ANNALS OF MATHEMATICS
Vol. 43, No. 1, January, 1942

HOMOTOPY PROPERTIES OF THE REAL ORTHOGONAL GROUPS

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(Received September 25, 1941)

1. Introduction

In this paper we propose to investigate the topological structure of the rotation group R_n of the n-sphere, with special emphasis on its homotopy properties. Of particular interest are the homotopy groups π_i of R_n. These groups, one for each dimension i, were first defined for a general space by Hurewicz [1].[1] Like the homology groups, they are topological invariants of a space; unlike the homology groups, however, no general method for computing them is known. Each space thus presents a problem in itself.

The computation of the groups $\pi_i(R_n)$ for $i \leqq 5$ and all n will be carried out by the method of fibre mappings and covering homotopies developed by Hurewicz and Steenrod [2]. We shall make extensive use of the results of Freudenthal [3], Hopf [4], and Pontrjagin [5] on the homotopy groups of spheres.

The groups $\pi_i(R_n)$ are useful not only in the study of the homotopy properties of spheres, but also are used by Whitney in his theory of sphere-bundles [6, 7], where they appear as coefficient groups for certain cocycle invariants.

Another application of our results appears in the theory of continuous vector fields over spheres. It is well known that no continuous field of unit vectors can be defined over the spheres of even dimension. Over the odd-dimensional spheres, however, one such vector field can always be defined; and if $n \equiv 3 \pmod 4$ or $n \equiv 7 \pmod 8$ it is possible to define three or seven independent vector fields, respectively, over the n-sphere S^n. These can be readily constructed by the use of the multiplication matrices for quaternions and Cayley numbers. For a general odd n, however, there is no known result on the maximum number of independent vector fields which can be defined over S^n.

In this paper the case $n \equiv 1 \pmod 4$ is resolved as follows: *Any two vector fields over* S^{4m+1} $(m = 0, 1, 2, \cdots)$ *are somewhere dependent.* As a corollary to this result it is observed that *the tangent sphere-bundle of* S^n *is not simple if* $n > 1$ *and* $n \equiv 1 \pmod 4$.

This investigation was carried out under the direction of Prof. N. E. Steenrod, to whom the author wishes to acknowledge his indebtedness for many valuable suggestions and criticisms.

2. Table of groups π_1 to π_5 of R_n

In this section the results of our computation of the homotopy groups $\pi_i(R_n)$ are exhibited, and a set of generators for these groups is given. Proofs will be deferred until Section 8.

[1] Numbers in square brackets refer to the bibliography at the end of the paper.

Let ∞ denote the free cyclic group, 2 the cyclic group of order two. If A and B are two abelian groups, $A + B$ denotes their direct sum. In terms of these notations, the groups $\pi_i(R_n)$ may be tabulated as follows:

	R_1	R_2	R_3	R_4	R_5	R_6	$\cdots$	R_n	$\cdots$
π_1	∞	2	2	2	2	2	$\cdots$	2	$\cdots$
π_2	0	0	0	0	0	0	$\cdots$	0	$\cdots$
π_3	0	∞	$\infty + \infty$	∞	∞	∞	$\cdots$	∞	$\cdots$
π_4	0	2	$2 + 2$	2	0	0	$\cdots$	0	$\cdots$
π_5	0	0	0	0	∞	0	$\cdots$	0	$\cdots$

The results of the first two rows were first obtained by Cartan [8].

A generator of $\pi_1(R_n)$ is given by the map of the circle $x_1^2 + x_2^2 = 1$ defined by

$$x \to \begin{Vmatrix} x_1 & -x_2 & 0 \\ x_2 & x_1 & 0 \\ 0 & 0 & I_{n-1} \end{Vmatrix},$$

where I_{n-1} is the $(n - 1)$-rowed identity matrix.

A generator of $\pi_3(R_n)$ $(n \geqq 3)$ is given by the map of the 3-sphere $\sum_{i=1}^{4} x_i^2 = 1$

$$x \to \begin{Vmatrix} x_1 & -x_2 & -x_3 & -x_4 & 0 \\ x_2 & x_1 & -x_4 & x_3 & 0 \\ x_3 & x_4 & x_1 & -x_2 & 0 \\ x_4 & -x_3 & x_2 & x_1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-3} \end{Vmatrix}.$$

The corner matrix is the linear transformation of Euclidean 4-space defined by multiplying every quaternion on the left by $x_1 + ix_2 + jx_3 + kx_4$.

The generator of $\pi_3(R_2)$ is the well-known double covering of R_2 by S^3. Bordering this matrix with a 1 in the lower right hand corner and zeros elsewhere, we obtain the extra generator of $\pi_3(R_3)$.

To obtain the generator of $\pi_4(R_2)$, we map S^4 on S^3 essentially, and then map S^3 into R_2 by means of the generator of $\pi_3(R_2)$ given above. Such an essential map was constructed by Freudenthal [3]. In a similar manner we obtain generators for $\pi_4(R_3)$ and $\pi_4(R_4)$.

Finally, the generator of $\pi_5(R_5)$ is determined by the map of S^5 into R_5 given by

$$x \to \| \delta_{ij} - 2x_i x_j \| \cdot \begin{Vmatrix} I_5 & 0 \\ 0 & -1 \end{Vmatrix}.$$

3. Preliminary notions

In this section we introduce notations and concepts which will occur throughout the paper. The relative and absolute homotopy groups of a space are introduced and two homomorphisms relating these groups are discussed.

Let points x in Euclidean $(n + 1)$-space be referred to coördinates $(x_1, x_2, \cdots, x_{n+1})$. The unit sphere $\sum x_i^2 = 1$ we denote by S^n. The *equatorial plane* $x_{n+1} = 0$ divides S^n into two hemispheres V_1^n and V_2^n defined by the inequalities $x_{n+1} \geqq 0$ and $x_{n+1} \leqq 0$ respectively. If $x = (x_1, x_2, \cdots, x_{n+1}) \in S^n$, the *antipodal point* $(-x_1, -x_2, \cdots, -x_{n+1})$ is denoted by $\tilde{x}$. We shall refer to the point $x^o = (0, 0, \cdots, 1)$ as the *north pole*, and to its antipode $\tilde{x}^o$ as the *south pole*.

The group R_n of all rotations of S^n may be represented as the group of all real square orthogonal matrices of order $n + 1$ with determinant $+1$. The subgroup of R_n consisting of all those rotations of S^n which leave the north pole fixed is isomorphic with the group R_{n-1}, and we shall denote the former group also by the symbol R_{n-1}.

Let Y be a topological space, F a closed subset of Y, and y_o a fixed point of F. Let $\mathfrak{X}$ denote the space of all maps of V_1^n into Y which carry the boundary $\partial V_1^n = S^{n-1}$ into F and the north pole x^o of S^{n-1} into y_o. We introduce an equivalence relation in $\mathfrak{X}$ as follows: two maps $f_1, f_2 \in \mathfrak{X}$ are said to be *equivalent* if they are homotopic, and during the homotopy S^{n-1} remains in F and x^o remains at y_o. In other words, two points of $\mathfrak{X}$ are equivalent if they can be joined by an arc in $\mathfrak{X}$. The relation of equivalence is easily seen to be reflexive, symmetric, and transitive, and thus divides $\mathfrak{X}$ into classes of equivalent maps, called *homotopy classes*. The homotopy class determined by a map f we denote by $\{f\}$; the set of all such homotopy classes by $\pi_n(Y, F)$. Hurewicz [1] has introduced an operation, called addition, in $\pi_n(Y, F)$, by means of which it becomes a group, the n^{th} *relative homotopy group of* Y *modulo* F. If the closed set F is specialized to consist only of the point y_o, the group $\pi_n(Y, y_o)$ so obtained is called the *absolute homotopy group* $\pi_n(Y)$. The latter group may also be defined by means of mappings of spheres into Y and we shall frequently find it convenient to use this definition.

We now introduce a homomorphism ω of $\pi_n(Y, F)$ into $\pi_{n-1}(F)$. This homomorphism is defined as follows: if $\alpha = \{f\} \in \pi_n(Y, F)$, then $\omega(\alpha)$ denotes the homotopy class of $\pi_{n-1}(F)$ determined by the map $f(S^{n-1}) \subset F$. Evidently $\{f\} = \{g\}$ implies $\omega(\{f\}) = \omega(\{g\})$. Thus ω maps $\pi_n(Y, F)$ into $\pi_{n-1}(F)$, and it follows from the definition of addition that ω is a homomorphism. Let $\pi_{no}(Y, F)$ denote the kernel of this homomorphism, $\pi_{n-1,o}(F)$ the image of $\pi_n(Y, F)$ under ω. Evidently $\pi_{no}(Y, F)$ consists of those homotopy classes determined by those relative n-cells in Y modulo F which are contractible into F, while $\pi_{n-1,o}(F)$ consists of the classes determined by those $(n - 1)$-spheres in F which are homotopic to points in Y.

Since each element of $\pi_n(Y)$, considered as a set, is a subset of an element of $\pi_n(Y, F)$, we have a natural mapping ψ of $\pi_n(Y)$ into $\pi_n(Y, F)$, which is a homomorphism. It is not hard to show that $\psi(\{f\}) = 0$ if and only if some f' in the class of f maps S^n into F. If $\{f_1\} = \{f_2\}$ and $\psi(\{f_1\}) = \psi(\{f_2\}) = 0$, then f_1' is homotopic in Y to f_2'. Thus f_1' and f_2' determine the same element of $\pi_n(F)/\pi_{no}(F)$. Conversely, if $f(S^n) \subset F$, then $\psi(\{f\}) = 0$. Hence the kernel of the homomorphism ψ is isomorphic to $\pi_n(F)/\pi_{no}(F)$. The image of $\pi_n(Y)$ under ψ is evidently the group $\pi_{no}(Y, F)$. We summarize these results in

THEOREM 1. *The natural homomorphic maps* $\omega[\pi_n(Y, F)] \subset \pi_{n-1}(F)$ *and* $\psi[\pi_n(Y)] \subset \pi_n(Y, F)$ *are related as follows: the kernel of the homomorphism* ψ *is isomorphic to* $\pi_n(F)/\pi_{no}(F)$, *while the image of* $\pi_n(Y)$ *under* ψ *is the group* $\pi_{n,o}(Y,F)$.

4. Homotopy relations in compact Lie groups

Let G be a topological group, H a closed subgroup of G, and $B = G/H$ the space of left (or right) cosets of H in G. Then there is a natural mapping π of G onto B defined as follows: for every $g \in G$, $\pi(g)$ is the coset of B containing g.

If G is a compact Lie group, then π is a fibre map in the sense of Hurewicz and Steenrod [2]. A slicing function can be defined as follows: a plane of maximum dimension independent of the tangent plane to H at the identity 1 meets each coset b in a sufficiently small neighborhood U of $b_o = \pi(1)$ just once. We denote this point by $\phi(1, b)$. Then if $g^{-1}b \in U$, let $\phi(g, b) = g\phi(1, g^{-1}b)$. Evidently ϕ has all the required properties of a slicing function.

The following theorem will be useful in our discussion of the rotation groups:

THEOREM 2. *If* G *is a topological group and* B *is the space of left (or right) cosets of a closed subgroup* H *of* G, *and if there exists a map* $f(B) \subset G$ *such that* $\pi f(b) = b$, *then* G *is homeomorphic with the product space* $H \times B$.

We may suppose B is a space of left cosets. Let $f'(b) = f(b)\cdot[f(b_o)]^{-1}$; then $\pi f'(b) = \pi f(b) = b$, and $f'(b_o) = 1$. We then set up the homeomorphism by means of two maps $p(G) = H \times B$ and $q(H \times B) = G$ defined as follows:

$$p(g) = [g^{-1}\cdot f'(\pi g), \pi g] \qquad (g \in G),$$

$$q(h, b) = f'(b)\cdot h^{-1} \qquad (h \in H, b \in B).$$

Then

$$p[q(h, b)] = \{h[f'(b)]^{-1}f'(b), b\} = (h, b),$$

$$q[p(g)] = f'(\pi g)\cdot[f'(\pi g)]^{-1}\cdot g = g,$$

and both maps are continuous. Hence G and $H \times B$ are homeomorphic.

5. Slicing functions for $R_n \to R_n/R_{n-1}$

In this section the results of Section 4 are applied to the special case $G = R_n$, $H = R_{n-1}$, and an explicit slicing function is constructed. The special cases $n = 1, 3, 7$ are treated separately, and for these values of n Theorem 2 is applied.

Let us consider the mapping $\pi(R_n) = S^n$ defined by $\pi(r) = r(x^o)$ $(r \in R_n)$. As shown by Hurewicz and Steenrod [2], π is a fibre map of R_n into S^n, the fibres being the left cosets of R_{n-1} in R_n. In terms of the matrix representation of R_n, $\pi(r)$ is the last column of the matrix r.

We now define the slicing function promised above: if $x \neq \bar{x}^o$, let $\phi(I, x)$ be the rotation carrying x^o along a great circle into x, and leaving the $(n - 2)$-sphere orthogonal to this great circle fixed. In terms of matrices

$$\phi(I, x) = A_n(x) = \left\| \delta_{ij} - \frac{(x_i + \delta_{i,n+1})(x_j + \delta_{j,n+1})}{x_{n+1} + 1} \right\| \cdot \left\| \begin{matrix} I_n & 0 \\ 0 & -1 \end{matrix} \right\| \qquad (1)$$

$$(i, j = 1, \cdots, n + 1),$$

where I_n denotes the n-rowed identity matrix. Then $\phi(r, x)$ is defined as is Section 2. Evidently it is impossible to extend $\phi(I, x)$ so as to be defined and continuous over all of S^n. We shall show later that there is no slicing function with this property for a general n.

For the dimensions 1, 3, and 7, however, it is possible to define such a slicing function. Let $\mathfrak{A}_{n+1}$ $(n = 1, 3, 7)$ denote the algebras of complex numbers, quaternions, and Cayley numbers, respectively, over the field of real numbers. By means of these algebras a multiplication $x \cdot y$ of points of Euclidean $(n + 1)$-space is defined. This multiplication has the property that $\| x \cdot y \| = \| x \| \cdot \| y \|$, where $\| x \|^2 = \sum x_i^2$ is the square of the distance of the point x from the origin. Hence S^n is closed under multiplication and for $x, y \in S^n$ we have $x \cdot y = B_n(x) \cdot y$, where $B_n(x) \in R_n$, and, if the coördinate system is chosen so that x^o is the unit of the algebra, $B_n(x^o) = I$, $\pi B_n(x) = x$. Since $B_n(x)$ is defined for all $x \in S^n$, we have

THEOREM 3. *For $n = 1, 3, 7$, R_n is a product space $R_{n-1} \times S^n$.*

Since the i^{th} homotopy group of a product space $X \times Y$ is the direct sum of the i^{th} homotopy groups of X and Y, we have

COROLLARY. *For $n = 1, 3, 7$, $\pi_i(R_n)$ is the direct sum $\pi_i(R_{n-1}) + \pi_i(S^n)$; in particular, $\pi_{n-1}(R_{n-1}) = \pi_{n-1}(R_n)$.*

6. The canonical map of S^n in R_n

We now introduce a mapping C_n of S^n into R_n which plays an important role in the following discussion. We shall refer to it as the *canonical map*. It is proved that this map is contractible into R_{n-1} if n is even; while for n odd it is not so contractible. The canonical map is then used to construct a generator for $\pi_n(R_n, R_{n-1})$.

In order to define the map C_n, let $\theta(x)(x \in S^n)$ denote the angular distance from x^o to x, and let x' be the point in the great circle through x^o and x with $\theta(x') = 2\theta(x)$. Let $C_n(x)$ $(x \neq \bar{x}^o)$ be the rotation which carries x^o along a great circle into x' and leaves the orthogonal $(n - 2)$-sphere fixed; and let $C_n(\bar{x}^o) = I$. Evidently

$$C_n(x) = [A_n(x)]^2 = \| \delta_{ij} - 2x_i x_j \| \cdot \left\| \begin{matrix} I_n & 0 \\ 0 & -1 \end{matrix} \right\| \qquad (i, j = 1, \cdots, n + 1). \qquad (2)$$

We observe that C_n, unlike A_n, is defined and continuous over all of S^n. Furthermore, antipodal points have the same image, while distinct pairs of antipodal points have distinct images. Thus the image of S^n in R_n is a projective n-space.

Let $g_n(S^n) = S^n$ denote the projection πC_n of the canonical map. If $g_{n,i}(x)$ is the i^{th} coördinate of $g(x)$, we have $g_{n,i}(x) = 2x_i x_{n+1} - \delta_{i,n+1}$ $(i = 1, \cdots, n+1)$.

THEOREM 4. *If n is even, g_n has degree zero; if n is odd, g_n has degree two.*

For g_n maps the equator S^{n-1} into $\bar{x}^o$ and maps $V_1^n - S^{n-1}$ topologically on $S^n - \bar{x}^o$; in fact, g_n can be obtained by a homotopy of S^n on itself in which each point moves along the great circle joining it to the north pole. Thus g_n maps V_1^n on S^n with degree 1. Furthermore, g_n maps V_2^n on S^n with degree $(-1)^{n+1}$; for $g_n(x) = g_n(\bar{x})$ $(x \in V_2^n)$ and the antipodal transformation $x \to \bar{x}$ has degree $(-1)^{n+1}$. Hence g_n maps S^n on itself with degree $1 + (-1)^{n+1}$.

If n is even, g_n has degree zero, and hence is homotopic to a point. We shall give a homotopy of g_n which will be useful in a later section. This homotopy $g_n(x, t)$ is given by the equations

$$\begin{aligned}
g_{n,2i-1}(x, t) &= 2\{(1-t)x_{2i-1}x_{n+1} + [t(1-t)]^{\frac{1}{2}}x_{2i}\}, \\
g_{n,2i}(x, t) &= 2\{(1-t)x_{2i}x_{n+1} - [t(1-t)]^{\frac{1}{2}}x_{2i-1}\} \qquad (i = 1, \cdots, n/2), \\
g_{n,n+1}(x, t) &= 1 - 2(1-t)(1 - x_{n+1}^2).
\end{aligned} \tag{3}$$

It is easy to verify that $g_n(x, t)$ contracts $g_n(S^n)$ over S^n into x^o.

If n is odd, g_n has degree two. We shall give a deformation of g_n into a second map g_n' of degree two. The latter map is defined by the equations

$$\begin{aligned}
g'_{n,i}(x) &= x_i \qquad (i = 1, \cdots, n-1); \\
g'_{n,n}(x) &= \frac{2x_n x_{n+1}}{(x_n^2 + x_{n+1}^2)^{\frac{1}{2}}}, \\
g'_{n,n+1}(x) &= \frac{x_{n+1}^2 - x_n^2}{(x_n^2 + x_{n+1}^2)^{\frac{1}{2}}} \qquad (x_n^2 + x_{n+1}^2 \neq 0), \\
g'_{n,n}(x) &= g'_{n,n+1}(x) = 0 \qquad (x_n = x_{n+1} = 0).
\end{aligned} \tag{4}$$

It is easily seen that g_n' is defined and continuous over all of S^n. The homotopy $g_n(x, t)$ of g_n over S^n into g_n' is given by

$$\begin{aligned}
g_{n,2i-1}(x, t) &= tx_{2i-1} + [t(1-t)]^{\frac{1}{2}}x_{2i} + 2x_{n+1}\frac{(1-t)x_{2i-1} - [t(1-t)]^{\frac{1}{2}}x_{2i}}{\{1 - t[1 - (x_n^2 + x_{n+1}^2)]\}^{\frac{1}{2}}}, \\
g_{n,2i}(x, t) &= tx_{2i} - [t(1-t)]^{\frac{1}{2}}x_{2i-1} + 2x_{n+1}\frac{(1-t)x_{2i} + [t(1-t)]^{\frac{1}{2}}x_{2i-1}}{\{1 - t[1 - (x_n^2 + x_{n+1}^2)]\}^{\frac{1}{2}}} \\
&\qquad \left(i = 1, \cdots, \frac{n-1}{2}\right), \\
g_{n,n}(x, t) &= \frac{2x_n x_{n+1}}{\{1 - t[1 - (x_n^2 + x_{n+1}^2)]\}^{\frac{1}{2}}}, \\
g_{n,n+1}(x, t) &= \frac{2x_{n+1}^2}{\{1 - t[1 - (x_n^2 + x_{n+1}^2)]\}^{\frac{1}{2}}} - \{1 - t[1 - (x_n^2 + x_{n+1}^2)]\}^{\frac{1}{2}}.
\end{aligned} \tag{5}$$

Although $g_n(x, t)$ is not defined everywhere for $t = 1$, it is easily verified that $\lim_{t \to 1} g_n(x, t) = g'_n(x)$ uniformly in x.

We now use the canonical map to construct a generator of $\pi_n(R_n, R_{n-1})$. We shall take as fixed reference points for this group the north pole of S^{n-1} and the identity matrix $I \in R_{n-1}$. Since ([2], Theorem 2) $\pi_n(R_n, R_{n-1})$ is isomorphic to $\pi_n(S^n)$ under the map $\pi(R_n) = S^n$, a generator of the former group is represented by any relative n-cell in R_n modulo R_{n-1} which projects into S^n with degree ± 1. To define such a relative n-cell, we observe that C_n maps V_1^n into R_n and S^{n-1} into the coset $\tilde{R}_{n-1}$ opposite to R_{n-1} and projects into S^n with degree 1. Hence the map $D_n(x) = C_n(x) \cdot \begin{Vmatrix} I_{n-1} & 0 \\ 0 & -I_2 \end{Vmatrix}$ $(x \in V_1^n)$ defines a relative cell in R_n modulo R_{n-1}, and it is easily verified that $D_n(x^o) = I$, while $d_n(x) = \pi D_n(x) = \tilde{g}_n(x)$ has degree $(-1)^{n+1}$. Hence D_n represents the required generator.

Since $\pi_{n-1}(R_n, R_{n-1}) = \pi_{n-1}(S^n) = 0$, it follows from the results of Section 3 that $\pi_{n-1}(R_n) = \pi_{n-1}(R_{n-1})/\pi_{n-1,o}(R_{n-1})$. Thus $\pi_{n-1}(R_n)$ is a factor group of $\pi_{n-1}(R_{n-1})$, the kernel of the homomorphism being the group $\pi_{n-1,o}(R_{n-1})$. But $\pi_{n-1,o}(R_{n-1}) = \omega[\pi_n(R_n, R_{n-1})]$; hence a generator of $\pi_{n-1,o}(R_{n-1})$ is given by the map ωD_n, which is easily shown to be the canonical map C_{n-1}. Hence

THEOREM 5. *The kernel of the homomorphism $\pi_{n-1}(R_{n-1}) \to \pi_{n-1}(R_n)$ is the subgroup of the former group generated by the canonical map.*

7. On the possibility of sectioning the cosets of R_{n-1} in R_n

In this section the following question is considered: Is there an n-sphere in R which projects into S^n with degree 1? It is shown that this question can be answered in the negative for certain values of n. For other dimensions the question remains open.

The first step toward the solution of this problem appears in

THEOREM 6. *The following conditions are equivalent:*

1) *there is a map $F(S^n) \subset R_n$ such that πF has degree one;*

2) *R_n can be represented as a product space $R_{n-1} \times S^n$;*

3) *the homomorphism $\pi_{n-1}(R_{n-1}) \to \pi_{n-1}(R_n)$ is an isomorphism;*

4) *the canonical map of S^{n-1} into R_{n-1} is homotopic to a point in R_{n-1}.*

The first condition implies the second. For let $F(S^n) \subset R_n$ be such that π has degree one. Since πF is homotopic to the identity, it follows from th covering homotopy theorem ([2], Theorem 1) that F is homotopic to a ma $F'(S^n) \subset R_n$ such that $\pi F'(x) = x$. Then by Theorem 2, $R_n = R_{n-1} \times S^n$.

That the second condition implies the third we have observed in the proof (the Corollary to Theorem 3; that the third implies the fourth follows fro Theorem 5.

The fourth implies the first; for during the homotopy of C_{n-1} to a point relative n-cell is swept out in R_{n-1} whose boundary is the $(n - 1)$-sphere defin by the canonical map. But the map D_n defines a relative n-cell in R_n with tl same boundary as the first one. Joining these two cells by identifying corr sponding points on the boundaries, we obtain a sphere in R_n whose projection has the same degree $(-1)^{n+1}$ as that of d_n. The required map is then easily constructed.

Theorem 6 enables us to answer immediately the question posed above for the case when n is even. For suppose that n is even and suppose that such a map exists. Then by the fourth condition in Theorem 6, the canonical map C_{n-1} is homotopic to a point in R_{n-1}. Hence $g_{n-1} = \pi C_{n-1}$ is inessential. But we have already shown that g_{n-1} has degree two. This contradiction completes the proof of

THEOREM 7. *If n is even, there is no map $F(S^n) \subset R_n$ such that πF has degree one.*

We now turn to the much more difficult case where n is odd. A partial answer to the question is obtained in

THEOREM 8. *If $n > 1$ and $n \equiv 1 \pmod 4$, there is no map $F(S^n) \subset R_n$ such that πF has degree one.*

The proof may be outlined as follows:

1) Since for n odd, g_{n-1} is homotopic to a point, it follows that C_{n-1} is homotopic to a map $G_{n-1}(S^{n-1}) \subset R_{n-2}$. Such a homotopy is exhibited, and it is proved that $k_{n-1} = \pi G_{n-1}$ is essential if $n \equiv 1 \pmod 4$, and inessential if $n \equiv 3 \pmod 4$.

2) A generator $P_n(V_1^n) \subset R_{n-1}$ of the group $\pi_n(R_{n-1}, R_{n-2})$ is constructed for all $n \geqq 5$, and the projection p_{n-1}^* of the map $\omega P_n(S^{n-1}) \subset R_{n-2}$ is shown to be inessential.

When these steps have been established, the proof of the theorem may be completed as follows: Suppose that $n > 1$ and $n \equiv 1 \pmod 4$ and suppose that the theorem is not true. Then C_{n-1}, and consequently G_{n-1}, is homotopic to a point in R_{n-1}. Since G_{n-1} is not homotopic to a point in R_{n-2}, the deformation of G_{n-1} defines a relative n-cell in R_{n-1} modulo R_{n-2}, and hence an essential element of $\pi_n(R_{n-1}, R_{n-2})$. This group has been shown by Freudenthal [3] to be the cyclic group of period 2 if $n \geqq 4$. Hence ωP_n and G_{n-1} are homotopic in R_{n-2}. But p_{n-1}^* and k_{n-1} are not homotopic, a contradiction.

LEMMA. *If n is odd, the canonical map C_{n-1} is homotopic in R_{n-1} to a map $G_{n-1}(S^{n-1}) \subset R_{n-2}$, whose projection into S^{n-2} is essential if $n \equiv 1 \pmod 4$ and inessential if $n \equiv 3 \pmod 4$.*

Let E^n denote the closed n-cell bounded by the unit sphere S^{n-1} in Euclidean n-space. H^n denotes the upper half $x_n \geqq 0$ of E^n. Let points $y \in H^n$ be represented by coördinates (x, r) where $x \in V_1^{n-1}$ is the central projection of the point y on V_1^{n-1}, and r is the distance of the point y from the origin. Let

$$H(r) = \begin{Vmatrix} c & s & 0 & 0 & \cdots & 0 \\ -s & c & 0 & 0 & \cdots & 0 \\ 0 & 0 & c & s & \cdots & 0 \\ 0 & 0 & -s & c & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{Vmatrix},$$

where $c = 1 - 2r^2$, $s = 2r(1 - r^2)^{\frac{1}{2}}$ $(0 \leqq r \leqq 1)$. Let

$$G(y) = G(x, r) = C_{n-1}(x)\cdot H(r)\cdot C_{n-1}(x)^{-1}\cdot\begin{Vmatrix} -I_{n-1} & 0 \\ 0 & 1 \end{Vmatrix} \tag{6}$$

$$(x \in V_1^n;\, 0 \leqq r \leqq 1)\text{s}$$

map H^n into R_{n-1}. Evidently $G(x) = C_{n-1}[g_{n-1}(x)]$ for $x \in V_1^{n-1}$, while G map, the equatorial plane into R_{n-2} and S^{n-2} into a single point. Deform V_1^{n-1} over H^n into E^{n-1} by letting each point move along the perpendicular joining it to the plane $x_n = 0$. The image of G under this homotopy gives a homotopy of $C_{n-1}[g_{n-1}(x)]$ to a map $K(V_1^{n-1}) \subset R_{n-2}$. Since under the homotopy S^{n-2} remains fixed, K maps S^{n-2} into a single point. Evidently $K(x)$ can be represented in the form $K(x) = G_{n-1}[g_{n-1}(x)]$, where $G_{n-1}(S^{n-1}) \subset R_{n-2}$ is defined and continuous over all of S^{n-1}, and G_{n-1} is homotopic in R_{n-1} to C_{n-1}.

By multiplying out the matrices in (6) and computing the homotopy, we find that the map $k_{n-1} = \pi G_{n-1}$ of S^{n-1} into S^{n-2} is represented by the equations

$$\tag{7} \begin{aligned} y_{2i-1} &= \frac{2(x_{2i-1}x_{n-2} + x_{2i}x_{n-1})}{(1 - x_n^2)^{\frac{1}{2}}}, \\ y_{2i} &= \frac{2(x_{2i}x_{n-2} - x_{2i-1}x_{n-1})}{(1 - x_n^2)^{\frac{1}{2}}} && \left(i = 1, \cdots, \frac{n-3}{2}\right), \\ y_{n-2} &= \frac{2(x_{n-2}^2 + x_{n-1}^2)}{(1 - x_n^2)^{\frac{1}{2}}} - (1 - x_n^2)^{\frac{1}{2}}, \\ y_{n-1} &= x_n && (1 - x_n^2 \neq 0); \\ y_i &= 0\ (i = 1, \cdots, n-2), \qquad y_{n-1} = x_n\ (1 - x_n^2 = 0). \end{aligned}$$

It follows easily that k_{n-1} is defined and continuous over all of S^{n-1} and maps S^{n-1} on S^{n-2}. In order to investigate this map, let us consider the map $m_{n-1}(S^{n-2}) = S^{n-3}$ obtained by setting $x_n = y_{n-1} = 0$ in (7). This map can be studied more easily in complex coördinates. Let $z_j = x_{2j-1} + ix_{2j}$, $w_j = y_{2j-1} + iy_{2j}$ $(j = 1, \cdots, (n-1)/2 = k)$, where $w_k = y_{n-2}$ is real. Then S^{n-2} and S^{n-3} are given by the equations $\sum z_j\bar{z}_j = 1$, $\sum w_j\bar{w}_j = 1$, respectively. In terms of these coördinates the equations representing the map m_{n-1} take the form

$$\tag{8} \begin{aligned} w_j &= 2z_j\bar{z}_k && (j = 1, \cdots, k-1), \\ w_k &= 2z_k\bar{z}_k - 1. \end{aligned}$$

If the complex coördinates occurring in (8) are formally replaced by real ones, the map g_{k-1} is obtained. In equations (3) and (5) we have constructed homotopies of g_n for n even and odd, respectively. The functions defining these homotopies can be extended so as to be defined for complex coördinates, as follows: if k is odd, i.e., if $n \equiv 3 \pmod 4$, let

$$w_{2j-1} = 2\{(1 - t)z_{2j-1}\bar{z}_k + [t(1 - t)]^{\frac{1}{2}}\bar{z}_{2j}\},$$

$$(9)\qquad \begin{aligned} w_{2j} &= 2\{(1-t)z_{2j}\bar{z}_k - [t(1-t)]^{\frac{1}{2}}\bar{z}_{2j-1}\} \qquad \left(j = 1, \cdots, \frac{k-1}{2}\right), \\ w_k &= 1 - 2(1-t)(1-z_k\bar{z}_k). \end{aligned}$$

Evidently the homotopy (9) deforms m_{n-1} over S^{n-3} to a point. Hence m_{n-1}, and consequently also k_{n-1}, is homotopic to a point. On the other hand, if k is even, i.e., if $n \equiv 1 \pmod 4$, let us consider the map $m'_{n-1}(S^{n-2}) \subset S^{n-3}$ defined by the equations

$$(10)\qquad \begin{aligned} w_j &= z_j \qquad (j = 1, \cdots, k-2), \\ w_{k-1} &= \frac{2z_{k-1}\bar{z}_k}{(z_{k-1}\bar{z}_{k-1} + z_k\bar{z}_k)^{\frac{1}{2}}}, \\ w_k &= \frac{2z_k\bar{z}_k}{(z_{k-1}\bar{z}_{k-1} + z_k\bar{z}_k)^{\frac{1}{2}}} - (z_{k-1}\bar{z}_{k-1} + z_k\bar{z}_k)^{\frac{1}{2}} \qquad (z_{k-1}\bar{z}_{k-1} + z_k\bar{z}_k \neq 0); \\ w_{k-1} &= w_k = 0 \quad \text{for} \quad z_{k-1} = z_k = 0. \end{aligned}$$

Setting $w_j = z_j = 0$ ($j = 1, \cdots, k \equiv 2$) in (10) and writing the result in real coördinates, we obtain a map of S^3 on S^2 of Hopf invariant ± 1 (cf. Hopf [4], §5). It follows from the results of Freudenthal [3] that m'_{n-1} is essential.

We shall now define a deformation of m_{n-1} to m'_{n-1} (for $n \equiv 1 \pmod 4$) by extending the functions in (5) so as to be defined for complex coördinates, as follows: let

$$(11)\qquad \begin{aligned} w_{2j-1} &= tz_{2j-1} + [t(1-t)]^{\frac{1}{2}}\bar{z}_{2j} + 2\,\frac{(1-t)z_{2j-1}\bar{z}_k - [t(1-t)]^{\frac{1}{2}}z_k\bar{z}_{2j}}{\{1 - t[1 - (z_{k-1}\bar{z}_{k-1} + z_k\bar{z}_k)]\}^{\frac{1}{2}}}, \\ w_{2j} &= tz_{2j} - [t(1-t)]^{\frac{1}{2}}\bar{z}_{2j-1} + 2\,\frac{(1-t)z_{2j}\bar{z}_k + [t(1-t)]^{\frac{1}{2}}z_k\bar{z}_{2j-1}}{\{1 - t[1 - (z_{k-1}\bar{z}_{k-1} + z_k\bar{z}_k)]\}^{\frac{1}{2}}} \\ &\qquad \left(j = 1, \cdots, \frac{k-2}{2}\right), \\ w_{k-1} &= \frac{2z_{k-1}\bar{z}_k}{\{1 - t[1 - (z_{k-1}\bar{z}_{k-1} + z_k\bar{z}_k)]\}^{\frac{1}{2}}}, \\ w_k &= \frac{2z_k\bar{z}_k}{\{1 - t[1 - (z_{k-1}\bar{z}_{k-1} + z_k\bar{z}_k)]\}^{\frac{1}{2}}} - \{1 - t[1 - (z_{k-1}\bar{z}_{k-1} + z_k\bar{z}_k)]\}^{\frac{1}{2}}. \end{aligned}$$

Although this map is not defined everywhere for $t = 1$, it is readily verified that as $t \to 1$, the functions in (11) converge to the corresponding ones in (10) uniformly in z. Thus m_{n-1} and m'_{n-1} are homotopic, so that m_{n-1}, and consequently also k_{n-1}, is essential. This completes the proof of the Lemma.

As indicated above, the next step in the proof is to construct a generator of the group $\pi_n(R_{n-1}, R_{n-2})$. This group is isomorphic with $\pi_n(S^{n-1})$ under the projection π ([2], Theorem 2). If $n \geqq 4$ we may take as generator of the latter group the map $p_n(S^n) = S^{n-1}$ defined by

$$p_{n,i}(x) = x_i \qquad (i = 1, \cdots, n-4),$$

$$
(12)\qquad
\begin{aligned}
p_{n,n-3}(x) &= \frac{2(x_{n-3}x_{n-1} + x_{n-2}x_n)}{(x_n^2 + x_{n-1}^2 + x_{n-2}^2 + x_{n-3}^2)^{\frac{1}{2}}},\\
p_{n,n-2}(x) &= \frac{2(x_{n-2}x_{n-1} - x_{n-3}x_n)}{(x_n^2 + x_{n-1}^2 + x_{n-2}^2 + x_{n-3}^2)^{\frac{1}{2}}},\\
p_{n,n-1}(x) &= \frac{x_n^2 + x_{n-1}^2 - x_{n-2}^2 - x_{n-3}^2}{(x_n^2 + x_{n-1}^2 + x_{n-2}^2 + x_{n-3}^2)^{\frac{1}{2}}},\\
p_{n,n}(x) &= x_{n+1} \qquad\qquad (x_n^2 + x_{n-1}^2 + x_{n-2}^2 + x_{n-3}^2 \neq 0),\\
p_{n,n-3}(x) &= p_{n,n-2}(x) = p_{n,n-1}(x) = 0 \qquad (x_n = x_{n-1} = x_{n-2} = x_{n-3} = 0).
\end{aligned}
$$

Evidently p_n maps V_1^n into V_1^{n-1} and S^{n-1} into S^{n-2}. The latter map is essential; in fact, it represents a generator of $\pi_{n-1}(S^{n-2})$. Let $P_n(x) = D_{n-1}[p_n(x)]$ $(x \in V_1^n)$. Then P_n represents the desired generator of $\pi_n(R_{n-1}, R_{n-2})$; for $\pi P_n(x) = d_{n-1}[p_n(x)]$, and the latter map represents a generator of $\pi_n(S^{n-1}, x^o) = \pi_n(S^{n-1})$.

Let $P_n^* = \omega P_n$, and $p_{n-1}^* = \pi P_n^*$ its projection into S^{n-2}. Since p_n maps S^{n-1} into S^{n-2}, and since $\omega D_{n-1} = C_{n-2}$, we have $P_n^*(x) = C_{n-2}[p_n(x)]$, and hence $p_{n-1}^*(x) = g_{n-2}[p_n(x)]$ $(x \in S^{n-1})$. Let α denote the element of $\pi_{n-2}(S^{n-2})$ determined by the identity map, β the element of $\pi_{n-1}(S^{n-2})$ defined by the map $p_n(S^{n-1}) = S^{n-2}$. Since $\{g_{n-2}\} = 0$ or 2α according as n is even or odd, we have $\{p_{n-1}^*\} = 0$ or 4β[2] respectively. But Freudenthal [3] has proved that $2\beta = 0$ if $n \geqq 5$; hence for all $n \geqq 5$ we have $\{p_{n-1}^*\} = 0$, so that p_{n-1}^* is inessential. This completes the proof of the theorem.

8. Computation of the homotopy groups

We are now in a position to establish the results exhibited in Section 2. The generators which we shall compute here will differ slightly from those already exhibited; however, they may be easily seen to be homotopic.

We have already observed that $\pi_n(R_{n+1})$ is a homomorphic image of $\pi_n(R_n)$. In a similar manner we can prove (cf. [2], Theorem 5) that $\pi_n(R_{n+k})$ $(k = 1, 2, \cdots)$ is isomorphic with $\pi_n(R_{n+1})$. Another useful result is the following:

THEOREM 9. *If n is even, $\pi_n(R_n)$ is a factor group of $\pi_n(R_{n-1})$; the kernel of the homomorphism is the group $\pi_{n,o}(R_{n-1})$.*

Since $\{D_n\}$ is a generator of $\pi_n(R_n, R_{n-1})$ and since $\omega\{D_n\} = \{C_{n-1}\} \neq 0$ for n even, it follows that $\pi_{n,o}(R_n, R_{n-1}) = 0$. Hence $\pi_n(R_n) = \pi_n(R_{n-1})/\pi_{n,o}(R_{n-1})$.

In order to compute $\pi_1(R_n)$, we first observe that, since R_1 is homeomorphic to S^1, its fundamental group is the free cyclic group generated by any map of S^1 on R_1 of degree 1. Such a generator α_1 is given by the map

$$
B_1(x) = \left\| \begin{matrix} x_2 & x_1 \\ -x_1 & x_2 \end{matrix} \right\|.
$$

Since R_2 is homeomorphic with projective 3-space P^3, it follows that $\pi_1(R_2)$

[2] This follows easily from **Theorem** II b′ of [4].

is the cyclic group of order two. But $\pi_1(R_2)$ is a factor group of $\pi_1(R_1)$; hence for a generator of $\pi_1(R_2)$ we may take the generator α_1 for $\pi_1(R_1)$ subject to the condition $2\alpha_1 = 0$ in R_2. The same result holds for $\pi_1(R_n)$.

Since all the higher homotopy groups of S^1 vanish, the same holds for R_1. In particular, $\pi_2(R_1) = 0$. Then, by Theorem 9, $\pi_2(R_n) = 0$ for all n.

The higher homotopy groups of R_2 are isomorphic to those of its covering space S^3. In particular, $\pi_3(R_2)$ is the free cyclic group, and a generator α_3 is represented by the covering map $H(S^3) = R_2$ given by the equation

$$H(x) = \begin{Vmatrix} x_4^2 - x_3^2 - x_2^2 + x_1^2 & 2(x_1x_2 - x_3x_4) & 2(x_1x_3 + x_2x_4) \\ 2(x_1x_3 + x_2x_4) & x_4^2 - x_3^2 + x_2^2 - x_1^2 & 2(x_2x_3 - x_1x_4) \\ 2(x_1x_3 - x_2x_4) & 2(x_1x_4 + x_2x_3) & x_4^2 + x_3^2 - x_2^2 - x_1^2 \end{Vmatrix}.$$

We observe that $\pi H(x)$ maps S^3 on S^2 with Hopf invariant ± 1.

Since R_3 is the product space of R_2 and the quaternion subgroup Q^3, the homotopy groups of R_3 are the direct sums of the groups of the same dimension of R_2 and $Q^3 = S^3$. In particular, $\pi_3(R_3) = \pi_3(R_2) + \pi_3(S^3)$ is a free group with two generators. For one of these generators we may take the generator α_3 of $\pi_3(R_2)$; the second generator β_3 is determined by the quaternion matrix

$$B_3(x) = \begin{Vmatrix} x_4 & x_3 & -x_2 & x_1 \\ -x_3 & x_4 & x_1 & x_2 \\ -x_2 & -x_1 & x_4 & x_3 \\ -x_1 & -x_2 & -x_3 & x_4 \end{Vmatrix} \qquad (x \in S^3).$$

To compute $\pi_3(R_4) = \pi_3(R_3)/\pi_{3,o}(R_3)$, we observe that

$$C_3(x) = B_3[g_3(x)] \cdot \begin{Vmatrix} H(x) & 0 \\ 0 & 1 \end{Vmatrix}$$

so that $\{C_3\} = 2\beta_3 + \alpha_3 = 0$ in R_4. Hence $\pi_3(R_4)$ is a free cyclic group with the one generator β_3, and the same is true of $\pi_3(R_n)$ $(n \geqq 4)$.

We now consider the groups π_4. Freudenthal [3] and Pontrjagin [5] have proved that $\pi_4(S^3)$ is the cyclic group of order two, a generator being determined by the map $p_4(S^4) = S^3$. Hence a generator α_4 of $\pi_4(R_2)$ is determined by the map Hp_4. The group $\pi_4(R_3)$ is the direct sum of two cyclic groups of order two; for generators we may take α_4 and $\beta_4 = \{B_3p_4\}$.

We have observed in the proof of Theorem 8 that $\pi_{4,o}(R_3)$ is generated by $\omega\{P_5\} = \{P_5^*\}$. But

$$P_5^*(x) = B_3[p_4^*(x)] \cdot \begin{Vmatrix} H[p_4(x)] & 0 \\ 0 & 1 \end{Vmatrix}$$

and $\{p_4^*\} = 0$, so that $\{P_5^*\} = \alpha_4 = 0$ in R_4. Hence $\pi_4(R_4)$ is the cyclic group of order two generated by β_4.

We have proved in Theorem 8 that $\{C_4\} \neq 0$ in R_4. Hence $\pi_{4,o}(R_4) = \pi_4(R_4)$, so that $\pi_4(R_5) = \pi_4(R_n) = 0$ $(n \geqq 5)$.

We conclude by computing the five-dimensional homotopy groups of R_n. Pontrjagin [5] has proved that $\pi_5(S^3) = 0$; hence $\pi_5(R_2) = \pi_5(R_3) = 0$. Hence $\pi_5(R_4) = \pi_{5,o}(R_4, R_3) = 0$ since $\pi_{4,o}(R_3) \neq 0$. This in turn implies that $\pi_5(R_5) = \pi_{5,o}(R_5, R_4)$. But $\pi_5(R_5)$ contains the essential element $\alpha_5 = \{C_5\}$, and since πC_5 has degree two, it follows from Theorem 8 that $\pi_5(R_5)$ is the free cyclic group generated by α_5. Since $\alpha_5 = 0$ in R_6, we have finally that $\pi_5(R_6) = \pi_5(R_n) = 0$ $(n \geqq 6)$.

9. Continuous vector fields over spheres

The above results will now be applied to the study of continuous vector fields over S^n. Geometrically, a continuous vector field may be thought of as a set of functions $V^i(x)$ $(i = 1, \cdots, n + 1; x \in S^n)$ defining a unit vector tangent to S^n at the point x, the functions $V^i(x)$ being continuous over all of S^n. A set of p such fields are said to be *independent* if the vectors of the fields at each point of S^n are independent vectors.

Let P_k denote the point of S^n whose coördinates are δ_{ik} $(i, k = 1, \cdots, n + 1)$. R_{n-p} denotes the subgroup of R_n consisting of all rotations leaving P_k fixed $(k = n - p + 2, \cdots, n + 1)$. The coset space R_n/R_{n-p} may be thought of as the space of all sets of p mutually orthogonal points of S^n; for two elements of R_n are in the same coset of R_{n-p} if and only if their last p columns are identical. These columns define the orthogonal p-tuple associated with the given coset. Conversely, given a set of orthogonal points $Q_1, Q_2, \cdots, Q_p$, the required coset is the set of all rotations carrying P_{n-p+2} into $Q_1, \cdots$, and P_{n+1} into Q_p.

Since $R_{n-p} \subset R_{n-1}$, each coset of R_{n-p} lies in some coset of R_{n-1}. Thus a natural mapping $R_n/R_{n-p} \rightarrow R_n/R_{n-1} = S^n$ is defined; each coset of R_{n-p} is mapped into the coset of R_{n-1} containing it. We refer to this map as the *projection*; it is easily verified that it is a fibre map. If we regard an element of R_n/R_{n-p} as being determined by a set of p mutually orthogonal points, the projection is the p^{th} point of the set.

Since the usual process of orthogonalizing a set of independent vectors can be carried out here, there is no loss of generality in assuming that the vectors of the fields we are dealing with are orthogonal at each point and of unit length.

THEOREM 10. *Every set of p orthogonal vector fields over S^n defines a mapping of S^n into R_n/R_{n-p-1} whose projection into S^n is the identity; conversely, any such map defines a set of p orthogonal vector fields.*

By translating the vectors at any point x to the center of S^n and adjoining the point x, we obtain an orthogonal $(p + 1)$-tuple whose projection is x. This process is continuous and gives the required map. Conversely, given such a map, we define the p orthogonal vectors at x as follows: the given map associates with x an orthogonal $(p + 1)$-tuple, the $(p + 1)^{\text{st}}$ point of which is x. Translation of radius vectors drawn to the first p points to the point x gives the set of vectors at x.

COROLLARY. *There exists a set of p independent vector fields over S^n if and only if the canonical map of S^{n-1} into R_{n-1} is contractible in R_{n-1} into R_{n-p-1}.*

It follows from Theorem 7 that there is no vector field over S^n if n is even. However, if n is odd, it follows from the Lemma used in the proof of Theorem 8 that there is at least one. Over S^3 and S^7 are three and seven orthogonal fields, respectively. These are defined by the mappings $S^3 \to R_3 \to S^3$ and $S^7 \to R_7 \to S^7$ of degree one given by the matrices B and B obtained in Section 5.

THEOREM 11. *If $n \equiv 3 \pmod 4$ there at least three orthogonal vector fields over S^n; if $n \equiv 7 \pmod 8$ there are at least seven.*

We shall prove the theorem for $n = 4m + 3$; the proof for $n = 8m + 7$ is entirely analogous. Let $V^i(x)$ ($i = 1, 2, 3$; $x \in S^n$) define a set of orthogonal vector fields over S^n; the components of the vector $V^i(x)$ will be denoted by $V^i_j(x)$ ($j = 1, 2, 3, 4$). We extend $V^i(x)$ to be defined over all of E^4 asfollows: if $y \in E^4$, let x be the central projection of y on S^3, r the distance of y from the origin. Then $V^i(y) = rV^i(x)$ defines a set of three orthogonal vectors at each point of E^4, and these vectors vanish only at the origin.

If z is a point of E^{4m+4} with coördinates $(z_1, \cdots, z_{4m+4})$, let x^j ($j = 1, \cdots, m + 1$) denote the point of E^4 with coördinates $(z_{4j-3}, z_{4j-2}, z_{4j-1}, z_{4j})$. Then we define three vector fields $W^i(z)$ over E^{4m+4} as follows:

$$W^i_{4j+k}(z) = V^i_k(x^{j+1}) \qquad (i = 1, 2, 3; j = 0, \cdots, m; k = 1, 2, 3, 4).$$

Evidently the vectors $W^i(z)$ are mutually orthogonal at each point z, and if $z \in S^{4m+3}$ they are of unit length. Thus the theorem is proved.

The problem of determining the maximum number of independent fields which can exist over S^n has not yet been solved for general odd n. For $n \equiv 1 \pmod 4$ the solution appears in

THEOREM 12. *If $n \equiv 1 \pmod 4$ any two vector fields over S^n are somewhere dependent.*

The theorem is evidently true for $n = 1$. Suppose $n > 1$ and suppose that the theorem were not true. Then by the Corollary to Theorem 10, the canonical map C_{n-1} would be contractible in R_{n-1} into R_{n-3}. Hence the map $G_{n-1}(S^{n-1}) \subset R_{n-2}$ would be homotopic in R_{n-1} to a map $F(S^{n-1}) \subset R_{n-3}$. Hence $\{G_{n-1}\} - \{F\} \in \pi_{n-1,o}(R_{n-2})$. But $\pi[\{G_{n-1}\} - \{F\}] = \pi\{G_{n-1}\} - \pi\{F\} = \pi\{G_{n-1}\} \neq 0$. In the proof of Theorem 8 we have shown that this is impossible.

COROLLARY. *If $n > 1$ and $n \equiv 1 \pmod 4$ the tangent sphere-bundle of S^n is not simple* ([7], p. 788).

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