ANNALS OF MATHEMATICS
Vol. 75, No. 3, May, 1962
Printed in Japan

KNOTTED ($4k-1$)-SPHERES IN $6k$-SPACE*

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(Received July 21, 1961)

It is proved in [3] that any differentiably imbedded $(4k-1)$-sphere in an m-sphere S^m is unknotted, i.e., bounds in S^m a differentiable $4k$-disk, provided that $m > 6k$. We want to show here that this is no longer true if $m = 6k$. In fact, there is an infinite number of isotopy classes of differentiably imbedded $(4k-1)$-spheres in $6k$-space. Nevertheless according to Zeeman [13], in the combinatorial case, any n-sphere is unknotted in S^m if $m > n+2$.

Following an idea of Fox and Milnor [2], we first define the group $\Sigma^{m,n}$ of h-cobordism classes of imbedded n-spheres in S^m (see definition in 1.3). An imbedded n-sphere in S^m is h-cobordant to zero if it bounds a differentiable homotopy $(n+1)$-disk in the unit disk D^{m+1}.

According to a new result of Smale [11], in most cases the elements of $\Sigma^{m,n}$ correspond to the isotopy classes of imbedded n-spheres in m-spheres (cf. 1.3).

The main result of this paper is that $\Sigma^{6k,4k-1}$ is isomorphic to the group of integers, if $k > 1$.

Here is an outline of the proof which relies heavily on results of Kervaire and methods of Milnor [7]. We first define a homomorphism $i : \Sigma^{6k,4k-1} \to Z$ as follows. An imbedded $(4k-1)$-sphere K^{4k-1} in S^{6k} is, according to Kervaire [7], the boundary of a manifold V of index 0 imbedded in a unit ball B^{6k+1}; moreover V admits a field F of normal $(2k+1)$-frames in B. We then define an integer $i(B, V, F)$ by means of linking numbers (cf. 2.5); it turns out that $i(B, V, F)$ depends only on the h-cobordism class of the imbedded K^{4k-1} in S^{6k} and provides a homomorphism $i : \Sigma^{6k,4k-1} \to Z$ (cf. § 2).

It is proved in § 3 that i is injective for $k > 1$, using surgery methods as in Milnor [7].

Finally a specific imbedding for which $i = \pm 1$ is constructed in 4.1, proving that i is surjective.

It is possible to prove, using methods of [3] and results of this paper, that the isotopy classes of differentiable imbeddings of S^{4k-1} in S^{6k} (in the strong sense of [3]) are in $1-1$ correspondence with the integers (even for $k = 1$).

In a subsequent paper we plan to give more information on the groups $\Sigma^{m,n}$.

* This work is partially supported by NSF contract G-11594

A conversation with M. A. Kervaire has helped me to simplify some of the proofs of § 2.

Terminology. All the manifolds, submanifolds, maps, imbeddings considered here are implicitly supposed to be differentiable of class C^∞.

S^m denotes the unit sphere in the euclidean space R^{n+1} and D^{n+1} the unit disk in R^{n+1} bounded by S^n. By an n-sphere we mean a differentiable manifold of dimension n having the same homotopy type as S^n; an n-disk is a contractible manifold of dimension n with a simply connected boundary.

1. h-Cobordism classes of pairs of manifolds

1.1. DEFINITION. We consider pairs (A, B) of a compact oriented manifold A and a compact oriented submanifold B of A. If A and B have boundaries ∂A and ∂B, we suppose that ∂B is a submanifold of ∂A and that B meets ∂A transversally along ∂B. The pair $(\partial A, \partial B)$ will also be denoted by $\partial(A, B)$. The pair $(-A, -B) = -(A, B)$ is obtained from (A, B) by reversing both orientations. An imbedding of a pair (A, B) in a pair (A', B') is an imbedding of A in A' whose restriction to B is an imbedding of B in B'.

Two pairs (M_1, N_1) and (M_2, N_2), without boundaries, are *h-cobordant* if there exists a pair (V, W) such that

(1) $\partial(V, W) = (M_1, N_1) - (M_2, N_2)$, i.e., there exists an orientation preserving diffeomorphism of $\partial(V, W)$ onto the disjoint union of (M_1, N_1) and $-(M_2, N_2)$,

(2) M_i and N_i are deformation retracts of V and W resp., $i = 1, 2$.

This is clearly an equivalence relation noted $(M_1, N_1) \overset{h}{\sim} (M_2, N_2)$. The definition can be easily extended to the case where M_i and N_i have boundaries.

Notice that M_1 and M_2 (resp. N_1 and N_2) are h-cobordant (J-equivalent in the old terminology, cf. Milnor [7]).

If the submanifold N_1 of M_1 is *isotopic* to the submanifold N_2 of M_2, i.e., if there exists an orientation preserving diffeomorphism of (M_1, N_1) onto (M_2, N_2), then these two pairs are h-cobordant. A recent result of Smale [11] asserts that the converse is true provided that $\dim N_i > 4$, $\dim M_i - \dim N_i \neq 2$, M_i, N_i simply connected.

1.2. *Connected sum of pairs.* Let (M_i, N_i) be two pairs of closed connected manifolds, $i = 1, 2$, $\dim M_i = m$, $\dim N_i = n$. The connected sum $(M_1, N_1) \# (M_2, N_2) = (M_1 \# M_2, N_1 \# N_2)$ is defined as follows. As in Milnor [8, p. 13], consider the pair (rD^m, rD^n) of the m-disk of radius r defined by $\sum x_i^2 \leqq r^2$ in $R^m(x_1, x_2, \cdots, x_m)$ and the n-disk intersection of rD^m with the plane $x_j = 0, n < j \leqq m$. Let p_1 and p_2 be imbeddings of

$(2D^m, 2D^n)$ in (M_1, N_1) and $-(M_2, N_2)$ respectively, which are orientation preserving. In the disjoint sum

$$(M_1 - p_1(1/2D^m), N_1 - p_1(1/2D^n)) + (M_2 - p_2(1/2D^m), N_2 - p_2(1/2D^n)) ,$$

identify $p_1(t\vec{u})$ with $p_2(1/t\vec{u})$, $\vec{u} \in S^{m-1}$, $1/2 < t < 2$. One then obtains a pair $(M_1 \# M_2, N_1 \# N_2)$ which is unique up to orientations preserving diffeomorphism of pairs. This follows from the fact that two orientations preserving imbeddings of (D^m, D^n) in a pair (M, N) of connected manifolds are isotopic (proof as in Milnor [8, Theorem 2.2]).

This sum operation, defined for classes of connected pairs of closed m-manifolds and n-manifolds up to orientations preserving diffeomorphisms, is associative and commutative. There is a unit element, the pair (S^m, S^n), where S^n is imbedded in S^m by the natural inclusion of R^{n+1} in R^{m+1}.

Moreover the h-cobordism class of the connected sum of two pairs depends only on the h-cobordism classes of these pairs; the proof as in Milnor [7, 2.3]. So that the h-cobordism classes of pairs of connected closed manifolds of dimension m and n respectively form also a semi-group.

This sum operation can also be defined for pairs $(M_i\, N_i)$, $i = 1, 2$, of manifolds with connected boundaries. In the preceding definition, (D^m, D^n) is replaced by the pair (D^m_+, D^n_+) of half disks $x_1 \geqq 0$; moreover the imbedded $p_i(2D^m_+, 2D^n_+)$ must intersect $(\partial M_i, \partial N_i)$ along $p_i(2D^{m-1}, 2D^{n-1})$, where (D^{m-1}, D^{n-1}) is the intersection of (D^m_+, D^n_+) with the plane $x_1 = 0$. The unit element is in this case the pair (D^m, D^n).

1.3. *The groups $\theta^{m,n}$ and $\Sigma^{m,n}$.* The h-cobordism classes of pairs (S^m, K^n) consisting of an oriented n-sphere K^n imbedded in the unit m-sphere S^m (or rather in an m-sphere h-cobordant to S^m) form an abelian group $\theta^{m,n}$. The inverse of (S^m, K^n) is $-(S^m, K^n)$; This is proved in the same way as Lemma 2.4 in Milnor [7]. A pair (S^m, K^n) represents the unit element of $\theta^{m,n}$ if and only if, in an $(m + 1)$-disk whose boundary is S^m, K^n bounds an $(n + 1)$-disk.

The correspondence $(S^m, K^n) \to K^n$ induces a homomorphism of $\theta^{m,n}$ in θ^n, the group of h-cobordism classes of n-spheres (cf. Milnor [7]). The kernel is the *subgroup* $\Sigma^{m,n}$ of $\theta^{m,n}$, of pairs represented by an n-sphere h-cobordant to the usual S^n imbedded in S^m. The image is the subgroup of θ^n of h-cobordism classes of n-spheres which admit a representative imbeddable in S^m.

It follows directly from the definition that $\Sigma^{n+1,n} = 0$ for all n. According to Fox-Milnor [2], the group $\Sigma^{3,1}$ is not finitely generated. From [3] it follows that $\Sigma^{m,n} = 0$ for $2m > 3n + 3$. In the limit case $2m =$

$3n + 3$, then $m = 3d$ and $n = 2d - 1$. We shall deal here with the case d even. When d is odd and > 1, we shall prove later that there exists a homomorphism of Z_2 onto $\Sigma^{3d,2d-1}$; we cannot decide in general if this homomorphism is injective.

1.4. *Framed submanifold.* A framed submanifold N of M is a triple (M, N, F), where (M, N) is a pair of manifolds and F a framing of the normal bundle of N in M: at each point x of N, $F(x)$ is an ordered basis $f_1(x), \cdots, f_k(x)$ of the normal space to N at x which depends differentiably on x. Moreover the orientation of M at x is the sum of the orientation of N at x and the orientation given by $F(x)$[1].

All the definitions of 1.1 and 1.2 can be extended to framed submanifolds. If M and N have boundaries, then $\partial(M, N, F)$ is the framed submanifold $(\partial M, \partial N, \partial F)$, where ∂F is the restriction of F to ∂N. By definition $-(M, N, F) = (-M, -N, F)$.

Two closed framed submanifolds (M_i, N_i, F_i), $i = 1, 2$, are h-cobordant if there exists a framed submanifold (V, W, F) such that

(1) $\partial(V, W, F) = (M_1, N_1, F_1) - (M_2, N_2, F_2)$

(2) M_i, N_i are deformation retracts of V and W respectively.

It follows from the result of Smale quoted above, that the framed submanifolds (M_i, N_i, F_i), $i = 1, 2$, are h-cobordant, if and only if there exists an orientation preserving diffeomorphism of (M_1, N_1) onto (M_2, N_2) which maps F_1 on F_2.

The connected sum $(M_1 \# M_2, N_1 \# N_2, F_1 \# F_2)$ of connected framed submanifolds (M_i, N_i, F_i), $i = 1, 2$, dim $M_i = m$, dim $N_i = n$, is also defined as before. One considers (in the case of closed submanifolds for instance) imbeddings of the standard framed D^n in $D^m : (D^m, D^n, F)$, where $F = (t_{n+1}, \cdots, t_m)$; t_k is the unit vector which defines the orientation of the k-axis in R^m.

The h-cobordism classes of connected closed framed n-submanifolds of m-manifolds also form a semi-group. The unit element is the standard framed S^n in $S^m : (S^m, S^n, F_0) = \partial(D^{m+1}, D^{n+1}, F)$.

In particular, the h-cobordism classes of framed n-spheres in an m-sphere h-cobordant to S^m form an abelian group $F^{m,n}$. There is a natural homomorphism of $F^{m,n}$ in $\theta^{m,n}$ by ignoring the framing; the kernel consists of the h-cobordism classes of the framings of the standard pair (S^m, S^n).

1.5 *The suspension.* The suspension of an m-disk B is the $(m + 1)$-disk

[1] In principle, the normal space of N at x will be the quotient of the tangent space M_x of M at x by the tangent space N_x of N at x. Nevertheless we shall often identify implicitly this normal space with a subspace of M_x complementary to N_x. This ambiguity is not important because only the homotopy class of the framing F is relevant.

SB obtained from the product $B \times D^1$ by smoothing (see Milonr [7, appendix]) corners along $\partial B \times \{1\}$ and $\partial B \times \{-1\}$ (recall that D^1 is the interval $[-1, +1]$); B is identified with the subspace $B \times \{0\}$ of SB.

The suspension of an n-sphere K^n h-cobordant to S^n, i.e., which bounds an $(n+1)$-disk B, is the boundary SK^n of SB. Its h-cobordism class does not depend on the particular choice of B. Again K^n is identified with a subspace of SK^n: the boundary of $B \subset SB$.

The suspension of a pair (K^m, V), where K^m is h-cobordant to S^m, is the pair (SK^m, V), where $V \subset K^m \subset SK^m$.

The suspension of a framed submanifold (K^m, V, F) is the framed submanifold (SK^m, V, SF), where SF is the field F completed by the vector field along V normal to K^m in SK^m (compare Kervaire [4]).

Similar definition if K^m is replaced by a disk,

The h-cobordism class of the suspension of a pair depends only on the h-cobordism class of this pair; thus suspension induces homomorphisms: $\theta^{m,n} \to \theta^{m+1,n}$ and $\Sigma^{m,n} \to \Sigma^{m+1,n}$.

The N-fold suspension is the suspension iterated N times.

2. The integer attached to an element of $\Sigma^{6k,4k-1}$

We first state a theorem which follows from results of Kervaire.

2.1. THEOREM. *Any* $(4k-1)$*-sphere* K, *h-cobordant to* S^{4k-1}, *imbedded in the boundary of a* $(6k+1)$*-disk* B, *bounds in* B *a framed submanifold* (B, V, F) *with* index $V = 0$.

Let $(\partial B, K, F)$ be a framed n-submanifold in the boundary of an $(m+1)$-disk B. The Thom construction (see Thom [12]) associates to $(\partial B, K, F)$ an element $\nu(F) \in \pi_m(S^{m-n})$. This element is 0 if and only if $(\partial B, K, F)$ bounds a framed submanifold in B.

On the other hand, according to Milnor-Kervaire [6], any framed $4k$-submanifold in an m-disk which bounds the standard framed S^{4k-1} in S^m (cf. 1.4) has index 0. This proves that 2.1 follows from 2.2.

2.2. THEOREM (Kervaire). *Let* K^n *be a sphere* h*-cobordant to* S^n, *imbedded in a sphere* K^{n+d} h*-cobordant to* S^{n+d}, *with* $n < 2d-1$. *Then* K^n *admits a field* F *of normal frames in* K^{n+d} *such that*

(1) $\nu(F) = 0$,

(2) *after* N*-fold suspension,* N *large,* (K^{n+d}, K^n, F) *becomes* h*-cobordant to the standard framed* S^n *in* S^{n+d+N}.

Statement (1) is proved in Kervaire [4] and [5, Lemma 4.1]. The fact that K^{n+d} and K^n are h-cobordant, instead of diffeomorphic, to S^{n+d} and S^n does not play any role in the proof. Before proving (2), we recall a few facts.

2.3. Let (K^{n+d}, K^n, F) be a framed n-sphere K^n in an $(n+d)$-sphere K^{n+d}. Given a map ξ or K^n in the rotation group SO_d, one gets a new framed submanifold $(K^{n+d}, K^n, \xi\cdot F)$ by letting $\xi(x)$ act on $F(x)$ for all $x \in K^n$ (see Kervaire [4]); the h-cobordism class of $(K^{n+d}, K^n, \xi\cdot F)$ depends only on the homotopy class of ξ, also denoted by $\xi \in \pi_n(\mathrm{SO}_d)$.

It is easy to check (compare Kervaire [5, p. 133]), that

$$(K^{n+d}, K^n, \xi\cdot F) \overset{h}{\sim} (S^{n+d}, S^n, \xi\cdot F_0) \# (K^{n+d}, K^n, F)$$

where (S^{n+d}, S^n, F_0) is the standard framed S^n in S^{n+d}.

The Thom construction ν is additive and, according to Kervaire [4], $\nu(\xi\cdot F_0) = \sigma J\xi$, where J is the Whitehead homomorphism $\pi_n(\mathrm{SO}_d) \to \pi_{n+d}(S^d)$ and σ an automorphism of $\pi_{n+d}(S^d)$. Hence (see Kervaire [5, Lemma 4.1])

$$\nu(\xi\cdot F) = \sigma J\xi + \nu(F) .$$

2.4. PROOF OF 2.2, (2). *By decreasing induction on d.* By 2.2, (1), there exists a framing F of K^n in K^{n+d} with $\nu(F) = 0$. Suppose the theorem true for $d+1$: there exists an element $\xi \in \pi_n(\mathrm{SO}_{d+1})$ such that, for the suspension SF of F, $\nu(\xi\cdot SF) = 0$ and the $(N-1)$-fold suspension of (SK^{n+d}, K^n, SF) is h-cobordant to the standard (S^{n+d+N}, S^n, F_0). By 2.3, $J\xi = 0$. From the exactness of the diagram on [4, p. 364] (see also [5, footnote 5]), there exists an element $\eta \in \pi_n(\mathrm{SO}_d)$ which gives ξ by suspension and such that $J\eta = 0$. Then $(K^{n+d}, K^n, \eta\cdot F)$ satisfies (1) and (2) of 2.2 because $\nu(\eta\cdot F) = 0$ by 2.3, and its N-fold suspension is h-cobordant to the $(N-1)$-fold suspension of $(SK^{n+d}, K^n, \xi\cdot SF)$.

2.5. *The integer $i(B, V, F)$.* Let (B, V, F) be a framed $4k$-submanifold of a $(6k+1)$-disk B, the boundary of V being a $(4k-1)$-sphere in ∂B. Let f_1 be the first vector of the field F. We slightly push each $2k$-cycle c of V away from V along the direction f_1. More precisely, consider a riemannian metric on B such that ∂B is totally geodesic. Let j be the map of V in $B - V$ defined by $j(x) = \exp \varepsilon f_1(x)$, where ε is a positive number so small that $t \exp \varepsilon f_1(x) \notin V$ for $0 < t \leqq 1$. The linking number of $j(c)$ with V in B defines a linear function on the $2k$-cycles of V, hence a cohomology class $\lambda \in H^{2k}(V, \partial V) = H^{2k}(V)$ with integral coefficients modulo torsion. The value $\lambda^2[V]$ of its square on the fundamental class of V is an even integer (cf. Milnor [7, p. 8]).

We define $i(B, V, F) = 1/2\, \lambda^2[V]$.

If (B_i, V_i, F_i), $i = 1, 2$, are two such framed submanifolds, their connected sum (cf. 1.2, 1.4) is again a framed $4k$-submanifold of a $(6k+1)$-disk whose boundary is a $(4k-1)$-sphere. It follows from the definitions that

$$i(B_1 \# B_2, V_1 \# V_2, F_1 \# F_2) = i(B_1, V_1, F_1) + i(B_2, V_2, F_2) .$$

2.6. THEOREM. *The integer $i(B, V, F)$ depends only on the h-cobordism class of the pair $(\partial B, \partial V)$ and on the index of V.*

2.7. LEMMA. *$i(B, V, F)$ depends only on the h-cobordism class of $\partial(B, V, F)$.*

Suppose that $\partial(B_1, V_1, F_1)$ is h-cobordant to $\partial(B_2, V_2, F_2)$. By definition 1.4, there exists a framed submanifold (B_0, V_0, F_0) and a diffeomorphism h of $\partial(B_0, V_0, F_0)$ onto $\partial(B_1, V_1, F_1) - \partial(B_2, V_2, F_2)$. If one identifies the boundary of the disjoint union $(B_1, V_1, F_1) - (B_2, V_2, F_2)$ with the boundary of $-(B_0, V_0, F_0)$ by h, one gets a framed $4k$-submanifold (M, N, F), where M is a $(6k + 1)$-sphere, because ∂B_1 and ∂B_2 are deformation retracts of B_0.

Define as before the integer $i(M, N, F) = 1/2\lambda^2[N]$; here $\lambda \in H^{2k}(N)$ associates to each $2k$-cycle c of N the linking number with N of c pushed away from N along a vector of F. As ∂V_1 and ∂V_2 are deformation retracts of V_0,

$$i(M, N, F) = i(B_1, V_1, F_1) - i(B_2, V_2, F_2) .$$

Hence 2.7 follows from 2.8.

2.8. PROPOSITION. *For any framed $4k$-submanifold (M, N, F) of a $(6k + 1)$-sphere M, the integer $i(M, N, F)$ is zero.*

We may assume that M is the boundary of a $(6k + 2)$-disk B by replacing M by $M \# -M$ (the value of $i(M, N, F)$ is not changed if one removes a disk in M far away from N).

Suppose that (M, N, F) is cobordant to zero (cf. Thom [12]), i.e., it bounds a framed submanifold (B, N', F'). Then $i(M, N, F) = 1/2\lambda^2[N] = 0$. This is because λ is the restriction to $N = \partial N'$ of the cohomology class $\lambda' \in H^{2k}(N')$ defined, like λ, by linking numbers; hence $\lambda^2[N] = \langle \lambda'^2, \partial N' \rangle = 0$.

Two framed $4k$-submanifolds (M_i, N_i, F_i), $i = 1, 2$, in the boundaries M_i of $(6k + 2)$-disks B_i are cobordant if $(M_1, N_1, F_1) \# -(M_2, N_2, F_2)$ is cobordant to zero (cf. Thom [12]). The cobordism classes of such framed submanifolds form a group $C^{6k+1,4k}$ with respect to the connected sum. The integer $i(M, N, F)$ behaves additively and vanishes on framed submanifolds cobordant to zero; hence the correspondence $(M, N, F) \to i(M, N, F)$ gives a homomorphism i of $C^{6k+1,4k}$ in Z, the group of integers.

On the other hand, the Thom construction (cf. Thom [12], Pontrjagin [9]) gives an isomorphism of $C^{6k+1,4k}$ on $\pi_{6k+1}(S^{2k+1})$ which is finite (Serre [10]). Hence the homomorphism i is trivial.

2.9. PROOF OF 2.6. Let (B_i, V_i, F_i), $i = 1, 2$, be a framed $4k$-submanifold V_i of a $(6k + 1)$-disk B_i whose boundary is a $(4k - 1)$-sphere. Suppose that index V_1 = index V_2 and also that $(\partial B_1, \partial V_1) \overset{h}{\sim} (\partial B_2, \partial V_2)$. Then $\partial(B_1, V_1, F_1) \# -\partial(B_2, V_2, F_2)$ is h-cobordant to $(S^{6k}, S^{4k-1}, \xi \cdot F_0)$, where $\xi \in \pi_{4k-1}(\mathrm{so}_{2k+1})$ (cf. 1.4 and 2.3). As Index $(V_1 \# - V_2) = 0$, $(S^{6k}, S^{4k-1}, \xi \cdot F_0)$ bounds in a $(6k+1)$-disk a framed submanifold of index 0. According to Milnor-Kervaire [6], this implies that by N-fold suspension, N large, ξ is mapped on the zero element of $\pi_{4k-1}(\mathrm{so}_{2k-1+N})$. As the kernel of the homomorphism $\pi_{4k-1}(\mathrm{so}_{2k+1}) \to \pi_{4k-1}(\mathrm{so}_{2k-1+N})$ is finite (cf. Borel-Hirzebruch [1, p. 348]), ξ is an element of finite order r.

It follows that the connected sum of r copies of $\partial(B_1, V_1, F_1)$ is h-cobordant to the sum of r copies of $\partial(B_2, V_2, F_2)$. By Lemma 2.7, $ri(B_1, V_1, F_1) = ri(B_2, V_2, F_2)$; hence $i(B_1, V_1, F_1) = i(B_2, V_2, F_2)$, q. e. d.

2.10. An element of $\Sigma^{6k,4k-1}$ represented by a pair (K^{6k}, K^{4k-1}), where K^{6k} is the boundary of a disk B, bounds by 2.1 a framed submanifold (B, V, F) of index 0. The integer $i(B, V, F)$ depends only on the h-cobordism class of (K^{6k}, K^{4k-1}) and behaves additively (cf. 2.5). We can now state the final result of this section.

THEOREM. *The integer $i(B, V, F)$ provides a homomorphism i of $\Sigma^{6k,4k-1}$ in the group Z of integers.*

3. Spherical modifications of framed submanifolds

The object of this section is to prove the following:

3.1. THEOREM. *The homomorphism $i: \Sigma^{6k,4k-1} \to Z$ is injective for $k > 1$.*

Our aim is to simplify as much as possible the homotopy type of the submanifold V of 2.1 by spherical modifications. The only obstruction to getting a disk will be the integer $i(B, V, F)$.

3.2. We begin with a few general remarks. Let (B, V, F) be a framed n-submanifold V in an $(n+d)$-disk B; $F = (f_1, f_2, \cdots, f_d)$. Given an element $\alpha \in \pi_r(V)$, we make the following three hypotheses:

(1) α is represented by an embedding $g: S^r \to V$ (away from the boundary of V);

(2) g can be extended as an imbedding of the unit disk D^{r+1} in B so that $g(D^{r+1} - S^r) \cap V = \varnothing$ and $g(D^{r+1})$ is tangent to f_1 along $g(S^r)$.

(3) Consider the isomorphism of $\tau(R^{r+1})$[2] restricted to S^r onto the subbundle of $\tau(B)$ generated by $\tau(gS^r)$ and f_1, which maps $\tau(S^r)$ on $\tau(gS^r)$ by dg and the unit vector normal to S^r, pointing inside D^{r+1}, on f_1. The natural trivialization of $\tau(R^{r+1})$ by the basis vectors gives, *via* this isomorphism, a field of $(r+1)$ independent vectors $e_1, e_2, \cdots, e_{r+1}$ along $g(S^r)$.

[2] $\tau(M)$ denotes the tangent bundle of the manifold M.

Choose a trivialization of $\tau(B)$; we can view it as a bundle map T of $\tau(B)$ on R^{n+d} whose restriction to each fiber is an isomorphism. Then $x \rightarrow Te_1(x), \cdots, Te_{r+1}(x), Tf_2(x), \cdots, Tf_d(x)$ is a map of S^r in the Stiefel manifold $V_{n+d,r+d}$ of $(d+r)$-frames in $(n+d)$-space. Its homotopy class $\xi \in \pi_r(V_{n+d,r+d})$ is well defined.

The third hypothesis is $\xi = 0$ and $2r + 2 < n + d$.

3.3. PROPOSITION. *If the three preceding conditions are verified for $\alpha \in \pi_r(V)$, there exists in $B \times I$, $I=$ interval* $[0, 1]$, *a framed submanifold $(B \times I, \bar{V}, \bar{F})$ such that*

(a) *its intersection with $B \times \{0\} = B$ is (B, V, F),*

(b) *its intersection with $B \times \{1\} = B$ is (B, V', F'), where V' is obtained from V by a spherical modification associated to α* (cf. Milnor [7]),

(c) *if $\partial V \neq \emptyset$, then $(\partial B \times I) \cap \bar{V} = \partial V \times I$.*

We shall say that the new framed submanifold (B, V', F') is obtained from (B, V, F) by a spherical modification associated to $\alpha \in \pi_r(V)$.

PROOF. We can find in a neighborhood U of $g(D^{r+1})$ local coordinates $(x, y, z) = (x_1, \cdots, x_{r+1}, y_1, \cdots, y_{n-r}, z_1, \cdots, z_{d-1})$ in which $g(D^{r+1})$ is defined by $y = z = 0$, $x^2 = \sum x_i^2 \leqq 1$. Denote by $f'_k(x)$ the projection of the vector $f_k(x)$ in the plane $x = 0$, parallel to $y = z = 0$. The homotopy class of the map $x \rightarrow (f'_2(x), \cdots, f'_d(x))$ of S^r into the Stiefel manifold $V_{n+d-r-1,d-1}$ is an element $\xi' \in \pi_r(V_{n+d-r-1,d-1})$. By the natural inclusion of $V_{n+d-r-1,d-1}$ in $V_{n+d,r+d}$, ξ' gives ξ. If $2r + 2 < n + d$, this inclusion induces an isomorphism for homotopy groups of dimension r, as can be seen from the homotopy sequence of the fibration associated to this inclusion. Hence (3) implies $\xi' = 0$.

We can suppose therefore that $f_i = \partial/\partial z_{i-1}$ on $g(S^r)$ for $1 < i \leqq d$. Finally, using the tubular neighborhood theorem (Milnor [8]), we can choose U and the local coordinates (x, y, z) such that (see fig. 1)

(i) $V \cap U$ is defined by $\varphi(x,y) = -x^2 + y^2 + a(\rho^2) = 0$, $z = 0$, where $a(\rho^2)$ is a decreasing function of $\rho^2 = x^2 + y^2$, equal to 1 for $\rho \leqq 1$ and 0 for $\rho \geqq 2$,

(ii) on $V \cap U$, $f_1 =$ gradient of φ,

(iii) f_i is the restriction to $V \cap U$ of $\partial/\partial z_{i-1}$, $1 < i \leqq d$.

Define in the product $B \times I$ the manifold $\bar{V}$ as follows. Outside of $U \times I$, $\bar{V}$ is $(V - U) \times I$. In $U \times I$, $\bar{V}$ is defined by $\varphi(x, y, t) = -x^2 + y^2 + a(\rho^2)(1 - 2t) = 0$, $t \in I$, and $z = 0$. Notice that $\bar{V}$ coincides with $V \times I$ outside the ball $2D^{n+d} \times I : x^2 + y^2 + z^2 \leqq 4$. The field $\bar{F} = (\bar{f}_1, \cdots, \bar{f}_d)$ will be defined as $\bar{f}_i = (f_i, 0)$ outside of $U \times I$ and in $(U \times I) \cap \bar{V}$ by $\bar{f}_i = \partial/\partial z_{i-1}$ for $1 < i \leqq d$ and $\bar{f}_1 =$ gradient of $\varphi(x, y, t)$.

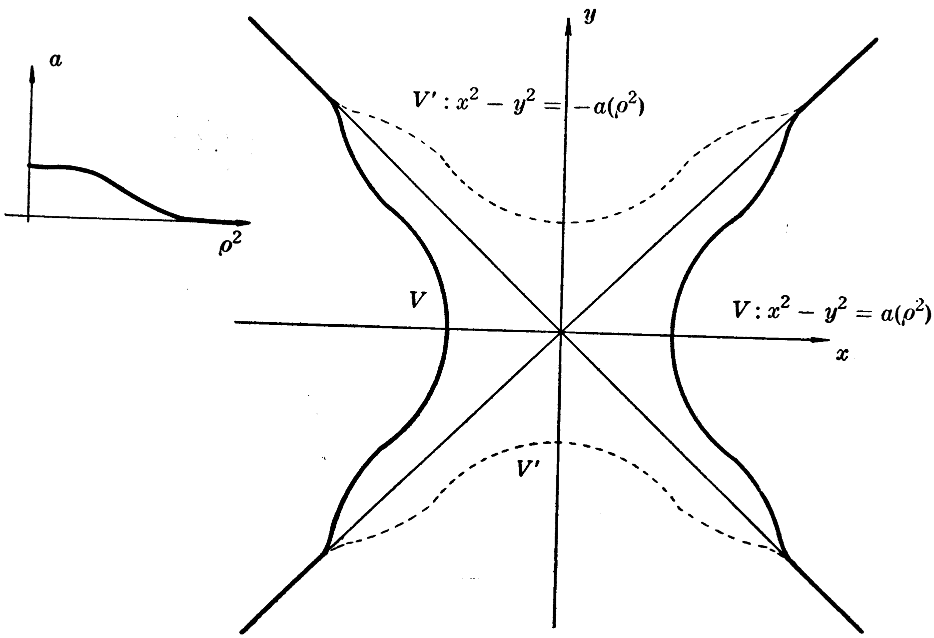

Fig. 1.

Notice that $V \cap 2D^{n+d}$ is diffeomorphic to $S^r \times D^{n-r}$ and $V' \cap 2D^{n+d}$ to $D^{r+1} \times S^{n-r-1}$; outside of $2D^{n+d}$, $V = V'$.

This completes the proof of 3.3.

3.4. Let $\xi \in \pi_r(V_{n+d,r+d})$ be defined as in 3.2, (3). Let ∂ be the boundary operator of the homotopy sequence of the bundle E: $\mathrm{so}_{n+d} \to V_{n+d,r+d}$, with fibre so_{n-r}.

LEMMA. *$\partial\xi \in \pi_{r-1}(\mathrm{so}_{n-r})$ is the obstruction to trivializing the normal bundle of $g(S^r)$ in V.*

This is because the bundle of normal frames to $g(S^r)$ in V is isomorphic to the inverse image of E by ξ.

3.5. PROOF OF 3.1. Any pair which represents an element of $\Sigma^{6k,4k-1}$ bounds a framed $4k$-submanifold (B, V, F), where B is a $(6k + 1)$-disk and index $V = 0$ (cf. 2.1).

First remark that the index of V is not changed by spherical modifications (cf. Thom [12]). The same is true for the integer $i(B, V, F)$ (cf. proof of 2.8).

After a sequence of spherical modifications, we can suppose that V is $(2k - 1)$-connected (Milnor [7]); indeed, the conditions (1), (2), (3) can always be verified if $r < 2k$.

According to Milnor [7], it is possible to find a basis $(\alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_s)$ of $H_{2k}(V, Z)$ with intersections $\langle\alpha_i, \alpha_j\rangle = 0, \langle\beta_i, \beta_j\rangle = 0, \langle\alpha_i, \beta_j\rangle = \delta_{ij}$. Moreover if $s > 1$, one can suppose, after a suitable change of basis which does not mix up the α's and the β's, that $\lambda(\alpha_s) = 0$; $\lambda \in H^{2k}(V)$ is the class which defines $i(B, V, F) = 1/2\lambda^2[V] = \sum\lambda(\alpha_i)\lambda(\beta_i)$ (cf. 2.5).

Now for a homology class $\alpha \in H_{2k}(V)$ such that $\langle\alpha, \alpha\rangle = 0$ and $\lambda(\alpha) = 0$, the three conditions of 3.2 can be verified if $k > 1$. First there exists an imbedding $g: S^{2k} \to V$ which represents α by Whitney's imbedding theorem, provided $k > 1$, (compare Milnor [7, 5.9]). The condition $\lambda(\alpha) = 0$ means that $g(S^{2k})$ pushed in the f_1-direction is homologous to zero in $B - V$; so it is also homotopic to zero in $B - V$ which is $(2k - 1)$-connected; the condition (2) is verified using Whitney's imbedding theorem $(2k + 2 < 6k + 1)$. Finally, consider the commutative diagram

$$\begin{array}{ccc} \pi_{2k}(S^{2k}) & \xrightarrow{\partial} & \pi_{2k-1}(\mathrm{SO}_{2k}) \\ \downarrow & & \downarrow \\ \pi_{2k}(V_{6k+1,4k+1}) & \xrightarrow{\partial} & \pi_{2k-1}(\mathrm{SO}_{2k}) \end{array}$$

associated to the inclusion $S^{2k} = \mathrm{SO}_{2k+1}/\mathrm{SO}_{2k} \to V_{6k+1,4k+1} = \mathrm{SO}_{6k+1}/\mathrm{SO}_{2k}$. The class ξ (as defined in 3.2, (3)) is the image of n times the generator of $\pi_{2k}(S^{2k})$, and $\partial\xi$ is the obstruction to trivializing the normal bundle of $g(S^{2k})$ in V (Lemma 3.4). According to Milnor [7, proof of 5.11], $\langle\alpha, \alpha\rangle = 0$ implies that $n = 0$. Hence $\xi = 0$.

From 3.3 and Milnor [7], it is then possible to get, after $s - 1$ spherical modifications, a framed submanifold (B, V, F) such that $H_{2k}(V)$ has two generators α, β with $\langle\alpha, \alpha\rangle = 0, \langle\beta, \beta\rangle = 0, \langle\alpha, \beta\rangle = 1$. Now if $i(B, V, F) = \lambda(\alpha)\lambda(\beta) = 0$, then $\lambda(\alpha) = 0$ or $\lambda(\beta) = 0$. A last spherical modification will lead to a framed submanifold (B, V, F), where V is a disk. This completes the proof of 3.1.

3.6. REMARK. The same argument shows, according to 2.8, that each element of $\pi_{6k}(S^{2k+1})$ is represented by a framed $4k$-submanifold of R^{6k} which is an homotopy sphere.

4. The generator of the group $\Sigma^{6k,4k-1}$

We describe here a specific imbedded $(4k - 1)$-sphere in $6k$-space which bounds in a $(6k + 1)$-space B a framed submanifold (B, V, F) with $i(B, V, F) = \pm 1$. This will prove that the homomorphism i of 2.10 is surjective.

4.1. We consider in the numerical space R^{3d} of coordinates $(x, y, z) = (x_1, \cdots, x_d, y_1, \cdots, y_d, z_1, \cdots, z_d)$ three imbedded $(2d - 1)$-spheres (see fig. 2):

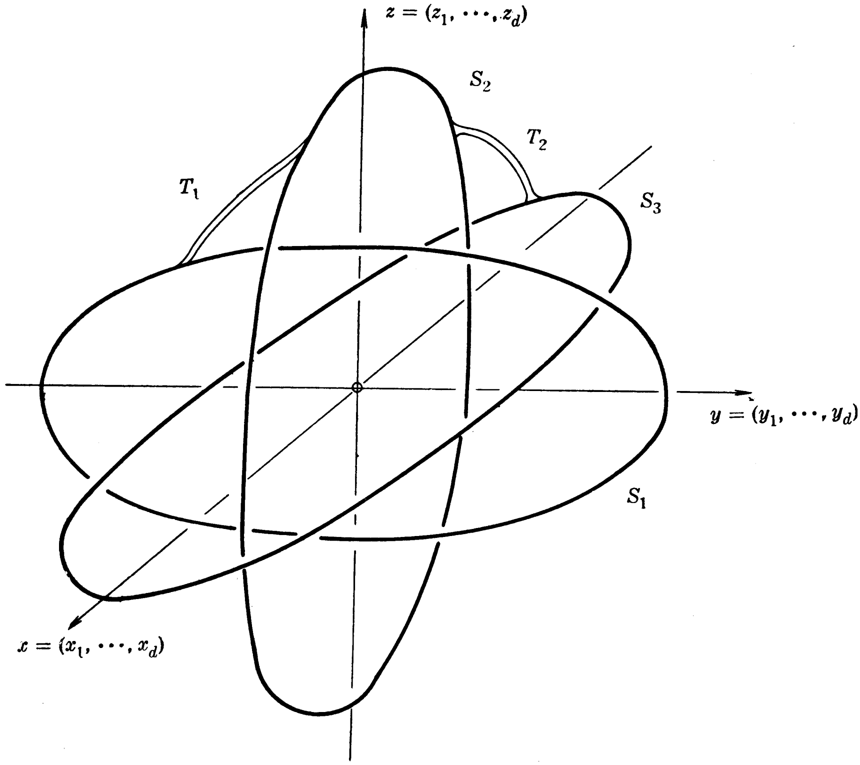

Fig. 2.

$$S_1 : x = 0 , \qquad \frac{y^2}{\alpha^2} + \frac{z^2}{\beta^2} = 1 ;$$

$$S_2 : y = 0 , \qquad \frac{z^2}{\alpha^2} + \frac{x^2}{\beta^2} = 1 ;$$

$$S_3 : z = 0 , \qquad \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 ;$$

α and β are real numbers, $\alpha > \beta > 0$.

To get an imbedded $(2d - 1)$-sphere S, we join S_1 to S_2 and S_2 to S_3 by thin tubes T_1 and T_2. More precisely, orient S_1, S_2 and S_3. Then construct an imbedding $\tau_1 : D^1 \times D^{2k} \to R^{6k}$ such that $\{-1\} \times D^{2k}$ is imbedded in S_1 with orientation preserved, $\{+1\} \times D^{2k}$ is imbedded in S_2 with orientation reversed, and $\tau_1(D^1 \times D^{2k})$ does not meet S_1, S_2 or S_3 elsewhere. Denote by D_0^{2k} the interior of the unit disk D^{2k}. Then remove $\tau_1(D^1 \times D_0^{2k})$ from the union $S_1 \cup \tau_1(D^1 \times D^{2k}) \cup S_2$ and smooth corners along $\tau_1(\partial D^1 \times \partial D^{2k})$. One gets a new imbedded $(2d - 1)$-sphere denoted by $S_1 \# S_2$ (this operation is just a spherical modification as described in 3.3). The h-cobordism class (or more precisely, the isotopy class) of $S_1 \# S_2$ does not depend on the particular choice of the imbedding τ_1 if $d > 1$, because two such imbeddings are isotopic. In the same way, piping a tube from $S_1 \# S_2$ to S_3, we get finally an imbedded $(2d-1)$-sphere $S = S_1 \# S_2 \# S_3$ in R^{3d}, or in S^{3d} if one adds the point at infinity.

4.2. PROPOSITION *This imbedded sphere S bounds in the unit disk B^{3d+1} a framed submanifold (B, V, F) such that $i(B, V, F) = \pm 1$ $(d = 2k)$.*

Removing the point at infinity, we replace B^{3d+1} by the half space $R_+^{3d+1}(x, y, z, t)$, $t \geqq 0$. To construct V (see fig. 3), consider the three

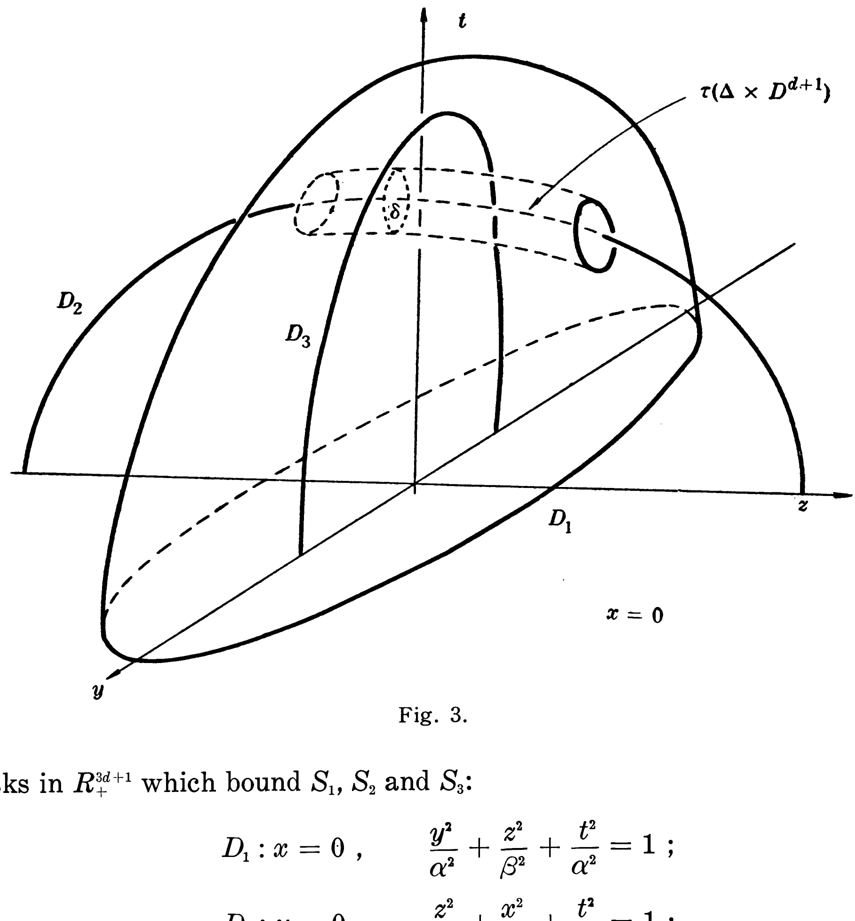

Fig. 3.

disks in R_+^{3d+1} which bound S_1, S_2 and S_3:

$$D_1 : x = 0 , \qquad \frac{y^2}{\alpha^2} + \frac{z^2}{\beta^2} + \frac{t^2}{\alpha^2} = 1 ;$$

$$D_2 : y = 0 , \qquad \frac{z^2}{\alpha^2} + \frac{x^2}{\beta^2} + \frac{t^2}{\beta^2} = 1 ;$$

$$D_3 : z = 0 , \qquad \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{t^2}{\gamma^2} = 1 , \qquad \beta < \gamma < \alpha .$$

The intersections $D_1 \cap D_3$ and $D_2 \cap D_3$ are empty and $D_1 \cap D_2$ is the $(d-1)$-sphere $\Sigma : x = y = 0; z^2 = t^2 = \alpha^2\beta^2/(\alpha^2 + \beta^2)$. Σ bounds in D_2 the d-disk $\Delta : x = y = 0, z^2/\alpha^2 + t^2/\beta^2 = 1, z^2 \leqq \alpha^2\beta^2/(\alpha^2 + \beta^2)$.

Now consider an imbedding τ of $\Delta \times D^{d+1}$ in the plane $x = 0$, such that $\tau(\partial\Delta \times D^{d+1}) \subset D_1$, $\tau(\Delta \times D^{d+1})$ intersects D_2 only on Δ and does not meet D_3. Moreover the intersection of $\tau(\Delta \times D^{d+1})$ with the plane $x = y = 0$ is $\tau(\Delta \times D^1)$, $D^1 \subset D^{d+1}$.

Replace D_1 by the submanifold $D_1' = D_1 - \tau(\Delta \times D_0^{d+1})$ with corners smoothed along $\tau(\partial\Delta \times \partial D^{d+1})$ (again D_0^{d+1} denotes the interior of D^{d+1}). In fact D_1' is obtained from D_1 by a spherical modification. Notice that $D_1' \cap D_2 = \varnothing$ and $D_1' \cap D_3 = \varnothing$.

D_1' is $(d-1)$-connected. $H_d(D_1', Z)$ has two generators: one is represented by the cycle $a = \tau(\{\delta\} \times \partial D^{d+1})$, $\delta \in \Delta$; the other one by the cycle b, union of $\tau(\Delta \times \{\xi\})$, $\xi \in \partial D^1$, and of the d-disk Δ' that $\tau(\partial\Delta \times \{\xi\})$ bounds in $D_1 \cap (x = y = 0)$.

The manifold V is obtained from D_1', D_2, D_3 by joining, in R_+^{3d+1}, D_1' to D_2 and D_2' to D_3 with half tubes (diffeomorphic to $D^1 \times S_+^{2d}$ with boundary $D_1 \times S^{2d-2}$ in R^{3d}) such that $\partial V = S$. Again V is $(d-1)$-connected; $H_d(V)$ has the two generators a and b with intersections $\langle a, a\rangle = 0$, $\langle b, b\rangle = 0$, $\langle a, b\rangle = \pm 1$.

The framing $F = (f_1, \cdots, f_{d+1})$ of V in R_+^{3d+1} is obtained by extending the natural framings of D_1, D_2, D_3 to D_1', D_2, D_3 and then to V along the half tubes. For instance f_1 restricted to D_1' will be a vector field normal to D_1' in the plane $x = 0$.

Now the cycle a bounds in R_+^{3d+1} the disk $\tau(\{\delta\} \times D^{d+1})$ which meets D_2 at $\tau(\delta)$; b bounds a disk in the plane $x = y = 0$ which intersects D_3 at the point $x = y = 0$, $t = \gamma$. Moreover these disks are tangent along their boundaries to a vector field homotopic to f_1. Hence $\lambda(a) = \pm 1$, and $\lambda(b) = \pm 1$.

As a consequence if $d = 2k$, $i(B, V, F) = \pm\, \lambda(a)\lambda(b) = \pm 1$.

Combining 3.1 and 4.2, we get the main result :

4.3. THEOREM. *The homomorphism* i: $\Sigma^{6k,4k-1} \to Z$ *is an isomorphism if* $k > 1$, *and is surjective if* $k = 1$. *The generator of* $\Sigma^{6k,4k-1}$ *is the imbedded sphere* S *described in* 4.1.

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