F_r^n to be the set of all mappings $h:(E^n, J^{n-1}) \rightarrow (S^r, y_*)$ which satisfy (6.3), topologized in the usual way by means of the metric in S^r. We choose as base point for the homotopy groups of F_r^n the constant map Y_* such that $Y_*(x) = y_*$ for $x \in E^n$.

Let $f \in F_r^n$, and let k be an integer with $1 \leqq k \leqq n-2$. For each point $x = (x_1, \cdots, x_k) \in E^k$, let f_x be the map defined by

$$f_x(y_1, \cdots, y_{n-k}) = f(x_1, \cdots, x_k, y_1, \cdots, y_{n-k}) \tag{6.4}$$

for $(y_1, \cdots, y_{n-k}) \in E^{n-k}$. Then $f_x \in F_r^{n-k}$, and $f_x = Y_*$ for $x \in \dot{E}^k$. Hence the map $\bar{f}:(E^k, \dot{E}^k) \rightarrow (F_r^{n-k}, Y_*)$ defined by

$$\bar{f}(x) = f_x \qquad (x \in E^k) \tag{6.5}$$

is an element of the function space $F^k(F_r^{n-k}, Y_*)$, and the correspondence $f \rightarrow \bar{f}$ is a 1:1 correspondence (in fact, an isometry) between F_r^n and $F^k(F_r^{n-k}, Y_*)$. This correspondence induces a 1:1 correspondence between the set π_r^n of path-components of F_r^n and the elements of $\pi_k(F_r^{n-k})$ and therefore induces a group structure on π_r^n. This group structure is easily seen to be independent of k; it will be convenient for later applications to take $k = 2$.

Let $f \in F_r^n$, and let $Df:(E^{n-2}, \dot{E}^{n-2}) \rightarrow (S^{r-1}, y_*)$ be the map such that

$$Df(x_1, \cdots, x_{n-2}) = f(x_1, \cdots, x_{n-2}, 0, 1), \tag{6.6}$$

so that f is a nullhomotopy of the suspension of Df. Clearly f homotopic to g implies Df homotopic to Dg, and therefore D induces a mapping $\Delta:\pi_r^n \rightarrow \pi_{n-2}(S^{r-1})$; it follows from the definition of addition in homotopy groups that Δ is a homomorphism. Clearly the image of Δ is the kernel of $E:\pi_{n-2}(S^{r-1}) \rightarrow \pi_{n-1}(S^r)$.

Let $f:(E^{n+1}, \dot{E}^{n+1}) \rightarrow (S^r \times S^r, S^r \vee S^r)$, and define a map $Bf:(E^{n+2}, \dot{E}^{n+2}) \rightarrow (S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$ by

$$Bf(x_1, \cdots, x_{n+2}) = \bar{\delta}_r(f(x_1, \cdots, x_{n+1}), x_{n+2}). \tag{6.7}$$

The map Bf represents the zero element of $\pi_{n+2}(S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$, as we show by exhibiting two nullhomotopies $B'f$, $B''f$ of Bf. These are maps of the half-cube $\{x_{n+3} \geqq 0\}$ of E^{n+3} into $S^{r+1} \times S^{r+1}$ and are defined as follows. Let $f' = p_1 \circ f$, $f'' = p_2 \circ f$ be the projections of f on the two factors of $S^{r+1} \times S^{r+1}$, and for $x \in E^{n+3}$, set $\bar{x} = (x_1, \cdots, x_{n+1})$, $t = x_{n+2}$, $u = x_{n+3}$. Then for $x \in E^{n+3}$, $u \geqq 0$,

$$(6.8) \qquad \begin{cases} B'f(x) = (d_r(f'(\bar{x}), t - u(1 + t)), d_r(f''(\bar{x}), t + u(1 - t))), \\ B''f(x) = (d_r(f'(\bar{x}), t + u(1 - t)), d_r(f''(\bar{x}), t - u(1 + t))). \end{cases}$$

Note that $B'f(x) = B''f(x) = y_*$ for $x_{n+3} = 1$, $Bf(x) = B'f(x) = B''f(x)$ for $x \in E^{n+2}$, while $B'f'(x)$ and $B''f(x)$ belong to $S^{r+1} \vee S^{r+1}$ for $x \in \dot{E}^{n+3}$. Hence $B'f$ and $B''f$ are nullhomotopies of Bf.

Now let $h:(E^{n+2}, J^{n+1}) \to (S^{r+1}, y_*)$ be a map representing $\xi \in \pi_{r+1}^{n+2}$. Let $h' = Dh:(E^n, \dot{E}^n) \to (S^r, y_*)$; and for each $t \in E^1$, let $h_t:(E^{n+1}, \dot{E}^{n+1}) \to (S^{r+1}, y_*)$ be the mapping defined by

$$(6.9) \qquad h_t(x_1, \cdots, x_{n+1}) = h(x_1, \cdots, x_{n+1}, t).$$

Define further $H_t:(E^{n+1}, \dot{E}^{n+1}) \to (S^{r+1} \vee S^{r+1}, y_* \times y_*)$ by $H_t = \varphi_{r+1} \circ h_t$. Note that

$$(6.10) \qquad QH_1 = Q(\varphi_{r+1} \circ h_1) = Q(\varphi_{r+1} \circ E_1 h').$$

From (3.53), (1.25), and (1.6) we have

$$(6.11) \qquad \begin{aligned} \varphi_{r+1}(E_1 h'(x_1, \cdots, x_{n+1})) = (d_r(\varphi_r'(h'(x_1, \cdots, x_n)), x_{n+1}), \\ d_r(\varphi_r''(h'(x_1, \cdots, x_n)), x_{n+1})). \end{aligned}$$

Let $H' = \varphi_r \circ h'$; then

$$(6.12) \qquad B(QH')(x_1, \cdots, x_n, t, x_{n+1}) = QH_1(x_1, \cdots, x_{n+1}, t).$$

For $|x_1| \leqq \frac{1}{3}$ this follows from (6.10), (6.11) and (4.13). For $\frac{1}{3} \leqq x_1 \leqq 1$ we have

$$(6.13) \qquad \begin{aligned} &B(QH')(x_1, \cdots, x_n, t, x_{n+1}) \\ &\quad = \bar{\delta}_r(y_*, \varphi_r''(h'(-\tfrac{3}{2}(1 + t)x_1 + \tfrac{1}{2}(1 + 3t), x_2, \cdots, x_n)), x_{n+1}) \\ &\quad = QH_1(x_1, \cdots, x_{n+1}, t) \end{aligned}$$

because $d_r(y_*, t) = y_*$. Similarly, (6.12) holds for $-1 \leqq x_1 \leqq -\frac{1}{3}$ and therefore in all cases.

Finally, define two mappings $K'h$, $K''h$ of $(E^{n+3}, \dot{E}^{n+3})$ into $(S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$ by

$$(6.14) \qquad \begin{cases} K'h(x_1, \cdots, x_{n-1}, t, u) = \begin{cases} QH_{2t+1}(x_1, \cdots, x_{n+1}, u) & (-1 \leqq t \leqq 0), \\ B'(QH')(x_1, \cdots, x_n, u, x_{n+1}, t) & (0 \leqq t \leqq 1), \end{cases} \\ K''h(x_1, \cdots, x_{n+1}, t, u) = \begin{cases} QH_{2t+1}(x_1, \cdots, x_{n+1}, u) & (-1 \leqq t \leqq 0), \\ B''(QH')(x_1, \cdots, x_n, u, x_{n+1}, t) & (0 \leqq t \leqq 1). \end{cases} \end{cases}$$

That $K'h$ and $K''h$ are well-defined and continuous follows from (6.12). Clearly, if h_1, h_2 are joined by a path in F_{r+1}^{n+2}, then $K'h_1$ is homotopic to $K'h_2$, and $K''h_1$ is homotopic to $K''h_2$. Thus K', K'' induce mappings Λ_0', Λ_0'' of π_{r+1}^{n+2} into $\pi_{n+3}(S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$.

(6.15) *Λ_0' and Λ_0'' are homomorphisms.*

For let h^1, h^2 be elements of F_{r+1}^{n+2} such that $h^1(x_1, \cdots, x_{n+2}) = y_*$ for $x_2 \leqq 0$, and $h^2(x_1, \cdots, x_{n+2}) = y_*$ for $x_2 \geqq 0$, and let $h^3 \in F_{r+1}^{n+2}$ be defined by

$$(6.16) \qquad h^3(x) = \begin{cases} h^1(x) & (x_2 \geqq 0), \\ h^2(x) & (x_2 \leqq 0). \end{cases}$$

Then for $x_2 \leqq 0$

$$(6.17) \quad \begin{cases} K'h^1(x_1, \cdots, x_{n+3}) = K'h^2(x_1, -x_2, x_3, \cdots, x_{n+3}) = y_*, \\ K'h^3(x_1, \cdots, x_{n+3}) = K'h^2(x_1, \cdots, x_{n+3}), \\ K'h^3(x_1, -x_2, x_3, \cdots, x_{n+3}) = K'h^1(x_1, -x_2, x_3, \cdots, x_{n+3}), \end{cases}$$

and similarly for K''. It follows from (2.8) that Λ_0' and Λ_0'' are homomorphisms.

If $n < 3r - 2$, $\overline{\Psi}_{r+1,r+1}$ is an isomorphism onto and we may define

$$(6.18) \qquad \begin{cases} \Lambda' = h \circ \partial \circ \overline{\Psi}_{r+1,r+1}^{-1} \circ \Lambda_0', \\ \Lambda'' = h \circ \partial \circ \overline{\Psi}_{r+1,r+1}^{-1} \circ \Lambda_0'', \end{cases}$$

(6.19). *Λ' and Λ'' are homomorphisms of π_{r+1}^{n+2} into $\pi_{n+2}(S^{2r+1})$.*

7. Properties of the Freudenthal invariants

We shall give in this section a proof of two formulas analagous to (6.1) connecting the Freudenthal invariants of an element $\xi \in \pi_{r+1}^{n+2}$. These formulas will be used as a basis for discussion of the kernel of $E:\pi_n(S^r) \to \pi_{n+1}(S^{r+1})$. At the end of the section we give a proof of the improvement of one of Freudenthal's theorems promised in §3.

We first define a homomorphism $A:\pi_{n+1}(S^r \times S^r, S^r \vee S^r) \to \pi_{n+3}(S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$ which is "parallel" to the suspension operation in a sense that will be explained below. Let $f:(E^{n+1}, \dot{E}^{n+1}, J^n) \to (S^r \times S^r, S^r \vee S^r, y_* \times y_*)$, and let $f' = p_1 \circ f$, $f'' = p_2 \circ f$. Define a map $A_0f:(E^{n+3}, \dot{E}^{n+3}) \to (S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$ by the formula

$$(7.1) \quad A_0f(x_1, \cdots, x_{n+3}) = (d_r(f'(x_1, \cdots, x_{n+1}), x_{n+2}), \\ d_r(f''(x_1, \cdots, x_{n+1}), x_{n+3})).$$

Clearly f homotopic to g implies A_0f homotopic to A_0g, and therefore the operation A_0 induces a mapping $A:\pi_{n+1}(S^r \times S^r, S^r \vee S^r) \to \pi_{n+3}(S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$, which is easily shown by the use of Theorem 2.11 to be a homomorphism.

Next, let $f:(E^{n+1}, \dot{E}^{n+1}) \to (E^{2r}, \dot{E}^{2r})$, and for $x \in E^{n+1}$ let $f_1(x), \cdots, f_{2r}(x)$ be the coordinates of $f(x)$. Let $E_0^* f$ be the map of $(E^{n+3}, \dot{E}^{n+3})$ into $(E^{2r+2}, \dot{E}^{2r+2})$ for which

$$(7.2) \quad E_0^* f(x_1, \cdots, x_{n+3})(f_1(x), \cdots, f_r(x), x_{n+2}, f_{r+1}(x), \cdots, f_{2r}(x), x_{n+3}),$$

where $x = (x_1, \cdots, x_{n+1}) \in E^{n+1}$. Again f homotopic to g implies $E_0^* f$ homotopic to $E_0^* g$; hence E_0^* induces a mapping $E^*: \pi_{n+1}(E^{2r}, \dot{E}^{2r}) \to \pi_{n+3}(E^{2r+2}, \dot{E}^{2r+2})$. The map of E^{2r+2} into itself which sends $(x_1, \cdots, x_{2r+2})$ into $(x_1, \cdots, x_r, x_{r+2}, \cdots, x_{2r}, x_{r+1}, x_{2r+2})$ has degree $(-1)^r$ and transforms $E_0^* f$ into $E_0'(E_0' f)$; this map, restricted to $\dot{E}^{2r+1}$, represents $(-1)^r \iota_{2r+1}$, and therefore E^* is a homomorphism equivalent to the one which sends $\alpha \in \pi_n(S^{2r-1})$ into $((-1)^r \iota_{2r+1}) \circ E(E(\alpha))$, which is equal to $(-1)^r E(E(\alpha))$ because of (3.64).

The sense in which A is parallel to the suspension operation is explained in

THEOREM 7.3. *In the diagram*

$$\begin{array}{ccc} \pi_{n+1}(E^{2r}, \dot{E}^{2r}) & \xrightarrow{E^*} & \pi_{n+3}(E^{2r+2}, \dot{E}^{2r+2}) \\ \downarrow \bar{\Psi}_{r,r} & & \downarrow \bar{\Psi}_{r+1,r+1} \\ \pi_{n+1}(S^r \times S^r, S^r \vee S^r) & \xrightarrow{A} & \pi_{n+3}(S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1}), \end{array}$$

the commutativity relation $\bar{\Psi}_{r+1,r+1} \circ E^* = A \circ \bar{\Psi}_{r,r}$ *holds.*

Let $\alpha \in \pi_{n+1}(E^{2r}, \dot{E}^{2r})$, and let $f:(E^{n+1}, \dot{E}^{n+1}, J^n) \to (E^{2r}, \dot{E}^{2r}, z^{2r})$ be a map representing α. Then $A(\bar{\Psi}_{r,r}(\alpha))$ is represented by the map $A_0(\bar{\psi}_{r,r} \circ f)$, while $\bar{\Psi}_{r+1,r+1}(E^*(\alpha))$ is represented by $\bar{\psi}_{r+1,r+1} \circ (E_0^* f)$. But

$$\begin{aligned} A_0(\bar{\psi}_{r,r} \circ f)(x_1, \cdots, x_{n+3}) &= (d_r(\psi_r(f_1(x), \cdots, f_r(x)), x_{n+2}), \\ &\qquad d_r(\psi_r(f_{r+1}(x), \cdots, f_{2r}(x)), x_{n+3})) \\ &= (\psi_{r+1}(f_1(x), \cdots, f_r(x), x_{n+2}), \\ (7.4) \qquad &\qquad \psi_{r+1}(f_{r+1}(x), \cdots, f_{2r}(x), x_{n+3})) \\ &= \bar{\psi}_{r+1,r+1}(f_1(x), \cdots, f_r(x), x_{n+2}, \\ &\qquad f_{r+1}(x), \cdots, f_{2r}(x), x_{n+3}) \\ &= \bar{\psi}_{r+1,r+1}(E_0^* f(x_1, \cdots, x_{n+3})) \end{aligned}$$

by (7.1), (1.30), (1.31), and (7.2).

Let w be the map of $(E^2, \dot{E}^2)$ into itself such that

$$(7.5) \qquad w(x, y) = \begin{cases} (x - (1 - x)y, x + (1 + x)y) & (-1 \leqq y \leqq 0), \\ (x - (1 + x)y, x + (1 - x)y) & (0 \leqq y \leqq 1), \end{cases}$$

for $(x, y) \in E^2$. The map w is easily seen to preserve orientation; hence the map of $(E^{n+3}, \dot{E}^{n+3})$ into itself which sends the point $\eta_{n+1,2}(x, z)$ into $\eta_{n+1,2}(x, w(z))$ for $x \in E^{n+1}$, $z \in E^2$, is homotopic to the identity. Hence if $f:(E^{n+1}, \dot{E}^{n+1}) \to$

$(S^r \times S^r, S^r \vee S^r)$, is a representative of $\alpha \in \pi_{n+1}(S^r \times S^r, S^r \vee S^r)$, and if $A'f$ is the map of $(E^{n+3}, \dot{E}^{n+3})$ into $(S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$ such that

$$A'f(\eta_{n+1,2}(x, z)) = A_0 f(\eta_{n+1,2}(x, w(z))), \tag{7.6}$$

then it is clear that

(7.7) *$A'f$ is a representative of $A(\alpha)$.*

THEOREM 7.8. *If $\xi \in \pi_{r+1}^{n+2}$, then*

$$\Lambda_0'(\xi) - \Lambda_0''(\xi) = A(Q(\varphi_r(\Delta(\xi)))). \tag{7.9}$$

Let $h:(E^{n+2}, J^{n+1}) \to (S^{r+1}, y_*)$ be a map representing ξ; then $\Lambda_0'(\xi) - \Lambda_0''(\xi)$ is represented by the map $L:(E^{n+3}, \dot{E}^{n+3}) \to (S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$ such that

$$L(x_1, \cdots, x_{n+1}, t, u) = \begin{cases} K'h(x, 2t - 1, u) & (0 \leqq t \leqq 1) \\ K''h(x, -2t - 1, u) & (-1 \leqq t \leqq 0). \end{cases} \tag{7.10}$$

Now for $0 \leqq t \leqq \frac{1}{2}$,

$$\begin{cases} L(x, t, u) = QH_{4t-1}(x_1, \cdots, x_{n+1}, u) \\ L(x, -t, u) = QH_{4t-1}(x_1, \cdots, x_{n+1}, u) = L(x, t, u) \end{cases} \tag{7.11}$$

hence $\Lambda_0'(\xi) - \Lambda_0''(\xi)$ is also represented by the map L':

$$\begin{aligned} L'(x_1, \cdots, x_{n+1}, t, u) &= \begin{cases} K'h(x, t, u) & (0 \leqq t \leqq 1), \\ K''h(x, -t, u) & (-1 \leqq t \leqq 0), \end{cases} \\ &= \begin{cases} B'(QH')(x_1, \cdots, x_n, u, x_{n+1}, t) & (0 \leqq t \leqq 1), \\ B''(QH')(x_1, \cdots, x_n, u, x_{n+1}, -t) & (-1 \leqq t \leqq 0), \end{cases} \\ &= A'(QH')(x_1, \cdots, x_n, u, x_{n+1}\, t). \end{aligned} \tag{7.12}$$

The mapping $(x_1, \cdots, x_{n+1}, t, u) \to (x_1, \cdots, x_n, u, x_{n+1}, t)$ has degree 1, and therefore L' represents the same element of $\pi_{n+3}(S^{r+1} \vee S^{r+1}, S^{r+1} \vee S^{r+1})$ as $A'(QH')$ and therefore L represents the same element as $A_0(QH')$. But QH' represents $Q(\varphi_r(\Delta(\xi)))$ and hence $\Lambda_0'(\xi) - \Lambda_0''(\xi) = A(Q(\varphi_r(\Delta(\xi)))$ as desired.

COROLLARY 7.13. *If $\xi \in \pi_{r+1}^{n+2}$ and $n < 3r - 3$, then*

$$\Lambda'(\xi) - \Lambda''(\xi) = (-1)^r E(E(H(\Delta(\xi)))). \tag{7.14}$$

In order to prove the second Freudenthal formula, we consider a mapping $h:(E^{n+2}, J^{n+1}) \to (S^{r+1}, y_*)$ representing $\xi \in \pi_{r+1}^{n+2}$ and consider the effect on $\Lambda_0'(\xi)$ and $\Lambda_0''(\xi)$ of replacing h by $\bar{h} = \rho_{r+1} \circ h \circ \theta_{n+2}$. We make the following observations:

(7.15) *If $f:(E^{n+1}, \dot{E}^{n+1}) \to (S^{r+1} \vee S^{r+1}, y_* \times y_*)$, and $f_1 = \sigma_{r+1}' \circ f \circ \theta_{n+1}$, then $\sigma_{r+1} \circ Qf_1 \circ \theta_{n+2} = Qf$.*

(7.16) *If* $g:(E^{n+1}, \dot{E}^{n+1}) \to (S^r \times S^r, S^r \vee S^r)$, *and* $g_2 = \sigma_r \circ g \circ \theta_{n+1}$, *then* $\sigma_{r+1} \circ B'g_2 \circ \theta_{n+3} = B''g$ *and* $\sigma_{r+1} \circ B''g_2 \circ \theta_{n+3} = B'g$.

(7.15) follows from (1.26), (1.28), (1.14), and (4.12), while (7.16) follows from (1.26), (1.14), and (6.8).

Now let $h \in F_{r+1}^{n+2}$, and let $\bar{h} = \rho_{r+1} \circ h \circ \theta_{n+2}$. Then

$$
\begin{aligned}
\bar{h}(x_1, \cdots, x_{n+1}, 1) &= \rho_{r+1}(h(-x_1, x_2, \cdots, x_{n+1}, 1)) \\
&= \rho_{r+1}(d_r(Dh(-x_1, x_2, \cdots, x_n), x_{n+1})) \\
&= d_r(\rho_r(Dh(-x_1, x_2, \cdots, x_n)), x_{n+1}) \\
&= d_r(\rho_r(h(-x_1, x_2, \cdots, x_n, 0, 1)), x_{n+1}) \\
&= d_r(\bar{h}(x_1, \cdots, x_n, 0, 1), x_{n+1})
\end{aligned} \tag{7.17}
$$

by (6.3), (6.6), (1.6), and (1.11), and hence $\bar{h} \in F_{r+1}^{n+2}$. Moreover

$$D\bar{h} = \rho_r \circ Dh \circ \theta_n,$$

while, in the notation used in the definition of the maps $K'h$ and $K''h$ in §6, we have

$$\bar{h}_t = \rho_{r+1} \circ h_t \circ \theta_{n+1}, \tag{7.18}$$

and hence

$$
\begin{aligned}
\bar{H}_t = \varphi_{r+1} \circ \bar{h}_t &= \varphi_{r+1} \circ \rho_{r+1} \circ h_t \circ \theta_{n+1} \\
&= \sigma'_{r+1} \circ \varphi_{r+1} \circ h_t \circ \theta_{n+1} \\
&= \sigma'_{r+1} \circ H_t \circ \theta_{n+1}
\end{aligned} \tag{7.19}
$$

by (1.27). Also

$$
\begin{aligned}
\bar{H}' = \varphi_r \circ D\bar{h} &= \varphi_r \circ \rho_r \circ Dh \circ \theta_n \\
&= \sigma'_r \circ \varphi_r \circ Dh \circ \theta_n \\
&= \sigma'_r \circ H' \circ \theta_n .
\end{aligned} \tag{7.20}
$$

Hence for $-1 \leqq t \leqq 0$,

$$
\begin{aligned}
K'\bar{h}(x_1, \cdots, x_{n+1}, t, u) &= Q\bar{H}_{2t+1}(x_1, \cdots, x_{n+1}, u) \\
&= \sigma_{r+1} \circ QH_{2t+1} \circ \theta_{n+2}(x_1, \cdots, x_{n+1}, u) \\
&= \sigma_{r+1} \circ QH_{2t+1}(-x_1, x_2, \cdots, x_{n+1}, u)
\end{aligned} \tag{7.21}
$$

by (6.14) and (7.15). For $0 \leqq t \leqq 1$,

$$
\begin{aligned}
K'\bar{h}(x_1, \cdots, x_{n+1}, t, u) &= B'(Q\bar{H}')(x_1, \cdots, x_n, u, x_{n+1}, t) \\
&= B'(Q(\sigma'_r \circ H' \circ \theta_n))(x_1, \cdots, x_n, u, x_{n+1}, t) \\
&= B'(\sigma_r \circ QH' \circ \theta_{n+1})(x_1, \cdots, x_n, u, x_{n+1}, t) \\
&= \sigma_{r+1} \circ B''(QH')(-x_1, x_2, \cdots, x_n, u, x_{n+1}, t)
\end{aligned} \tag{7.22}
$$

by (6.14), (7.15), and (7.16). It follows from (7.21) and (7.22) that

(7.23) $$K'\bar{h} = \sigma_{r+1} \circ K''h \circ \theta_{n+3}$$

and similarly that

(7.24) $$K''\bar{h} = \sigma_{r+1} \circ K'h \circ \theta_{n+3}\,.$$

We next observe that

(7.25) *If $r > 1$, the map $\bar{h}$ represents the element $-\xi \in \pi_{r+1}^{n+2}$.*

Clearly $h \circ \theta_{n+2}$ represents $-\xi$. For $0 \leqq t \leqq 1$, let $\beta_t : S^{r+1} \to S^{r+1}$ be the rotation of S^{r+1} through an angle πt, about the r-dimensional plane $x_2 = x_3 = 0$, so that β_0 the identity, $\beta_1 = \rho_{r+1}$, β_t is given by

(7.26) $$\beta_t(x_1, \cdots, x_{r+2}) = (x_1, x_2 \cos \pi t - x_3 \sin \pi t, x_2 \sin \pi t - x_3 \cos \pi t, x_4, \cdots, x_{r+2}).$$

Then $\beta_t \circ h \circ \theta_{n+2}$ deforms $h \circ \theta_{n+2}$ into $\bar{h} = \rho_{r+1} \circ h \circ \theta_{n+2}$, and for each t, it follows from (6.3), (7.26), and (1.6) that

(7.27) $$\begin{aligned}\beta_t \circ h \circ \theta_{n+2}(x_1, \cdots, x_{n+1}, 1) &= \beta_t \circ d_r(h(\theta_{n+2}(x_1, \cdots, x_n, 0, 1)), x_{n+1})\\ &= d_r(\beta_t(h(\theta_{n+2}(x_1, \cdots, x_n, 0, 1))), x_{n+1}),\end{aligned}$$

so that $\beta_t \circ h \circ \theta_{n+2} \in F_{r+1}^{n+2}$. Hence $\bar{h}$ represents the same element $-\xi$ of π_{r+1}^{n+2} as does $h \circ \theta_{n+2}$.

THEOREM 7.28. *If $n < 3r - 3$ and $r > 1$, then $\Lambda_0'(\xi) = (-1)^{r+1}\Lambda_0''(\xi)$.*

The maps θ_{n+3} maps $(E^{n+3}, \dot{E}^{n+3})$ on itself with degree -1, and hence $K'h \circ \theta_{n+3}$ represents $-\Lambda_0'(\xi)$, and it follows from (7.23) and (7.25) that

(7.29) $$\Lambda_0'(-\xi) = \bar{\sigma}_{r+1}(-\Lambda_0''(\xi))$$

But from the proof of Theorem 4.22 we see that

(7.30) $$\bar{\sigma}_{r+1}(-\Lambda_0''(\xi)) = (-1)^r\Lambda_0''(\xi).$$

Combining (7.30) and (7.29) gives the proof of Theorem 7.28.

COROLLARY 7.31. *If $n < 3r - 3$, $r > 1$, $\xi \in \pi_{r+1}^{n+2}$, then $\Lambda'(\xi) = (-1)^{r+1}\Lambda''(\xi)$.*

If $\alpha \in \pi_n(S^r)$ and $E(\alpha) = 0$, and if ξ is an element of π_{r+1}^{n+2} such that $\Delta(\xi) = \alpha$, then

(7.32) $$\begin{aligned}(-1)^r E(E(H(\alpha))) &= \Lambda'(\xi) - \Lambda''(\xi)\\ &= \Lambda'(\xi) + (-1)^r\Lambda'(\xi).\end{aligned}$$

Since $E \circ E$ is an isomorphism onto, we conclude

THEOREM 7.33. *If $\alpha \in \pi_n(S^r)$ and $E(\alpha) = 0$, and if $n < 3r - 3$, $r > 1$, then*

$$\begin{cases} H(\alpha) = 0 & \text{if } r \text{ is odd,}\\ H(\alpha) \in 2\pi_n(S^{2r+1}) & \text{if } r \text{ is even.}\end{cases}$$

This generalizes a part of one of Freudenthal's theorems.

Conversely, suppose that r is even, and let $\beta \in 2\pi_n(S^{2r+1})$, and choose $\gamma \in \pi_n(S^{2r-1})$ such that $2\gamma = \beta$. If $n < 3r - 3$, then there is an element $\gamma' \in \pi_{n-r}(S^{r-1})$ such that $\gamma' * \iota_{r-1} = \gamma$, since $n - r < 2r - 3 = 2(r - 1) - 1$, and therefore the r-fold suspension: $\pi_{n-r}(S^{r-1}) \to \pi_n(S^{2r-1})$ is onto. By Theorem 5.30, $H([E(\gamma'), \iota_r]) = 2(\gamma' * \iota_{r-1}) = 2\gamma = \beta$, and $E([E(\gamma'), \iota_r]) = 0$ by 3.66. Hence

THEOREM 7.34. *If $\gamma \in \pi_n(S^{2r-1})$ and r is even, there exists an $\alpha \in \pi_n(S^r)$ such that $E(\alpha) = 0$ and $H(\alpha) = 2\gamma$.*

In the special case $n = 2r - 1$, there are integers d' and d'' such that

$$\Lambda'(\xi) = d'\iota_{2r+1}, \quad \Lambda''(\xi) = d''\iota_{2r+1}. \tag{7.35}$$

We also have

$$E(E(H(\alpha))) = H(\alpha) * \iota_1 = \varepsilon H_0(\alpha)(\iota_{2r-1} * \iota_1) = \varepsilon H_0(\alpha)\iota_{2r+1}, \tag{7.36}$$

where $\varepsilon = \pm 1$. Therefore we have

$$\begin{cases} d' - d'' = (-1)^r \varepsilon H_0(\alpha), \\ d' = (-1)^{r+1} d''. \end{cases} \tag{7.37}$$

Comparing with the similar equations (6.1) for the Freudenthal invariants c', c'', we conclude that

$$\begin{cases} d' = \varepsilon(-1)^r c', \\ d'' = \varepsilon(-1)^r c''. \end{cases} \tag{7.38}$$

Thus the homomorphisms Λ', Λ'' generalize the Freudenthal invariants.

Let $\xi_1, \xi_2 \in \pi_{r+1}^{n+2}$ and suppose that $\Delta(\xi_1) = \Delta(\xi_2) = \alpha$. Let $f: (E^n, \dot{E}^n) \to (S^r, y_*)$ be a representative of α, and let $h_1, h_2 \in F_{r+1}^{n+2}$ such that h_i is a representative of ξ_i for $i = 1, 2$, and such that $Dh_1 = Dh_2 = f$. Define a map $T: (E^{n+3}, \dot{E}^{n+3}) \to (S^{r-1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$ by

$$T(x_1, \cdots, x_{n+3}) = \begin{cases} K'h_1(x_1, \cdots, x_{n+1}, 2x_{n+2} + 1, x_{n+3}) \\ \qquad (-1 \leqq x_{n+2} \leqq 0), \\ K'h_2(x_1, \cdots, x_{n+1}, 1 - 2x_{n+2}, x_{n+3}) \\ \qquad (0 \leqq x_{n+2} \leqq 1). \end{cases} \tag{7.39}$$

For $0 \leqq x_{n+2} \leqq \frac{1}{2}$,

$$\begin{cases} T(x_1, \cdots, x_{n+3}) = B'(QH_2')(x_1, \cdots, x_n, x_{n+3}, x_{n+1}, 1 - 2x_{n+2}), \\ T(x_1, \cdots, x_{n+1}, -x_{n+2}, x_{n+3}) \\ \qquad = B'(QH_1')(x_1, \cdots, x_n, x_{n+3}, x_{n+1}, 1 - 2x_{n+2}) = T(x_1, \cdots, x_{n+3}) \end{cases} \tag{7.40}$$

since $H_1' = H_2'$. Hence T is homotopic to the map $T_0: (E^{n+3}, \dot{E}^{n+3}) \to (S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1})$ such that

$$(7.41) \quad T_0(x_1, \cdots, x_{n+1}, t, u) = \begin{cases} K'h_1(x_1, \cdots, x_{n+1}, t, u) \\ = QH_{1,2t+1}(x_1, \cdots, x_{n+1}, u) & (-1 \leqq t \leqq 0), \\ K'h_2(x_1, \cdots, x_{n+1}, -t, u) \\ = QH_{2,1-2t}(x_1, \cdots, x_{n+1}, u) & (0 \leqq t \leqq 1). \end{cases}$$

Let $h: (E^{n+2}, \dot{E}^{n+2}) \to (S^{r+1}, y_*)$ be the map such that

$$(7.42) \quad h(x_1, \cdots, x_{n+2}) = \begin{cases} h_1(x_1, \cdots, x_{n+1}, 2x_{n+2} + 1) & (-1 \leqq x_{n+2} \leqq 0), \\ h_2(x_1, \cdots, x_{n+1}, 1 - 2x_{n+2}) & (0 \leqq x_{n+2} \leqq 1). \end{cases}$$

Then it is clear that

$$(7.43) \qquad Q(\varphi_{r+1} \circ h) = T_0 .$$

The map T is a representative of $\Lambda_0'(\xi_1) - \Lambda_0'(\xi_2)$, and $Q(\varphi_{r+1} \circ h)$ is a representative of $Q(\boldsymbol{\varphi}_{r+1}(\alpha'))$, where α' is the element of $\pi_{n+2}(S^{r+1})$ represented by h. We have proved that

$$(7.44) \qquad \Lambda_0'(\xi_1) - \Lambda_0'(\xi_2) = Q(\boldsymbol{\varphi}_{r+1}(\alpha'))$$

and therefore

THEOREM 7.45. *If $\xi_1, \xi_2 \in \pi_{r+1}^{n+2}$ and $\Delta(\xi_1) = \Delta(\xi_2)$, and if $n < 3r - 2$, then there is an $\alpha \in \pi_{n+2}(S^{r+1})$ such that*

$$(7.46) \qquad \Lambda'(\xi_1) - \Lambda'(\xi_2) = H(\alpha).$$

Conversely, let $\xi_1 \in \pi_{r+1}^{n+2}$, and let $\alpha \in \pi_{n+2}(S^{r+1})$. Let $h_1 \in F_{r+1}^{n+2}$ be a representative of ξ_1. Then[5] there exists a representative $h: (E^{n+2}, \dot{E}^{n+2}) \to (S^{r+1}, y_*)$ of α such that

$$(7.47) \quad h(x_1, \cdots, x_{n+2}) = h_1(x_1, \cdots, x_{n+1}, 2x_{n+2} + 1) \qquad (-1 \leqq x_{n+2} \leqq 0).$$

Let $h_2 \in F_{r+1}^{n+2}$ be the map such that

$$(7.48) \qquad h_2(x_1, \cdots, x_{n+2}) = h(x_1, \cdots, x_{n+1}, \tfrac{1}{2}(1 - x_{n+2})),$$

and let ξ_2 be the element of π_{r+1}^{n+2} represented by h_2. Then from the above discussion it is clear that

$$(7.49) \qquad \Lambda'(\xi_1) - \Lambda'(\xi_2) = H(\alpha).$$

Since $h_1' = h_2'$, we have $\Delta(\xi_1) = \Delta(\xi_2)$. If $\xi_1 = 0$, then $-\xi_2 \in$ Kernel Δ, and therefor the image of H is contained in the image under Λ' of the kernel of Δ. Conversely if $\xi_1 \in$ Kernel Δ and h_1 is chosen so that $h_1'(x) = y_*$ for $x \in E^n$, then h_1 is a map of $(E^{n+2}, \dot{E}^{n+2})$ into (S^{r+1}, y_*) which represents an element $\alpha \in \pi_{n+2}(S^{r+1})$.

[5] Cf. [21, Lemma 1].

If we choose $h_2 = Y_*$, so that $\xi_2 = 0$, we find that h is also a representative of α, and therefore $\Lambda'(\xi_1) = H(\alpha)$, so that Λ' maps Kernel Δ into Image H. We have proved

THEOREM 7.50. *Λ' maps Kernel Δ onto Image H.*

Let Ω be the projection of $\pi_{n+2}(S^{2r+1})$ onto its factor group $\pi_{n+2}(S^{2r+1})$/Image H. Since Λ' maps Kernel Δ into Image H, there is a unique homomorphism Φ:Kernel $E \to \pi_{n+2}(S^{2r+1})$/Image H such that the commutativity relation

$$\begin{array}{ccc} \pi_{r+1}^{n+2} & \xrightarrow{\Lambda'} & \pi_{n+2}(S^{2r+1}) \\ \Big\downarrow \Delta & & \Big\downarrow \Omega \\ \text{Kernel } E & \xrightarrow{\Phi} & \pi_{n+2}(S^{2r+1})/\text{Image } H \end{array} \tag{7.51}$$

$\Omega \circ \Lambda' = \Phi \circ \Delta$ shown in the diagram holds.

Freudenthal, [9, §8.6], has proved that if $n = 2r - 1$, then $\Lambda'(\xi) = 0$ implies $\Delta(\xi) = 0$, so that Kernel $\Lambda' \subset$ Kernel. Δ. It follows that Φ is an isomorphism into. For if $\alpha \in$ Kernel Φ, let $\xi \in \pi_{r+1}^{n+2}$ such that $\Delta(\xi) = \alpha$. Then $0 = \Phi(\alpha) = \Phi(\Delta(\xi)) = \Omega(\Lambda'(\xi))$, so that $\Lambda'(\xi) \in$ Image H. By Theorem 7.50, there is an element $\xi' \in$ Kernel Δ such that $\Lambda'(\xi') = \Lambda'(\xi)$. Then $\xi - \xi' \in$ Kernel Λ', and therefore $\xi - \xi' \in$ Kernel Δ. But $0 = \Delta(\xi - \xi') = \Delta(\xi) - \Delta(\xi') = \alpha$, and Φ is an isomorphism.

We now prove (3.49). If r is even, it has been proved by Freudenthal in [9] that Kernel E is infinite cyclic, and by the author in [18, p. 470] that $[\iota_r, \iota_r]$ generates Kernel E. Suppose now that r is odd, and that there is an element $\alpha \in \pi_{2r+1}(S^{r+1})$ such that $H_0(\alpha) = 1$. Then Image $H = \pi_{2r+1}(S^{2r+1})$, and the fact that Φ is an isomorphism implies that Kernel $E = 0$, i.e., E is an isomorphism. Since $[\iota_r, \iota_r] \in$ Kernel E by 3.66, we have $[\iota_r, \iota_r] = 0$. Finally, suppose that r is odd and no element of $\pi_{2r+1}(S^{r+1})$ with Hopf invariant 1 exists. Then $[\iota_r, \iota_r] \neq 0$ and $[\iota_r, \iota_r] \in$ Kernel E. Now Image $H = 2\pi_{2r+1}(S^{2r+1})$ and therefore Φ maps Kernel E isomorphically into the group $\pi_{2r+1}(S^{2r+1})$/Image H, which is cyclic of order 2. It follows from the fact that Φ is an isomorphism of the non-zero group Kernel E into a cyclic group of order 2 that Kernel E is cyclic of order 2 and that $[\iota_r, \iota_r]$ is the generator of this group.

8. Construction of some essential mappings of spheres on spheres

The following homotopy groups of spheres are known explicitly.

(8.1) *$\pi_n(S^n)$ is an infinite cyclic group generated by ι_n, for $n \geqq 1$.* [1].

(8.2) *$\pi_n(S^1) = 0$ for $n > 1$.* [1].

(8.3) *$\pi_3(S^2)$ is an infinite cyclic group generated by an element ν_2 with Hopf invariant* 1. [13].

(8.4). *$\pi_{n+1}(S^n)$ is a cyclic group of order 2 generated by $\nu_n = E^{n-2}(\nu_2)$ for $n \geqq 3$.* [9].

(8.5) *$\pi_4(S^2)$ is a cyclic group of order 2 generated by $\nu_2 \circ \nu_3$.* [14].

Pontrjagin (C.R. Acad. Sci. URSS, 19(1938), pp. 147–149, 361–363) announced that $\pi_{n+2}(S^n) = 0$ for $n \geqq 3$, but a complete proof of this result has not appeared.

In the following table we give a list of the groups $\pi_n(S^r)$ which have been proved to be different from zero, but which have not been explicitly determined, and exhibit in each case a non-zero element.

(8.6)

n	r	non-zero element
$4k - 1$	$2k$	$[\iota_{2k}, \iota_{2k}]$
7	4	ν_4'
$7 + k$	$4 + k$	$\nu_{4+k}' = E^k(\nu_4')$
8	4	$\nu_4' \circ \nu_7$
10	4	$\nu_4' \circ \nu_7'$
15	8	ν_8''
$15 + k$	$8 + k$	$\nu_{8+k}'' = E^k(\nu_8'')$
16	8	$\nu_8'' \circ \nu_{15}$
18	8	$\nu_8'' \circ \nu_{15}'$
22	8	$\nu_8'' \circ \nu_{15}''$

In the table, k ranges over all positive integers; ν_4' and ν_8 are the homotopy classes of the Hopf maps $h_4: S^7 \to S^4$ and $h_8: S^{15} \to S_8$ referred to in §5.

In this section we proceed to extend the table (8.6). We first construct non-zero elements of $\pi_p(R_k)$, and then apply the results of §5 to obtain non-zero elements of $\pi_{p+k+1}(S^{k+1})$. We then apply the results of §7 to prove that the suspensions of some of these elements are different from zero.

THEOREM 8.7. $\pi_n(S^r) \neq 0$ *for the following values of* n *and* r:

n	14	14	$8k$	$16k + 2$	$8k + 1$	$16k + 3$
r	7	4	$4k$	$8k$	$4k + 1$	$8k + 1$

Again k ranges over all positive integers.

We have defined R_k as the group of all rotations of S^k. It will be convenient to extend each element of R_k to a mapping of the whole infinite-dimensional cartesian space C into itself in an obvious fashion. Thus, for the purposes of the present section, R_k is the set of all orthogonal linear transformations of C into itself which act as the identity on the subspace

$$C(k) = \{x \in C \mid x_1 = \cdots = x_{k+1} = 0\} \tag{8.8}$$

of C. Each element of R_k may be represented as a matrix $r = (r_{ij})(i, j = 1, 2, \cdots)$ with infinitely many rows and columns of the form

$$\begin{pmatrix} r_0 & 0 \\ 0 & I \end{pmatrix} \tag{8.9}$$

where r_0 is an orthogonal matrix with $(k + 1)$ rows and columns, the zeros represent rectangular $(k + 1) \times \infty$ and $\infty \times (k + 1)$ matrices of zeros, and I is the identity $\infty \times \infty$ matrix, and where the matrix r_0 has determinant 1. For ease in writing formulas, however, we shall frequently replace r by r_0. We note that $R_1 \subset R_2 \subset R_3 \subset \cdots \subset R_n \subset R_{n+1} \subset \cdots$ and that

$$R_n = \{r \in R_{n+1} \mid r_{n+2,\, n+2} = 1\}.$$

Denote by y^k the point of C whose k^{th} coordinate is 1 and whose other coordinates are zero. Instead of the mapping κ considered in §5 we shall define a map $\kappa_k: R_k \to S^k$ by

$$\kappa_k(r) = r(y^{k+1}). \tag{8.10}$$

It is not difficult to see that $\boldsymbol{\kappa}_k = \boldsymbol{\kappa}$. Denote also by κ_k' the map κ_k regarded as a mapping: $(R_k, R_{k-1}) \to (S^k, y^{k+1})$.

We first prove that $\pi_{14}(S^8) \neq 0$ by proving the existence of an element $\gamma_2 \in \pi_7(R_6)$ such that $\boldsymbol{\kappa}_6(\gamma_2) \neq 0$. For this purpose recall that C^8 can be made into a (non-associative) algebra $\mathfrak{C}^8$ without zero-divisors over the reals with the following multiplication table:

$$\begin{array}{ll} y^1 \cdot y^2 = y^4,\ y^2 \cdot y^4 = y^1,\ y^4 \cdot y^1 = y^2 & \\ y^2 \cdot y^3 = y^5,\ y^3 \cdot y^5 = y^2,\ y^5 \cdot y^2 = y^3 & \\ \cdots & \\ y^7 \cdot y^1 = y^3,\ y^1 \cdot y^3 = y^7,\ y^3 \cdot y^7 = y^1 & \\ y^i \cdot y^j = -y^i \cdot y^j & (i \neq j;\ i, j = 1, \cdots, 7) \\ y^8 \cdot y^i = y^i \cdot y^8 = y^i & (i = 1, \cdots, 8) \\ y^i \cdot y^i = -y^8 & (i = 1, \cdots, 7) \end{array} \tag{8.11}$$

and that for $x, y \in C^8$, $\| x \cdot y \| = \| x \| \cdot \| y \|$. For each $x \in C^8$ we let $\bar{x} = -x + 2x_8 y^8$; then $x \cdot \bar{x} = \bar{x} \cdot x = (\| x \|)^2 \cdot y^8$. It is known that the subalgebra of $\mathfrak{C}^8$ generated by the identity y^8 and anyother two elements of C^8 is associative, and therefore, for each $x, y \in C^8$, the element $x \cdot y \cdot \bar{x}$ is well-defined. The mapping $f: C^8 \times C^8 \to C^8$ defined by

$$f(x, y) = x \cdot y \cdot \bar{x} \tag{8.12}$$

carries $S^7 \times S^7$ into S^7, since $\| \bar{x} \| = \| x \|$ and therefore $\| f(x, y) \| = \| x \| \cdot \| y \| \cdot \| \bar{x} \| = 1$ if $\| x \| = \| y \| = 1$. Moreover, it is easily seen that, for each fixed $x \in S^7$, the mapping $y \to f(x, y)$ is a rotation $\bar{f}(x)$: since $f(x, y^8) = x \cdot y^8 \cdot \bar{x} = x \cdot \bar{x} = y^8$, $\bar{f}(x)$ belongs to R_6. Thus the correspondence $x \to \bar{f}(x)$ is a mapping $\bar{f}$: $S^7 \to R_6$. Now

$$(8.13) \quad \begin{aligned} \kappa_6(\bar{f}(x)) = x \cdot y^7 \cdot \bar{x} = (2(x_1x_7 + x_3x_8),\ 2(x_2x_7 + x_6x_8),\ 2(x_3x_7 - x_1x_8),\\ 2(x_4x_7 + x_5x_8),\ 2(x_3x_7 - x_4x_8),\ 2(x_6x^7 - x_2x_8),\ 2(x_7^2 + x_8^2) - 1). \end{aligned}$$

The author has constructed elsewhere [16, pp. 140–1] a map g: $S^7 \to S^6$ given by

$$(8.14) \quad \begin{aligned} g(x) = (2(x_1x_7 + x_2x_8),\quad 2(x_2x_7 - x_1x_8),\quad 2(x_3x_7 + x_4x_8),\\ 2(x_4x_7 - x_3x_8), 2(x_5x_7 + x_6x_8),\quad 2(x_6x_7 - x_5x_8),\ 2(x_7^2 + x_8^2) - 1), \end{aligned}$$

and proved that g represents the non-zero element ν_6 of $\pi_7(S^6)$.
Now if h: $S^7 \to S^7$ is the map such that

$$(8.15) \qquad h(x_1, \cdots, x_8) = (x_1, x_3, x_2, x_6, x_4, x_5, x_7, x_8),$$

and if h': $S^6 \to S^6$ is the map such that

$$(8.16) \qquad h'(x_1, \cdots, x_7) = (x_1, x_3, x_2, x_5, x_6, x_4, x_7),$$

then $h' \circ g \circ h = \kappa_6 \circ \bar{f}$. But h and h' both have degree -1, and therefore $\kappa_6 \circ \bar{f}$ represents $(-\iota_1) \circ \nu_6 \circ (-\iota_7) = \nu_6 \neq 0$. Letting γ_2 be the element of $\pi_7(R_6)$ represented by $\bar{f}$, we have $\kappa_6(\gamma_2) \neq 0$, and therefore $\pi_{14}(S^7)$ contains the non-zero element $J(\gamma_2)$. Therefore [14] the element $\nu_4' \circ J(\gamma_2)$ of $\pi_{14}(S^4)$ is also different from zero.

Eckmann [4] and the author [16] have constructed for each $k \geqq 1$, an element $\gamma_{4k-1} \in \pi_{4k}(R_{4k-1})$ such that $\kappa_{4k-1}(\gamma_{4k-1})$ is the non-zero element $\nu_{4k-1} \in \pi_{4k}(S^{4k-1})$. Hence $J(\gamma_{4k-1})$ is a non-zero element of $\pi_{8k}(S^{4k})$.

We now exhibit a map f: $S^{8k+2} \to R_{8k-1}$ representing an element $\gamma_{8k-1} \in \pi_{8k+2}(R_{8k-1})$. In order to do this, we represent S^{8k+2} as the set of all $(2k + 1)$-tuples of quaternions $(x_0, x_1, \cdots, x_{2k})$ such that $\bar{x}_0 = -x_0$ is a "pure imaginary" quaternion and $\sum_{i=0}^{2k} x_i\bar{x}_i = 1$; and we represent S^{8k-1} as the set of all $2k$-tuples $(x_1, \cdots, x_{2k})$ with x_i quaternions and $\sum_{i=1}^{k2} x_i\bar{x}_i = 1$. Now is x is a quaternion, let $L(x)$ be the matrix of the linear transformation $y \to x \cdot y$, so that, if

$$x = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k},$$

then

$$(8.17) \qquad L(x) = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix}$$

If $A = (a_{\alpha\beta})$ is an $r \times r$ matrix with quaternion elements, let $L(A)$ be the $4r \times 4r$ real matrix obtained by replacing each element $a_{\alpha\beta}$ of A by the 4×4 matrix

$L(a_{\alpha\beta})$. If A is a matrix with quaternion elements, we denote by A^* the conjugate transpose of A. Then, if A has the property that A^*A = the identity, it follows that $L(A)$ is a proper orthogonal matrix.

If $x \in S^{8k+2}$, define a matrix $F(x)$ with quaternion elements by

$$F_{\alpha\beta}(x) = \delta_{\alpha\beta} - 2x_\alpha(1 - x_0)^{-2}\bar{x}_\beta \qquad (\alpha, \beta = 1, \cdots, 2k) \tag{8.18}$$

where $\delta_{\alpha\beta} = 0$ or 1 according as $\alpha \neq \beta$ or $\alpha = \beta$. An easy computation shows that $(F(x))^* \cdot F(x)$ = the identity. Hence the function f defined by

$$f(x) = L(F(x)) \qquad (x \in S^{8k+2}) \tag{8.19}$$

is a mapping $f\colon S^{8k+2} \to R_{8k-1}$. Let γ'_{8k-1} be the element of $\pi_{8k+2}(R_{8k-1})$ represented by f.

To compute $\kappa_{8k-1} \circ f$, observe that y^{8k} is the point $(0, \cdots, 0, \boldsymbol{k})$, and that, for $x \in S^{8k+2}$, the point $(f(x))(y^{8k})$ is obtained from the last column of the matrix $F(x)$ by right-multiplying each of its elements by the quaternion $\boldsymbol{k}$. We have therefore:

$$\begin{aligned}\kappa_{8k-1}(f(x)) = (-2x_1(1 - x_0)^{-2}\bar{x}_{2k}\boldsymbol{k}, \cdots, -2x_{2k-1}(1 - x_0)^{-2}\bar{x}_{2k}\boldsymbol{k},\\ (1 - 2x_{2k}(1 - x_0)^{-2}\bar{x}_{2k})\boldsymbol{k}).\end{aligned} \tag{8.20}$$

The author has constructed elsewhere [19] a map $g\colon S^{8k+2} \to S^{8k-1}$ given by

$$g(x) = (1 - 2x_1(1 - x_0)^{-2}\bar{x}_1, -2x_2(1 - x_0)^{-2}\bar{x}_1, \cdots, -2x_{2k}(1 - x_0)^{-2}\bar{x}_1) \tag{8.21}$$

and proved that g represents an element $\bar{\nu}'_{8k+1}$ of $\pi_{8k+2}(S^{8k-1})$ which is the $(8k - 5)$-fold suspension of an element of $\pi_7(S^4)$ whose Hopf invariant is ± 1. Hence the element $\bar{\nu}'_{8k-1}$ does not belong to $2\pi_{8k+2}(S^{8k-1})$. Let $h\colon S^{8k+2} \to S^{8k+2}$ and $h'\colon S^{8k-1} \to S^{8k-1}$ be the maps such that

$$\begin{aligned} h(x_0, x_1, \cdots, x_{2k}) &= (x_0, x_{2k}, \cdots, x_1),\\ h'(x_1, \cdots, x_{2k}) &= (x_{2k}\boldsymbol{k}, \cdots, x_1\boldsymbol{k}).\end{aligned} \tag{8.22}$$

Then $h' \circ g \circ h = \kappa_{8k-1} \circ f$. Since h and h' both have degree $(-1)^k$, $\kappa_{8k-1} \circ f$ represents $((-1)^k\iota_{8k-1}) \circ \bar{\nu}'_{8k-1} \circ ((-1)^k\iota_{8k+2}) = \bar{\nu}'_{8k-1}$, since $\bar{\nu}'_{8k-1}$ is the suspension of an element of $\pi_{8k+1}(S^{8k-2})$. We have proved that $\boldsymbol{k}_{8k-1}(\gamma'_{8k-1}) = \bar{\nu}'_{8k-1} \neq 0$, and therefore $\pi_{16k+2}(S^{8k})$ contains the non-zero element $J(\gamma'_{8k-1})$.

Since $H(J(\gamma_{4k-1})) = \boldsymbol{k}_{4k-1}(\gamma_{4k-1}) * \iota_{4k-1} = \nu_{8k-1}$ is the non-zero element of $\pi_{8k}(S^{8k-1})$, we conclude from Theorem 7.33 that $E(J(\gamma_{4k-1})) \neq 0$. Hence $\pi_{8k+1}(S^{4k+1}) \neq 0$. In the same way, $H(J(\gamma'_{8k-1})) = \boldsymbol{k}_{8k-1}(\gamma'_{8k-1}) * \iota_{8k-1} = \bar{\nu}'_{8k-1} * \iota_{8k-1}$ does not belong to $2\pi_{16k+2}(S^{16k-1})$, and therefore $E(J(\gamma'_{8k-1})) \neq 0$. Hence $\pi_{16k+3}(S^{8k+1}) \neq 0$. Since E maps $\pi_{14}(S^7)$ isomorphically [2, Theorem 15] into $\pi_{16}(S^8)$, $E(\gamma_2)$ is a non-zero element of $\pi_{15}(S^8)$. Since $E\colon \pi_{15}(S^8) \to \pi_{16}(S^9)$ maps the image of $E\colon \pi_{14}(S^7) \to \pi_{15}(S^8)$ isomorphically [9, Theorem II], $E^2(\gamma_2) \neq 0$, and therefore $E^k(\gamma_2) \neq 0$ for $k > 2$. Although the groups $\pi_{14+k}(S^{7+k})$ were known to be non-zero for $k > 0$, the non-zero elements $E^k(\gamma_2)$ constructed here are different from the known ones. In particular, $E(\gamma_2)$ lies in the direct summand

$E(\pi_{14}(S^7))$, and ν_8'' generates the infinite cyclic direct summand $h_8(\pi_{15}(S^{15}))$, of $\pi_{15}(S^8)$. Hence $\pi_{15}(S^8)$ is not a cyclic group.

9. Proof of the non-existence of elements of $\pi_{8k+3}(S^{4k+2})$ with Hopf invariant 1

In this section we answer in the negative a question proposed by Hopf [11] by proving the assertion made in the title of the section for $k > 0$. As a corollary, we observe that S^{4k+1} does not admit a continuous multiplication with two-sided identity if $k > 0$.

For the proof, we require some preliminaries. Let G^r be the space of all maps of S^r into S^r of degree 1, and for each $y \in S^r$, let $F_y^r = \{f \in G^r \mid f(y^{r+1}) = y\}$; and let $F^r = F_{y^{r+1}}^r$. For $f \in G^r$, define $\tilde{\kappa}(f) = f(y^{r+1})$. Then $\tilde{\kappa}: G^r \to S^r$ is a fibre mapping [7], the fibres being the sets F_y^r. Let $\tilde{\kappa}'$ denote the map $\tilde{\kappa}$ considered as a map: $(G^r, F^r) \to (S^r, y^{r+1})$. Note that $R_r \subset G^r$, and that $R_r \cap F^r = R_{r-1}$, while $\tilde{\kappa} \mid R_r = \kappa_r$.

Let $I: \pi_i(F^r) \to \pi_{i+r}(S^r)$ be the Hurewicz isomorphism [18]; I is an isomorphism onto, and if $J': \pi_i(R_{r-1}) \to \pi_i(F^r)$ is the injection homomorphism, we have [18]

$$I \circ J' = J. \tag{9.1}$$

The homomorphism $\tilde{\kappa}': \pi_{i+1}(G^r, F^r) \to \pi_{i+1}(S^r)$ is an isomorphism onto [14]. The author has proved [18, Theorem 3.2] that if $\alpha \in \pi_{i-1}(S^r)$, then $I(\partial((\tilde{\kappa}')^{-1} \cdot (\alpha))) = [\alpha, \iota_r]$.

$$\begin{array}{ccc} \pi_{i+1}(G^r, F^r) & \xrightarrow{\partial} & \pi_i(F^r) \\ \tilde{\kappa}' \downarrow & & \downarrow I \\ \pi_{i+1}(S^r) & & \pi_{i+r}(S^r). \end{array} \tag{9.2}$$

Let $f: S^{r-1} \to R_{r-1}$ be the map such that

$$f(x) = f'(x) \cdot \begin{pmatrix} I_{r-1} & 0 \\ 0 & -1 \end{pmatrix} \qquad (x \in S^{r-1}) \tag{9.3}$$

where

$$f'(x) = (\delta_{ij} - 2x_i x_j) \qquad (x \in S^{r-1}). \tag{9.4}$$

The map f represents an element $\alpha \in \pi_{r-1}(R_{r-1})$ which is the image under ∂: $\pi_r(R_r, R_{r-1}) \to \pi_{r-1}(R_{r-1})$ of a generator β of the infinite cyclic group $\pi_r(R_r, R_{r-1}) \approx \pi_r(S^r)$ [16].

$$\begin{array}{ccc} \pi_r(R_r, R_{r-1}) & \xrightarrow{\partial} & \pi_{r-1}(R_{r-1}) \\ J'' \downarrow & & \downarrow J' \\ \pi_r(G^r, F^r) & \xrightarrow{\partial} & \pi_{r-1}(F^r) \\ \tilde{\kappa}' \downarrow & & \downarrow I \\ \pi_r(S^r) & & \pi_{2r-1}(S^r) \end{array} \tag{9.5}$$

Let $J'': \pi_r(R_r, R_{r-1}) \to \pi_r(G^r, F^r)$ be the homomorphism induced by the identity map; then $\tilde{\kappa}' \circ J'' = \kappa_r'$. Since κ_r' and $\tilde{\kappa}'$ are isomorphisms onto, J'' is an isomorphism onto. Since $J''(\beta)$ is a generator of $\pi_r(G^r, F^r)$, we have $\tilde{\kappa}'(J''(\beta)) = \pm\iota_r$, and therefore $J'(\alpha) = J'(\partial(\beta)) = \partial(J''(\beta)) = \pm\partial((\tilde{\kappa}')^{-1}(\iota_r))$, and therefore $J(\alpha) = I(J'(\alpha)) = \pm[\iota_r, \iota_r]$.

The author has proved [17, Theorem 2] that $J \circ \alpha_{r-1} = E \circ J$,

(9.6)
$$\begin{array}{ccc} \pi_{r-1}(R_{r-2}) & \xrightarrow{\alpha_{r-1}} & \pi_{r-1}(R_{r-1}) \\ J\downarrow & & \downarrow J \\ \pi_{2r-2}(S^{r-1}) & \xrightarrow{E} & \pi_{2r-1}(S^r) \end{array}$$

where α_{r-1} is the identity map of R_{r-2} into R_{r-1}. Suppose now that $r = 4k + 1$ and $k > 0$. Then it has been proved by the author [16, pp. 139–40] that $\alpha_{4k}(\gamma_{4k-1}) = \alpha$, and therefore we have

$$E(J(\gamma_{4k-1})) = J(\alpha_{4k}(\gamma_{4k-1})) = J(\alpha) = \pm[\iota_{4k+1}, \iota_{4k+1}].$$

We have shown that $E(J(\gamma_{4k-1})) \neq 0$, and therefore $[\iota_{4k+1}, \iota_{4k+1}] \neq 0$. By (3.71) and (3.72), no element of $\pi_{8k+3}(S^{4k+2})$ with Hopf invariant 1 can exist.

10. Concluding remarks

Consider the following sequence of groups and homomorphisms:

(10.1)
$$\pi_{3r-2}(S^r) \to \cdots \to \pi_n(S^r) \xrightarrow{E_n} \pi_{n+1}(S^{r+1}) \xrightarrow{H_n} \pi_{n+1}(S^{2r+1}) \xrightarrow{P_n} \pi_{n-1}(S^r) \to \cdots$$

The homomorphism P_n is defined as follows. The $(r + 1)$-fold suspension $E^{r+1}: \pi_{n-r}(S^r) \to \pi_{n+1}(S^{2r+1})$ is an isomorphism onto for $n < 3r - 1$. Let $P_n' : \pi_{n-r}(S^r) \to \pi_{n-1}(S^r)$ be defined by

(10.2)
$$P_n'(\alpha) = [\alpha, \iota_r].$$

Then $P_n = P_n' \circ (E^{r+1})^{-1}$.

We have shown that $H_n \circ E_n$ is the zero homomorphism, and the author has shown elsewhere [18, Theorem 3.11] that $E_{n-1} \circ P_n'$ is zero, and therefore $E_{n-1} \circ P_n$ is likewise trivial. On the other hand, if $\alpha \in \pi_{n-r}(S^r)$ and $P_n'(\alpha) = 0$, then there is a map $f: \dot{E}^{n-r+1} \times \dot{E}^{r+1}$ into S^r of type (α, ι_r). The map $Gf: \dot{E}^{n+2} \to S^{r+1}$ represents an element β of $\pi_{n+1}(S^{r+1})$ such that $H_n(\beta) = \alpha * \iota_r = E^{r+1}(\alpha)$. Thus $P_n(\alpha) = 0$ implies that there exists $\beta \in \pi_{n+1}(S^{r+1})$ such that $H_n(\beta) = \alpha$. We have

(10.3)
$$\begin{cases} \text{Kernel } H_n \supset \text{Image } E_n \\ \text{Kernel } P_n \subset \text{Image } H_n \\ \text{Kernel } E_{n-1} \supset \text{Image } P_n . \end{cases}$$

It is not known whether the opposite inclusions hold.

An examination of the Freudenthal theorems shows that they can be formulated as follows:

$$\text{(10.4)}\qquad \begin{cases} \text{Kernel } H_n = \text{Image } E_n & \text{for } n \leqq 2r, \\ \text{Kernel } P_n = \text{Image } H_n & \text{for } n \leqq 2r, \\ \text{Kernel } E_{n-1} = \text{Image } P_n & \text{for } n \leqq 2r. \end{cases}$$

The problem of generalizing the Freudenthal theorems might then be formulated as that of determining whether or not the inclusions opposite to those in (10.3) hold.

Let $\alpha \in \pi_p(S^r)$, and let $G_\alpha^{p,r}$ be the space of all mappings: $S^p \to S^r$ which represent α. Let $F_\alpha^{p,r}$ be the set of all $f \in G_\alpha^{p,r}$ such that $f(y_*) = y_*$. The space $G_\alpha^{p,r}$ can be imbedded in a natural way in $F_{E(\alpha)}^{p+1,r+1}$, and then the injection homomorphisms: $\pi_q(F_\alpha^{p,r}) \to \pi_q(F_{E(\alpha)}^{p+1,r+1})$ is equivalent [18, Theorem 3.10] under the Hurewicz isomorphism to the suspension homomorphism $E:\pi_{p+q}(S^r) \to \pi_{p+q+1}(S^{r+1})$. The homomorphism $\Lambda' : \pi_{r+1}^{p+q+1} \to \pi_{p+q+1}(S^{2r+1})$ can be modified slightly to yield a homomorphism $I'_\alpha : \pi_q(F_{E(\alpha)}^{p+1,r+1}, F_\alpha^{p,r}) \to \pi_{p+q+1}(S^{2r+1})$. In the diagram

$$\text{(10.5)}\qquad \begin{array}{ccccccccc} \cdots \to \pi_q(F_\alpha^{p,r}) & \xrightarrow{i'} & \pi_q(G_\alpha^{p,r}) & \xrightarrow{j'} & \pi_q(G_\alpha^{p,r}, F_\alpha^{p,r}) & \xrightarrow{\partial} & \pi_{q-1}(F_\alpha^{p,r}) \to \cdots \\ \downarrow e & & \downarrow e' & & \downarrow e'' & & \downarrow e \\ \cdots \to \pi_q(F_\alpha^{p,r}) & \xrightarrow{i} & \pi_q(F_{E(\alpha)}^{p+1,r+1}) & \xrightarrow{j} & \pi_q(F_{E(\alpha)}^{p+1,r+1}, F_\alpha^{p,r}) & \xrightarrow{\partial} & \pi_{q-1}(F_\alpha^{p,r}) \to \cdots \\ \downarrow I_\alpha & & \downarrow I_{E(\alpha)} & & \downarrow I'_\alpha & & \downarrow I_\alpha \\ \cdots \to \pi_{p+q}(S^r) & \xrightarrow{E} & \pi_{p+q+1}(S^{r+1}) & \xrightarrow{H} & \pi_{p+q+1}(S^{2r+1}) & \xrightarrow{P} & \pi_{p+q-1}(S^r) \to \cdots \end{array}$$

the top sequence is the homotopy sequence of the pair $(G_\alpha^{p,r}, F_\alpha^{p,r})$, the middle sequence is the homotopy sequence of the pair $(F_{E(\alpha)}^{p+1,r+1}, F_\alpha^{p,r})$, and the bottom sequence is (10.1). The homomorphisms e, e', e'' are induced by identity maps, while the homomorphisms I_α and $I_{E(\alpha)}$ are the Hurewicz isomorphisms onto. All the commutativity relations suggested by the diagram are known to hold, except possibly for the relations $I'_\alpha \circ j = H \circ I_{E(\alpha)}$ and $I_\alpha \circ \partial = P \circ I'_\alpha$. Since the Hurewicz homomorphisms are known to be isomorphisms onto, it is natural to ask whether I'_α is an isomorphism onto. Examination of the diagram

$$\text{(10.6)}\qquad \begin{array}{ccc} \pi_q(G_\alpha^{p,r}, F_\alpha^{p,r}) & \xrightarrow{k} & (\pi_q S^r) \\ \downarrow e'' & & \downarrow l \\ \pi_q(F_{E(\alpha)}^{p+1,r+1}, F^{p,r}) & \xrightarrow{I'_\alpha} & \pi_{p+q+1}(S^{2r+1}) \end{array}$$

where $l(\beta) = \beta * \alpha$, suggests the conjecture that $l \circ \kappa = I'_\alpha \circ e''$. If this is so, then the fact that in the case $p = r$, $\alpha = \iota_r$, the homomorphism l is an isomorphism onto for $q < 2r - 1$, would show that I'_α is onto and e'' an isomorphism into.

One may also ask whether the Hopf homomorphism can be further generalized to the case $n \geqq 3r - 3$. A crucial case seems to be $n = 3r - 2$. Investigation of the homomorphism of $\pi_{3r-1}(E^r \times E^r, (E^r \times E^r)^{\cdot})$ into $\pi_{3r-1}(S^r \times S^r, S^r \vee S^r)$ induced by $\bar{\Psi}'_{r,r}$ suggests the possibility of a new invariant essentially distinct from the Hopf invariant. For let $f:(E^{3r-1}, \dot{E}^{3r-1}) \to (S^r \times S^r, S^r \vee S^r)$ be a mapping representing $\alpha \in \pi_{3r-1}(S^r \times S^r, S^r \vee S^r)$. The element α is the image under $\bar{\Psi}_{r,r}$ of an element $\beta \in \pi_{3r-1}(E^{2r}, \dot{E}^{2r})$ if and only if f is homotopic to a mapping g such that $g^{-1}(u)$ is a single point for some $u \in (S^r \times S^r) - (S^r \vee S^r)$. The set $f^{-1}(u)$ carries an $(r-1)$-dimensional cycle z_{r-1} interior to E^{3r-1}; if c_r is a chain bounded by z_{r-1}, then the image of c_r under f is an r-cycle of $S^r \times S^r$ which is homologous to $a(S^r \times y_*) + b(y_* \times S^r)$ for some integers a, b. This pair of integers is easily seen to be a homotopy invariant of f; and the results of Eilenberg [5] suggest that f is homotopic to a map g as above if and only if $a = b = 0$. Thus to each element $\alpha \in \pi_{3r-2}(S^r)$ it is possible to associate a pair of integers (a, b), which measure in some sense the impossibility of defining a Hopf invariant of α as we have done for $n < 3r - 3$. It is not known to the author whether there exists an element $\alpha \in \pi_{3r-2}(S^r)$ such that $(a, b) \neq (0, 0)$.

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