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ON PRODUCTS IN HOMOTOPY GROUPS[1]

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1. Introduction

One of the outstanding problems in homotopy theory is that of determining the homotopy groups of simple spaces. Even for as simple a space as the n-sphere very little is known. In fact, in most cases, it is not known whether or not the homotopy groups are zero.

J. H. C. Whitehead [16][2] has defined a product between two of the homotopy groups π_p and π_q of a space X with values in π_{p+q-1}. In some instances this product affords a method for constructing non-zero elements of $\pi_{p+q-1}(X)$. He also defined generalized products, involving the homotopy groups of the rotation groups.

Hurewicz [10] originally defined the group $\pi_n(X)$ as the fundamental group of a certain function space over X. The elements of $\pi_n(X)$ may also be regarded as equivalence classes of mappings of the n-sphere S^n into X, and it is the latter definition which has been used in most of the applications.

In this paper the original point of view adopted by Hurewicz is combined with the second approach. The method of fibre spaces of Hurewicz and Steenrod [11] is used to study the interrelations between the homotopy groups of a space X and those of certain function spaces over X. In Section 2 we state preliminary results, many of which are known, and establish the necessary homomorphisms between the homotopy groups of the spaces under consideration. In Section 3 the products of J. H. C. Whitehead are characterized as operations in function spaces. Using this characterization, we are able to prove that the Freudenthal "Einhängung" of a product is always inessential. A partial converse to this result is obtained.

In Section 5 the generalized products defined by J. H. C. Whitehead are characterized in terms of known operations. This characterization is used to verify a conjecture of J. H. C. Whitehead.

2. Preliminaries

We introduce here the function spaces needed in the sequel and discuss the interrelations among their homotopy groups. Most of the results presented here are known, but proofs are given when they are necessary for an understanding of the results to follow.

Let I^p denote the set of points $y = (y_1, \cdots, y_p)$ in Euclidean p-space, such that $0 \leqq y_i \leqq 1$, $(i = 1, \cdots, p)$. Denote by S^p the set of points y in Euclidean $(p+1)$-space such that $|y| = [\sum y_i^2]^{\frac{1}{2}} = 1$. It will frequently be desirable to

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[2] Numbers in square brackets refer to the bibliography.

make use of sets homeomorphic with I^p; if X is such a set, choose a fixed homeomorphism h_X of I^p on X. An arbitrary homeomorphism h'_X of I^p on X will then be called *admissible* if the map $h_X^{-1}h'_X$ of I^p on itself has degree $+1$. A similar remark applies to S^p.

Let X be a connected locally contractible metric space. Define $G^p(X)$ to be the space of all mappings[3] f of I^p into X such that the image of the boundary $\sum^{p-1}$ of I^p under f is a single point of X (depending on f). Let $\tau(f) = f(\sum^{p-1})$ ($f \in G^p(X)$), and $F^p(X,x) = \tau^{-1}(x)$ ($x \in X$). It is easy to see that τ is a mapping of $G^p(X)$ on the whole of X.

If f, $g \in F^p(X,x)$, define $f + g$ to be the mapping $h \in F^p(X,x)$ such that

$$h(y_1, \cdots, y_{p-1}, y_p) = \begin{cases} f(y_1, \cdots, y_{p-1}, 2y_p) & (0 \leqq y_p \leqq \tfrac{1}{2}); \\ g(y_1, \cdots, y_{p-1}, 2y_p - 1) & (\tfrac{1}{2} \leqq y_p \leqq 1). \end{cases}$$

It is clear that h depends continuously on f and g. Hence the component γ of h in $F^p(X,x)$ depends only on the components α and β of f and g, respectively. Define $\alpha + \beta = \gamma$; then $+$ is a well defined operation in the set $\pi_p(X,x)$ of components of $F^p(X,x)$. Moreover, under $+$, the set $\pi_p(X,x)$ forms a group, the p^{th} *homotopy group of X at the point x* [10]. The group $\pi_p(X,x)$ is abelian if $p > 1$; while $\pi_1(X,x)$ is the fundamental group of X at x.

The homotopy group $\pi_p(X,x)$ can also be defined using maps of spheres into X. Let y_0 be a fixed reference point $\in S^p$, and define $\bar{G}^p(X)$ to be the space of all maps of S^p into X. Let $\bar{\tau}(f) = f(y_0)$ ($f \in \bar{G}^p(X)$, and $\bar{F}^p(X,x) = \bar{\tau}^{-1}(x)(x \in X)$. Choose a fixed mapping $\psi \in F^p(S^p,y_0)$ such that $\psi \mid I^p - \sum^{p-1}$ is topological. Any map $\psi' \in F^p(S^p,y_0)$ such that $\psi' \mid I^p - \sum^{p-1}$ is topological will be said to be *admissible* if $\psi'\psi^{-1}$ maps S^p on itself with degree 1.

If ψ' is an admissible map of I^p on S^p, then the correspondence Ψ' of $\bar{G}^p$ with G^p defined by

$$[\Psi'(g)](y) = g[\psi'(y)] \qquad (g \epsilon \bar{G}^p,\ y \epsilon I^p)$$

is a homeomorphism between $\bar{G}^p$ and G^p such that $\Psi'[\bar{F}^p(X,x)] = F^p(X,x)$.

(2.1)*[4] *If X is a compact absolute neighborhood retract, then τ and $\bar{\tau}$ are fibre mappings* [11] *while $F^p(X,x)$ and $\bar{F}^p(X,x)$ are fibres in $G^p(X)$ and $\bar{G}^p(X)$, respectively.* [1,7].

The components of $\bar{F}^p(X,x)$ form a group $\bar{\pi}_p(X,x)$ isomorphic with $\pi_p(X,x)$, the group operation being that induced by an admissible mapping ψ of I^p on S^p. Since any two admissible mappings are homotopic, the group operation does not depend on the particular ψ chosen. It can be described explicitly as

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[3] By a *mapping* of X into Y is understood a *continuous* function on X to Y. The space Y^X of all mappings of X into Y is metrized by the usual formula $\rho(f, g) = \sup_{x \in X} \rho\ [f(x), g(x)]$ if X is compact.

[4] In the starred theorems it is assumed that X is a compact absolute neighborhood retract (ANR).

follows: let K_1^p, K_2^p be the hemispheres of S^p defined by the inequalities $y_{p+1} \geqq 0$, $y_{p+1} \leqq 0$, respectively and let $y_0 = (1, 0, \cdots 0) \in K_1^p \cap K_2^p$. If $f_i \in \bar{F}^p(X,x)$ $(i = 1, 2)$, then there exists f_i' in the same component of $\bar{F}^p(X,x)$ as f_i such that $f_1'(K_2^p) = f_2'(K_1^p) = x$. The sum of the components of f_1 and f_2 is the component of the map $f \in \bar{F}^p(X,x)$ such that

$$f(y) = \begin{cases} f_1'(y) & (y \in K_1^p), \\ f_2'(y) & (y \in K_2^p). \end{cases}$$

If Y is a set homeomorphic with I^p, then any map f of Y into X such that f maps the boundary of Y into a point x determines an element α of $\pi_p(X,x)$: α is the component of $F^p(X,x)$ containing the map fh_Y. We shall say that α *is represented by* f. A similar remark holds for sets homeomorphic with S^p.

Let $x, x' \in X$, $\alpha \in \pi_p(X,x)$ and let g be a path in X from x to x' (so that g is a map of I^1 into X such that $g(0) = x$, $g(1) = x'$; such a path exists in virtue of our assumptions on X). Let E^p be the set of points y in Euclidean p-space such that $|y| \leqq 1$; then E^p is homeomorphic with I^p, and we define h_{E^p} to be any homeomorphism such that, if E^p and I^p are assigned similar orientations with respect to a given orientation of p-space, then h_{E^p} has degree $+1$. Let $f(E^p) \subset X$ be a representative of α, and define $\theta_g(\alpha)$ to be the element of $\pi_p(X,x')$ represented by the map f' such that

$$f'(y_1, \cdots, y_p) = \begin{cases} f(2y_1, 2y_2, \cdots, 2y_p) & (0 \leqq |y| \leqq \tfrac{1}{2}), \\ g(2|y| - 1) & (\tfrac{1}{2} \leqq |y| \leqq 1). \end{cases}$$

Then $\theta_g(\alpha)$ depends only on α and on the homotopy class of paths from x to x' determined by g. Eilenberg [4] has shown

(2.2) *θ_g is an isomorphism of $\pi_p(X,x)$ with the whole of $\pi_p(X,x')$. If g' is a path from x' to $x'' \in X$, then $\theta_{g'g} = \theta_{g'}\theta_g$.*

It follows that the system $\{\pi_p(X,x) \mid x \in X\}$ is a system of local groups in the sense of Steenrod [13]. X is said to be *p-simple* if the system of groups $\{\pi_p(X,x)\}$ is simple.

If $\xi \in \pi_1(X,x)$ is represented by a closed path g from x to x, let $\alpha^\xi = \theta_g(\alpha)$ $(\alpha \in \pi_p(X,x))$. Then the correspondence $\alpha \rightarrow \alpha^\xi$ is an automorphism of $\pi_p(X,x)$.

(2.3) *If $\alpha, \beta \in \pi_p(X,x)$, then α and β belong to the same component of $G^p(X)$ if and only if there exists $\xi \in \pi_1(X,x)$ such that $\beta = \alpha^\xi$* [4].

The notion of homotopy group has been relativized by Hurewicz [10]. Let A be a closed, arcwise connected subset of X, x a point of A, y_0 a fixed reference point in S^{p-1}. Then if $p \geqq 2$ define $F^p(X,A,x)$ to be the space of maps f of E^p into X such that $f(S^{p-1}) \subset A$ and $f(y_0) = x$. The components of $F^p(X,A,x)$ form a group $\pi_p(X,A,x)$, the group operation being defined as follows: let E_1^p, E_2^p be the subsets of E^p defined by $y_p \geqq 0$, $y_p \leqq 0$ respectively, and let $y_0 = (1, 0, \cdots 0) \in E_1^p \cap E_2^p \cap S^{p-1}$. If f_i $(i = 1, 2) \in F^p(X,A,x)$, then there exists f_i' in the same component of $F^p(X,A,x)$ as f_i such that $f_1'(E_2^p) = f_2'(E_1^p) = x$.

The sum of the components of f_1 and f_2 is the component of the map $f \in F^p(X,A,x)$ such that

$$f(y) = \begin{cases} f_1'(y) & (y \in E_1^p), \\ f_2'(y) & (y \in E_2^p). \end{cases}$$

The group $\pi_p(X,A,x)$ is called the p^{th} *relative homotopy group of* X mod A at x. It is abelian if $p > 2$. The system of groups $\{\pi_p(X,A,x) \mid x \in A\}$ forms a system of local groups in A. Moreover, $\pi_p(X,A,x)$ is isomorphic with $\pi_p(X,x)$ if A consists of the single point x, the isomorphism being that induced by h_{E^p}.

The following notational conventions will be adopted hereafter: if $\alpha \in \pi_p(X,x)$ then $G_\alpha^p(X)$ will denote the component of $G^p(X)$ containing α, and the component α will frequently be denoted by $F_\alpha^p(X,x)$. We shall not distinguish between $\pi_p(X,x)$ and $\bar{\pi}_p(X,x)$, nor between $\pi_p(X,x,x)$ and $\pi_p(X,x)$. If X is a set homeomorphic with E^p or S^{p-1}, and X is a subset of the same Euclidean p-space, then h_X is to be chosen so that, if X and E^p (or S^{p-1}) are assigned similar orientations with respect to a given orientation of p-space, then h_X has degree $+1$.

Let $A \subset X$ be as above, $x \epsilon A$, and consider the *homotopy sequence* $\mathfrak{N}$ (X,A,x) of groups and homomorphisms [6]

$$\cdots \to \pi_p(X,x) \to \pi_p(X,A,x) \to \pi_{p-1}(A,x) \to \pi_{p-1}(X,x) \to \cdots \to \pi_1(X,x),$$

defined as follows: if $f \in F^p(X,x)$, then $fh_{E^p}^{-1} \in F^p(X,A,x)$, and the correspondence $f \to fh_{E^p}^{-1}$ induces the first homomorphism exhibited. If $f \in F^p(X,A,x)$, then $f \mid S^{p-1} \in \bar{F}^p(A,x)$, and the second homomorphism is that induced by the correspondence $f \to f \mid S^{p-1}$. Since $A \subset X$, $F^p(A,x) \subset F^p(X,x)$; the identity correspondence induces the third homomorphism.

The principal property of $\mathfrak{N}$ (X,A,x) is

(2.4) *The kernel of each homomorphism is the image of the preceding.* [6]

Since X is arcwise connected, the mapping τ $(G^p) = X$ defined above maps each component of G^p on the whole of X, and $\tau_\alpha = \tau \mid G_\alpha^p$ is a fibre map with fibre $F = \tau^{-1}(x)$ = the union of the sets $F_{\alpha\xi}^p(X,x)$ for all $\xi \in \pi_1(X,x)$. Let $a_0 \in F^p(X,x)$; since S^q is connected $(q > 0)$, $\pi_{q+1}(G^p[X],F,a_0) = \pi_{q+1}(G^p[X], F^p[X,x], a_0)$. The map τ_α induces a natural homomorphism (also denoted by τ_α) of $\pi_{q+1}(G^p[X], F^p[X,x], a_0)$ into $\pi_{q+1}(X,x)$ as follows: if $f \in F^{q+1}(G^p[X], F^p[X,x], a_0)$, then $\tau_\alpha f h_{E^{q+1}} \in F^{q+1}(X,x)$; the homomorphism τ_α is that induced by $f \to \tau_\alpha f h_{E^{q+1}}$

(2.5)* τ_α *is an isomorphism of* $\pi_{q+1}(G^p[X], F^p[X,x], a_0)$ *with the whole of* $\pi_{q+1}(X,x)$ [11].

Two spaces A and B are said to have the same homotopy type [10] if there exist mappings $\varphi(A) \subset B$, $\psi(B) \subset A$ such that $\varphi\psi$ and $\psi\varphi$ are homotopic to the identity maps of B and A respectively. Under these circumstances the homotopy groups of A and B are isomorphic; more precisely, φ and ψ induce isomorphisms

$$\pi_p(A,a) \to \pi_p(B,\varphi[a]) \to \pi_p(A, \psi[\varphi(a)])$$

$$\pi_p(B,b) \to \pi_p(A,\psi[b]) \to \pi_p(A, \varphi[\psi(b)])$$

of the local group systems $\{\pi_p(A,a),a\}$ and $\{\pi_p(B,b),b\}$.

(2.6) *Any two components of $F^p(X,x)$ have the same homotopy type.* [10].

It is sufficient to prove that $F_0^p(X,x)$ and $F_\alpha^p(X,x)$ have the same homotopy type. For this purpose let $a \in F_\alpha^p(X,x)$, and let $-a$ be the map $\in F_{-\alpha}^p(X,x)$ such that

$$-a(y_1, \cdots, y_p) = a(y_1, \cdots, y_{p-1}, 1 - y_p) \qquad ((y_1 \cdots y_p) \in I^p).$$

Let $\varphi(f) = f + a$ $(f \in F_0^p)$, $\psi(g) = g + (-a)$ $(g \in F_\alpha^p)$. Then $\varphi\psi(g) = (g + [-a]) + a \in F_\alpha^p$, $\psi\varphi(f) = (f + a) + (-a) \in F_0^p$. A deformation of $\psi\varphi$ to the identity is then defined by

$$f_t(y_1, \cdots, y_{p-1}, y_p) = \begin{cases} f\left(y_1, \cdots, y_{p-1}, \dfrac{4y_p}{1+3t}\right) & \left(0 \leqq y_p \leqq \dfrac{1+3t}{4}\right), \\ a(y_1, \cdots, y_{p-1}, 4y_p - [1+3t]) & \left(\dfrac{1+3t}{4} \leqq y_p \leqq \dfrac{1+t}{2}\right), \\ a(y_1, \cdots, y_{p-1}, 2 - 2y_p) & \left(\dfrac{1+t}{2} \leqq y_p \leqq 1\right) \end{cases}$$

Then $f_0 = (f + a) + (-a) = \psi\varphi(f)$ and $f_1 = f$, while f_t is continuous in the two variables f and t. Similarly $\varphi\psi$ can be deformed to the identity. We shall denote by H_α the natural isomorphism of $\pi_{q-1}[F_\alpha^p(X,x),a]$ with $\pi_{q-1}[F_0^p(X,x), a_0]$ induced by ψ.

On the other hand, the components of G^p almost never have the same homotopy type. (For a counter-example, see the Appendix). If, however, there is a mapping $\lambda(X) \subset G^p$ such that $\tau_\alpha\lambda$ is the identity, then a construction similar to that given above with $\varphi(f) = f + \lambda[\tau_0(f)]$ and $\psi(g) = g + (-\lambda[\tau_\alpha(g)])(f \in G_0^p$, $g \in G_\alpha^p)$ shows that G_0^p and G_α^p satisfy the even stronger condition formulated in

DEFINITION (2.7). *Two fibre spaces X_1 and X_2 over the same space B with projections $\tau_i(X_i) = B$ $(i = 1, 2)$ are equivalent over B if there exist mappings $\varphi_1(X_1) \subset X_2$, $\varphi_2(X_2) \subset X_1$ such that*

(1) *$\varphi_1\varphi_2$ and $\varphi_2\varphi_1$ are homotopic to the identity maps of X_2 and X_1, respectively;*

(2) *$\tau_2\varphi_1$ is homotopic to τ_1 and $\tau_1\varphi_2$ is homotopic to τ_2.*

Conversely, we can show

THEOREM (2.8)* *If $G_\alpha^p(X)$ is equivalent to $G_0^p(X)$ over X, then there exists a map $\lambda(X) \subset G_\alpha^p(X)$ such that $\tau_\alpha\lambda$ is the identity.*

For let $\lambda_0(x) \in G_0^p(X)$ be the constant map of I^p into x, so that $\tau_0\lambda_0$ is the identity. Under the hypotheses of the theorem there is a map $\varphi[G_0^p(X)] \subset G_\alpha^p(X)$ such that $\tau_\alpha\varphi$ is homotopic to τ_0. Let $\lambda' = \varphi\lambda_0$. Then $\tau_\alpha\lambda' = \tau_\alpha\varphi\lambda_0$ is homotopic to $\tau_0\lambda_0 =$ the identity. By the covering homotopy theorem [11, Theorem 1], λ' is homotopic to a map λ such that $\tau_\alpha\lambda =$ the identity.

Let a_0 be the constant map of E^p into x. We shall define a map of $\pi_{q-1}[F_0^p(X,x), a_0]$ into $\pi_{p+q-1}(X,x) = \pi_n(X,x)$. The boundary $E^p \times S^{q-1} \cup S^{p-1} \times E^q$ of $E^p \times E^q$ is a sphere S_0^n. Let y_0, z_0 be the reference points of S^{p-1} and S^{q-1} respectively, and $w_0 = (y_0, z_0)$ the reference point ϵS_0^n. Let $\varphi \epsilon \pi_{q-1} [F_0^p(X,x), a_0]$ be represented[5] by a map $f(E^p \times S^{q-1}) \subset X$ such that $f(S^{p-1} \times S^{q-1}) = f(E^p \times z_0) = x$. Let $I_0(f)$ be the map of S_0^n into X such that

$$[I_0(f_0)](y,z) = \begin{cases} f(y,z) & ((y,z) \epsilon E^p \times S^{q-1}), \\ x & ((y,z) \epsilon S^{p-1} \times E^q). \end{cases}$$

Then $I_0(f)$ represents an element $I_0(\varphi) \epsilon \pi_n(X,x)$. Since g depends continuously on f, $I_0(\varphi)$ depends only on φ and not on the choice of representative.

THEOREM (2.9). *I_0 maps $\pi_{q-1}[F_0^p(X, x), a_0]$ isomorphically on the whole of $\pi_n(X,x)$.*

Let E_1^q, E_2^q be the subsets of E^q defined by $z_q \geqq 0$, $z_q \leqq 0$, respectively, $z_0 = (1, 0, \cdots 0) \epsilon E_1^q \cap E_2^q \cap S^{q-1}$, $E^{q-1} = E_1^q \cap E_2^q$, $K_i^{q-1} = S^{q-1} \cap E_i^q$ $(i = 1,2)$. Then $E^n = E^p \times E^{q-1}$ separates S_0^n into two cells K_1^n and K_2^n, where $K_i^n = E^p \times K_i^{q-1} \cup S^{p-1} \times E_i^q$ $(i = 1, 2)$. Let $\varphi_i \epsilon \pi_{q-1}[F_0^p(X, x), a_0]$ be represented by maps $f_i(E^p \times S^{q-1}) \subset X$ with $f_i(S^{p-1} \times S^{q-1}) = f_1(E^p \times K_2^{p-1}) = f_2(E^p \times K_1^{q-1}) = x$. Then $\varphi_0 = \varphi_1 + \varphi_2$ is represented by f_0, where $f_0 \mid E^p \times K_i^{q-1} = f_i \mid E^p \times K_i^{q-1}$ $(i = 1, 2)$. Since $I_0(f_0) \mid K_i^n = I_0(f_i) \mid K_i^n$, while $[I_0(f_1)](K_2^n) = [I_0(f_2)](K_1^n)] = x$, it follows that $I_0(f_0)$ represents $I_0(\varphi_1) + I_0(\varphi_2)$, so that $I_0(\varphi_0) = I_0(\varphi_1) + I_0(\varphi_2)$ and I_0 is a homomorphism.

To complete the proof of the theorem, we first observe that the set $H = E^p \times z_0 \cup S^{p-1} \times E^q$ is contractible over itself to a point. Hence there exists a deformation $\rho(S_0^n \times I^1) \subset S_0^n$ which contracts H over itself to w_0 keeping w_0 fixed. Let $g_0(S_0^n) \subset X$ with $g_0(w_0) = x$ represent an arbitrary element of $\pi_n(X,x)$, and define $g_t(w) = g[\rho(w,t)]$ $(w \epsilon S_0^n, 0 \leqq t \leqq 1)$. Then g_t defines a homotopy of g_0 keeping w_0 at x to g_1. But $g_1 = I_0[g_1 \mid E^p \times S^{q-1}]$. Finally suppose that $f(E^p \times S^{q-1}) \subset X$ is a representative of $\varphi \epsilon \pi_{q-1} [F_0^p(X, x), a_0]$ such that $I_0(\varphi) = 0$, and let $h(S_0^n \times I^1) \subset X$ be a homotopy of $I_0(f)$ to x keeping w_0 at x. Then

$$h'(w,t) = \begin{cases} h[\rho(w,2t),0] & (0 \leqq t \leqq \tfrac{1}{2}, w \epsilon S_0^n) \\ h[\rho(w,1), 2t-1] & (\tfrac{1}{2} \leqq t \leqq 1, w \epsilon S_0^n), \end{cases}$$

deforms $I_0(f)$ to x keeping H at x. This concludes the proof of the theorem.

Let $I_\alpha[\pi_{q-1}(F_\alpha^p(X,x)), a] \subset \pi_n(X,x)$ be defined by $I_\alpha = I_0 H_\alpha$. Then

(2.10) *I_α is an isomorphism of $\pi_{q-1}[F_\alpha^p(X,x), a]$ with the whole of $\pi_n(X,x)$*

We shall refer to I_α as the *Hurewicz isomorphism.*

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[5] Any map $f(Z) \subset G^p(X)$ determines a map $f^*(E^p \times Z) \subset X$ by the rule $f^*(y, z) =$ the image of y in X under the mapping $f(z)$. For notational convenience we shall not distinguish between f and f^*.

3. The products $[\alpha, \beta]$

We now introduce the products defined by J. H. C. Whitehead [16] and characterize them as operations in the function spaces G^p. Applications are made to the homotopy groups of spheres; in particular, it is shown that the "Einhängung" [8] of a product is always zero.

Let $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$ be represented by maps $a(E^p) \subset X$, $b(E^q) \subset X$, such that $a(S^{p-1}) = b(S^{q-1}) = x_0$. Then $[\alpha,\beta]$ is the element of $\pi_n(X)$ represented by

$$h(y,z) = \begin{cases} a(y) & (y \in E^p, z \in S^{q-1}), \\ b(z) & (y \in S^{p-1}, z \in E^q). \end{cases}$$

(3.1) *The products satisfy the distributive laws:*

$$[\alpha,\beta_1 + \beta_2] = [\alpha,\beta_1] + [\alpha,\beta_2] \qquad (q > 1),$$

$$[\alpha_1 + \alpha_2 , \beta] = [\alpha_1 , \beta] + [\alpha_2 , \beta] \qquad (p > 1);$$

Moreover,

$$[\beta, \alpha] = (-1)^{pq}[\alpha, \beta]. \quad [16]$$

Thus for fixed α, the correspondence $\beta \to [\alpha, \beta]$ is a homomorphism ρ_α of $\pi_q(X,x)$ into $\pi_n(X,x)$.

Let η_α be the natural homomorphism of $\pi_q[G_\alpha^p(X), F_\alpha^p(X, x), a]$ into $\pi_{q-1}[F_\alpha^p(X, x), a]$. Consider the following diagram:

$$\begin{array}{ccc} \pi_q(G_\alpha , F_\alpha) & \xrightarrow{\eta_\alpha} & \pi_{q-1}(F_\alpha) \\ \updownarrow \tau_\alpha & & \updownarrow I_\alpha \\ \pi_q(X) & \xrightarrow{\rho_\alpha} & \pi_n(X) \end{array}$$

The principal result of this section is:

THEOREM (3.2). $I_\alpha \eta_\alpha = \rho_\alpha \tau_\alpha$

We first state a preliminary lemma. Let $\varphi \in \pi_{q-1}(F_\alpha^p)$ be represented by $f(E^p \times S^{q-1}) \subset X$ with $f(S^{p-1} \times S^{q-1}) = x_0$ and $f \mid E^p \times z_0 = a$, the reference point for $\pi_{q-1}(F_\alpha^p)$. Define

$$g(y,z) = \begin{cases} f(y,z) & (y \in E^p, z \in S^{q-1}), \\ x_0 & (y \in S^{p-1}, z \in E^q). \end{cases}$$

Then

LEMMA (3.3). *The map $g(S_0^n) \subset X$ represents $I_\alpha \varphi \in \pi_n(X)$.*

We can assume $f(z) = f_0(z) + a$, where f_0 represents the element $H_\alpha \varphi \in \pi_{q-1}$-$(F_0^p)$. By a repetition of the argument in Theorem (2.8) with p and q interchanged, it is easy to see that the element $\theta \in \pi_n(X)$ represented by g is the sum of $I_0 H_\alpha \varphi = I_\alpha \varphi$ and the element $\psi \in \pi_n(X)$ represented by the map

$$h(y,z) = \begin{cases} x & (y \in E_1^p,\ z \in S^{q-1}), \\ a(y) & (y \in E_2^p,\ z \in S^{q-1}), \\ x & (y \in S^{p-1},\ z \in E^q). \end{cases}$$

Clearly $\psi = [\alpha,0] = 0$. This establishes the lemma.

To prove Theorem (3.2), let $\nu \in \pi_q(G_\alpha^p, F_\alpha^p)$, $\beta = \tau_\alpha \nu$, $\varphi = I_\alpha^{-1}[\alpha, \beta]$. Choose for reference point in F_α^p a map such that $a(y) = x_0(\,|\,y\,|\, \geqq \frac{1}{2})$. Let $a_0(y) = a(y/2)$ $(0 \leqq |\,y\,| \leqq 1)$. Choose a representative b of β such that b maps the segment $(0,z_0)$ into x_0. Then $[\alpha,\beta]$ is represented by the map

$$f(y,z) = \begin{cases} a_0(y) & (y \in E^p,\ z \in S^{q-1}), \\ b(z) & (y \in S^{p-1},\ z \in E^q). \end{cases}$$

Let ζ be the mapping of S_0^n on itself defined by

$$\zeta(y,z) = \begin{cases} (2y,z) & (\,|\,y\,| \leqq \frac{1}{2}, |\,z\,| = 1) \\ (y/\,|\,y\,|\,,\ [2 - 2\,|\,y\,|\,]z) & (\,|\,y\,| \geqq \frac{1}{2}, |\,z\,| = 1), \\ (y,0) & (\,|\,y\,| = 1, |\,z\,| \leqq 1). \end{cases}$$

Then ζ is homotopic to the identity so that during the homotopy the point w moves along the segment $y_0 \times (0,z_0)$. Hence $f\zeta$ is homotopic keeping w_0 at x_0 to f, so that $f\zeta$ represents $[\alpha,\beta]$. Since $f\zeta\ (S^{p-1} \times E^q) = x_0$, it follows from the lemma that $g = f\zeta \mid E^p \times S^{q-1}$ represents $I_\alpha^{-1}[\alpha,\beta]$. We now extend g to a map $h(E^p \times E^q) \subset X$ by defining

$$h(y, [1 - t]z) = \begin{cases} a_0(2y) & (\,|\,y\,| \leqq \frac{1}{2},\quad |\,z\,| = 1,\quad 0 \leqq t \leqq 1) \\ x_0 & \left(\frac{1}{2} \leqq |\,y\,| \leqq \dfrac{1+t}{2},\quad |\,z\,| = 1\right), \\ b([2 - 2\,|\,y\,| + t]z) & \left(\dfrac{1+t}{2} \leqq |\,y\,| \leqq 1,\quad |\,z\,| = 1\right). \end{cases}$$

Then $h(S^{p-1}, [1-t]z) = b(tz)$, so that h defines a map h^* of E^q into G_α^p, while $h^*(S^{q-1}) \subset F_\alpha^p$ and $h^*(z_0) = a$. Moreover, $\tau_\alpha h^* = b$, so that h represents the element $\tau_\alpha^{-1}\beta = \nu$. Hence $\eta_\alpha(\nu) = I_\alpha^{-1}[\alpha, \tau_\alpha(\nu)]$, as was to be proved.

Theorem (3.2) can be restated as follows:

(3.4)* *If* $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, *then* $[\alpha, \beta] = I_\alpha[\eta_\alpha(\tau_\alpha^{-1}(\beta))]$. Since $\tau_\alpha^{-1}(\beta)$ is defined and unique, this gives a new characterization of $[\alpha,\beta]$.

A mapping $f(S^p \times S^q) \subset X$ is said to be of *type* (α,β) $(\alpha \in \pi_p(X), \beta \in \pi_q(X))$ (Cf [9]) if $f \mid S^p \times z_0$ is a representative of α and $f \mid y_0 \times S^q$ is a representative of β.

Corollary (3.5). *In order that* $[\alpha,\beta] = 0$ *it is necessary and sufficient that there exist a map* $f(S^p \times S^q) \subset X$ *of type* (α,β).

COROLLARY (3.6). *If X admits a continuous multiplication with two-sided identity e (e.g., if X is a topological group), then all products $[\alpha,\beta]$ vanish.*

Let $\alpha \in \pi_p(X,e)$, $\beta \in \pi_q(X,e)$ be represented by maps $a(S^p) \subset X$, $b(S^q) \subset X$ with $a(y_0) = b(z_0) = e$. Then the map f^* of $S^p \times S^q$ into X defined by

$$f^*(y,z) = a(y) \cdot b(z)$$

has type (α,β). By the preceding Corollary, $[\alpha,\beta] = 0$.

If X is taken to be an r-sphere S^r, further results can be obtained. Freudenthal [8] has defined an operation associating with each $f \in \bar{G}^p(S^r)$ a map $E f \in \bar{F}^{p+1}(S^{r+1},\bar{x}_0)$. This element will be referred to as the "Einhängung" of f, and can be described as follows: S^{r+1} is the set of points $\{(x,t) \mid x \in E^r,\ t \in E^1, |x|^2 + t^2 = 1\}$, and S^{p+1} has a similar parametrization. Let the reference points be $\bar{y}_0 = (0, -1) \in S^{p+1}$ and $\bar{x}_0 = (0,-1) \in S^{r+1}$. Then if $f \in \bar{G}^p(S^r)$, Ef is the map $g \in \bar{F}^{p+1}(S^{r+1}, \bar{x}_0)$ defined by

$$g(y,t) = \begin{cases} ([1-t^2]^{\frac{1}{2}} f[y(1-t^2)^{-\frac{1}{2}}],t) & (|t| \neq 1), \\ (0,t) & (|t| = 1). \end{cases}$$

If K_1^{r+1} and K_2^{r+1} are the hemispheres of S^{r+1} defined by $t \geqq 0$ and $t \leqq 0$ respectively, and K_1^{p+1} and K_2^{p+1} are defined similarly, then $g(K_i^{p+1}) \subset K_i^{r+1}$ ($i = 1, 2$), while $g \mid S^p = f$. Obviously any map of S^{p+1} into S^{r+1} having these properties represents the same element of $\pi_{p+1}(S^{r+1}, \bar{x}_0)$ as g. The mapping E is an isometric imbedding of $\bar{G}^p(S^r)$ in $\bar{F}^{p+1}(S^{r+1}, \bar{x}_0)$. Since S^r is p-simple, it follows that the element $E\alpha \in \pi_{p+1}(S^{r+1}, \bar{x}_0)$ represented by Ef depends only on the element $\alpha \in \pi_p(S^r)$ represented by f. Freudenthal [8] has proved:

(3.7) *The map $E[\pi_p(S^r)] \subset \pi_{p+1}(S^{r+1})$ is a homomorphism.*

(3.8) *If $p \leqq 2r-1$ or $p = 2r$ and r is even, then $E[\pi_p(S^r)] = \pi_{p+1}(S^{r+1})$.*

(3.9) *If $p < 2r-1$, then $E^{-1}(0) = 0$.*

The imbedding $E[\bar{G}^p(S^r)] \subset \bar{F}^{p+1}(S^{r+1}, \bar{x}_0)$ defines (in many ways, depending on the choice of the correspondence between G^p and $\bar{G}^p$ on the one hand, and between F^{p+1} and $\bar{F}^{p+1}$ on the other) an imbedding of $G^p(S^r)$ in $F^{p+1}(S^{r+1}, \bar{x}_0)$. Since $F^{p+1}(S^{r+1},\bar{x}_0)$ and $F^{p+1}(S^{r+1},x_0)$ are homeomorphic and S^{r+1} is $(p+1)$-simple, it does no harm to replace $\bar{x}_0$ by a point $x_0 \in S^r$. We shall give an explicit representation of such an imbedding. The process can be described intuitively as follows: a point $u_2 \in \operatorname{Int} I^{p+1}$ is first chosen. Then if $f \in G^p(S^r)$, the map f is extended to a map f^* of I^{p+1} in S^{r+1} such that $f^*(\sum^p) = f(\sum^{p-1})$ is a single point depending on f, while $f^*(u_2) = x_0$. The cell I^{p+1} is then "turned inside out" to obtain a map E^*f such that $E^*f(\sum^{p-1}) = x_0$, while $E^*f(u_2) = \sum^{p-1}$. The correspondence $f \to E^*f$ yields the desired imbedding. Let $u_1 = (0, 0, \cdots, \frac{1}{8})$, $u_2 = (0, 0, \cdots, \frac{7}{8}) \in I^{p+1}$. Define a map $d(S^r \times S^r \times I^1) \subset E^{r+1}$ by setting $d(x, x', t) =$ the point of E^{r+1} which separates in the ratio $t{:}1-t$ the segment xx'. For fixed x', d defines a deformation of S^r over E^{r+1} to the point x'; and the deformation d depends continuously on x'. Let

$d_i(x,x',t)$ $(i = 1, 2)$ be the vertical projection of the point $d(x, x', t)$ on K_i^{r+1}. Let $M = \sum^p \cup I^p \times (\frac{1}{4}, \frac{3}{4}) \subset I^{p+1}$, and if $f \in G^p(S^r)$, define $f^*(M) \subset S^{r+1}$ by

$$f^*(u_1, \cdots, u_{p+1}) = \begin{cases} d_1[f(u_1, \cdots, u_p), f(\sum^{p-1}), 2-4u_{p+1}] & (\frac{1}{4} \leqq u_{p+1} \leqq \frac{1}{2}), \\ d_2[f(u_1, \cdots, u_p), f(\sum^{p-1}), 4u_{p+1} - 2] & (\frac{1}{2} \leqq u_{p+1} \leqq \frac{3}{4}), \\ f(\sum^{p-1}) & ((u_1, \cdots, u_{p+1}) \in \sum^p). \end{cases}$$

If $u \in \sum^{p-1} \times (0,\frac{1}{4}) \cup I^p \times 0 \cup I^p \times \frac{1}{4}$, let f^* map the point dividing the segment uu_1 in the ratio $t{:}1-t$ into the point $d_1[f(\sum^{p-1}), x_0, t]$, and define f^* similarly over $I^p \times (\frac{3}{4},1)$. Then f^* is defined and continuous over I^{p+1}, and $f^*(\sum^p) = f(\sum^{p-1})$, while $f^*[I^p \times (0,\frac{1}{2})] \subset K_1^{r+1}$ and $f^*[I^p \times (\frac{1}{2},1)] \subset K_2^{r+1}$, and $f^*(u_2) = x_0$. Moreover, f^* depends continuously on f, and $f^* \in G^{p+1}(S^{r+1})$.

Let $\psi \in F^{p+1}(S^{p+1}, y_0)$ be an admissible map such that $\psi[I^p \times (0,\frac{1}{2})] \subset K_1^{p+1}$, $\psi[I^p \times (\frac{1}{2},1)] \subset K_2^{p+1}$, and $\psi(u_2) = \bar{y}_0$. Let $\psi' \in F^{p+1}(S^{p+1},\bar{y}_0)$ be an admissible map such that ψ' maps a cell I_0^{p+1} with $u_2 \in I_0^{p+1} \subset \operatorname{Int} I^{p+1}$ into K_1^{p+1}, ψ'-$(I^{p+1}-I_0^{p+1}) \subset K_2^{p+1}$, and $\psi'(u_2) = y_0$. Let $\psi_0 = \psi\psi'^{-1}$. Then ψ_0 is a topological map of S^{p+1} on itself of degree 1 which interchanges y_0 and $\bar{y}_0$. Since y_0 and $\bar{y}_0$ are in the arcwise connected subset $\psi_1[I^p \times (\frac{3}{4}, 1)]$, it follows that there is a homotopy $\{\psi_{0t} \mid 0 \leqq t \leqq 1\}$ of ψ_0 to the identity such that the image of $\bar{y}_0$ remains in $\psi[I^p \times (\frac{3}{4}, 1)]$ during the homotopy.

Now define $E^*f = f^*\psi^{-1}\psi'$. Then E^* is easily verified to be a topological map of $G^p(S^r)$ into $F^{p+1}(S^{r+1}, x_0)$. Moreover, E^* is equivalent to E in the sense that the mappings of $\bar{G}^p(S^r)$ into $\bar{F}^{p+1}(S^{r+1},\bar{x}_0)$ given by the following diagram are homotopic:

$$\begin{array}{ccc} \bar{G}^p(S^r) & \xrightarrow{E} & \bar{F}^{p+1}(S^{r+1}, \bar{x}_0) \\ \Big\updownarrow & & \updownarrow \\ & & F^{p+1}(S^{r+1}, x_0) \\ & & \updownarrow \\ G^p(S^r) & \xrightarrow{E^*} & F^{p+1}(S^{r+1}, x_0) . \end{array}$$

Here the map $\bar{G}^p(S^r) \leftrightarrow G^p(S^r)$ is that induced by $\psi^* = \psi \mid I^p \times \frac{1}{2}$; the map $\bar{F}^{p+1}(S^{r+1},x_0) \leftrightarrow F^{p+1}(S^{r+1},x_0)$ is that induced by ψ'; and the map $\bar{F}^{p+1}(S^{r+1},\bar{x}_0) \leftrightarrow \bar{F}^{p+1}(S^{r+1},x_0)$ is that induced by any path from $\bar{x}_0$ to x_0 (the homotopy class of the latter map is independent of the path, since S^{r+1} is simply connected). The proof of the above statement is continued in the fact that if $f \in \bar{G}^p(S^r)$, then the map g defined by $g = E^*(f\psi \mid I^p \times \frac{1}{2})$. ψ'^{-1} has the property that $g(K_i^{p+1}) \subset K_i^{r+1}$ while $g \mid S^p = f$. Hence there is a homotopy of Ef to g which is continuous in f.

The map E^* induces a homomorphism J of $\pi_{q-1}[G_\alpha^p(S_r)]$ into $\pi_{q-1}[F_{E\alpha}^{p+1}(S^{r+1})]$. Combining this with the natural homomorphism ζ of $\pi_{q-1}[F_\alpha^p(S^r)]$ into

$\pi_{q-1}[G^p_\alpha(S^r)]$ we have a homomorphism $J_0 = J\zeta$ of $\pi_{q-1}[F^p_\alpha(S^r)]$ into $\pi_{q-1}[F^{p+1}_{E\alpha}(S^{r+1})]$. This homomorphism is equivalent

$$\begin{array}{ccccc}
\pi_{q-1}[F^p_\alpha(S^r)] & \xrightarrow{\zeta} & \pi_{q-1}[G^p_\alpha(S^r)] & \xrightarrow{J} & \pi_{q-1}[F^{p+1}_{E\alpha}(S^{r+1})] \\
\updownarrow I_\alpha & & & & \updownarrow I_{E\alpha} \\
\pi_n(S^r) & & \xrightarrow{\quad E \quad} & & \pi_{n+1}(S^{r+1})
\end{array}$$

to the Freudenthal homomorphism $E[\pi_n(S^r)] \subset \pi_{n+1}(S^{r+1})$ in the sense that

THEOREM (3.10). $I_{E\alpha}J_0 = EI_\alpha$.

The first step in the proof is the observation that the imbedding of $G^p(S^r)$ in $G^{p+1}(S^{r+1})$ defined by the correspondence $f \to f^*$ maps $F^p(S^r, x_0)$ into $F^{p+1}(S^{r+1}, x_0)$ and that the induced homomorphism of $\pi_{q-1}[F^p(S^r)]$ into $\pi_{q-1}[F^{p+1}(S^{r+1})]$ is J_0. For if $f \in F^p(S^r, x_0)$, let $f^*_t = f^*\psi^{-1}\psi_{0t}\psi'$. Then $f^*_0 = f^*\psi^{-1}\psi\psi'^{-1}\psi' = f^*$, while $f^*_1 = f^*\psi^{-1}\psi' = E^*f$; and since $f \in F^p(S^r, x_0)$, then $f^*[I^p \times (0, \frac{1}{4})] = x_0$, and consequently $f^*_t(\sum^p) = f^*\psi^{-1}\psi_{0t}(\bar{y}_0) \in f^*\psi^{-1}(\psi'[\mathrm{I}^p \times (0, \frac{1}{4})]) = f^*[I^p \times (0, \frac{1}{4})] = x_0$.

Now let $\varphi \in \pi_{q-1}[F^p_\alpha(S^r, x_0)]$ be represented by a map $f(I^p \times S^{q-1}) \subset S^r$ with $f(\sum^{p-1} \times S^{q-1}) = x_0$ and $f \mid E^p \times z_0 = a$. Then $EI_\alpha\varphi$ is represented by a map $h(S^{n+1}) \subset S^{r+1}$ with $h(K^{n+1}_i) \subset K^{r+1}_i$, while $h \mid S^n_0 = I_\alpha(f)$. On the other hand, it follows from the remarks of the preceding paragraph that $J_0\varphi$ is represented by f^*, so that $I_{E\alpha}J_0\varphi$ is represented by $h' = I_{E\alpha}(f^*)$. But from the representation of $I_{E\alpha}$ given in Lemma (3.3) it follows that $h'(K^{n+1}_i) \subset K^{r+1}_i$ and $h' \mid S^n_0 = I_\alpha f$. Hence $I_{E\alpha}J_0\varphi = EI_\alpha\varphi$ as desired.

THEOREM (3.11). *If* $\alpha \in \pi_p(S^r)$, $\beta \in \pi_q(S^r)$, *then* $E[\alpha, \beta] = 0$.

Consider the diagram:

$$\begin{array}{ccccccc}
\pi_q[G^p_\alpha(S^r), F^p_\alpha(S^r)] & \xrightarrow{\eta_\alpha} & \pi_{q-1}[F^p_\alpha(S^r)] & \xrightarrow{\zeta} & \pi_{q-1}[G^p_\alpha(S^r)] & \xrightarrow{J} & \pi_{q-1}[F^{p+1}_{E\alpha}(S^{r+1})] \\
\updownarrow T_\alpha & & \updownarrow I_\alpha & & & & \updownarrow I_{E\alpha} \\
\pi_q(S^r) & \xrightarrow{\rho_\alpha} & \pi_n(S^r) & & \xrightarrow{\quad E \quad} & & \pi_{n+1}(S^{r+1})
\end{array}$$

The statement to be proved is: $E\rho_\alpha(\beta) = 0(\beta \in \pi_q(S^r)$. But, by Theorems (3.10) and (3.2) $E\rho_\alpha = (I_{E\alpha}J_0I_\alpha^{-1})(I_\alpha\eta_\alpha\tau_\alpha) = I_{E\alpha}J\zeta\eta_\alpha\tau_\alpha^{-1} = 0$, since ζ and η_α are two successive homomorphisms in $\mathfrak{N}[G^p_\alpha(S^r), F^p_\alpha(S^r), a]$, so that $\zeta\eta_\alpha = 0$.

Theorem (3.11) can be restated: $\rho_\alpha[\pi_q(S^r)] \subset E^{-1}(0)$. The converse inclusion is trivially true if $n < 2r - 1$; for then E is an isomorphism. It is also true if $n = 2r - 1$, $q = r$ is even, and $\alpha = \iota$, the element of $\pi_r(S^r)$ determined by the identity map. For the author has proved [15] that then $E^{-1}(0)$ is a cyclic group generated by an element of $\pi_{2r-1}(S^r)$ of Hopf invariant 2 [9]. But, since r is even, the element $[\iota, \iota]$ has Hopf invariant 2, and $E[\iota, \iota] = 0$. Hence any element of $E^{-1}(0)$ has the form $k[\iota, \iota] = \rho_\iota[k_\iota]$.

The equality $E^{-1}(0)$ = the union of the subgroups $\rho_\alpha[\pi_q(S^r)]$ ($\alpha \in \pi_p(S^r)$, $p + q = n + 1$) is in general false if $n > 2r - 1$. For a counter-example, see the Appendix.

THEOREM (3.12). *There exists a map of S^{2r+1} on S^{r+1} of Hopf invariant 1 if and only if $[\iota, \iota] = 0$.*

This follows from (3.5); for Eilenberg has shown [5] that there is a map of S^{2r+1} on S^{r+1} of Hopf invariant 1 if and only if there exists a map of $S^r \times S^r$ on S^r of type (ι, ι).

4. A generalization

We now define a new product which includes the Hurewicz isomorphisms and the J. H. C. Whitehead products as special cases. This product associates with an element of $\pi_{q-1}(F^p_\alpha)$ and an element of $\pi_{p-1}(F^q_\beta)$ ($\alpha \in \pi_p(X)$, $\beta \in \pi_q(x)$) a third element in $\pi_n(X)$. The new product will then be characterized in terms of the I_α and $[\beta, \nu]$.

Let $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, $\varphi \in \pi_{q-1}(F^p_\alpha, a)$, $\psi \in \pi_{p-1}(F^q_\beta, b)$, and choose representatives $a \in F^p_\alpha$, $b \in F^q_\beta$ for α, β respectively. Let φ and ψ be represented by maps $f(E^p \times S^{q-1}) \subset X$, $g(S^{p-1} \times E^q) \subset X$, with $f(S^{p-1} \times S^{q-1}) = g(S^{p-1} \times S^{q-1}) = X_0$ and $f \mid E^p \times z_0 = a$, $g \mid y_0 \times E^q = b$. Let

$$h(y, z) = \begin{cases} f(y, z) & (y \in E^p,\ z \in S^{q-1}), \\ g(y, z) & (y \in S^{p-1},\ z \in E^q). \end{cases}$$

Then $h(S^n_0) \subset X$, $h(y_0, z_0) = x_0$, so that h defines an element $\theta = (\alpha, \beta; \varphi, \psi) \in \pi_n(X)$. Evidently θ depends only on $\alpha, \beta, \varphi, \psi$, and not on the representatives chosen. Since α, β depend on φ, ψ, we may regard $(\alpha, \beta; \varphi, \psi)$ as defining a multiplication between $\pi_{q-1}(F^p_\alpha)$ and $\pi_{p-1}(F^q_\beta)$ with values in $\pi_n(X)$.

It is obvious from the definition that

$$(\alpha, \beta; 0, 0) = [\alpha, \beta]. \tag{4.1}$$

while

$$(\alpha, 0; \varphi, 0) = I_\alpha(\varphi) \tag{4.2}$$

follows from (3.3); and

$$(0, \beta; 0, \psi) = (-1)^{pq} I_\beta(\psi) \tag{4.3}$$

follows from (4.2) and the fact that $E^p \times E^q$ can be mapped on $E^q \times E^p$ with degree $(-1)^{pq}$ so that the positively oriented cells $E^p \times 0$ and $0 \times E^q$ are mapped on $0 \times E^p$ and $E^q \times 0$ with degree 1.

THEOREM (4.4). $(\alpha, \beta; \varphi, \psi) = [\alpha, \beta] + I_\alpha\varphi + (-1)^{pq} I_\beta\psi$.

We first observe that the theorem is true for $\alpha = 0$, $\beta = 0$. For let a and b be the constant maps of E^p and E^q into x_0, and choose representatives f for φ

and g for ψ such that $f \mid S^{p-1} \times S^{q-1} = g \mid S^{p-1} \times S^{q-1} = f \mid E^p \times K_2^{q-1} = g \mid S^{p-1} \times E_1^q = x_0$. Let

$$f^*(w) = \begin{cases} f(w) & (w \in E^p \times S^{q-1}), \\ x_0 & (w \in S^{p-1} \times E^q); \end{cases}$$

$$g^*(w) = \begin{cases} x_0 & (w \in E^p \times S^{q-1}), \\ g(z) & (w \in S^{p-1} \times E^q); \end{cases}$$

$$h(w) = \begin{cases} f(w) & (w \in E^p \times S^{q-1}), \\ g(w) & (w \in S^{p-1} \times E^q). \end{cases}$$

Then $f^* \mid K_2^n = g^* \mid K_1^n = x_0$, while $h \mid K_1^n = f^* \mid K_1^n$, $h \mid K_2^n = g^* \mid K_2^n$. Hence h represents the sum $I_0\varphi + (-1)^{pq} I_0\psi$ of the elements of $\pi_n(X)$ represented by f^* and g^*. On the other hand, the third equation above shows that h represents $(0, 0; \varphi, \psi)$.

Let F be a map of $S^n \cup E^n = K_1^n \cup E^n \cup K_2^n$ into X with $F(w_1) = x_0$, where $w_1 \in S^{p-1} \times S^{q-1} \cap E^{q-1}$. Let $h_i (i = 1, 2)$ be an admissible map of S^n on $K_i^n \cup E^n$ which is the identity on K_i^n and such that $h_i(E^p \times K_i^{q-1}) = E^p \times E^{q-1}$. Let $F_0 = F \mid S^n$, $F_i = Fh_i$ $(i = 1, 2)$ map S^n into X and let α_0, α_1, α_2 be the elements of $\pi_n(X)$ represented by F_0, F_1, F_2 respectively. Then it is well-known (cf. [16]) that $\alpha_0 = \alpha_1 + \alpha_2$.

To complete the proof of the theorem, choose the reference point for S^n as above. Choose a representative f for φ such that $f(y, z) = a(y)$ for $y \in E^p$, $z \in K_1^{q-1}$, and a representative g for ψ such that $g(y, z) = b(z)$ for $(y, z) \in S^{p-1} \times E_1^q$, where a is the reference point for F_α^p and b, the reference point of F_β^q, is such that $b(E_2^q) = x_0$. Then if σ is a topological map of E^q on E_1^p, and $g' = g \mid S^{p-1} \times E_1^q$, $g''(y, z) = g'(y, \sigma(z))$, it is evident that g'' represents the element $H_\beta\psi$.

Let

$$h(y, z) = \begin{cases} f(y, z) & ((y, z) \in E^p \times S^{q-1}), \\ g(y, z) & ((y, z) \in S^{p-1} \times E^q), \\ (a(y) & ((y, z) \in E^p \times E^{q-1}). \end{cases}$$

Then $h(K_1^n \cup K_2^n \cup E^n) \subset X$, $h(w_1) = x_0$. On $K_1^n \cup E_n$,

$$h(y, z) = \begin{cases} a(y) & ((y, z) \in E^p \times K_1^{q-1}) \\ b(z) & ((y, z) \in S^{p-1} \times E_1^q) \\ a(y) & ((y, z) \in E^p \times E^{q-1}). \end{cases}$$

Clearly hh_1 represents $[\alpha, \beta]$. On $K_2^n \cup E^n$,

$$h(y, z) = \begin{cases} f(y, z) & ((y, z) \in E^p \times K_2^{q-1}), \\ g(y, z) & ((y, z) \in S^{p-1} \times E_2^q), \\ a(y) & ((y, z) \in E^p \times E^{q-1}) \end{cases}$$

Evidently $hh_2 \mid E^p \times S^{q-1}$ represents φ, while $hh_2 \mid S^{p-1} \times E^q$ represents $H_\beta\psi$. Hence hh_2 represents $(\alpha, 0;\ \varphi, H_\beta\psi)$, and it follows that $h \mid K_1^n \cup K_2^n$ represents $[\alpha, \beta] + (\alpha, 0; \varphi, H_\beta\psi)$. Hence $[\alpha, \beta] + (\alpha, 0; \varphi, H_\beta\psi) = (\alpha, \beta; \varphi, \psi)$. Similarly, $(\alpha, 0; \varphi, H_\beta\psi) = [\alpha, 0] + (0, 0; H_\alpha\varphi, H_\beta\psi) = I_0H_\alpha\varphi + (-1)^{pq}I_0H_\beta\psi$. Combining these results we have the desired equation

$$(\alpha, \beta; \varphi, \psi) = [\alpha, \beta] + I_\alpha\varphi + (-1)^{pq}I_\beta\psi.$$

5. The Generalized Products of J. H. C. Whitehead

In this section we shall prove that the generalized products of J. H. C. Whitehead can be characterized in terms of known operations. Using this characterization we can then verify a conjecture of Whitehead's.

Let R_{p-1} be the proper orthogonal group in p variables; R_{p-1} may be considered as a group of transformations of E^p (or alternatively of S^{p-1}). Let $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, $\varphi \in \pi_{q-1}(R_{p-1})$, $\psi \in \pi_{p-1}(R_{q-1})$. Then the generalized product $\{\alpha, \beta; \varphi, \psi\}$ is the element of $\pi_n(X)$ defined by

$$h(y, z) = \begin{cases} a[f(y, z)] & (y \in E^p,\ z \in S^{q-1}) \\ b[g(y, z)] & (y \in S^{p-1},\ z \in E^q), \end{cases}$$

where a, b, f, g are representatives of α, β, φ, ψ respectively. [16]

Since each element of R_{p-1} transforms S^{p-1} on itself with degree 1, we may assume $R_{p-1} \subset G_\iota^{p-1}(S^{p-1})$. Let L be the natural homomorphism of $\pi_{q-1}(R_{p-1})$ in $\pi_{q-1}[G_\iota^{p-1}(S^{p-1})]$, $H = I_{\iota'}JL$. The homomorphism H has been considered by the author [15] in another connection. If $\varphi \in \pi_{q-1}(R_{q-1})$ is represented by $f(S^{p-1} \times S^{q-1}) \subset S^{p-1}$, then $H\varphi$ is represented by a map $f^*(S^n) \subset S^r$ with $f^*(E^p \times S^{q-1}) \subset K_1^r$, $f^*(S^{p-1} \times E^q) \subset K_2^r$, and $f^* \mid S^{p-1} \times S^{q-1} = f$.

Theorem (5.1). $\{\alpha, \beta; \varphi, \psi\} = [\alpha, \beta] + \alpha\cdot H\varphi + (-1)^{pq}\beta\cdot H\psi$.[6]

Let $f(E^p \times S^{q-1}) \subset E^p$ be a representative of φ; let $\bar\zeta$ be the map $\zeta \mid E^p \times S^{q-1}$, where ζ is the mapping defined in the proof of (3.2); finally, let ψ be a map of E^p on S^p of degree 1 such that $\psi(S^{p-1}) = \zeta_0$ and $\psi \mid E^p - S^{p-1}$ is topological. Define $f_0 = \psi f\cdot\bar\zeta^{-1}$. Although $\bar\zeta^{-1}$ is not defined everywhere, f_0 is a well-defined and continuous map of S^n into S^p which represents the element $H\varphi$. Let $a_0 = a\psi^{-1}$; then a_0 is well-defined and continuous and maps S^p into X. Let $\bar f = a_0f_0$; then $\bar f$ represents $\alpha\cdot H\varphi$. There is a homotopy ζ_t of ζ to the identity such that ζ_t is a homeomorphism $(0 < t \leqq 1)$ and such that $\zeta_t(S^{p-1} \times E^q) \subset S^{p-1} \times E^q$. Hence $\bar f = af\bar\zeta^{-1}$ is homotopic to the map h_1 defined by

$$h_1 = \begin{cases} a[f(y, z)] & ((y, z) \in E^p \times S^{q-1}), \\ x_0 & ((y, z) \in S^{p-1} \times E^q). \end{cases}$$

[6] If $\alpha \in \pi_n(S^r)$, $\beta \in \pi_r(X)$, then $\beta\cdot\alpha$ denotes the element of $\pi_n(X)$ represented by the composite map gf, where g, f are representatives of β, α respectively.

Then $h_1 \mid E^p \times S^{q-1} = h \mid E^p \times S^{q-1}$ represents $I_\alpha^{-1}[\alpha \cdot H(\varphi)]$. Similarly $h \mid S^{p-1} \times E^q$ represents $(-1)^{pq} I_\beta^{-1}[\beta \cdot H(\psi)]$. Hence, by (4.4), h represents the element

$$(\alpha, \beta, I_\alpha^{-1}[\alpha \cdot H(\varphi)], I_\beta^{-1}[\beta \cdot H(\psi)]) = [\alpha, \beta] + \alpha \cdot H(\varphi) + (-1)^{pq}\beta \cdot H(\psi).$$

Consider the special case $X = S^p$, $\alpha = \iota'$, $\beta = 0$, $q = 2$, $p > 1$. The author has shown [15] that H maps $\pi_1(R_{p-1})$ isomorphically on $\pi_{p+1}(S^p)$. Hence

(5.2) *The correspondence* $\varphi \rightarrow \{\iota', 0; \varphi, 0\}$ *is an isomorphism of* $\pi_1(R_{p-1})$ *with* $\pi_{p+1}(S^p)$.

This verifies a conjecture of J. H. C. Whitehead [16].

Appendix

Example 1. We shall prove that $\pi_2[G_0^2(S^2)]$ is the direct sum of an infinite cyclic group and a group of order 2, while $\pi_2[G_\iota^2(S^2)]$ is a group with at most two elements. Hence $G_0^2(S^2)$ and $G_\iota^2(S^2)$ do not have the same homotopy type.

We first observe that $\pi_2[F_0^2(S^2)] = \pi_2[F_\iota^2(S^2)] = \pi_4(S^2)$ is a group with two elements [8; 12]. Since there is a map λ of S^2 into $G_0^2(S^2)$ such that $\tau_0\lambda$ is the identity, it follows [2, p. 170] that $\pi_2[G_0^2(S^2)] = \pi_2[F_0^2(S^2)] + \pi_2(S^2)$, and $\pi_2(S^2)$ is infinite cyclic. This proves the first assertion.

It is now sufficient to prove that every map of S^2 into G_ι^2 is contractible into F_ι^2. Choose for reference point the identity map i and consider the homomorphism $\zeta[\pi_2(F_\iota^2)] \subset \pi_2(G_\iota^2)$, $\eta[\pi_2(G_\iota^2)] \subset \pi_2(G_\iota^2, F_\iota^2)$, $\xi[\pi_2(G_\iota^2, F_\iota^2)] \subset \pi_1(F_\iota^2)$, associated with $\mathfrak{N}(G_\iota^2, F_\iota^2, i)$. To prove that $\xi[\pi_2(\mathrm{F}_\iota^2)] = \pi_2(G_\iota^2)$ as desired, it is sufficient, by (2.3) to prove that $\eta[\pi_2(G_\iota^2)] = 0$; again by (2.3) it is sufficient to prove that ζ is an isomorphism. The group $\pi_2(G_\iota^2, F_\iota^2) = \pi_2(S^2)$ is infinite cyclic, and since $R_2 \subset G_\iota^2$, $R_1 \subset F_\iota^2$, a generator for the former group is determined by a generator α of $\pi_2(R_2, R_1)$. This has been constructed independently by Eckmann [3] and the author [15], and it is proved in the references cited that $\zeta(\alpha) = 2\beta$, where β is a generator of the infinite cyclic group $\pi_1(R_1)$. But $\pi_1(R_1)$ is mapped isomorphically on $\pi_1(F_\iota^2) = \pi_3(S^2)$[15]. The second assertion then follows immediately. (It would not be difficult to prove more; namely, that $\pi_2(G_\iota^2)$ contains exactly two elements).

2. We shall construct an element $\alpha \in \pi_4(S^2)$ such that $E\alpha = 0$, and $\alpha \neq [\beta, \iota]$ for any $\beta \in \pi_3(S^2)$. The group $\pi_4(S^2)$ is cyclic of order 2; let α be its non-trivial element. Since $\pi_5(S^3) = 0$ [12], $E\alpha = 0$. We must then prove $[\beta, \iota] = 0$ for all $\beta \in \pi_3(S^2)$. This is equivalent by (3.5) to proving that for every $\beta \in \pi_3(S^2)$ there is a map $S^3 \times S^2$ into S^2 of type (β, ι). But there exists a map ν of S^3 into R_2 (the double covering of the projective 3-space R_2 by S^3) [3; 14] whose projection by $\tau_\iota \mid R_2$ into S^2 is the Hopf map of invariant 1. Then if β has Hopf invariant k, the element $k\nu \in \pi_3(R_2)$ defines the required map.

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