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Study of modules over formal triangular matrix rings

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Abstract

In this paper we carry out a systematic study of modules over a formal triangular matrix ring

$$T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}.$$

Using the alternative description of right T-modules as triples $(X,Y)_f$ with $X \in Mod - A$, $Y \in Mod - B$ and $f : Y \bigotimes_B M \to X$ in $Mod - A$, we shall characterize respectively uniform, hollow, finitely embedded, projective, generator or progenerator modules over T. For projective modules an explicit method for constructing a dual basis is described. Also necessary and sufficient conditions are found for a T-module to admit a projective cover. When the conditions are fulfilled we give an explicit method for constructing a projective cover.

MSC: 16D40; 16D90; 16S99

0. Introduction

All the rings we consider will be associative rings with $1 \neq 0$ and all the modules will be unital modules. Unless otherwise mentioned we will be working with right modules. For any ring R, the category of right R-modules is denoted by $Mod-R$. Given a formal triangular matrix ring $T = \left[\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right]$, it is well known [5] that the category $Mod - T$ is equivalent to a category Ω of triples $(X,Y)_f$ where $X \in Mod - A$, $Y \in Mod - B$ and $f : Y \bigotimes_B M \to X$ is a map in $Mod - A$.

Denoting the right T-module associated to $(X,Y)_f$ by $(X \oplus Y)_T$, in Section 1, we describe the triples in Ω which correspond to submodules and quotient modules of

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$(X \oplus Y)_T$. We determine necessary and sufficient conditions for a submodule $(X' \oplus Y')_T$ of $(X \oplus Y)_T$ to be essential (resp. small) in $(X \oplus Y)_T$. We then use these to characterize respectively uniform, hollow or local modules over T. In Section 2 the Jacobson radical as well as the socle of $(X \oplus Y)_T$ will be determined. Using the description of the socle we find necessary and sufficient conditions for T-modules to be finitely embedded. In [9] one of the authors proved that any quasi-injective finitely embedded module is co-hopfian. Applying our characterization of finitely embedded modules over T we construct examples of finitely embedded modules which are not co-hopfian, thus showing that quasi-injectivity is needed for the validity of the result in [9]. In Section 3 we give necessary and sufficient conditions for $(X \oplus Y)_T$ to be projective. When it is projective we give an explicit method for obtaining a dual basis. In Section 4 we first determine necessary and sufficient conditions for $(X \oplus Y)_T$ to be a generator. When $(X \oplus Y)_T$ is projective, these conditions take a particularly simple form. Using this result we obtain necessary and sufficient conditions for a T-module to be a progenerator. In Section 5 we obtain necessary and sufficient conditions for a T-module to admit a projective cover in $Mod - T$. When these conditions are satisfied we give an explicit method for constructing a projective cover of $(X \oplus Y)_T$. As an easy corollary of our results on projective covers, we obtain the well-known result that T is semi-perfect (resp. right perfect) if and only if A and B are so. Using description of left T-modules in terms of suitable triples, one sees that an analogous result is valid in the case of left perfectness as well.

Finally we wish to point out that formal triangular matrix rings play an important role in the representation theory of algebras.

1. Uniform, hollow, respectively local modules over *T*

We will first explain the notations that we will be adopting. For any left B, right A bimodule ${}_BM_A$ we write T for the formal triangular matrix ring

$$\begin{bmatrix} A & 0 \\ M & B \end{bmatrix}.$$

Let Ω denote the category whose objects are triples $(X, Y)_f$ where $X \in Mod - A$, $Y \in Mod - B$ and $f : Y \otimes_B M \to X$ is a map in $Mod - A$. If $(X, Y)_f$ and $(U, V)_g$ are objects in Ω, the morphisms from $(X, Y)_f$ to $(U, V)_g$ in Ω are pairs (φ_1, φ_2) where $\varphi_1 : X \to U$ is a map in $Mod - A$, $\varphi_2 : Y \to V$ is a map in $Mod - B$ satisfying the condition $\varphi_1 \circ f = g \circ (\varphi_2 \otimes Id_M)$. It is well-known [5] that the category Ω is equivalent to the category $Mod - T$. The right T-module corresponding to the triple $(X, Y)_f$ is the additive group $X \oplus Y$ with the right T-action given by

$$(x, y) \begin{bmatrix} a & 0 \\ m & b \end{bmatrix} = (xa + f(y \otimes m), yb).$$

To make the notation less cumbersome we write $(X \oplus Y)_T$ for this right T-module. It not only depends on X and Y but also on f. Often the map f occurring in a triple will be clear from the context. If $(\varphi_1, \varphi_2) : (X, Y)_f \to (U, V)_g$ is a map in Ω the associated map $\varphi : (X \oplus Y)_T \to (U \oplus V)_T$ in $Mod - T$ is given by $\varphi(x, y) = (\varphi_1(x), \varphi_2(y))$ for any $x \in X, y \in Y$. It is clear that φ is injective (resp. surjective)$\Leftrightarrow$ $\varphi_1 : X \to U$, $\varphi_2 : Y \to V$ are injective (resp. surjective).

Let $(X, Y)_f \in Obj\,\Omega$ and $(X \oplus Y)_T$ be the associated right T-module. We describe triples in $Obj\,\Omega$ which correspond to submodules (resp. quotient modules) of $(X \oplus Y)_T$. Note that under the right T-action on $X \oplus Y$ we have

$$(0 \oplus Y)\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} = (f(Y \otimes M), 0).$$

We will denote the submodule $f(Y \otimes M)$ of X_A by YM. Now consider $Y' \leq Y_B$ and let $j_2 : Y' \to Y$ denote the inclusion. Then

$$(0 \oplus Y')\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} = (f \circ (j_2 \otimes Id_M)(Y' \otimes M), 0).$$

The submodule $f \circ (j_2 \otimes Id_M)(Y' \otimes M)$ of X_A will be denoted by $Y'M$. Let $X' \leq X_A$ satisfy $Y'M \leq X'$. Writing f' for $f \circ (j_2 \otimes Id_M)$ and denoting the inclusion $X' \to X$ by j_1 we see that $(X', Y')_{f'}$ is in Ω and $(j_1, j_2) : (X', Y')_{f'} \to (X, Y)_f$ is a map in Ω realizing $(X' \oplus Y')_T$ as a T-submodule of $(X \oplus Y)_T$. Also it is clear that every T-submodule of $(X \oplus Y)_T$ is obtained in this way. Let X'' (resp. Y'') be a quotient of X_A (resp. Y_B) with $\eta_1 : X \to X''$ (resp. $\eta_2 : Y \to Y''$) the canonical quotient maps. Let $X' = \ker \eta_1$ and $Y' = \ker \eta_2$. Suppose $Y'M \leq X'$. Denoting the inclusions $X' \to X, Y' \to Y$ by j_1 and j_2, respectively, we get a map $f'' : Y'' \otimes_B M \to X''$ rendering the following diagram commutative

$$\begin{tikzcd}
Y' \otimes M \arrow[r, "j_2 \otimes Id_M"] \arrow[d, "f'"] & Y \otimes M \arrow[r, "\eta_2 \otimes Id_M"] \arrow[d, "f"] & Y'' \otimes M \arrow[r] \arrow[d, "f''"] & 0 \\
X' \arrow[r, "j_1"] & X \arrow[r, "\eta_1"] & X'' \arrow[r] & 0
\end{tikzcd}$$

In this diagram $f' = f \circ (j_2 \otimes Id_M)$ and the rows are exact. Also it is clear that $(\eta_1, \eta_2) : (X, Y)_f \to (X'', Y'')_{f''}$ is a map in Ω realizing $(X'' \oplus Y'')_T$ as a quotient of $(X \oplus Y)_T$. The kernel of the associated map $\eta : (X \oplus Y)_T \to (X'' \oplus Y'')_T$ is precisely $(X' \oplus Y')_T$. When we talk of a submodule $(X' \oplus Y')_T$ of $(X \oplus Y)_T$ we have $X' \leq X_A$, $Y' \leq Y_B, f \circ (j_2 \otimes Id_M)(Y' \otimes M) \leq X'$. The map $f' : Y' \otimes M \to X'$ is completely determined; it has to be $f \circ (j_2 \otimes Id_M)$. Similarly, when we deal with a quotient $(X'' \oplus Y'')_T$ of $(X \oplus Y)_T$ the map $f'' : Y'' \otimes M \to X''$ is completely determined. Because of these facts we will not specifically mention the maps f' and f'' in these situations.

Let $(X,Y)_f \in Obj\,\Omega$ and $L=\{y\in Y \mid f(y\otimes m)=0$ for all $m\in M\}$. Then clearly $L\le Y_B$ and $(0\oplus L)_T$ is a submodule of $(X\oplus Y)_T$. The following proposition gives necessary and sufficient conditions for a submodule $(X'\oplus Y')_T$ of $(X\oplus Y)_T$ to be essential in $(X\oplus Y)_T$.

Proposition 1.1. *$(X'\oplus Y')_T$ is essential in $(X\oplus Y)_T$ if and only if X' is essential in X_A and $Y'\cap L$ is essential in L_B.*

Proof. Assume that $(X'\oplus Y')_T$ is essential in $(X\oplus Y)_T$. Let $0\ne x\in X$. Then $(x,0)$ is non-zero in $(X\oplus Y)_T$. We can find an element

$$\begin{bmatrix} a & 0\\ m & b\end{bmatrix}\in T \quad\text{with } (0,0)\ne(x,0)\begin{bmatrix} a & 0\\ m & b\end{bmatrix}\in(X'\oplus Y')_T.$$

But

$$(x,0)\begin{bmatrix} a & 0\\ m & b\end{bmatrix}=(xa,0).$$

Thus $0\ne xa\in X'$. This proves that X'_A is essential in X_A. Let $0\ne y\in L$. Then $(0,y)\ne(0,0)$ in $(X\oplus Y)_T$. We can find an element

$$\begin{bmatrix} a & 0\\ m & b\end{bmatrix}\in T \quad\text{with } (0,0)\ne(0,y)\begin{bmatrix} a & 0\\ m & b\end{bmatrix}\in(X'\oplus Y')_T.$$

But

$$(0,y)\begin{bmatrix} a & 0\\ m & b\end{bmatrix}=(0,yb)\quad\text{since } f(y\otimes m)=0.$$

Thus $0\ne yb\in Y'$. Since $yb\in L$ we see that $0\ne yb\in Y'\cap L$, showing that $(Y'\cap L)_B$ is essential in L_B.

Conversely assume that X'_A is essential in X_A and that $(Y'\cap L)_B$ is essential in L_B. Let $(0,0)\ne(x,y)\in(X\oplus Y)_T$. In case $x\ne0$, we can find an $a\in A$ with $0\ne xa\in X'$. It follows that

$$(x,y)\begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix}=(xa,0)$$

is a non-zero element of $(X'\oplus Y')_T$. Suppose $x=0$. Then $y\ne0$. If $y\notin L$, we can find an $m\in M$ with $f(y\otimes m)\ne0$ in X. Hence there exists an $a\in A$ with $0\ne f(y\otimes m)a\in X'$. Thus

$$(0,y)\begin{bmatrix} 0 & 0\\ m & 0\end{bmatrix}\begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix}=(f(y\otimes m)a,0)$$

is a non-zero element of $(X'\oplus Y')_T$. If on the other hand $y\in L$ we have $f(y\otimes m)=0$ for all $m\in M$. Since $(Y'\cap L)_B$ is essential in L_B, we can find an element $b\in B$ with

$0 \neq yb \in Y' \cap L$. In this case,

$$(0, y)\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = (0, yb) \neq (0, 0) \quad \text{and} \quad (0, yb) \in (X' \oplus Y')_T.$$

This proves that $(X' \oplus Y')_T$ is essential in $(X \oplus Y)_T$. □

As already observed, $(0, L)_0$ defines a submodule $(0 \oplus L)_T \subseteq (X \oplus Y)_T$. Using this observation, from Proposition 1.1, we immediately get the following:

Corollary 1.2. *$(X \oplus Y)_T$ is uniform if and only if* (a) *or* (b) *mentioned below holds.*
(a) *$X_A = 0$ and $L_B = Y_B$ is uniform.*
(b) *$L_B = 0$ and X_A is uniform.*

We will now give an example to show that $(X \oplus Y)_T$ can be uniform, even when Y_B has infinite Goldie dimension.

Example 1.1. Let K be any field, $A = K(X_1, X_2, X_3, \ldots)$ the field of rational functions in countably many indeterminates; $B = K(X_1^2, X_2^2, X_3^2, \ldots)$ the field of rational functions in $X_1^2, X_2^2, X_3^2, \ldots$. Let $M = K(X_1, X_2, X_3, \ldots)$ be regarded as a left B, right A bimodule in the usual way. Let

$$T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}, \quad X_A = A_A, \quad Y_B = M_B.$$

Since X_A is a vector space of dimension 1 over A, X_A is uniform. Also Y_B is a vector space of infinite dimension over B. Hence Y_B has infinite Goldie dimension. The map $(\alpha(X_1, X_2, X_3, \ldots), \beta(X_1, X_2, X_3, \ldots)) \to \alpha(X_1, X_2, X_3, \ldots)\beta(X_1, X_2, X_3, \ldots)$ of $Y \times M$ in X gives rise to a map $f : Y \otimes_B M \to X$ in $Mod - A$. In the case of the module $(X \oplus Y)_T$ associated to $(X, Y)_f$ we have $L_B = 0$ and X_A uniform. From Proposition 1.1 we see that $(X \oplus Y)_T$ is uniform, yet Y_B has infinite Goldie dimension.

If $V \in Mod - R$ and $W \leq V_R$ we write $W \ll V$ to indicate that W is a small (or superfluous) submodule of V. The following proposition gives necessary and sufficient conditions for $(X' \oplus Y')_T$ to be small in $(X \oplus Y)_T$.

Proposition 1.3. *$(X' \oplus Y')_T$ is small in $(X \oplus Y)_T$ if and only if Y' is small in Y_B and $\eta(X')$ is small in $(X/f(Y \otimes M))_A$ where $\eta : X \to (X/f(Y \otimes M))$ is the canonical quotient map.*

Proof. Assume that $(X' \oplus Y')_T \ll (X \oplus Y)_T$. Let $H \leq Y_B$ satisfy $Y' + H = Y$. If $\mu : H \to Y$ denotes the inclusion, then with $g = f \circ (\mu \otimes Id_M)$ we have $(X, H)_g$ giving rise to a T-submodule $(X \oplus H)_T$ which satisfies $(X' \oplus Y')_T + (X \oplus H)_T = (X \oplus Y)_T$. The assumption $(X' \oplus Y')_T \ll (X \oplus Y)_T$ yields $H = Y$. Thus $Y' + H = Y \Rightarrow H = Y$. This proves that $Y' \ll Y_B$.

Let $E \leq (X/f(Y \otimes M))_A$ satisfy $\eta(X') + E = \eta(X)$ and $D = \eta^{-1}(E)$. Then $f(Y \otimes M) \subseteq D$. Hence $(D, Y)_f$ gives rise to a submodule $(D \oplus Y)_T$ of $(X \oplus Y)_T$. From $\eta(X') + E = \eta(X)$ we get $X' + D = X$. Hence $(X' \oplus Y')_T + (D \oplus Y)_T = (X \oplus Y)_T$. The assumption $(X' \oplus Y')_T \ll (X \oplus Y)_T$ yields $D = X$. In turn, this yields $E = \eta(X)$. Thus $\eta(X') + E = \eta(X) \Rightarrow E = \eta(X)$. This gives $\eta(X') \ll \eta(X)_A$.

Conversely, assume that $\eta(X') \ll \eta(X)_A$ and $Y' \ll Y_B$. Let $U \leq X_A, V \leq Y_B$ satisfy the condition $f \circ (v \otimes Id_M)(V \otimes M) \subseteq U$ where $v: V \to Y$ denotes the inclusion. Writing h for $f \circ (v \otimes Id_M)$, suppose $(U, V)_h$ satisfies $(X' \oplus Y')_T + (U \oplus V)_T = (X \oplus Y)_T$. Then $Y' + V = Y$ and $X' + U = X$. From $Y' \ll Y_B$ we see that $V = Y$. It follows that $v = Id_Y$ and $f(Y \otimes M) \subseteq U$. From $X' + U = X$ we get $\eta(X') + \eta(U) = \eta(X)$. The hypothesis $\eta(X') \ll \eta(X)_A$ now yields $\eta(U) = \eta(X)$. Since $f(Y \otimes M) \subseteq U$ we get $U = X$. Thus $(X' \oplus Y')_T + (U \oplus V)_T = (X \oplus Y)_T \Rightarrow U = X$ and $V = Y$. This proves that $(X' \oplus Y')_T \ll (X \oplus Y)_T$. $\square$

Recall that a module is said to be *hollow* if it is non zero and in it every proper submodule is small.

Corollary 1.4. *The right T-module $(X \oplus Y)_T$ determined by $(X, Y)_f$ is hollow if and only if* (a) *or* (b) *mentioned below is true.*

(a) *Y_B is hollow and $X = f(Y \otimes M)$.*

(b) *$Y = 0$ and X_A is hollow.*

Proof. Assume $(X \oplus Y)_T$ hollow. Suppose $Y \neq 0$. Then $\exists Y' \leq Y_B$ with $Y' \neq Y_B$. If $j: Y' \to Y$ denotes the inclusion and $f' = f \circ (j \otimes Id_M)$, the submodule $(X \oplus Y')_T$ determined by $(X, Y')_{f'}$ is a proper submodule of $(X \oplus Y)_T$. As such, $(X \oplus Y')_T \ll (X \oplus Y)_T$. From Proposition 1.3 we see that $Y' \ll Y_B$ and $\eta(X) \ll \eta(X)_A$. But the latter means $\eta(X) = 0$ or $X = f(Y \otimes M)$. Thus if $Y \neq 0$, we see that Y_B is hollow and $X = f(Y \otimes M)$.

In case $Y = 0$, since $(X \oplus Y)_T$ is hollow by assumption, $X \neq 0$. For any $X' \leq X_A$ with $X' \neq X$, $(X' \oplus 0)_T$ is a proper submodule of $(X \oplus 0)_T$. Hence $(X' \oplus 0)_T \ll (X \oplus 0)_T$. From Proposition 1.3 we get $X' \ll X_A$. Hence X_A is hollow.

Conversely assume either (a) or (b) is valid. In case (a), if $(X' \oplus Y')_T$ is a proper submodule of $(X \oplus Y)_T$ we should necessarily have $Y' \subsetneqq Y$. Because, if $Y' = Y$ then $X' \supset f(Y \otimes M) = X$ and $(X' \oplus Y')_T = (X \oplus Y)_T$. If $Y' \subsetneqq Y, (X' \oplus Y')_T \subseteq (X \oplus Y')_T$. We need only prove that $(X' \oplus Y')_T \ll (X \oplus Y)_T$. This follows immediately from Proposition 1.3 since $Y' \ll Y$ and $\eta(X) = 0$. In case (b), any proper submodule of $(X \oplus 0)_T$ is of the form $(X' \oplus 0)_T$ with $X' \subsetneqq X$. Since $X' \ll X$, from Proposition 1.3 we get $(X' \oplus 0)_T \ll (X \oplus 0)_T$. $\square$

We now construct an example to show that $(X \oplus Y)_T$ can be hollow even when X_A has infinite dual Goldie dimension in the sense of [8].

Example 1.2. Let K be a field; $A=K$, $B=K$ and M an infinite dimensional vector space over K regarded as a (K,K) bimodule in the usual way. Let

$$T=\begin{bmatrix} A & 0 \\ M & B \end{bmatrix}.$$

Let $X_A=M_A, Y_B=B_B$, and suppose $f:B\otimes_B M\to M$ is given by $f(b\otimes m)=bm$. Then from Proposition 1.3 we see that $(X\oplus Y)_T$ corresponding to $(X,Y)_f$ is hollow, because Y_B is hollow and $X=f(Y\otimes M)$. However the dual Goldie dimension of X_A is infinite.

Recall that a module L is said to be *local* if it has a unique maximal submodule containing every proper submodule of L. Any local L is cyclic and hollow. If a hollow module H admits a maximal submodule then H is local. In particular a module L is local $\Leftrightarrow$ L is a finitely generated hollow module. These comments allow us to characterize local modules over a formal triangular matrix ring.

Corollary 1.5. *The right T-module $(X\oplus Y)_T$ determined by $(X,Y)_f$ is local if and only if* (a) *or* (b) *mentioned below is true.*

(a) *Y_B is local and $X=f(Y\otimes M)$.*

(b) *$Y=0$ and X_A is local.*

Proof. Immediate consequence of Corollary 1.4 and the well-known result (Exercise 1D(b) on p. 7 of [4]) that $(X\oplus Y)_T$ is finitely generated $\Leftrightarrow$ the modules Y_B and $(X/f(Y\otimes M))_A$ are finitely generated. $\square$

2. Determination of $Rad(X\oplus Y)_T$ and $Soc(X\oplus Y)_T$

Let $(X,Y)_f\in Obj\,\Omega$ and $(X\oplus Y)_T$ the right T-module determined by $(X,Y)_f$. In this section we will determine the Jacobson radical $Rad(X\oplus Y)_T$ and the socle $Soc(X\oplus Y)_T$ of $(X\oplus Y)_T$. For this purpose we will first describe the maximal (resp. simple) submodules of $(X\oplus Y)_T$. Let $L=\{y\in Y\mid f(y\otimes m)=0 \text{ for all } m\in M\}$.

Proposition 2.1. *Let $\mathscr{F}_1=\{(X'\oplus Y)_T\mid X'$ a maximal submodule of X_A with $f(Y\otimes M)\le X'\}$ and $\mathscr{F}_2=\{(X\oplus Y')_T\mid Y'$ a maximal submodule of $Y_B\}$. Let $\mathscr{S}_1=\{(X'\oplus 0)_T\mid X'$ a simple submodule of $X_A\}$ and $\mathscr{S}_2=\{(0\oplus L')_T\mid L'$ a simple submodule of $L_B\}$. Then*

(a) *The family $\mathscr{F}$ of maximal submodules of $(X\oplus Y)_T$ is precisely $\mathscr{F}_1\cup\mathscr{F}_2$.*

(b) *The family $\mathscr{S}$ of minimal submodules of $(X\oplus Y)_T$ is precisely $\mathscr{S}_1\cup\mathscr{S}_2$.*

Proof. Let $(X'\oplus Y')_T$ be any maximal submodule of $(X\oplus Y)_T$. If $Y'\subsetneqq Y_B$, since $(X'\oplus Y')_T\subseteq(X\oplus Y')_T\subsetneqq(X\oplus Y)_T$ we conclude that $X'=X$. Also from $(X\oplus Y')_T\subseteq(X\oplus Y'')_T\subsetneqq(X\oplus Y)_T$ for any $Y'\subseteq Y''\subsetneqq Y_B$ we see that $Y'=Y''$ whenever $Y'\subseteq Y''\subsetneqq Y_B$. Hence Y' is a maximal submodule of Y_B thereby showing that $(X'\oplus Y')_T=(X\oplus Y')_T$ is in $\mathscr{F}_2$. Now suppose $Y'=Y$. Then $f(Y\otimes M)\le X'$. Since $(X'\oplus Y')_T=(X'\oplus Y)_T$ is a maximal submodule of $(X\oplus Y)_T$ we immediately see that X'

is a maximal submodule X_A. Thus $(X' \oplus Y')_T = (X' \oplus Y)_T$ is in $\mathscr{F}_1$. Conversely, it is straightforward to see that any submodule of $(X \oplus Y)_T$ belonging to $\mathscr{F}_1 \cup \mathscr{F}_2$ is a maximal submodule of $(X \oplus Y)_T$. This proves (a).

Let $(X' \oplus Y')_T$ be any simple submodule of $(X \oplus Y)_T$. If $X' \neq 0$, for any $0 \neq X'' \subsetneqq X_A$ since $(X'' \oplus 0)_T \subseteq (X' \oplus Y')_T$ we see that $Y' = 0$ and X' is a simple submodule of X_A. If on the other hand $X' = 0$, we should have $f \circ (j \otimes Id_M)(Y' \otimes M) = 0$, hence $j(Y') \subseteq L_B$ where $j : Y' \to Y$ denotes the inclusion. Thus $Y' \leq L_B$. Also $0 \neq Y'' \subseteq Y' \subsetneqq L_B \Rightarrow (0 \oplus Y'')_T \subseteq (0 \oplus Y')_T$. It follows that Y' is a simple submodule of L_B. Thus $\mathscr{S} \subseteq \mathscr{S}_1 \cup \mathscr{S}_2$. Conversely, it is easily seen that any submodule of $(X \oplus Y)_T$ belonging to $\mathscr{S}_1 \cup \mathscr{S}_2$ is simple. This proves (b). $\square$

Corollary 2.2. *Let $\eta : X_A \to (X/f(Y \otimes M))_A$ denote the canonical quotient map. Then*
(a) $Rad(X \oplus Y)_T = (\eta^{-1}(Rad(X/f(Y \otimes M))_A) \oplus Rad(Y_B))_T$.
(b) $Soc(X \oplus Y)_T = (Soc(X_A) \oplus Soc(L_B))_T$.

Proof. Immediate consequence of Proposition 2.1. $\square$

Recall that a module V is said to be finitely embedded (or finitely co-generated; see [1, 7, 10]) if $Soc V$ is finitely generated and essential in V. Proposition 2.1 enables us to obtain the following.

Theorem 2.3. (1) *$Soc(X \oplus Y)_T$ is finitely generated if and only if $Soc(X_A)$ and $Soc(L_B)$ are finitely generated.*
(2) *$Soc(X \oplus Y)_T$ is essential in $(X \oplus Y)_T$ if and only if $Soc(X_A)$ is essential in X_A and $Soc(L_B)$ is essential in L_B.*
(3) *$(X \oplus Y)_T$ is finitely embedded if and only if X_A and L_B are finitely embedded.*

Proof. From Corollary 2.2(b) we have $Soc(X \oplus Y)_T = (Soc(X_A) \oplus Soc(L_B))_T$. Note that $(Soc(X_A) \oplus Soc(L_B))_T$ corresponds to the triple $(Soc(X_A), Soc(L_B))_0$. From a well-known result (Exercise 1D(b) on p. 7 of [4]) we see that $Soc(X \oplus Y)_T$ is finitely generated $\Leftrightarrow Soc X_A/((Soc L_B) \otimes M)$ and $Soc(L_B)$ are finitely generated. This proves (1). (2) is an immediate consequence of Proposition 1.1. (3) is immediate from (1) and (2). $\square$

Often one is interested in finding conditions implying $Rad(V) \ll V$. In this connection we have the following.

Proposition 2.4. *$Rad(X \oplus Y)_T$ is small in $(X \oplus Y)_T$ if and only if $Rad Y_B$ is small in Y_B and $Rad(X/f(Y \otimes M))_A$ is small in $(X/f(Y \otimes M))_A$.*

Proof. Immediate consequence of Corollary 2.2(a) and Proposition 1.3. $\square$

Let R be a ring and $V \in Mod - R$. Recall that V is said to be *co-hopfian* if every injective endomorphism $f : V \to V$ is automatically an isomorphism. In [9] one of the authors of the present paper has shown that any quasi-injective finitely embedded module V is co-hopfian. Using (3) of Theorem 2.3 which characterizes finitely embedded modules over formal triangular matrix rings, we will construct an example of a finitely embedded module which is not co-hopfian, thus showing that quasi-injectivity of V cannot be dispensed with for the validity of the above result.

Example 2.1. Let

$$T = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_{p^\infty} & \mathbb{Z} \end{bmatrix}$$

where p is a prime. Consider the T-module $(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z})_T$ associated to the triple $(\mathbb{Z}_{p^\infty}, \mathbb{Z})_f$ where $f : \mathbb{Z} \otimes \mathbb{Z}_{p^\infty} \to \mathbb{Z}_{p^\infty}$ is the identity map of $\mathbb{Z}_{p^\infty}$; equivalently $f(k \otimes x) = kx$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{Z}_{p^\infty}$. In this case $\mathbb{Z}_{p^\infty}$ is finitely embedded in $Mod - \mathbb{Z}$. Also $L = \{k \in \mathbb{Z} \mid kx = 0 \text{ for all } x \in \mathbb{Z}_{p^\infty}\} = 0$ is finitely embedded in $Mod - \mathbb{Z}$. From (3) of Theorem 2.3 we see that $(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z})_T$ is finitely embedded. Let n be an integer ≥ 2 and relatively prime to p. Let $\sigma_1 : \mathbb{Z}_{p^\infty} \to \mathbb{Z}_{p^\infty}$ and $\sigma_2 : \mathbb{Z} \to \mathbb{Z}$ be both given by multiplication by n. Clearly

$$\begin{array}{ccc} \mathbb{Z} \otimes \mathbb{Z}_{p^\infty} & \xrightarrow{f} & \mathbb{Z}_{p^\infty} \\ \Big\downarrow{\scriptstyle \sigma \otimes Id} & & \Big\downarrow{\scriptstyle \sigma_1} \\ \mathbb{Z} \otimes \mathbb{Z}_{p^\infty} & \xrightarrow{f} & \mathbb{Z}_{p^\infty} \end{array}$$

is a commutative diagram. The map $\sigma = (\sigma_1, \sigma_2) : (\mathbb{Z}_{p^\infty} \oplus \mathbb{Z})_T \to (\mathbb{Z}_{p^\infty} \oplus \mathbb{Z})_T$ is an injective endomorphism. This is because σ_1 is an isomorphism (since $(p, n) = 1$) and σ_2 is injective. However σ is not surjective, because σ_2 is not. Thus $(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z})_T$ is not co-hopfian.

3. Projective modules and dual bases over T

Projective right ideals over

$$T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$$

are completely characterized in [3]. (See Proposition 4.5 on p. 110.) Also in case T happens to be an Artin algebra, finitely generated projective modules are completely characterized in [2]. Actually a similar characterization is valid for arbitrary projective modules over any formal triangular matrix ring T. In this section we obtain such a characterization and describe a method of obtaining a “dual basis” for projective modules over T. Let $(X,Y)_f \in \Omega$ and let $V_T = (X \oplus Y)_T$. Writing I for the two-sided ideal

$$\begin{bmatrix} 0 & 0 \\ M & B \end{bmatrix}$$

of T it is straightforward to see that $VI = (f(Y \otimes M) \oplus Y)_T$. Using the isomorphism

$$\begin{bmatrix} a & 0 \\ M & B \end{bmatrix} \rightarrowtail a \text{ of } T/I$$

with A when we regard V/VI as a right A-module, it is clear that $V/VI \cong (X/f(Y \otimes M))_A$. Similarly

$$J = \begin{bmatrix} A & 0 \\ M & 0 \end{bmatrix}$$

is a two-sided ideal of T and $VJ = (X \oplus 0)_T$. When we regard V/VJ as a right B-module via the isomorphism

$$\begin{bmatrix} A & 0 \\ M & b \end{bmatrix} \rightarrowtail b \text{ of } T/J$$

with B it is clear that $V/VJ \cong Y_B$ in $Mod - B$. Under these circumstances we have the following.

Theorem 3.1. *$(X \oplus Y)_T$ is projective if and only if $(X/f(Y \otimes M))_A$ and Y_B are projective and $f : Y \otimes M \to X$ is one-one.*

Proof. Suppose $(X \oplus Y)_T$ is projective. Writing V for $(X \oplus Y)_T$ we know that V/VI is projective in $Mod - (T/I)$. Thus, $(X/f(Y \otimes M))_A$ and Y_B are projective. Writing E for the left ideal

$$\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \text{ of } T,$$

the sequence $0 \to E \hookrightarrow T$ is exact in $T - Mod$. Since V_T is projective (in particular flat) the sequence $0 \to V \otimes_T E \xrightarrow{Id \otimes j} V \otimes_T T$ is an exact sequence of abelian groups where $j : E \to T$ denotes the inclusion. Let

$$e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in T.$$

Then for any $m \in M$ we have

$$e_2 \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}.$$

Hence for any $(x, y) \in (X \oplus Y)_T = V_T$ we have

$$(x, y) \otimes_T \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix} = (x, y) \otimes_T e_2 m = (x, y) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes_T \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$$
$$= (0, y) \otimes_T \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}.$$

There is an isomorphism $Y \otimes_B M \xrightarrow{\cong} V \otimes_T E$ of the abelian groups carrying

$$y \otimes_B m \text{ to } (0, y) \otimes_T \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}.$$

Also $V \otimes_T T$ is isomorphic to V in Mod-$\mathbb{Z}$ under $v \otimes t \xrightarrow{\theta} vt$. The composite map $Y \otimes_B M \xrightarrow{\cong} V \otimes_T E \xrightarrow{Id \otimes j} V \otimes_T T \overset{\theta}{\cong} V$ carries $y \otimes m$ to $(f(y \otimes m), 0)$. Since $Id \otimes j$ is a monomorphism it follows that f is a monomorphism.

Conversely, assume that $(X/f(Y \otimes M))_A$ and Y_B are projective and that $f : Y \otimes M \to X$ is a monomorphism. The projectivity of $(X/f(Y \otimes M))_A$ is equivalent to the projectivity of V/VI in $Mod - (T/I)$ where $V = (X \oplus Y)_T$. Also $I = e_2 T$ and e_2 is an idempotent in T. Hence T/I is projective in $Mod - T$. It follows that V/VI is projective in $Mod - T$. Also, as observed already $VI = (f(Y \otimes M) \oplus Y)_T$. Since $f : Y \otimes M \to f(Y \otimes M)$ is an isomorphism it follows that $(f(Y \otimes M) \oplus Y)_T \cong ((Y \otimes M) \oplus Y)_T$ where $((Y \otimes M) \oplus Y)_T$ is a T-module corresponding to $(Y \otimes M, Y)_{Id}$. By assumption Y_B is projective. Hence $\exists$ some Y'_B satisfying $Y_B \oplus Y'_B = \bigoplus_\alpha B_\alpha$ where each $B_\alpha = B$. Writing $((Y \oplus Y') \otimes M \oplus (Y \oplus Y'))_T$ for the T-module corresponding to the triple $((Y \oplus Y') \otimes M, Y \oplus Y')_{Id}$ we see that $((Y \otimes M) \oplus Y)_T$ is a direct summand of $((Y \oplus Y') \otimes M \oplus (Y \oplus Y'))_T = ((\bigoplus_\alpha B_\alpha \otimes M) \oplus (\bigoplus_\alpha B_\alpha))_T = \bigoplus_\alpha (M \oplus B)_T = \bigoplus_\alpha e_2 T$ (direct sum of a family of copies of $e_2 T$). Since $e_2 T$ is projective in $Mod - T$ we see that $VI \cong ((Y \otimes M) \oplus Y)_T$ is projective in $Mod - T$.

Since V/VI is projective in $Mod - T$, the exact sequence $0 \to VI \to V \to V/VI \to 0$ splits, yielding $V \cong VI \oplus (V/VI)$ projective in $Mod - T$. This completes the proof. $\square$

Let $(X, Y)_f \in \Omega$. From Theorem 3.1, the module $(X \oplus Y)_T$ is projective $\Leftrightarrow$ Y_B is projective, $f : Y \otimes M \to X$ is monic and $X = P \oplus f(Y \otimes M)$ with P_A projective. Given any map $\varphi : Y \to B$ in $Mod - B$, we can associate a unique map $h : f(Y \otimes M) \to M$ in $Mod - A$ satisfying $h(f(y \otimes m)) = \varphi(y)m$. For y, y_1, y_2 in Y, m, m_1, m_2 in M we have

$$\varphi(y_1 + y_2)m = \varphi(y_1)m + \varphi(y_2)m; \quad \varphi(y)(m_1 + m_2) = \varphi(y)m_1 + \varphi(y)m_2.$$

Also for any $b \in B, \varphi(yb)m = \varphi(y)bm$. Since $f : Y \otimes M \to f(Y \otimes M)$ is an isomorphism it follows that there exists a unique homomorphism $h : f(Y \otimes M) \to M$ of Abelian groups satisfying $h(f(y \otimes m)) = \varphi(y)m$. For any $a \in A$, clearly $hf(y \otimes ma)) = \varphi(y)(ma) = (\varphi(y)m)a$. Thus automatically h is a map in $Mod - A$.

Let $\{\varphi_i, y_i\}_{i \in \Delta_2}$ be a dual basis for Y_B; namely $\varphi_i : Y \to B$ are maps in $Mod - B$ such that:

(i) $\forall y \in Y$, $\varphi_i(y) = 0$ for i outside a finite subset F_y of Δ_2,
(ii) $y = \sum_{i \in \Delta_2} y_i \varphi_i(y)$ for any $y \in Y$.

As described above we obtain maps $h_i : f(Y \otimes M) \to M$ in $Mod - A$ satisfying $h_i(f(y \otimes m)) = \varphi_i(y)m$ for $y \in Y$, $m \in M$ and any $i \in \Delta_2$. Let $\{\theta_j, p_j\}_{j \in \Delta_1}$ be a dual basis for P_A. Note that T_T corresponds to the triple $(A \oplus M, B)_g$ where $g : B \otimes M \to A \oplus M$ is given by $g(b \otimes m) = (0, bm)$. Let Δ be the disjoint union of Δ_1 and Δ_2. For any $j \in \Delta_1$ let $x_j = (p_j, 0) \in P \oplus f(Y \otimes M) = X$ and $\beta_j : P \oplus f(Y \otimes M) \to A \oplus M$ be given by $\beta_j = \theta_j \oplus 0$. Let $y_j = 0$ in Y for every $j \in \Delta_1$ and $\gamma_j : Y \to B$ be the zero homomorphism. For any $i \in \Delta_2$, let $x_i = (0, 0) \in P \oplus f(Y \otimes M) = X$ and $\beta_i : P \oplus f(Y \otimes M) \to A \oplus M$ be given by $\beta_i = 0 \oplus h_i$. Let $y_i \in Y$ be the elements appearing in the dual basis $\{\varphi_i, y_i\}_{i \in \Delta_2}$ of Y_B and $\gamma_i = \varphi_i : Y \to B$. We then have the following:

Theorem 3.2. *For any $\mu \in \Delta, (\beta_\mu, \gamma_\mu) : (X, Y)_f \to (A \oplus M, B)_g$ is a map in Ω. Let $\alpha_\mu : (X \oplus Y)_T \to T = (A \oplus M \oplus B)_T$ be the associated map in $Mod - T$. Then $\{\alpha_\mu, (x_\mu, y_\mu)\}_{\mu \in \Delta}$ is a dual basis for $(X \oplus Y)_T$.*

Proof. First we check that $(\beta_\mu, \gamma_\mu) : (X, Y)_f \to (A \oplus M, B)_g$ is a map in Ω for every $\mu \in \Delta$. For this we need to show that $\beta_\mu f(y \otimes m) = g(\gamma_\mu(y) \otimes m)$ for all $y \in Y$, $m \in M$ and $\mu \in \Delta$. If $\mu \in \Delta_1$, we have $\beta_\mu f(y \otimes m) = (0, 0) = g(\gamma_\mu(y) \otimes m)$ in $A \oplus M$. If $\mu \in \Delta_2$, we have $\beta_\mu f(y \otimes m) = (0, h_\mu f(y \otimes m)) = (0, \varphi_\mu(y)m) \in A \oplus M$ and $g\gamma_\mu(y \otimes m) = g(\varphi_\mu(y) \otimes m) = (0, \varphi_\mu(y)m) \in A \oplus M$. This proves that $\beta_\mu f(y \otimes m) = g(\gamma_\mu(y) \otimes m)$ for all $y \in Y$, $m \in M$.

To show that $\{\alpha_\mu, (x_\mu, y_\mu)\}_{\mu \in \Delta}$ is a dual basis for $(X \oplus Y)_T$ we need to show that

$$\sum_{\mu \in \Delta} (x_\mu, y_\mu)\alpha_\mu(x, y) = (x, y) \tag{3.1}$$

for all $(x, y) \in (X \oplus Y)_T$. Elements of the form $f(y \otimes m)$ generate $f(Y \otimes M)$ in $Mod - A$. It suffices to check (3.1) separately in the following three cases:

(a) $x = (p, 0) \in P \oplus f(Y \otimes M) = X$ and $y = 0$ in Y.
(b) $x = (0, f(y' \otimes m)) \in P \oplus f(Y \otimes M) = X$ with $y' \in Y$, $m \in M$ and $y = 0$ in Y.
(c) $x = (0, 0) \in P \oplus f(Y \otimes M) = X$ and $y \in Y$ arbitrary.

Before dealing with these cases, observe that $(A \oplus M \oplus B)_T$ is identified with T_T under the bijective correspondence

$$((a, m), b) \to \begin{bmatrix} a & 0 \\ m & b \end{bmatrix}.$$

For any $p \in P$ note that

$$(\beta_\mu, \gamma_\mu)((p, 0), 0) = \begin{cases} \begin{bmatrix} \theta_\mu(p) & 0 \\ 0 & 0 \end{bmatrix} & \text{if } \mu \in \Delta_1 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } \mu \in \Delta_2. \end{cases}$$

Also $x_\mu = (p_\mu, 0) \in P \oplus f(Y \otimes M) = X$ and $y_\mu = 0 \in Y$ for all $\mu \in \Delta_1$. Hence

$$\sum_{\mu\in\Delta}(x_\mu, y_\mu)(\beta_\mu, \gamma_\mu)((p,0),0) = \sum_{\mu\in\Delta_1}((p,0),0)\begin{bmatrix}\theta_\mu(p) & 0\\ 0 & 0\end{bmatrix}$$
$$= \sum_{\mu\in\Delta_1}((p_\mu\theta_\mu(p),0),0)$$
$$= \left(\left(\sum_{\mu\in\Delta_1} p_\mu\theta_\mu(p), 0\right), 0\right) = ((p,0),0).$$

This proves (3.1) in case (a). For any $y' \in Y$ and $m \in M$ we have

$$(\beta_\mu, \gamma_\mu)((0, f(y'\otimes m)), 0) = \begin{cases}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \text{if } \mu\in\Delta_1,\\[2ex] \begin{bmatrix}0 & 0\\ \varphi_\mu(y')m & 0\end{bmatrix} & \text{if } \mu\in\Delta_2.\end{cases}$$

Also $x_\mu = (0,0) \in P \oplus f(Y \otimes M) = X$ for $\mu \in \Delta_2$. Hence

$$\sum_{\mu\in\Delta}(x_\mu, y_\mu)(\beta_\mu, \gamma_\mu)((0, f(y'\otimes m)), 0) = \sum_{\mu\in\Delta_2}((0,0), y_\mu)\begin{bmatrix}0 & 0\\ \varphi_\mu(y')m & 0\end{bmatrix}$$
$$= \sum_{\mu\in\Delta_2}\left(\left(0, f(y_\mu\otimes\varphi_\mu(y')m)\right), 0\right)$$
$$= \left(0, \sum_{\mu\in\Delta_2} f(y_\mu\otimes\varphi_\mu(y')m), 0\right)$$
$$= \left(\left(0, \sum_{\mu\in\Delta_2} f(y_\mu\varphi_\mu(y')\otimes m)\right), 0\right)$$
$$= \left(\left(0, f(\sum_{\mu\in\Delta_2} y_\mu\varphi_\mu(y')\otimes m)\right), 0\right)$$
$$= ((0, f(y'\otimes m)), 0).$$

This proves (1) in case (b). For any $y \in Y$ we have

$$(\beta_\mu, \gamma_\mu)((0,0), y) = \begin{cases}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \text{if } \mu\in\Delta_1,\\[2ex] \begin{bmatrix}0 & 0\\ 0 & \varphi_\mu(y)\end{bmatrix} & \text{if } \mu\in\Delta_2.\end{cases}$$

Also $x_\mu = (0,0) \in P \oplus f(Y \otimes M) = X$ for $\mu \in \Delta_2$. Hence

$$\begin{aligned}
\sum_{\mu\in\Delta} (x_\mu, y_\mu)(\beta_\mu, \gamma_\mu)((0,0), y) &= \sum_{\mu\in\Delta_2} ((0,0), y_\mu) \begin{bmatrix} 0 & 0 \\ 0 & \varphi_\mu(y) \end{bmatrix} \\
&= \sum_{\mu\in\Delta_2} ((0,0), y_\mu\varphi_\mu(y)) \\
&= \left((0,0), \sum_{\mu\in\Delta_2} y_\mu\varphi_\mu(y) \right) \\
&= ((0,0), y).
\end{aligned}$$

This proves (1) in case (c), thus completing the proof of Theorem 3.2. □

4. Generators, projective generators, respectively pro-generators in *Mod* − *T*

In this section we first determine necessary and sufficient conditions for $(X \oplus Y)_T$ to be a generator. When $(X \oplus Y)_T$ is projective, these take on a particularly simple form. Observing that $(X \oplus Y)_T$ is a progenerator $\Leftrightarrow$ it is finitely generated projective generator, we obtain necessary and sufficient conditions for $(X \oplus Y)_T$ to be a progenerator. In all the result stated below, T_T will be identified with the right T-module associated to $(A \oplus M, B)_g$ where $g : B \otimes M \to A \oplus M$ is given by $g(b \otimes m) = (0, bm)$ for all $b \in B$, $m \in M$.

Theorem 4.1. *$(X \oplus Y)_T$ is a generator if and only if the following are valid*:

(i) *$X/f(Y \otimes M)$ is a generator in Mod − A.*

(ii) *for some set J, there exists a surjective map $Y^{(J)} \xrightarrow{q} B$ in Mod − B and a map $\varphi : X^{(J)} \to M$ in Mod − A satisfying $\varphi \circ f^{(J)} = h \circ (q \otimes Id_M)$. Here $X^{(J)}$ (resp. $Y^{(J)}$) denote direct sum of X (resp. Y) indexed by the set J. Furthermore $f^{(J)} : Y^{(J)} \otimes M = (Y \otimes M)^{(J)} \to X^{(J)}$ is induced by f and $h : B \otimes M \to M$ is given by $h(b \otimes m) = bm$.*

Proof. Suppose $(X \oplus Y)_T$ is a generator. Then for some set J, there exists a surjective map $(X \oplus Y)_T^{(J)} \to T_T = ((A \oplus M) \oplus B)_T$. Observing that $(X \oplus Y)_T^{(J)}$ corresponds to the triple $(X^{(J)}, Y^{(J)})_{f^{(J)}}$, then the above yields surjective maps $\pi_1 : X^{(J)} \to A \oplus M$ in *Mod* − *A* and $\pi_2 : Y^{(J)} \to B$ in *Mod* − *B* satisfying $\pi_1 \circ f^{(J)} = g \circ (\pi_2 \otimes Id_M)$. If $p_A : A \oplus M \to A$, $p_M : A \oplus M \to M$ denote the respective projections, then $p_A \circ \pi_1 : X^{(J)} \to A$, $p_M \circ \pi_1 : X^{(J)} \to M$ are surjective maps in *Mod* − *A* satisfying $p_A \circ \pi_1 \circ f^{(J)} = 0$ and $p_M \circ \pi_1 \circ f^{(J)} = h \circ (\pi_2 \otimes Id_M)$. From $p_A \circ \pi_1 \circ f^{(J)} = 0$ we conclude that $p_M \circ \pi_1 : X^{(J)} \to A$ induces a surjection $(X/f(Y \otimes M))^{(J)} \to A$ in *Mod* − *A*. Hence $X/f(Y \otimes M)$ is a generator in *Mod* − *A*. If we denote $p_M \circ \pi_1$ by φ then $\varphi : X^{(J)} \to M$ is a map in *Mod* − *A* satisfying $\varphi \circ f^{(J)} = h \circ (\pi_2 \otimes Id_M)$. This yields (i) and (ii).

Conversely assume (i) and (ii). From (i) we get a surjective map in *Mod* − *A*, $(X/f(Y \otimes M))^{(J_1)} \xrightarrow{\theta} A$ for some indexing set J_1. Let $\eta : X \to X/f(Y \otimes M)$ denote the quotient map. From (ii), there exist an indexing set J_2, a surjective map $q:Y^{(J_2)}\to B$ in $Mod-B$, a map $\varphi:X^{(J_2)}\to M$ in $Mod-A$ satisfying $\varphi\circ f^{(J_2)}=h\circ(q\otimes Id_M)$. Let $I=J_1\dot{\cup}J_2$ the disjoint union of J_1 and J_2. We write X^I as $X^{(J_1)}\oplus X^{(J_2)}$ and Y^I as $Y^{(J_1)}\oplus Y^{(J_2)}$. Let $\pi_1:X^{(I)}=X^{(J_1)}\oplus X^{(J_2)}\to A\oplus M$ and $\pi_2:Y^{(I)}=Y^{(J_1)}\oplus Y^{(J_2)}\to B$ be the maps satisfying the following conditions:

$$\pi_1(u)=\begin{cases}(\theta\eta^{(J_1)}(u),0)\in A\oplus M & \text{for any } u\in X^{(J_1)},\\ (0,\varphi(u))\in A\oplus M & \text{for any } u\in X^{(J_2)}.\end{cases}$$

$$\pi_2(v)=\begin{cases}0\in B & \text{for any } v\in Y^{(J_1)},\\ q(v)\in B & \text{for any } v\in Y^{(J_2)}.\end{cases}$$

Also note that $f^{(I)}:Y^{(I)}\otimes M\to X^{(I)}$ is the same as $f^{(J_1)}\oplus f^{(J_2)}$. Since $\eta^{(J_1)}f^{(J_1)}(Y^{(J_1)}\otimes M)=0$ we see immediately that $g\circ((\pi_2|_{Y^{(J_1)}})\otimes Id_M)=0=(\pi_1|_{X^{(J_1)}})\circ f^{(J_1)}$. Also, for any $v\in Y^{(J_2)}$ and $m\in M$ we have

$$\begin{aligned}g\circ(\pi_2\otimes Id_M)(v\otimes m)&=g(q(v)\otimes m)=(0,q(v)m)\\&=(0,h\circ(q\otimes Id_M)(v\otimes m))\\&=(0,\varphi\circ f^{(J_2)}(v\otimes m))=\pi_1(f^{(I)}(v\otimes m)).\end{aligned}$$

So $g\circ(\pi_2\otimes Id_M)=\pi_1\circ f^{(I)}$, thus $(\pi_1,\pi_2):(X^{(I)},Y^{(I)})_{f^{(I)}}\to(A\oplus M,B)_g$ is a morphism in Ω. Since $q:Y^{(J_2)}\to B$ is surjective and $h:B\otimes M\to M$ is an isomorphism, we see that $h\circ(q\otimes Id_M):Y^{(J_2)}\otimes M\to M$ is surjective. From $\varphi\circ f^{(J_2)}=h\circ(q\otimes Id_M)$ it follows that $\varphi:X^{(J_2)}\to M$ is surjective. Also $\theta\circ\eta^{(J_1)}:X^{(J_1)}\to A$ is surjective. It follows that $\pi_1:X^{(I)}\to A\oplus M$ is surjective. Since $q:Y^{(J_2)}\to B$ is surjective, we see that $\pi_2:Y^{(I)}\to B$ is surjective. It follows that the map $\pi:(X\oplus Y)_T^{(I)}\to((A\oplus M)\oplus B)_T=T_T$ induced by (π_1,π_2) is surjective. Hence $(X\oplus Y)_T$ is a generator. □

Theorem 4.2. *Assume that $(X\oplus Y)_T$ is projective. Then $(X\oplus Y)_T$ is a generator if and only if $(X/f(Y\otimes M))_A$ and Y_B are generators.*

Proof. From Theorem 3.1 we see that $(X/f(Y\otimes M))_A$ and Y_B are projective and that $f:Y\otimes M\to f(Y\otimes M)$ is an isomorphism. In particular, it follows that $X_A=f(Y\otimes M)\oplus P_A$ with P_A projective. To prove Theorem 4.2 we have only to show that condition (ii) of Theorem 4.1 is satisfied. Since Y_B is a generator, there exists a surjective map $q:Y^{(J)}\to B$ in $Mod-B$ for a suitable indexing set J. We have $X^{(J)}=f^{(I)}(Y^{(J)}\otimes M)\oplus P^{(J)}$ with $f^{(J)}:Y^{(J)}\otimes M\to f^{(I)}(Y^{(J)}\otimes M)$ an isomorphism. Now $h\circ(q\otimes Id_M):Y^{(J)}\otimes M\to M$ is a map in $Mod-A$. If we define $\varphi:X^{(J)}\to M$ by $\varphi|_{f^{(J)}(Y^{(J)}\otimes M)}=h\circ(q\otimes Id_M)\circ(f^{(J)})^{-1}$ and $\varphi|_{P^{(J)}}=0$ it is clear that $\varphi\circ f^{(J)}=h\circ(q\otimes Id_M)$. □

As an immediate consequence of Theorems 3.1, 4.2 of the present paper and Exercise 1D(b) on p. 7 of [4] we get:

Corollary 4.3. *$(X \oplus Y)_T$ is a progenerator if and only if $(X/f(Y \otimes M))_A$ and Y_B are progenerators and $f : Y \otimes M \to X$ is monic.*

5. Projective covers in *Mod* − *T*

In this section we will give necessary and sufficient conditions for $(X \oplus Y)_T$ associated to $(X, Y)_f \in Obj\,\Omega$ to admit a projective cover in $Mod - T$ and explicitly describe the projective cover of $(X \oplus Y)_T$ when these conditions are satisfied.

Assume Y admits a projective cover $H \xrightarrow{\varepsilon_2} Y$ in $Mod - B$ and that $X/f(Y \otimes M)$ admits a projective cover $\delta_1 : P \to X/f(Y \otimes M)$ in $Mod - A$. Let $\eta : X \to X/f(Y \otimes M)$ denote the quotient map. Since P_A is projective we get a map $\gamma_1 : P \to X$ in $Mod - A$ satisfying $\eta \circ \gamma_1 = \delta_1$. Let $\theta = f \circ (\varepsilon_2 \otimes Id_M) : H \otimes M \to X$ and $\varepsilon_1 : (H \otimes M) \oplus P \to X$ be given by $\varepsilon_1 |_{H \otimes M} = \theta$ and $\varepsilon_1 |_P = \gamma_1$. From Theorem 3.1 we see that $(((H \otimes M) \oplus P) \oplus H)_T$ associated to $((H \otimes M) \oplus P, H)_j$ where $j : H \otimes M \to (H \otimes M) \oplus P$ denotes the inclusion as the first summand, is indeed a projective right T-module. Since $\varepsilon_1 \circ j = \theta = f \circ (\varepsilon_2 \otimes Id_M)$ we see that $(\varepsilon_1, \varepsilon_2) : ((H \otimes M) \oplus P, H)_j \to (X, Y)_f$ is a map in Ω. We write ε for the associated map $(((H \otimes M) \oplus P) \oplus H)_T \to (X \oplus Y)_T$ in $Mod - T$. With these conventions we now state and prove the main result of this section.

Theorem 5.1. *$(X \oplus Y)_T$ admits a projective cover if and only if $(X/f(Y \otimes M))_A$ and Y_B admit projective covers. When these conditions are satisfied, let $\delta_1 : P \to X/f(Y \otimes M)$ and $\varepsilon_2 : H \to Y$ denote projective covers in Mod − A and Mod − B respectively. Then $\varepsilon : (((H \otimes M) \oplus P) \oplus H)_T \to (X \oplus Y)_T$ described in the above paragraph yields a projective cover in Mod − T.*

Proof. Assume $(X \oplus Y)_T$ has a projective cover. As seen already in Theorem 3.1 any projective module in $Mod - T$ is associated to a triple $(\varphi(H \otimes M) \oplus P, H)_\varphi$ with H_B and P_A projective and $\varphi : H \otimes M \to \varphi(H \otimes M)$ an isomorphism in $Mod - A$. Let $(\pi_1, \pi_2) : (\varphi(H \otimes M) \oplus P, H)_\varphi \to (X, Y)_f$ yield a projective cover in $Mod - T$. Let $K = \ker \pi_1 : \varphi(H \otimes M) \oplus P \to X$ and $L = \ker \pi_2 : H \to Y$. Then $\pi_1 : \varphi(H \otimes M) \oplus P \to X$ and $\pi_2 : H \to Y$ are surjective and $(K \oplus L)_T \ll ((\varphi(H \otimes M) \oplus P) \oplus H)_T$. Let $v : \varphi(H \otimes M) \oplus P \to P$ denote the projection map. From Proposition 1.3 we see that $v(K) \ll P$ and $L \ll H$. The latter fact shows that $\pi_2 : H \to Y$ is a projective cover in Y_B.

Also from $\pi_1 \circ \varphi = f \circ (\pi_2 \otimes Id_M)$ and the surjectivity of $\pi_2 \otimes Id_M : H \otimes M \to Y \otimes M$ we see that $\pi_1(\varphi(H \otimes M)) = f(Y \otimes M)$. It follows that $\pi_1^{-1}(f(Y \otimes M)) = K + \varphi(H \otimes M)$. By passage to quotients, π_1 induces a surjection $\delta_1 : P \to X/\pi_1(\varphi(H \otimes M)) = X/f(Y \otimes M)$ with $\ker \delta_1 = v(\pi_1^{-1}(f(Y \otimes M)) = v(K)$. (Here we are regarding P as the quotient of $(\varphi(H \otimes M) \oplus P$ by $\varphi(H \otimes M))$. Since $v(K) \ll P$, it follows that $\delta_1 : P \to X/f(Y \otimes M)$ is a projective cover of $(X/f(Y \otimes M))_A$.

Conversely, assume $\delta_1 : P \to X/f(Y \otimes M)$ and $\varepsilon_2 : H \to Y$ are projective covers in $Mod - A$ and $Mod - B$ respectively. To prove that $\varepsilon : (((H \otimes M) \oplus P) \oplus H)_T \to (X \oplus Y)_T$

is a projective cover we have to prove the following:

(i) $\varepsilon_1 : (H \otimes M) \oplus P \to X$ and $\varepsilon_2 : H \to Y$ are surjective.

(ii) If $K = \ker \varepsilon_1$ and $L = \ker \varepsilon_2$ then $v(K) \ll P_A$ and $L \ll Y_B$ where $v : (H \otimes M) \oplus P \to P$ is the projection onto P.

Since $\varepsilon_2 : H \to Y_B$ is a projective cover, ε_2 is surjective and $L \ll Y_B$. By construction, $\varepsilon_1 |_{H \otimes M} = f \circ (\varepsilon_2 \otimes Id_M)$. Since $\varepsilon_2 : H \to Y$ is surjective, we get $\varepsilon_1(H \otimes M) = f(Y \otimes M)$. Also $\varepsilon_1 |_P = \gamma_1$ is a lift of $\delta_1 : P \to X/f(Y \otimes M)$ to a map $P \to X$. Since δ_1 is surjective we get $\varepsilon_1(P) + f(Y \otimes M) = X$. It follows that $\varepsilon_1((H \otimes M) \oplus P) = f(Y \otimes M) + \varepsilon_1(P) = X$. Thus $\varepsilon_1 : (H \otimes M) \oplus P \to X$ is surjective. Since $\varepsilon_1(H \otimes M) = f(Y \otimes M)$ we get $\varepsilon_1^{-1}(f(Y \otimes M)) = H \otimes M + K$. From the modular law, we have $H \otimes M + K = (H \otimes M) \oplus (P \cap K)$. This immediately yields, $v(K) = P \cap K$. But $\ker \delta_1 = \gamma_1^{-1}(f(Y \otimes M)) = P \cap \varepsilon_1^{-1}(f(Y \otimes M)) = P \cap \{(H \otimes M) \oplus (P \cap K)\} = P \cap (H \otimes M) \oplus (P \cap K)$ (again by the modular law) $= P \cap K = v(K)$.

Since $\delta_1 : P \to X/f(Y \otimes M)$ is a projective cover we get $v(K) \ll P_A$. This proves (i) and (ii) and completes the proof of Theorem 5.1. □

The following well-known result is an immediate consequence of Theorem 5.1.

Corollary 5.2. *Let*

$$T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}.$$

(i) *T is semi-perfect if and only if A and B are semi-perfect.*

(ii) *T is right perfect if and only if A and B are right perfect.*

Proof. (ii) is immediate from Theorem 5.1, while (i) needs the characterization of finitely generated T-modules. □

Remark 5.3. In (ii) we can replace “right perfect” by “left perfect” throughout.

Finally we wish to record that in [6] injective modules and injective hulls of modules over a ring of a Morita context (thus in particular over a formal triangular matrix ring) have been characterized.

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