Stiefel-Whitney homology classes

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1. Introduction

In this paper M is a smooth n-manifold without boundary; its tangent bundle is $\tau_M\colon T_M \xrightarrow{\pi} M$, and $T_x(M)$ is the tangent space at x. The q^{th} Stiefel-Whitney class of M, W^q, is the primary obstruction to finding $n-q+1$ linearly independent vector fields on M. W^q is an element of $H^q(M; Z)$ (q odd or $q = n$) and an element of $H^q(M; Z_2)$ (q even and $< n$); in the first case we use twisted coefficients. Thus the Poincaré dual of W^q is a homology class

$$W_{n-q} \in \begin{cases} H_{n-q}(M;\, Z) & q \text{ odd or } q = n \\ H_{n-q}(M;\, Z_2) & q \text{ even and } < n \end{cases}$$

where we use infinite chains if M is not compact. W_p will be called the p^{th} *Stiefel-Whitney homology class.*

Throughout the paper (K, φ) denotes a smooth triangulation of M (K is a simplicial complex and $\varphi\colon |K| \to M$ is a homeomorphism from the geometric realization of K to M; further φ is a smooth embedding on each simplex). K' denotes the first barycentric subdivision of K.

The simplices of K will be denoted by $\mathfrak{a}, \mathfrak{a}_i, \mathfrak{b}, \mathfrak{b}_i, \cdots$; their barycentres are denoted by $\underline{\mathfrak{a}}, \underline{\mathfrak{a}}_i, \underline{\mathfrak{b}}, \underline{\mathfrak{b}}_i$. If $\mathfrak{a}$ is a p-simplex we write $|\mathfrak{a}| = p$. The simplices of K' are denoted by $\sigma, \tau, \cdots$; and their barycentres are denoted by $\underline{\sigma}, \underline{\tau}, \cdots$. Each p-simplex $\sigma \leqq K'$ is uniquely of the form $\sigma = \langle \underline{\mathfrak{a}}_0 \cdots \underline{\mathfrak{a}}_p \rangle$ where $\mathfrak{a}_0 < \cdots < \mathfrak{a}_p \in K$. ($\mathfrak{a} \leqq \mathfrak{b}$ (resp. $\mathfrak{a} < \mathfrak{b}$) means $\mathfrak{a}$ is a face (resp. a proper face) of $\mathfrak{b}$.)

Each σ is given that orientation for which $\underline{\mathfrak{a}}_0, \cdots, \underline{\mathfrak{a}}_p$ is a positive ordering of the vertices.

An infinite integral simplicial p-chain on M will mean a formal infinite integral combination, $\sum \lambda_\sigma \sigma$, where the sum runs over the distinct p-simplices of K', ordered as described above. These chains form a complex $C_*(M) = \sum_p C_p(M)$ whose homology is the standard infinite integral homology of M.

In this paper we present a proof of the following theorem of Whitney [9].

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THEOREM 1. *The (infinite) chain*

$$C_p = \sum_{a_0 < \cdots < a_p \leqq K} (-1)^{|a_0| + \cdots + |a_p|} \langle \underline{a}_0 \cdots \underline{a}_p \rangle \qquad (0 \leqq p < n)$$

is a cycle (integral if $n - p$ odd or $p = 0$; mod 2 *if $n - p$ even) and represents the Stiefel-Whitney homology class W_p.*

(Note that C_p is the sum of all the p-simplices of K', with appropriate signs.)

The theorem has the obvious

COROLLARY. *The (infinite) chain*

$$\sum_{a_0 < \cdots < a_p \leqq K} \langle \underline{a}_0 \cdots \underline{a}_p \rangle$$

is a (mod 2)*-cycle and represents the p^{th}* (mod-2) *Stiefel-Whitney homology class.*

Remarks 1. The theorem was conjectured by Stiefel [7]. It was then proved by Whitney, who wrote up his proof for a book; unfortunately this never appeared. A proof using different techniques has recently been obtained by Cheeger, and a sketch appears in [3]. However, no complete proof seems to have appeared in print.

Rourke has observed that our proof works in the P.L. category. Sullivan has used the corollary to define (mod 2) Stiefel-Whitney classes for more general spaces [8].

2. $C_0 = \sum_{a < K} (-1)^{|a|} \underline{a}$ represents (if M is compact and connected) the Euler characteristic of M, $\chi(M) \in H_0(M; Z) = Z$. In this case the theorem reduces to the Hopf theorem ($\chi(M)$ = the index sum of a vector field with finitely many zeros).

The paper is organized as follows: in Section 2 we establish properties of the chains C_p for a wider class of spaces than manifolds. These have recently been obtained independently by Sullivan. In Section 3 we review basic facts about twisted chains and Poincaré duality.

The rest of the paper is devoted to the proof of Theorem 1. In Section 4 we construct vector fields $F_1, \cdots, F_n$ on M with the property that the first p are linearly independent of the $(p - 1)$-skeleton of K'. These vector fields are essentially the same as those given by Whitney in [9]. With the aid of these we can write down a representative $\Sigma m_\sigma \sigma$ of W_p, where each integer m_σ is the index of F_{p+1} (mod $F_1, \cdots, F_p$) at the barycentre of σ.

In Sections 5 and 6 the integers m_σ are computed. This is done by deforming F_{p+1} to more tractable local sections Z_σ. The index of Z_σ is computed by expressing its flow as a product of explosions and implosions.

We are grateful to J. Munkres for pointing out to us his theorem on

smooth cell decompositions [5] which greatly simplifies the computation of m_σ.

Section 7 contains the proof of a lemma used in Section 5.

2. Euler spaces

The chains C_p introduced above have a wider application. Let L be any locally-finite n-dimensional simplicial complex, with first barycentric subdivision L'. We use L also to denote the geometric realization of L (and L'); a simplex $\alpha \leqq L$ is also considered as a closed subset of L. $\dot{\alpha}$ denotes the boundary of α. $\overset{\circ}{\alpha}$ denotes its interior, and $Lk\ (\alpha; L)$ denotes the link of α in L.

Define (infinite) simplicial chains

$$C_p(L') = \sum\nolimits_{\alpha_0 < \cdots < \alpha_p \leqq L} (-1)^{|\alpha_0| + \cdots + |\alpha_p|} \langle \underline{\alpha}_0 \cdots \underline{\alpha}_p \rangle$$

$(0 \leqq p \leqq n)$. Sullivan and E. Akin [8] showed that these chains are (mod 2)-cycles if and only if for each $x \in L$, $\chi(L, L - x) \equiv 1 \pmod 2$; and they called such spaces (mod 2)-Euler spaces. (For a subspace B of a topological space A,

$$\chi(A, B) = \sum\nolimits_p (-1)^p \dim H_p(A, B; Q)$$

whenever the right hand side makes sense.)

An n-dimensional locally finite polyhedron L will be called an *integral Euler space* if

$$\chi(L, L - x) = (-1)^n \qquad \text{for all } x \in L\,.$$

Sullivan has shown that complex analytic spaces are integral Euler spaces [8].

PROPOSITION 1. *L is an integral Euler space if and only if*

$$\partial C_p(L') = \chi(S^{n-p}) C_{p-1}(L')\,. \qquad p \geqq 1\,.$$

Proof. For any $x \in L$ there is a simplex $\alpha < L$ such that $x \in \overset{\circ}{\alpha}$, and there is a homeomorphism carrying x to $\underline{\alpha}$. Moreover,

$$H_*(L, L - \underline{\alpha}; Q) \cong \tilde{H}_*(Lk(\underline{\alpha}; L'); Q)$$

where the isomorphism shifts degrees by 1. Thus L is an integral Euler space precisely when

$$1 - \chi(Lk(\underline{\alpha}; L')) = (-1)^n\,, \qquad \text{for all } \alpha < L\,.$$

Note that $Lk(\underline{\alpha}; L')$ is homeomorphic to the join $\dot{\alpha} * Lk(\alpha; L)$. Since $\dot{\alpha}$ is an $(|\alpha| - 1)$-sphere $(S^{-1} = \phi)$ and $1 - \chi$ behaves multiplicatively on joins, L is an integral Euler space if and only if

$$(-1)^{|\alpha|}(1 - \chi(Lk(\alpha; L))) = (-1)^n\,, \qquad \text{for all } \alpha < L\,.$$

On the other hand, write

$$\partial C_p = \sum\nolimits_{\alpha_0 < \cdots < \alpha_{p-1} \leqq L} \lambda(\alpha_0 \cdots \alpha_{p-1}) \langle \underline{\alpha}_0 \cdots \underline{\alpha}_{p-1} \rangle\,.$$

Then

$$\begin{aligned}\lambda(\mathfrak{a}_0, \cdots, \mathfrak{a}_{p-1}) &= \sum_{\mathfrak{a}<\mathfrak{a}_0} (-1)^{|\mathfrak{a}|+|\mathfrak{a}_0|+\cdots+|\mathfrak{a}_{p-1}|} \\ &\quad + \sum_{i=0}^{p-2} (-1)^{i+1} \sum_{\mathfrak{a}_i<\mathfrak{a}<\mathfrak{a}_{i+1}} (-1)^{|\mathfrak{a}|+|\mathfrak{a}_0|+\cdots+|\mathfrak{a}_{p-1}|} \\ &\quad + \sum_{\mathfrak{a}_{p-1}<\mathfrak{a}} (-1)^p(-1)^{|\mathfrak{a}|+|\mathfrak{a}_0|+\cdots+|\mathfrak{a}_{p-1}|} \\ &= (-1)^{|\mathfrak{a}_0|+\cdots+|\mathfrak{a}_{p-1}|}\{\chi(\dot{\mathfrak{a}}_0) + \sum_{i=0}^{p-2} (-1)^{i+1}(-1)^{|\mathfrak{a}_i|+1}\chi(Lk(\mathfrak{a}_i;\ \dot{\mathfrak{a}}_{i+1})) \\ &\quad + (-1)^p(-1)^{|\mathfrak{a}_{p-1}|+1}\chi(Lk(\mathfrak{a}_{p-1}:\ L))\}\ .\end{aligned}$$

Since the link of $\mathfrak{a}_i$ in $\dot{\mathfrak{a}}_{i+1}$ is an $(|\mathfrak{a}_{i+1}| - |\mathfrak{a}_i| - 2)$-sphere its Euler characteristic is $1 + (-1)^{|\mathfrak{a}_{i+1}|-|\mathfrak{a}_i|}$. Thus

$$\begin{aligned}&(-1)^{|\mathfrak{a}_0|+\cdots+|\mathfrak{a}_{p-1}|}\lambda(\mathfrak{a}_0, \cdots, \mathfrak{a}_{p-1}) \\ &= 1 - (-1)^{|\mathfrak{a}_0|} + \sum_{i=0}^{p-2} [(-1)^{i+|\mathfrak{a}_i|} - (-1)^{i+1+|\mathfrak{a}_{i+1}|}] \\ &\quad - (-1)^{p+|\mathfrak{a}_{p-1}|}\chi(Lk(\mathfrak{a}_{p-1};\ L)) \\ &= 1 + (-1)^{p+|\mathfrak{a}_{p-1}|}(1 - \chi(Lk(\mathfrak{a}_{p-1};\ L)))\ .\end{aligned}$$

This integer is $\chi(S^{n-p})$ if and only if

$$(-1)^n = (-1)^{|\mathfrak{a}_{p-1}|}(1 - \chi(Lk(\mathfrak{a}_{p-1};\ L)))\ .$$

The proposition follows. q.e.d.

Proposition 1, in conjunction with Theorem 1, yields the well known fact ([6], p. 195) that if q is even then $W^{q+1} \in H^{q+1}(M, Z)$ is the Bockstein of $W^q \in H^q(M;\ Z_2)$.

3. Poincaré duality

In this section we recall the definitions of twisted cochains and Poincaré duality.

Let K'' denote the second barycentric subdivision of K. For each $\sigma \leqq K'$ with barycentre $\underline{\sigma}$ we denote by $D\sigma$ and $\dot{D}\sigma$ the full subcomplexes of K'' defined by:

$\underline{\tau}$ is a vertex of $D\sigma$ (resp. $\dot{D}\sigma$) if and only if $\sigma \leqq \tau$ (resp. $\sigma < \tau$).

$\dot{D}\sigma$ is the boundary of $D\sigma$. The interiors of $D\sigma$ and σ are denoted by $\overset{\circ}{D}\sigma$ and $\overset{\circ}{\sigma}$.

Because K is a smooth triangulation of M, for each p-simplex $\sigma \leqq K'$, $D\sigma$ is an $(n - p)$-cell and $\dot{D}\sigma$ is an $n - p - 1$ sphere. The collection $\{D\sigma\}_{\sigma \leqq K'}$ decomposes M as a cell complex whose *cellular q-skeleton* consists of all the cells $D\sigma$ of dimension $\leqq q$. We denote it by M_q.

Note that for each σ the map

$$(x, y) \longmapsto \frac{1}{2}x + \frac{1}{2}y \qquad \begin{matrix} x \in \overset{\circ}{\sigma} \\ y \in \overset{\circ}{D}\sigma \end{matrix}$$

(convex combination in K') defines a homeomorphism ψ_σ of $\overset{\circ}{\sigma} \times \overset{\circ}{D}\sigma$ onto an open neighborhood, U_σ of $\underline{\sigma}$ in M.

The twisted integral cellular cochain complex $C^*(M)$ corresponding to the cell decomposition above of M is described as follows: Let $G(\sigma)$ denote the group of global sections of the integral orientation sheaf of $\tau_M|_{D\sigma}$; thus $G(\sigma) \cong \mathbf{Z}$ and its two generators are the two orientations in $\tau_M|_{D\sigma}$. A twisted integral cochain is a map, Φ, which assigns to each oriented q-cell $(D\sigma, \mu)$ an element of $G(\sigma)$, and which satisfies

$$\Phi(D\sigma, -\mu) = -\Phi(D\sigma, \mu)$$

($-\mu$ denotes the orientation of $D\sigma$ opposite to μ).

To define the coboundary operator, δ, observe that the boundary of a $(q+1)$-cell, $D\tau$, is the union of q-cells $D\sigma_i$. With an orientation, ν, in $D\tau$ associate the orientations μ_i in $D\sigma_i$ which satisfy

$$\nu = (\text{outward normal}) \times \mu_i \ .$$

Then set

$$(\delta\Phi)(D\tau, \nu) = \textstyle\sum_i \rho^{\tau}_{\sigma_i}(\Phi(D\sigma_i, \mu_i))$$

where $\rho^{\tau}_{\sigma_i}\colon G(\tau) \to G(\sigma_i)$ is the restriction isomorphism.

The canonical projections $G(\sigma) \to \mathbf{Z}_2$ $(\sigma < K')$ define a homomorphism from this cochain complex to the (mod 2)-cochain complex; it is called reduction mod 2.

Finally, we recall the definition of the Poincaré isomorphism $\mathcal{D}\colon C_*(M) \to C^*(M)$ which induces the standard Poincaré isomorphism of homology.

Let $(D\sigma, \mu)$ be an oriented $(n-p)$-cell. It determines a generator, $\langle \sigma, (D\sigma, \mu)\rangle$ of $G(\sigma)$ as follows: Identify the generators of $G(\sigma)$ with the orientations in U_σ in the usual way, and let $\langle \sigma, (D\sigma, \mu)\rangle$ be the orientation of U_σ for which $\psi_\sigma\colon \mathring{\sigma} \times \mathring{D\sigma} \to U_\sigma$ is orientation preserving. (Here $\mathring{\sigma}$ is oriented as described in Section 1, and $\mathring{\sigma} \times D\mathring{\sigma}$ is given the product orientation.)

Now $\mathcal{D}\colon C_p(M) \to C^{n-p}(M)$ is defined by

$$\mathcal{D}\left(\textstyle\sum_\tau \lambda_\tau \tau\right)(D\sigma, \mu) = \lambda_\sigma \langle \sigma, (D\sigma, \mu)\rangle \ .$$

The Poincaré isomorphism between (mod 2) chains and (mod 2) cochains is defined analogously; and the Poincaré duality commutes with reduction mod 2.

4. Stiefel-Whitney homology classes

Consider K (and hence K') as a P.L. subset of $\mathbf{R}^N$ (N sufficiently large). Each p-simplex $\sigma \leqq K'$ spans an affine p-plane $T(\sigma) \subset \mathbf{R}^N$. If $x \in \sigma$, its *tangent space* is the linear space $T_x(\sigma)$ obtained from the affine space $T(\sigma)$ by making x the origin. The *tangent bundle of* σ is the product bundle $\sigma \times T(\sigma)$ ($x \times T(\sigma)$ is identified with $T_x(\sigma)$). Note that

$$T_x(\sigma) = T(\sigma) = T_y(\sigma) \qquad x, y \in \sigma$$

identifies $T_x(\sigma)$ and $T_y(\sigma)$ as affine, but not as linear spaces. The orientation of σ defined in Section 1 determines an orientation in each $T_x(\sigma)$.

Since (K, φ) is a smooth triangulation of M, so is (K', φ). Thus the restriction, φ_σ, of φ to a simplex $\sigma \leqq K'$ is smooth, and its derivative

$$d\varphi_\sigma \colon \sigma \times T(\sigma) \longrightarrow T_M$$

is a bundle map which restricts to a linear injection in each fibre. *Henceforth we identify the geometric realization of* K *(and of* K'*) with* M *via* φ. *Further, for* $x \in \sigma \leqq K$ *we identify* $T_x(\sigma)$ *with the subspace of* $T_x(M)$ *to which it is mapped by* $d\varphi_\sigma$.

Let $x \in M$. Its closed star (in K') is denoted by $St\,x$. If $\sigma \leqq St\,x$ is a maximal simplex then $x \in T(\sigma)$. Thus a continuous cross-section $X\colon St\,x \to T_M$ is defined by

$$X(y) = x \qquad y \in St\,x\ .$$

X is smooth in each simplex and is called the *radial vector field generated by* x; $X(y) = 0$ if and only if $y = x$. The radial vector fields generated by the points $\underline{a}$, $\underline{a}_i$, $\underline{b}$, $\underline{b}_j$, $\underline{\sigma}$, will be denoted by A, A_i, B, B_j, and S. If $x = \sum \lambda_i \underline{a}_i$ then

$$X = \sum \lambda_i A_i\ .$$

For $x \in \langle \underline{a}_0 \cdots \underline{a}_p \rangle$ we write

$$x = \sum_{i=0}^{p} \lambda_{\underline{a}_i}(x) \underline{a}_i\ .$$

λ_{a_i} is extended to a continuous function in M by setting $\lambda_{a_1}(x) = 0$, $x \notin St\,\underline{a}_i$.

LEMMA 1. *Let* $\sigma = \langle \underline{a}_0 \cdots \underline{a}_p \rangle \leqq K'$ *and let* $x \in \mathring{\sigma}$. *Then*

(i) $A_1(x), \cdots, A_p(x)$ *is a positively ordered basis of* $T_x(\sigma)$.

(ii) $\sum_0^p \lambda_{a_i}(x) A_i(x) = 0$.

Proof. Clear.

Definition 1. The *fundamental vector fields* $F_1, \cdots, F_n$ on M are the vector fields given by

$$F_p(x) = \sum_{a_0 < \cdots < a_p \leqq K} \lambda_{a_0}(x) \cdots \lambda_{a_p}(x) A_p(x)\ .$$

LEMMA 2. *The* F_p *are well-defined continuous vector fields on* M, *smooth on each simplex. Moreover,*

(i) *If* $x \in \sigma$ *then* $F_p(x) \in T_x(\sigma)$, $1 \leqq p \leqq n$.

(ii) $F_p(x) = 0$ *if* x *is in the* $(p-1)$*-skeleton of* K'.

(iii) *If* $x \in \mathring{\sigma}$ $(\sigma = \langle \underline{a}_0 \cdots \underline{a}_p \rangle)$ *then* $F_1(x), \cdots, F_p(x)$ *is a positively ordered basis for* $T_x(\sigma)$.

Proof. (i) and (ii) are obvious. To prove (iii) observe that

$$F_i(x) = \sum_{j=i}^{p} \gamma_{ij}(x)\, A_j(x)$$

where $\gamma_{ii}(x) = \lambda_{a_0}(x) \cdots \lambda_{a_i}(x) > 0$. Thus (iii) follows from Lemma 1, (i).

q.e.d.

COROLLARY. *For any* y *not in the* $(q-1)$*-skeleton of* K' *the vectors* $F_1(y), \cdots, F_q(y)$ *are linearly independent.*

M^p will denote the p-skeleton of K' (recall that M_q denotes the q-skeleton of the dual cell complex). The corollary to Lemma 2 yields a trivial subbundle of $\tau_M|_{M-M^{p-1}}$:

$$\xi^p: E^p \xrightarrow{\pi} (M - M^{p-1})$$

whose fibre E^p_x at x is the space spanned by $F_1(x), \cdots, F_p(x)$. Orient ξ^p so that $F_1(x), \cdots, F_p(x)$ is a positive basis of E^p_x.

Let $\eta^{n-p}: N^{n-p} \xrightarrow{\pi} (M - M^{p-1})$ be the orthogonal complement of ξ^p with respect to some Riemannian metric in τ_M. Define $F^{\perp}_{p+1}$ to be the unique cross-section of η^{n-p} which satisfies

$$F^{\perp}_{p+1}(x) - F_{p+1}(x) \in E^p_x \qquad x \in M - M^{p-1}\,.$$

Then (again by the corollary to Lemma 2) $F^{\perp}_{p+1}(x) \neq 0$ if $x \notin M^p$.

$F^{\perp}_{p+1}$ determines a twisted integral $n-p$ cochain, Φ, which represents W^{n-p}, as follows: Fix an oriented $(n-p)$-cell $(D\sigma, \mu)$. Then $D\sigma \subset M - M^{p-1}$, and the decomposition

$$\xi^p|_{D\sigma} \oplus \eta^{n-p}|_{D\sigma} = \tau_M|_{D\sigma}$$

establishes a $1-1$ correspondence between orientations in $\eta^{n-p}|_{D\sigma}$ and orientations in $\tau_M|_{D\sigma}$. Moreover, $D\sigma \cap M^p = \{\underline{\sigma}\}$; hence $F^{\perp}_{p+1}$ defines a cross-section in $\eta^{n-p}|_{D\sigma}$ with a single zero at $\underline{\sigma}$.

Now choose an orientation in $\eta^{n-p}|_{D\sigma}$. Use the local product structure to obtain from $F^{\perp}_{p+1}$ a continuous map

$$\gamma_\sigma: (D\sigma, D\sigma - \{\underline{\sigma}\}) \longrightarrow (\mathbf{R}^{n-p}, \mathbf{R}^{n-p} - \{0\})$$

between pairs of oriented spaces. Set

$$\Phi(D\sigma, \mu) = (\text{degree of } \gamma_\sigma) g(\sigma)$$

where $g(\sigma) \in G(\sigma)$ is the orientation of $\tau_M|_{D\sigma}$ corresponding to the chosen orientation of $\eta^{n-p}|_{D\sigma}$.

The degree of γ_σ is called the *index* of $F^{\perp}_{p+1}$; it depends only on $F^{\perp}_{p+1}$ and the orientations of $D\sigma$ and $\eta^{n-p}|_{D\sigma}$. Thus the right hand side of the equation above depends only on $F^{\perp}_{p+1}$ and $(D\sigma, \mu)$; and so it defines an $(n - p)$ twisted cochain, Φ.

Moreover, Φ (or its (mod 2)-reduction if $(n - p)$ is even and $p > 0$) represents the Stiefel-Whitney cohomology class W^{n-p}. This follows easily from the obstruction definition of these classes—cf. [6]. For the equivalence of this and other definitions see Milnor [4].

An orientation of $\eta^{n-p}|_{D\sigma}$ determines an orientation of $\tau_M|_{D\sigma}$ and hence it determines an orientation in the open set U_σ of Section 3. Orientations in $\eta^{n-p}|_{D\sigma}$ and in $D\sigma$ will be called *compatible* if the homeomorphism $\psi_\sigma\colon \mathring{\sigma} \times \mathring{D}\sigma \to U_\sigma$ is orientation preserving. The following proposition is an immediate consequence of the discussion above, and Section 3.

PROPOSITION 2. *Let m_σ be the index of $F^{\perp}_{p+1}$ at $\underline{\sigma}$, defined with respect to compatible orientations of $D\sigma$ and $\eta^{n-p}|_{D\sigma}$. Then the (infinite) chain*

$$\sum\nolimits_{\sigma\ a\ p\text{-simplex of } K'} m_\sigma \sigma$$

(or its (mod 2)*-reduction if $n - p$ is even and $p > 0$) represents W_p.*

It remains to compute the integers m_σ. W_p is either an integral 2-torsion class or a (mod 2)-class if $p > 0$. In either case if c represents W_p so does $-c$. Thus Theorem 1 follows from Proposition 2 as soon as we have proved

THEOREM 2. *The integers m_σ of Proposition 2 are given by*

$$(-1)^{p(p+1)/2} m_\sigma = (-1)^{|a_0|+\cdots+|a_p|}$$

if $\sigma = \langle \underline{a}_0 \cdots \underline{a}_p \rangle$.

5. Deformation of F_{p+1}

By a theorem of Munkres ([5], 1.4 and 1.5) any smooth triangulation of M is isotopic to a smooth triangulation whose dual cells form a *smooth* cell decomposition of M. If φ_1, $\varphi_2\colon K' \to M$ are isotopic smooth triangulations, then the indices m_σ of the vector fields $F^{\perp}_p$ (defined via φ_1 and φ_2) are the same.

Thus to compute the m_σ we may and do assume that the dual cells $\varphi(D\sigma)$ $(\sigma < K')$ form a smooth cell decomposition of M. Then for a p-simplex $\sigma < K'$,

$$T_{\underline{\sigma}}(\sigma) \oplus T_{\underline{\sigma}}(D\sigma) = T_{\underline{\sigma}}(M) .$$

It follows that for some neighbourhood, W_σ, of $\underline{\sigma}$ in $D\sigma$ we have

$$\xi^p|_{W_\sigma} \oplus \tau_{W_\sigma} = \tau_M|_{W_\sigma} .$$

Hence we may and do choose the Riemannian metric in τ_M so that the restriction of η^{n-p} to W_σ coincides with the tangent bundle of W_σ. Moreover, the orientation induced in τ_{W_σ} (i.e. in $\eta^{n-p}|_{W_\sigma}$) by an orientation of $D\sigma$ is compatible with the orientation of $D\sigma$ in the sense of Section 4.

Henceforth in this section $\sigma = \langle \underline{\mathfrak{a}}_0 \cdots \underline{\mathfrak{a}}_p \rangle$ denotes a fixed p-simplex of K', and $\mathfrak{a}_0 < \cdots < \mathfrak{a}_p$. L_0 denotes the full subcomplex of K' spanned by the vertices $\underline{\mathfrak{b}}_j$ with $\mathfrak{b}_j < \mathfrak{a}_0$; L_i $(i = 1, \cdots, p)$ is the full subcomplex spanned by the $\underline{\mathfrak{b}}_j$ with $a_{i-1} < b_j < \mathfrak{a}_i$; L_{p+1} is the full subcomplex spanned by the $\underline{\mathfrak{b}}_j$ with $\mathfrak{b}_j > \mathfrak{a}_p$. Each L_i, $i \geqq 1$ is a combinatorial $(|\mathfrak{a}_i| - |\mathfrak{a}_{i-1}| - 2)$-sphere, and L_0 is an $(|\mathfrak{a}_0| - 1)$-sphere.

D_i will denote the cone on L_i with vertex 0; a point w in D_i is then of the form $\sum_j \lambda_j \underline{\mathfrak{b}}_j$ where the $\underline{\mathfrak{b}}_j$ are vertices of L_i, $\lambda_j \geqq 0$, and $\sum \lambda_j \leqq 1$. If $w = \sum \lambda_j \underline{\mathfrak{b}}_j \in D_i$ we write $\sum_j \lambda_j = |w|$. Then

$$w \longmapsto \sum_{i=0}^{p} \frac{1 - |w|}{p + 1} \underline{\mathfrak{a}}_i + \sum_j \lambda_j \underline{\mathfrak{b}}_j$$

defines a continuous embedding $\psi_i \colon D_i \to M$.

Remark. The letters $\underline{\mathfrak{b}}_j$ may denote any vertices of K'; they will *not* be restricted to vertices of the L_i's, unless explicitly stated (as above). In particular, unless the $\underline{\mathfrak{b}}_j$ are restricted to be vertices of L_i, the formal sum $\sum \lambda_j \underline{\mathfrak{b}}_j$ does not make sense as a point of D_i.

For each sequence $y_0, \cdots, y_{p+1}$ $(y_i \in D_i)$ there is some simplex $\tau < St\, \underline{\sigma}$ which contains all the $\psi_i y_i$. A continuous map

$$\psi \colon D_0 \times \cdots D_{p+1} \longrightarrow M$$

is given by

$$(y_0, \cdots, y_{p+1}) \longmapsto \sum_i \frac{1}{p + 2} \psi_i(y_i)$$

(convex combination in τ).

We can (and do) choose W_σ and neighbourhoods W_i of $\underline{\sigma}$ in D_i so that ψ restricts to a homeomorphism

$$\psi \colon W_0 \times \cdots \times W_{p+1} \xrightarrow{\cong} W_\sigma .$$

The restriction of ψ to $W_0 \cap \gamma_0 \times \cdots \times W_{p+1} \cap \gamma_{p+1}$ (γ_i a simplex of D_i) is a smooth embedding.

We shall deform $F^{\perp}_{p+1}$ to a vector field on $D\sigma$ whose flow is a product of implosions and explosions along the cone lines of the D_i. As a first step, define a vector field V_σ tangent to $D\sigma$ by

$$V_\sigma(x) = \sum_{\underline{\mathfrak{b}}_j \neq \underline{\mathfrak{a}}_0, \cdots, \underline{\mathfrak{a}}_p} \varepsilon_j \lambda_j B_j(x)$$

where $x = \sum_0^n \lambda_j \underline{\mathfrak{b}}_j \in D\sigma$ and $\varepsilon_j = (-1)^{p+1+i}$ if $\underline{\mathfrak{b}}_j \in L_i$. Thus if $x \in \operatorname{Im} \psi_i$ then

$V_\sigma(x) = (-1)^{p+1+i} \sum \lambda_j B_j(x)$. Since V_σ is tangent to $D\sigma$, its restriction to W_σ is a cross-section in η^{n-p}.

PROPOSITION 3. *Suppose $x \in W_\sigma - \{\underline{\sigma}\}$ and let t, s be non-negative numbers, not both zero. Then*

$$tV_\sigma(x) + sF^{\perp}_{p+1}(x) \neq 0 .$$

In particular V_σ is a vector field on $D\sigma$ with an isolated zero at $\underline{\sigma}$, and index m_σ at $\underline{\sigma}$.

LEMMA 3. *If $\tau = \langle \underline{\mathfrak{b}}_0 \cdots \underline{\mathfrak{b}}_q \rangle \leqq K'$ and $x = \sum_{i=0}^q \lambda_i \underline{\mathfrak{b}}_i \in \mathring{\tau}$, then*

(i) $B_0(x) = \sum_{j=1}^q \dfrac{(-1)^j}{\lambda_0^{j+1}} F_j(x)$

and

(ii) $B_i(x) = \dfrac{1}{\lambda_0 \cdots \lambda_i} \{F_i(x) + \sum_{j=i+1}^q \gamma_{ij} F_j(x)\}$

where

$$\gamma_{ij} = (-1)^{j-i} \sum_{0 \leq \nu_1 \leq \cdots \leq \nu_{j-i} \leq i} \frac{1}{\lambda_{\nu_1} \cdots \lambda_{\nu_{j-i}}}$$

and B_i is the radial field corresponding to $\underline{\mathfrak{b}}_i$.

Proof. Deferred to Section 7.

COROLLARY. $B_i(x) \equiv (-1)^{q-i} F_q(x) \pmod{F_1(x), \cdots, F_{q-1}(x)}$.

Proof of Proposition 3. Fix $x \in D\sigma - \{\underline{\sigma}\}$. Let $\tau = \langle \underline{\mathfrak{b}}_0 \cdots \underline{\mathfrak{b}}_q \rangle < K'$ $(\mathfrak{b}_0 < \cdots < \mathfrak{b}_q)$ be its carrier; write $x = \sum_0^q \lambda_i \underline{\mathfrak{b}}_i$. Then $p < q$ and $\sigma < \tau$. We must show that

$$tV_\sigma(x) + sF_{p+1}(x) \not\equiv 0 \pmod{F_1(x), \cdots, F_p(x)}$$

and we proceed by induction on q.

Suppose first that $q = p + 1$. Then for some i $(0 \leqq i \leqq p + 1)$

$$\mathfrak{a}_j = \mathfrak{b}_j \quad (j < i) \quad \text{and} \quad \mathfrak{a}_j = \mathfrak{b}_{j+1} \quad (j \geqq i) .$$

Thus the corollary to Lemma 3 yields

$$V_\sigma(x) = (-1)^{p+1+i} \lambda_i B_i(x) \equiv \lambda_i F_{p+1}(x) \pmod{F_1(x), \cdots, F_p(x)} .$$

Clearly the proposition must hold in this case.

Next assume that $q \geqq p + 2$ and that the proposition is proved for $q - 1$. Fix an integer r so that $\underline{\mathfrak{b}}_r$ is not a vertex of σ. Set $\tau_1 = \underline{\mathfrak{b}}_0 \cdots \hat{\underline{\mathfrak{b}}}_r \cdots \underline{\mathfrak{b}}_q$ and let $\theta: \mathring{\tau} \to \mathring{\tau}_1$ be the projection from $\underline{\mathfrak{b}}_r$,

$$\theta: \sum \lambda_i \underline{\mathfrak{b}}_i \longmapsto \sum_{i \neq r} \frac{\lambda_i}{1 - \lambda_r} \underline{\mathfrak{b}}_i .$$

Its derivative $d\theta\colon T_x(\tau) \to T_y(\tau_1)$, $\big(y = \theta(x)\big)$ satisfies

$$d\theta\big(B_i(x)\big) = B_i(y) \qquad i \in \{0, 1, \cdots, \hat{r}, \cdots, q\}$$

and

$$d\theta\big(B_r(x)\big) = 0 .$$

It follows that for each j

$$d\theta\big(F_j(x)\big) = (1 - \lambda_r)^j F_j(y) + \lambda_r(1 - \lambda_r)^{j-1} F_{j-1}(y)$$

(set $F_0(y) = 0$); moreover

$$d\theta\big(V_\sigma(x)\big) = (1 - \lambda_r) V_\sigma(y) .$$

Thus a relation of the form

$$tV_\sigma(x) + sF_{p+1}(x) \equiv 0 \pmod{F_1(x), \cdots, F_p(x)}$$

would yield (after $d\theta$ was applied to both sides) the relation

$$t(1 - \lambda_r) V_\sigma(y) + s(1 - \lambda_r)^{p+1} F_{p+1}(y) \equiv 0 \pmod{F_1(y), \cdots, F_p(y)} .$$

This would contradict the induction hypothesis; hence the proposition is proved. q.e.d.

Let S be the radial vector field in $St\,\underline{\sigma}$ which is generated by $\underline{\sigma}$. If $\underline{\mathfrak{b}}_j$ is a vertex of $St\,\underline{\sigma}$ we write $\bar{B}_j = B_j - S$; it is a vector field in $St\,\underline{\sigma} \cap St\,\mathfrak{b}_j$. With respect to the standard parallelism in the ambient $\mathbf{R}^N$, $\bar{B}_j(x)$ is the *constant* vector field $\underline{\mathfrak{b}}_j - \underline{\sigma}$.

We define a vector field, Z_σ, tangent to $D\sigma$ by

$$Z_\sigma(x) = \sum_{\underline{\mathfrak{b}}_j \neq \underline{\mathfrak{a}}_0, \cdots, \underline{\mathfrak{a}}_p} \varepsilon_j \lambda_j \bar{B}_j(x)$$

where $x = \sum_0^n \lambda_j \underline{\mathfrak{b}}_j \in D\sigma$, and $\varepsilon_j = (-1)^{p+1+i}$ if $\underline{\mathfrak{b}}_j \in L_i$.

PROPOSITION 4. *$tZ_\sigma(x) + (1 - t) V_\sigma(x) \neq 0$ if $x \in D\sigma - \underline{\sigma}$ and $0 \leqq t \leqq 1$. In particular, Z_σ has an isolated zero at $\underline{\sigma}$ with index m_σ.*

Proof. Suppose $x = \sum_0^n \lambda_j \underline{\mathfrak{b}}_j \in D\sigma$. Let $J = \{j/\underline{\mathfrak{b}}_j \neq \text{any } \underline{\mathfrak{a}}_i\}$. Then

$$Z_\sigma(x) = V_\sigma(x) - \big(\textstyle\sum_{j \in J} \varepsilon_j \lambda_j\big) S(x) .$$

Since $\underline{\sigma} = \sum_0^p (1/(p + 1))\underline{\mathfrak{a}}_i$, $S = \sum_0^p (1/(p + 1)) A_i$. Hence

$$tZ_\sigma(x) + (1 - t) V_\sigma(x) = \sum_{j \in J} \varepsilon_j \lambda_j B_j(x) - t\big(\textstyle\sum_{j \in J} \varepsilon_j \lambda_j\big) \sum_{i=0}^p \frac{1}{p + 1} A_i(x) .$$

Suppose $tZ_\sigma(x) + (1 - t) V_\sigma(x) = 0$. Since $\lambda_j = 0$ if $\underline{\mathfrak{b}}_j$ is not in the carrier of X, we obtain a relation among scalar multiples of the vectors corresponding to the vertices of the carrier of x. In view of Lemma 1, Section 4, all the coefficients must have the same sign, which is clearly impossible. q.e.d.

6. The flow of Z_σ

We retain the conventions of Section 5. A local flow $\zeta_t^i\colon W_i \to D_i$ (t sufficiently small) is defined by

$$\zeta_t^i(w) = \exp\left((-1)^{p+1+i}t\right)\cdot w\ .$$

ζ_t^i is an explosion (resp. implosion) along the cone lines of D_i if $p+1+i$ is even (resp. odd). The equation

$$\psi \circ (\zeta_t^0 \times \cdots \times \zeta_t^{p+1}) = \zeta_t \circ \psi$$

determines a local flow $\zeta_t\colon W_\sigma \to D\sigma$.

LEMMA 4. *Fix $x \in W_\sigma$. Then $t \mapsto \zeta_t(x)$ is a smooth curve in W_σ and its tangent vector at x is $Z_\sigma(x)$. In particular, ζ_t is the flow generated by Z_σ.*

Proof. Let $x = \sum_{j=0}^n \lambda_j \underline{\mathfrak{b}}_j \in W_\sigma$. As in Section 5 let $J = \{j/\underline{\mathfrak{b}}_j \neq \text{any } \underline{\mathfrak{a}}_i\}$ and set $\varepsilon_j = (-1)^{p+1+i}$ if $\underline{\mathfrak{b}}_j \in L_i$. Then

$$\zeta_t(x) = \sum\nolimits_{j \in J} e^{\varepsilon_j t}\lambda_j \underline{\mathfrak{b}}_j + \sum\nolimits_{i=0}^p \lambda(t)\underline{\mathfrak{a}}_i$$

where $\lambda(t)$ is chosen so that the coefficients sum to 1. The lemma follows.

Proof of Theorem 2. From Propositions 3 and 4, $m_\sigma = \text{index } Z_\sigma$. By Lemma 4 this is the product of the indices of the flows ζ_t^i. (For the fix-point index of a flow see [1], Chapter XIV). Since ζ_t^i is an explosion if $p+1+i$ is even, and an implosion if $p+1+i$ is odd,

$$\text{index } \zeta_t^i = \begin{cases} 1 & p+1+i \text{ even} \\ (-1)^{\dim D_i} & p+1+i \text{ odd}\ . \end{cases}$$

Hence

$$m_\sigma = \begin{cases} (-1)^{(|\mathfrak{a}_p|-|\mathfrak{a}_{p-1}|-1)+(|\mathfrak{a}_{p-2}|-|\mathfrak{a}_{p-3}|-1)+\cdots+(|\mathfrak{a}_0|)} & p \text{ even} \\ (-1)^{(|\mathfrak{a}_p|-|\mathfrak{a}_{p-1}|-1)+(|\mathfrak{a}_{p-2}|-|\mathfrak{a}_{p-3}|-1)+\cdots+(|\mathfrak{a}_1|-|\mathfrak{a}_0|-1)} & p \text{ odd} \end{cases}$$

and in either case this is $(-1)^{p(p+1)/2+|\mathfrak{a}_0|+\cdots+|\mathfrak{a}_p|}$. This finishes the proof of Theorem 2; Theorem 1 is now proved as well.

7. Lemma 3

If $\tau = \langle \underline{\mathfrak{b}}_0 \cdots \underline{\mathfrak{b}}_q \rangle \leq K'$ *and* $x = \sum_{i=0}^q \lambda_i \underline{\mathfrak{b}}_i \in \mathring{\tau}$, *then*

(i) $B_0(x) = \sum_{j=1}^q \dfrac{(-1)^j}{\lambda_0^{j+1}} F_j(x)$

and

(ii) $B_i(x) = \dfrac{1}{\lambda_0 \cdots \lambda_i}\{F_i(x) + \sum_{j=i+1}^q \gamma_{ij} F_j(x)\}$

where

$$\gamma_{ij} = (-1)^{j-i} \sum_{0 \leqq \nu_1 \leqq \cdots \leqq \nu_{j-i} \leqq i} \frac{1}{\lambda_{\nu_1} \cdots \lambda_{\nu_{j-i}}}$$

and B_i is the radial field corresponding to $\underline{\mathfrak{b}}_i$.

Proof. (i) Note that

$$\begin{aligned}
&\sum_{j=1}^{q} \frac{(-1)^j}{\lambda_0^j} F_j(x) \\
&= \sum_{i=1}^{q} \Big(\sum_{j=1}^{i} \frac{(-1)^j}{\lambda_0^j} \sum_{0 \leqq \nu_0 < \cdots < \nu_{j-1} < i} \lambda_{\nu_0} \cdots \lambda_{\nu_{j-1}} \Big) \lambda_i B_i(x) \\
&= \sum_{i=1}^{q} \Big(\sum_{j=1}^{i} \sum_{0 \leqq \nu_0 < \cdots < \nu_{j-1} < i} \Big(\frac{-\lambda_{\nu_0}}{\lambda_0} \Big) \cdots \Big(\frac{-\lambda_{\nu_{j-1}}}{\lambda_0} \Big) \Big) \lambda_i B_i(x) \\
&= \sum_{i=1}^{q} \Big(\Big(1 - \frac{\lambda_0}{\lambda_0}\Big)\Big(1 - \frac{\lambda_1}{\lambda_0}\Big) \cdots \Big(1 - \frac{\lambda_{i-1}}{\lambda_0}\Big) - 1 \Big) \lambda_i B_i(x) \\
&= -\sum_{i=1}^{q} \lambda_i B_i(x) = \lambda_0 B_0(x) \ .
\end{aligned}$$

(ii) The expression on the right can be re-expressed in terms of the vectors $B_k(x)$ $(k \geqq i)$. The coefficient of $B_i(x)$ (in the resulting expression) is obviously 1; it must be shown that the other coefficients vanish. Fix $k > i$. The coefficient of $B_k(x)$ is given by the formula

$$\begin{aligned}
&\frac{1}{\lambda_0 \cdots \lambda_i} \Big\{ \sum_{0 \leqq \mu_0 < \cdots < \mu_{i-1} < k} \lambda_{\mu_0} \cdots \lambda_{\mu_{i-1}} \\
&\quad + \sum_{j=i+1}^{k} (-1)^{j-i} \sum_{0 \leqq \nu_1 \leqq \cdots \leqq \nu_{j-i} \leqq i} \frac{1}{\lambda_{\nu_1} \cdots \lambda_{\nu_{j-1}}} \sum_{0 \leqq \mu_0 < \cdots < \mu_{j-1} < k} \lambda_{\mu_0} \cdots \lambda_{\mu_{j-1}} \Big\} \lambda_k \ .
\end{aligned}$$

To show that this is zero we divide the expression inside { } by $\lambda_0 \cdots \lambda_k$ and set $m = k - i$, $l = j - i$; it then reads

$$\begin{aligned}
&\sum_{0 \leqq \mu_1 < \cdots < \mu_m < k} \frac{1}{\lambda_{\mu_1} \cdots \lambda_{\mu_m}} \\
&\quad + \sum_{l=1}^{m-1} (-1)^l \sum_{0 \leqq \nu_1 \leqq \cdots \leqq \nu_l \leqq i} \frac{1}{\lambda_{\nu_1} \cdots \lambda_{\nu_l}} \sum_{0 \leqq \mu_1 < \cdots < \mu_{m-l} < k} \frac{1}{\lambda_{\mu_1} \cdots \lambda_{\mu_{m-l}}} \qquad (6.1) \\
&\quad + (-1)^m \sum_{0 \leqq \nu_1 \leqq \cdots \leqq \nu_m \leqq i} \frac{1}{\lambda_{\nu_1} \cdots \lambda_{\nu_m}}
\end{aligned}$$

and we must show that it vanishes.

Let E be a vector space with basis $e_0, \cdots, e_{k-1}$; let F be the subspace spanned by $e_0, \cdots, e_i$; and let F_1 be the subspace spanned by $e_{i+1}, \cdots, e_{k-1}$. ΛE, ΛF, and ΛF_1 are the exterior algebras over E, F, and F_1; and VF is the symmetric algebra over F.

Grade $W = \Lambda E \otimes VF$ by setting $W_s = \sum_{\nu + 2\mu = s} \Lambda^\nu E \otimes V^\mu F$. Let ρ: $E \to F$ be the projection with kernel F_1, and define an operator d in W, homogeneous of degree $+1$ by setting

$$d(1 \otimes z) = 0$$

and

$$d\{x_0 \wedge \cdots \wedge x_s \otimes z) = \sum_{j=0}^{s} (-1)^j x_0 \wedge \cdots \hat{x}_j \cdots \wedge x_s \otimes \rho(x_j) \vee z \, .$$

Then $d^2 = 0$ and so (W, d) is a chain complex. Evidently

$$(W, d) \cong \big(\Lambda F_1 \otimes (\Lambda F \otimes VF)\ ,\ \text{identity} \otimes d\big)$$

and since (cf. [2]) $(\Lambda F \otimes VF, d)$ is acyclic, it follows that the inclusion $\Lambda F_1 \otimes 1 \to W$ induces an isomorphism

$$\Lambda F_1 \xrightarrow{\cong} H(W, d) \, .$$

In particular, $H_j(W, d) = 0$ if $j > \dim F_1 = k - i - 1$.

On the other hand (W, d) is the direct sum of subcomplexes W^s given by

$$W^s = \sum_{\nu+\mu=s} \Lambda^\nu E \otimes V^\mu F \, .$$

Thus $H(W, d)$ is a direct sum of the $H(W^s)$; since $W^{k-i} \cap W_j = 0$ $(j \leqq k - i - 1)$ it follows that

$$H(W^{k-i}, d) = 0 \, .$$

Denote W^{k-i} by C and consider (C, d) as a complex with the gradation of C given by $C = \sum_{l=0}^{m} C_l$ where $m = k - i$, and

$$C_l = \Lambda^{m-l} E \otimes V^l F$$

(this is different from the other gradations!)

Finally, define a linear map $\Phi\colon E \to E$ by setting

$$\Phi(e_i) = \frac{1}{\lambda_i} e_i \, .$$

Φ restricts to a transformation of F, and these maps extend to a unique automorphism of the algebra W, which we again denote by Φ. It satisfies

$$\Phi \circ d = d \circ \Phi \, .$$

Moreover, Φ restricts to an automorphism of (C, d); denote the restriction of Φ to C_l by Φ_l. Then the number

$$\sum_{l=0}^{m} (-1)^l \operatorname{trace} \Phi_l$$

is precisely given by (6.1). Since $H(C, d) = 0$, and Φ is a chain map; this number vanishes.

Exactly the same argument as given in the proof of Lemma 3, (ii) establishes

PROPOSITION 5. *Let R be a commutative ring with identity. Let $\lambda_0, \cdots, \lambda_q$*

be invertible elements R. Then the $q \times q$ upper triangular matrices $A = (a_{ij})$ and $B = (b_{ij})$ given by

$$a_{ij} = \sum\nolimits_{0 \leqq \mu_0 < \cdots < \mu_{i-1} < j} \lambda_{\mu_0} \cdots \lambda_{\mu_{j-1}} \lambda_i \qquad i \leqq j$$

and

$$b_{ij} = \frac{1}{\lambda_0 \cdots \lambda_i} \left\{ \delta_{ij} + \sum\nolimits_{0 \leqq \nu_1 \leqq \cdots \leqq \nu_{j-i} \leqq i} \frac{(-1)^{j-i}}{\lambda_{\nu_1} \cdots \lambda_{\nu_{j-i}}} \right. \qquad i \leqq j$$

are inverse:

$$AB = I = BA$$

(δ_{ij} is the Kronecker delta).

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(Received August 23, 1971)
(Revised May 1972)