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IMMERSIONS IN THE STABLE RANGE

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Introduction

Let M and N be differentiable manifolds of dimensions m and n, and $T(M)$, $T(N)$ their tangent bundles. An *immersion* of M in N is a differentiable map $f: M \to N$ such that the differential $f_*: T(M) \to T(N)$ has rank m on each fibre. A *regular homotopy* is a homotopy $f_t: M \to N$ such that each f_t is an immersion, and such that $f_{t*}: T(M) \to T(N)$ is a (continuous) homotopy.

In [1] it is proved that the regular homotopy classes of immersions of M in N, $m < n$, are in one-one correspondence with the homotopy classes of fibre maps $T(M) \to T(N)$ whose restriction to each fibre of $T(M)$ is a linear map of rank m into a fibre of $T(N)$. Such maps will be called *linear*.

Our purpose is to give, in view of [1], another classification of immersions in the range $2n > 3m + 1$, by proving the following result: the homotopy classes of linear maps $T(M) \to T(N)$ are in one-one correspondence with the homotopy classes of those fibre maps $\varphi: T(M) \to T(N)$, called here *skew* maps, which have the property that $\varphi(-X) = -\varphi(X)$, and $\varphi(X) \neq 0$ if $X \neq 0$. This is proved by showing that the homotopy groups of the Stiefel manifold $V_{n,m}$ of m-frames in n-space are the same, up to a certain dimension, as those of the space $X_{n,m}$ of maps $\varphi: R^m \to R^n$ with the property that $\varphi^{-1}(0) = 0$ and $\varphi(-X) = -\varphi(X)$. This fact is easily proved using the Freudenthal suspension theorems.

A skew map $T(M) \to R^n$ is essentially the same as a map $\delta: U - \Delta \to S^{n-1}$, where U is a neighborhood of the diagonal Δ of $M \times M$, with the property that $\delta(x, y) = -\delta(y, x)$ (Theorem 2.1.). Such a map δ is called equivariant. We apply this to immersions $f: M \to R^n$, replacing the differential $f_*: T(M) \to R^n$ by $\delta_f: U - \Delta \to S^{n-1}$ defined by $\delta_f(x, y) = (f(x) - f(y))/||f(x) - f||$. In the case where f is one-one, δ_f is defined on $M \times M - \Delta$. This permits us to exploit our knowledge of the topology of M. As a result, we obtain a new proof of Kervaire's theorems (2.6) that the Smale invariant of an imbedding $f: S^m \to R^n$ vanishes if $2n > 3m + 1$, and $f(S^m)$ has a trivial normal bundle. Other applications are (2.8), (2.9) and (2.10).

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In the general case, where N is not necessarily R^n, there is a correspondence between skew maps $T(M) \to T(N)$ and equivariant maps $\theta\colon U \to N \times N$ which is one-one on homotopy classes. (Here "equivariant" means that if $\theta(x, y) = (u, v)$, then $\theta(y, x) = (v, u)$, and $u \neq v$ if $x \neq y$; "isovariant" would be more precise.) Since "equivariant map" is a topological concept, we obtain (4.3) a topologically invariant classification of immersions $M \to N$ in the "stable range" of dimensions, $2n > 3m + 1$. If $2n > 3m$, the existence of immersions approximating a given map $g : M \to N$ is topologically invariant; if g is a topological immersion, it can always be approximated by a differentiable one (5.1).

In the first three sections we treat mainly linear maps $T(M) \to T(N)$, particularly the differential of an immersion. In the last two sections we consider the existence and regular homotopy of immersions. For convenience, however, some results (2.10) of this nature are in § 2. In § 6 we apply the material of §§ 1 and 3 to obtain the topological invariance of the existence of certain tangent frame fields on a differentiable manifold.

By *manifold* we always mean a differentiable manifold of class C^∞. We assume all manifolds to have a complete riemannian metric. Throughout the paper, M is a manifold of dimension m, and N a manifold of dimension n. We denote the *tangent bundle* of M by $T(M)$, and the sub-bundle of null vectors by $T_0(M)$, which we may identify with M. The diagonal $\Delta \subset M \times M$ is the set of points (x, x) and is also identified with M. We use R^m for euclidean m-space; the unit sphere in R^m is S^{m-1}. By translation, we identify $T(R^m)$ with R^m, the tangent space of R^m at the origin.

A *differentiable imbedding* is an immersion which is a homeomorphism onto its image.

1. Maps of plane bundles

Let $\pi : E \to B$ and $\pi' : E' \to B'$ be fibre bundles. A map $\varphi\colon E \to E'$ is a *fibre map* if φ takes each fibre of E into a single fibre of E'. In this case there is a unique map $\bar\varphi\colon B \to B'$ such that $\bar\varphi\pi = \pi'\varphi$. We call $\bar\varphi$ the map induced by φ.

Now assume that E is a bundle of m-planes and E' a bundle of n-planes, with structural groups $\mathrm{GL}(m)$ and $\mathrm{GL}(n)$ respectively. A fibre map $\varphi\colon E \to E'$ is *linear* if for each $x \in B$, $\varphi \mid \pi^{-1}(x)$ is a linear map of rank m into the corresponding fibre of E'. *A linear homotopy* is a homotopy $\varphi_t\colon E \to E'$ such that each φ_t is linear. A fibre map $\varphi\colon E \to E'$ is a *skew map* if $\varphi(X) \neq 0$ whenever $X \neq 0$ and $\varphi(-X) = -\varphi(X)$. Every linear map is a skew map. The term *skew homotopy* has the natural definition.

Let $L_{n,m}$ denote the space of linear maps $R^m \to R^n$ of rank m, and $X_{n,m}$

the space of skew maps $R^m \to R^n$; these spaces have the compact open topology. Let ρ: $L_{n,m} \to X_{n,m}$ be the inclusion.

(1.1) LEMMA. *If* $0 < i < 2n - 2m - 1$, *then* $\rho_\#$: $\pi_i(L_{n,m}) \to \pi_i(X_{n,m})$ *is an isomorphism. If* $i = 2n - 2m - 1$, *then* $\rho_\#$ *is onto.*

PROOF. Since we are dealing with homotopy groups, we may replace $L_{n,m}$ with its deformation retract $V_{n,m}$, the space of linear maps of rank m which preserve length. Thus $V_{n,m}$ is the Stiefel manifold of orthonormal m-frames in R^n. Likewise we replace $X_{n,m}$ by the subspace $Y_{n,m}$ of radial maps that preserve length along each radius; $Y_{n,m}$ is a deformation retract of $X_{n,m}$. Since each $\varphi \in Y_{n,m}$ is completely determined by $\varphi \mid S^{m-1}$: $S^{m-1} \to S^{n-1}$, we see that $Y_{n,m}$ is the space of maps φ: $S^{m-1} \to S^{n-1}$ which commute with the antipodal map. We shall prove (1.1) by induction on m. We denote the inclusion $V_{n,m} \subset L_{n,m}$ by ρ_m. If $m = 1$, $V_{n,m} = Y_{n,m} = S^{n-1}$ and proof is complete. Assume now that $m > 1$, and assume inductively that

$$(1.1_{m-1}) \qquad (\rho_{m-1})_\# : \pi_i(V_{n,m-1}) \to \pi_i(Y_{n,m-1})$$

is an isomorphism for $0 < i < 2n - 2m - 3$, *and is onto for* $i = 2n - 2m - 3$.

Let p: $V_{n,m} \to V_{n,m-1}$ and q: $Y_{n,m} \to Y_{n,m-1}$ assign to each linear (respectively, skew) map its restrictions to R^{m-1}. It is well known that p defines a fibre bundle, and it is easy to see that q defines a fibration in the sense of Serre. Moreover, $q\rho_m = \rho_{m-1}p$. Choose $v \in V_{n,m-1}$ and put $\psi = \rho_{m-1}(v)$. To prove (1.1_m) it suffices to prove that $\rho_m \mid p^{-1}(v)$ induces an isomorphism $\pi_i(p^{-1}(v)) \to \pi_i(q^{-1}(\psi))$ for $i < 2n - 2m - 1$, and an epimorphism for $i = 2n - 2m - 1$; then (1.1_m) is proved by looking at the exact homotopy sequences of the fibrations defined by p and q. Now $p^{-1}(v)$ is homeomorphic to S^{n-m}. We identify $q^{-1}(\psi)$ with the space of maps f: $E^{m-1} \to S^{n-1}$ such that $f \mid S^{m-2} = \psi$, where E^{m-1} is the northern hemisphere of S^{m-1}. By contracting $f(S^{m-2})$ over the southern hemisphere of S^{n-1} to the south pole, we see that this space has the homotopy type of the iterated loop space $\Omega_{m-1}S^{n-1}$, and it is easy to see that $\rho_m \mid q^{-1}(v)$ induces a map $u_\#$: $S^{n-m} \to \Omega^{m-1}S^{n-1}$ with the property that $u_\#$: $\pi_i(S^{n-m}) \to \pi_i(\Omega^{m-1}S^{n-1}) = \pi_{i+m-1}(S^{n-1})$ is the iterated suspension; see [2, Ch. XI]. Since this suspension is an isomorphism for $i < 2n - 2m - 1$ and is onto for $i = 2n - m - 1$, proof is established.

Now let $\pi : E \to B$ be an m-plane bundle and $\pi' : E' \to B'$ an n-plane bundle, with respective structural groups GL(m), GL(n). We assume B is a simplicial complex.

(1.2) THEOREM.

(a) *Assume* $\dim B \leqq 2n - 2m - 1$. *Let* φ: $E \to E'$ *be a skew map.*

There is a linear map $\psi: E \to E'$ *such that* $\bar{\psi} = \bar{\varphi}$ *and* ψ *is skew homotopic to* φ.

(b) *Assume* $\dim B < 2n - 2m - 1$. *Let* $\varphi_0, \varphi_1: E \to E'$ *be linear maps. If* φ_0 *and* φ_1 *are skew homotopic, they are linearly homotopic; the linear homotopy may be chosen to cover the homotopy between* $\bar{\varphi}_0$ *and* $\bar{\varphi}_1$.

PROOF. The skew maps E into E' that cover $\bar{\varphi}$ are in one-one correspondence with the cross-sections of the bundle S over B, whose fibre over $x \in B$ is the space $(X_{n,m})_x$ of skew maps of the fibre E_x into the fibre $E'_{\bar{\varphi}(x)}$. The linear maps that cover $\bar{\varphi}$ correspond to the cross-sections of the sub-bundle $L \subset S$ consisting of linear maps; the fibre of L over x is $(L_{n,m})_x$. An equivalent formulation of (1.1) is that $\pi_i(X_{n,m}, L_{n,m}) = 0$ for $0 < i < 2n - 2m - 1$. Since the obstructions to deforming a cross-section of S into a cross-section of L lie in these groups, (a) is proved. The proof of (b) is similar, and is left to the reader.

2. Linear maps $T(M) \to R^n$.

If $f: M \to N$ is an immersion, there is a neighborhood U of the diagonal Δ in $M \times M$ such that if $(x, y) \in U - \Delta$, $f(x) \neq f(y)$. If $N = R^n$, we may therefore define $\delta_f: U - \Delta \to S^{n-1}$ by $\delta_f(x, y) = f(x) - f(y)/\|f(x) - f(y)\|$. We shall investigate the connection between δ_f and $f_*: T(M) \to R^n$.

There is a neighborhood O_M of Δ such that if $(x, y) \in O_M$, there is a unique shortest geodesic joining x to y. We denote by exp the exponential map $T(M) \to M$, and by $T_0(M)$ the set of null vectors of $T(M)$. Then there is a neighborhood A_M of $T_0(M)$ in $T(M)$ such that the map $T(M) \to M \times M$ given by $X \to (\exp X, \exp -X)$ maps A_M topologically onto O_M. We denote this homeomorphism by $e_M: A_M \to O_M$. Observe that if $e_M(X) = (x, y)$, $e_M(-X) = (y, x)$, and $e_M(X) \in \Delta$ if and only if $X \in T_0(M)$.

If U is a neighborhood of Δ, a map $\delta : U - \Delta \to S^{n-1}$ is *equivariant* if $\delta(y, x) = -\delta(x, y)$. Given such a δ, we shall define a skew map $\Phi(\delta) = \varphi: T(M) \to R^n$. (The definition of Φ is based on the identification of $T(M)$ with the normal bundle of Δ in $M \times M$ by means of the exponential map.) Let ε be a positive continuous function on $T(M)$ such that $\varepsilon(X) = \varepsilon(-X)$, $\varepsilon(X)$ $X \in A_M \cap e_M^{-1}(U)$, and $\varepsilon = 1$ in a neighborhood of $T_0(M)$. Now define $\varphi: T(M) \to R^n$ by $\varphi(X) = \|X\| \,\delta e_M(\varepsilon(X)X)$ if $X \neq 0$, and $\varphi(X) = 0$ if $X = 0$; clearly φ is skew. (We use $\|X\|$ for the norm of X.)

Let us define two equivariant maps $\delta_i: U_i - \Delta \to S^n - 1$, $i = 0, 1$ to be *germ homotopic* if for some symmetric neighborhood $V \subset U_0 \cap U_1$ of Δ, $\delta_0 \mid U_0 \cap V - \Delta$ is equivariantly homotopic to $\delta_1 \mid U_1 \cap V - \Delta$. It is not hard to prove

(2.1) THEOREM. *Φ induces a one-one correspondence between germ homotopy classes of equivariant maps $U - \Delta \to S^{n-1}$ and skew homotopy classes of skew maps $T(M) \to R^n$.*

This can be deduced easily from (3.1), below.

The following elementary fact is crucial to the theory.

(2.2) LEMMA. *Let f: $M \to R^n$ be an immerision. Then f_* and $\Phi(\delta_f)$ are skew homotopic.*

PROOF. Let γ be a geodesic segment in M with midpoint a. Let $x, y \in \gamma$ be distinct points equi-distant from a. Let X be the tangent to γ at a such that $\exp X = x$, $\exp - X = y$. Then $\Phi(\delta_f)(X) = [\|X\|(f(x) - f(y))]/(\|f(x) - f(y)\|)$ if X is small enough. Keeping X fixed and letting $x, y \to a$, the term on the right approaches $\|X\|(f_*X)/(\|f_*X\|)$. This provides a skew homotopy from $\Phi(\delta_f)$ to the skew map $\psi(X) = (\|X\|f_*X)/(\|fX_*\|)$. (If $X = 0$, at each stage of the homotopy $X \to 0$). Then the homotopy

$$(X, t) \to \{((1 - t)\|X\|/\|f_*X\|) + t\}f_*X$$

is a skew homotopy from ψ to f_*. We are now ready to prove

(2.3) THEOREM.

(a) *Assume $2n > 3m$. If there is an equivariant map δ: $U - \Delta \to S^{n-1}$, for some neighborhood U of the diagonal $\Delta \subset M \times M$, then there is a linear map ψ: $T(M) \to R^n$.*

(b) *Assume $2n > 3m + 1$. Let f, g: $M \to R^n$ be immersions. The linear maps f_*, g_*: $T(M) \to R^n$ are linearly homotopic if the maps δ_f, δ_g: $U - \Delta \to S^{n-1}$ are equivariantly homotopic for some neighborhood U of Δ.*

To prove (a), let φ: $T(M) \to R^n$ be the skew map $\Phi(\delta)$. The existence of a linear map follows from (1.3a). To prove (b), we apply (1.3b) and conclude that it suffices to show that f_* and g_* are skew homotopic. By (2.2) it is enough to prove that $\Phi(\delta_f)$ and $\Phi(\delta_g)$ are skew homotopic. This follows from (2.1).

(2.4) COROLLARY. *The existence of a linear map $T(M) \to R^n$ is independent of the structure of $T(M)$ if $2n > 3m$.*

In view of (2.3), the question arises as to when equivariant maps exist, and when they are homotopic. The following result is well known.

Let X, Y be spaces on which a group G acts in such a way that only the identity element of G leaves any element of X fixed. Let X/G be the orbit space of X under the action of G, and let $E \to X/G$ be the bundle with fibre Y associated to the principal G-bundle $X \to X/G$. (We assume

$X \to X/G$ is locally trivial.)

(2.5) LEMMA. *The cross-setions of E are in one-one correspondence with G-equivariant maps $X \to Y$. Two cross-sections are homotopic if and only if the corresponding maps are equivariantly homotopic.*

Thus if $U \subset M \times M$ is a symmetric neighborhood of Δ, to study equivariant maps $U - \Delta \to S^{n-1}$, we examine cross sections of the bundle over $(U - \Delta)/Z_2$ with fibre S^{n-1} associated to the Z_2-bundle $U - \Delta \to (U - \Delta)/Z_2$. If $U = M \times M$, we put $(M \times M - \Delta)/Z_2 = M^*$, and we denote this bundle by $Q_n(M) \to M^*$. The obstructions to a cross-section of $Q_n(M)$ lie in groups $H^{i+1}(M^*, G_{i,n})$ where $G_{i,n}$ denotes $\pi_i(S^{n-1})$ if $n - 1$ is odd, and the following local system otherwise. Let $F \subset \pi_1(M^*)$ be the image of $\pi_1(M \times M - \Delta)$. If $g \notin F$ and $\alpha \in \pi_i(S^{n-1})$, define $g\alpha \in \pi_i(S^{n-1})$ to be the homotopy class of the composite $S^i \to S^{n-1} \to S^{n-1}$ where the first map represents α, and the second is the antipodal map. If $h \in F$, put $h\alpha = \alpha$. If $i \leqq 2n - 2$, then $g\alpha = -\alpha$ (cf. [5, 23. 8]). This is the case if $2n > 3m$. Thus if $n - 1$ is even, and $2n > 3m$, the local system $G_{i,n}$ is the tensor product of $\pi_i(S^{n-1})$ and the twisted integer system Z_T associated to the covering $M \times M - \Delta \to M^*$. The obstructions to making two cross-sections homotopic lie in the groups $H^i(M^*; G_{i,n})$.

(2.6) THEOREM (Kervaire). *If $2n > 3m + 1$ and g, f: $S^m \to R^n$ are differentiable imbeddings, f_*, g_*: $T(S^m) \to R^n$ are linearly homotopic.*

(2.7) COROLLARY.

(a) *The Smale invariant C_f vanishes.*

(b) *$f(S^m)$ has a trivial normal bundle.*

PROOF. To prove (2.6), it suffices by (2.3b) to show that the cross-sections $S^{m*} \to Q_n(S^m)$ corresponding to δ_f, δ_g: $S^m \times S^m - \Delta \to S^{n-1}$ are homotopic. It is well known that S^{m*} admits real projective m-space as a deformation retract. (To see this, let the pair (x, y) of distinct points of S^m move uniformly to (x', y'), where $x'y'$ is the diameter parallel to xy, and x' is nearer to x than to y.) Therefore $H^i(S^{m*}) = 0$ for $i > m$ with any coefficient system. Since $m > n - 1$, there can be no obstruction to such a homotopy. The Smale invariant $C_f \in \pi_m(V_{n,m})$ can be defined as the obstruction to making f_* and i_* linearly homotopic, where i: $S^m \to R^n$ is the inclusion. Thus (2.6) is equivalent to (2.7a). To prove (2.7b), we observe that the normal bundle of $f(S^m)$ is determined by the linear homotopy class of f; for the $(n - m)$-plane normal to $f(S^m)$ at $f(x)$ is normal to the m-plane containing $f_*(T(M/x))$.

The results in (2.7) are due originally to M. Kervaire [3, 4].

(2.8) THEOREM. *Let M be a compact unbounded m-manifold such that*

$H_i(M) = 0$ for $0 < i \leqq k$.

(a) *If* $k < \frac{1}{2}m$, *there exists a linear map* $T(M) \to R^{2m-1}$.

(b) *Assume* $k \leqq \frac{1}{2}m$ *and* $n \geqq 2m - k + 1$. *If* f, g: $M \to R^n$ *are differentiable imbeddings,* f_*, g_*: $T(M) \to R^n$ *are linearly homotopic.*

(2.9) COROLLARY. $f(m)$ *and* $g(M)$ *have equivalent normal bundles.*

PROOF. We may assume that M is connected. To prove (a), we construct a cross-section of $Q_{2m-k}(M) \to M^*$ and apply (2.3). To prove (b), it will suffice to show that any two cross-sections of $Q_n(M)$ are homotopic.

Both these results are proved by showing that all obstructions vanish. This, in turn, is proved by computing the cohomology of M^*. It suffices to show that $H^i(M^*; G_{i,n}) = 0$, and $H^i(M; {}^*G_{i-1,n}) = 0$ for $i \geqq 2m - k$. Let Z' be either integer coefficients Z or the twisted integer system Z_T associated to the covering $M \times M - \Delta \to M^*$, and let Z'' be the other. Now $\pi_j(S^{n-1})$ is a finitely generated abelian group, hence a direct sum of cyclic groups. Therefore it suffices to show that $H^i(M^*; Z' \otimes Z_m) = 0$ for $m = 2, 3, \cdots, \infty$. By examining the exact cohomology sequence [5, 38. 5] corresponding to the sequence $0 \to Z' \to Z' \to Z' \otimes Z_m \to 0$, we find it suffices to prove $H^i(M^*; Z') = 0$ for $i \geqq 2m - k$. Consider the Thom-Gysin sequence of the covering $M \times M - \Delta \to M^*$: $\cdots \to H^i(M \times M - \Delta) \to H^i(M^*; Z') \to H^{i+1}(M^*; Z'') \to H^{i+1}(M \times M - \Delta) \to \cdots$. This sequence is exact and is described in [6, Ch. I, III]. If we can show $H^i(M \times M - \Delta) = 0$ for $i \geqq 2m - k$, proof will be complete; for, by exactness, then $H^i(M^*; Z') \approx H^{i+1}(M^*; Z'')$ for $i \geqq 2m - k$, and both groups vanish for $i > 2m$ since dim $M^* = 2m$. But $H^i(M \times M - \Delta) \approx H_{2m-i}(M \times M, \Delta)$ by Lefschetz duality, and this last is 0 for $2m \geqq i \geqq 2m - k$ by the Künneth formula and the connectedness of M. This completes the proof.

We remark that if M is either non-compact or bounded, this result can be improved by elementary means, for there is no obstruction to constructing a linear map $T(M) \to R^{2m-k-1}$, and no obstruction to a linear homotopy between linear maps $T(M) \to R^n$, if $n \geqq 2m - k$. This is because such obstructions lie in $H^{i+1}(M; \pi_i(V_{2m-k-1,m}))$ and $H^i(M; \pi_i(V_{n,m}))$ respectively, and $\pi_i(V_{p,q}) = 0$ for $i < p - q$.

Combining (4.1) and (2.8) proves

(2.10) THEOREM. *Let* M *be as in* (2.8). *If* $k < \frac{1}{2}m$, *there exists an immersion* $M \to R^{2m-k}$. *If* $k \leqq \frac{1}{2}m$ *and* $n \leqq 2m - k + 1$, *any two differentiable imbeddings* $M \to R^n$ *are regularly homotopic.*

3. Linear maps $T(M) \to T(N)$.

Let U be a neighborhood of Δ in $M \times M$. We call a map θ: $U \to N \times N$

equivariant if $\theta(x, y) = (u, v)$ whenever $\theta(y, x) = (v, u)$, and $u \neq v$ if $x \neq y$. We let $\theta \mid \Delta: M \to N$ be denoted by $\bar{\theta}$. If $f: M \to N$ is an immersion, there is some neighborhood U of Δ such that the map $\theta_f: U \to N \times N$, defined by $\theta_f(x, y) = \theta(f(x), f(y))$ is equivariant. Obviously $\bar{\theta}_f = f$.

We describe next an operator Φ assigning to every equivariant map $\theta: U \to N \times N$ a skew $\Phi(\theta): T(M) \to T(N)$. We first assign to θ a map $\xi: T(M) \to T(N)$ which is not necessarily a fibre map, but satisfies the other conditions for a skew map, namely $\xi(-X) = -\xi(X)$ an $\xi(X) \neq 0$ if $X \neq 0$. This is done as follows. Let V be a neighborhood of $T_0(M)$ in $T(M)$ such that $V \subset A_M$ and $e_M(V) \subset U \cap G^{-1}(O_N)$. (See §2 for the definition of e_M, A_M, O_M). Let ε be a positive continuous function on $T(M)$ with the properties that $\varepsilon(X) = \varepsilon(-X)$, $\varepsilon(X)X \in V$, and $\varepsilon = 1$ in a neighborhood of $T_0(M)$. Define $\xi: T(M) \to T(N)$ by $\xi(X) = 1/\varepsilon(X) e_N^{-1} \theta e_M(\varepsilon(X)X)$. Given such a map ξ, we construct a skew map φ as follows. Let $\bar{\xi}: M \to N$ be the restriction of ξ to $T_0(M)$ with M and $T_0(N)$ with N in the obvious way. Let W be a neighborhood of $T_0(M)$ such that if $X \in W$ is based at x, and $\xi(X)$ is based at y, then $(\bar{\xi}(x), y) \in O_N$. Let ε be a positive function on $T(M)$ such that $\varepsilon(-X) = \varepsilon(X)$, $\varepsilon(X)X \in W$, and $\varepsilon(X) = 1$ in a neighborhood of $T_0(M)$. If $(x, y) \in O_N$, let $\tau_{xy}: T(N/y) \to T(N/x)$ be the operation of parallel translation along the minimal geodesic joining x to y. Now define $\varphi: T(M) \to T(N)$ by $\varphi(X) = 1/(\varepsilon(X)) \tau_{\bar{\xi}(x)y} \xi(\varepsilon(X)X)$. It is easily verified that φ is skew, and $\bar{\varphi} = \bar{\xi}$. Moreover if ξ is skew, $\varphi = \xi$ because $\bar{\xi}(x) = y$ and τ_{yy} is the identity.

Starting with an equivariant map θ, the construction $\theta \to \xi \to \varphi$ produces a skew map $\Phi(\theta)$. Conversely, starting with a skew map $\varphi: T(M) \to T(N)$, we define an equivariant $\Theta(\varphi) = \theta: O_M \to N \times N$ by $\theta(x, s) = e_N \varphi e_M^{-1}(x, y)$. It is easy to see that if $\theta = \Theta(\varphi_0)$, then $\varphi(\theta)$ and φ_0 are skew homotopic; while if $\varphi = \varphi(\theta_0)$, then $\Theta(\varphi)$ and θ_0 are equivariantly homotopic on some neighborhood of Δ. Furthermore, Θ carries skew homotopic maps into equivariantly homotopic maps, while if θ_0 and θ_1 are equivariantly homotopic on some neighborhood of Δ, then $\Phi(\theta_0)$ and $\Phi(\theta_1)$ are skew homotopic. Finally if $\varphi_0 = \Phi(\theta_0)$ or $\theta_0 = \Theta(\varphi_0)$, then $\bar{\varphi} = \bar{\theta}_0$.

We define two equivariant maps $\varphi_0: U_0 \to N \times N$, $\varphi_1: U_1 \to N \times N$ to be *germ homotopic* if there is an equivariant homotopy $\psi_t: V \to N \times N$ such that $\psi_i \mid V \cap U_i = \varphi_i \mid V \cap U_i$ for $i = 0, 1$.

The above results are summarized in

(3.1) THEOREM. Θ *induces a one-one correspondence between skew homotopy classes of skew maps* $T(M) \to T(N)$ *and germ homotopy classes of equivariant maps* $\theta: U \to N \times N$. *If* $\Theta(\varphi) = \theta$, *then* $\bar{\theta} = \bar{\varphi}$.

(3.2) COROLLARY. *If* $2n > 3m$, *the existence of a linear map*

$T(M) \to T(N)$ is independent of the structures of $T(M)$ and $T(N)$.

PROOF OF (3.2). Clearly the existence of equivariant maps is independent of any differential considerations. The proof follows from (3.1) and (1.3).

4. Existence and classification of immersions

The results of this section are based on

(4.1) THEOREM. *Assume $n > m$. If φ: $T(M) \to T(N)$ is linear, $\bar{\varphi}$ can be approximated by immersions f: $M \to N$ such that f_* and φ are linearly homotopic. Two immersions f, g: $M \to N$ are regularly homotopic if and only if f_* and g_* are linearly homotopic.*

PROOF. See [1, § 5].

Applying (1.3) we have

(4.2) THEOREM.

(a) *Assume $2n > 3m$. If φ: $T(M) \to T(N)$ is skew, $\bar{\varphi}$ can be approximated by immersion f: $M \to N$ such that f_* and φ are skew homotopic.*

(b) *Assume $2n > 3m + 1$. Two immersions f, g: $M \to N$ are regularly homotopic if and only if f_* and g_* are skew homotopic.*

From this and (3.1) we obtain

(4.3) THEOREM.

(a) *Assume $2n > 3m$. If U is a neighborhood of Δ in $M \times M$, and δ: $U \to N \times N$ is equivariant, $\bar{\delta}$ can be approximated by immersions f: $M \to N$ such that δ_f and δ are germ homotopic.*

(b) *Assume $2n > 3m + 1$. Two immersions f, g: $M \to N$ are regularly homotopic if and only if δ_f and δ_g are germ homotopic.*

(4.4) COROLLARY. *If $2n > 3m$, the existence of immersions $M \to N$ is independent of the differential structures of M and N.*

5. Topological immersions

A map f: $M \to N$ is called a *topological immersion* if some neighborhood of each point of M is mapped topologically.

The equivariant map δ_f is defined as before. A *topological regular homotopy* is a homotopy f_t: $M \to N$ such that for each point x of M, there is a neighborhood U which is mapped topologically by each f_t. The independence of U from t is important here. It follows that δ_{f_t} is an equivariant homotopy on some neighborhood of Δ.

(5.1) THEOREM.

(a) *Assume $2n > 3m$. A topological immersion f: $M \to N$ can be* *approximated by differentiable immersions* $g: M \to N$ *such that* δ_f *and* δ_g *are germ homotopic.*

(b) *Assume* $2n > 3m + 1$. *Two differentiable immersions* $f, g: M \to N$ *are differentiably regularly homotopic if and only if they are topologically regularly homotopic.*

PROOF. Apply (4.3).

(5.2) COROLLARY. *If* M *is homeomorphic to a* π*-manifold,* M *can be immersed in* R^n, *where* $2n > 3m$.

PROOF. [1, 6.5], a π-manifold of dimension m can be immersed in R^{m+1}. Now apply (5.1a).

REMARK.. Let d be an integer such that $H^i(M) = 0$ for $i > d$ (and any group of coefficients). J. Milnor has pointed out to us that in § 4 and § 5, $3m$ could be replaced by $2m + d$ (also in (2.3), (2.4) and (3.2)). This is because in the hypothesis of (1.3), dim B can be replaced, without any change in the proof, by any integer d such that $H^i(B) = 0$ for $i > d$.

6. Tangent vector fields

In this section (which is independent of immersion theory) we show how to apply §§ 1 and 3 to obtain results which prove that certain invariants of the tangent bundle of a manifold are homeomorphism invariants. (It is not known whether the tangent bundle itself is a homeomorphism invariant. *Added in Proof*: Milnor has recently proved it is not.)

Let M and M' be differentiable m-manifolds with the same underlying topological space K, which we assume to be triangulated. No assumptions as to the compatibility of the differentiable and combinatorial structures are made; K can equally well be considered as a CW-complex.

Let λ be a field of tangent k-frames on M, over the $(i-1)$-skeleton K_{i-1}; briefly, a *k-field in M over* K_{i-1}. Then there is an obstruction cochain $\omega(\lambda) \in C^i(K; \pi_{i-1}(V_{m,q}))$ whose vanishing is necessary and sufficient for the extension of λ over K_i.

(6.1) THEOREM. *Let* λ *be a k-field in* M *over* K_{i-1}, *with obstruction cochain* $\omega(\lambda)$. *If* $i \leqq 2m - 2k - 1$ *there is a k-field* λ' *in* M' *over* K_{i-1} *such that* $\omega(\lambda') = \omega(\lambda)$.

Before proving (6.1), we point out some consequences.

(6.2) COROLLARY. *If* M *admits a k-field with* $k \leqq \frac{1}{2}(m-1)$, *so does* M'.

PROOF OF (6.2). Take $i = m$ in (6.1). By assumption, there is a k-field λ in M over K_{m-1} such that $\omega(\lambda) = 0$. By (6.1), there is a k-field λ' in M' over K_{m-1} with $\omega(\lambda') = 0$. Thus λ' can be extended over K_m.

As a special case of (6.1), pointed out to us by Milnor, suppose M is closed (i.e., connected, compact, and unbounded) and $(k-1)$-connected.

Then $H^i(M; \pi_{i-1}(V_{m,k})) = 0$ for $i < m$; hence there is a k-field λ in M over $M - x$, for any $x \in M$. The obstruction to extending λ lies in $\pi_{m-1}(V_{m,k})$, and is independent of the choice of λ. Thus to each $(k-1)$-connected closed differentiable manifold M is associated an element $\omega(M) \in \pi_{m-1}(V_{m,k})$. (Actually ω depends on k as well as M.) Applying (6.1) yields

(6.3) COROLLARY. *If M and M' are homeomorphic closed $(k-1)$-connected m-manifolds, then $\omega(M) = \omega(M')$.*

PROOF OF (6.1). Since M and M' are homeomorphic, by (3.1) there is a skew map φ: $T(M) \to T(M')$ covering the identity map of K. Now a k-field λ in M over K_{i-1} is nothing but a linear map λ: $K_{i-1} \times R^k \to T(M)$ covering the identity map of K_{i-1}, and likewise for M'. Thus $\varphi\lambda$: $K_{i-1} \times R^k \to T(M')$ is a skew map covering the identity map of K_{i-1}. By (1.2), there is a linear map λ': $K_{i-1} \times R^k \to T(M')$ which is skew homotopic to $\varphi\lambda$, and which also covers the identity map of K_{i-1}. It follows from (1.1) that $\omega(\lambda') = \omega(\lambda)$.

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