SURGERY OBSTRUCTIONS ON CLOSED MANIFOLDS AND THE INERTIA SUBGROUP

IAN HAMBLETON

Abstract. The Wall surgery obstruction groups have two interesting geometrically defined subgroups, consisting of the surgery obstructions between closed manifolds, and the inertial elements. We show that the inertia group $I_{n+1}(\pi, w)$ and the closed manifold subgroup $C_{n+1}(\pi, w)$ are equal in dimensions $n+1 \geq 6$, for any finitely-presented group π and any orientation character $w\colon \pi \to \mathbb{Z}/2$. This answers a question from [9, p. 107].

1. Introduction

Let π be a finitely-presented group, and let $L_n(\mathbb{Z}\pi, w)$ denote Wall's surgery obstruction group for oriented surgery problems up to simple homotopy equivalence, where $w\colon \pi \to \mathbb{Z}/2$ is an orientation character (see [32, Chap. 5-6]). We work with topological (rather than smooth) manifolds throughout, so rely on the work of Kirby-Siebenmann [15] for the extension of surgery theory to the topological category.

Let X^n be a closed, topological n-manifold, $n \geq 5$, and let $c\colon X \to B\pi$ denote the classifying map of its universal covering, so that $c_*\colon \pi_1(X, x_0) \xrightarrow{\approx} \pi$ is a given isomorphism. The orientation class $w_1(X) \in H^1(X; \mathbb{Z}/2)$ induces an orientation character $w\colon \pi \to \mathbb{Z}/2$. The surgery exact sequence

$$[\Sigma(X), G/TOP] \xrightarrow{\sigma_{n+1}(X)} L_{n+1}(\mathbb{Z}\pi, w) \longrightarrow \mathscr{S}(X) \longrightarrow [X, G/TOP] \xrightarrow{\sigma_n(X)} L_n(\mathbb{Z}\pi, w)$$

developed by Browder, Novikov, Sullivan and Wall [32, Chap. 9] relates the classification of manifolds which are simple homotopy equivalent to X to the calculation of the surgery obstruction maps $\sigma_{n+1}(X)$ and $\sigma_n(X)$.

In the surgery exact sequence $\Sigma(X) = (X \times I)/\partial(X \times I)$, and $\mathscr{S}(X)$ denotes the the s-cobordism classes of pairs (M, f), where $f\colon M \to X$ is a simple homotopy equivalence. For a suitable H-space structure on G/TOP (see [15], [18], [24]), these surgery obstruction maps are homomorphisms between abelian groups.

For a fixed (X, w), let $C_n(X, w) \subseteq L_n(\mathbb{Z}\pi, w)$ denote the image of $\sigma_n(X)$. This is the subgroup of $L_n(\mathbb{Z}\pi, w)$ given by the surgery obstructions of all degree 1 normal maps $(f, b)\colon M \to X$ from some closed n-manifold M. By varying X over all closed manifolds with the same orientation character w, we define the *closed manifold subgroup*

$$C_n(\pi, w) \subseteq L_n(\mathbb{Z}\pi, w) \tag{1.1}$$

Date: Oct. 26, 2009 (v2).

Research partially supported by NSERC Discovery Grant A4000. The author would like to thank the Max Planck Institut für Mathematik in Bonn for its hospitality and support while this work was done.

as the subgroup of the L-group generated by all of the closed manifold subgroups $C_n(X, w)$. In the oriented case ($w \equiv 1$), $C_n(\pi)$ is just the image of the Sullivan-Wall homomorphism [32, 13B.3]

$$\Omega_n(B\pi \times G/TOP, B\pi \times *) \to L_n(\mathbb{Z}\pi)$$

defined by the surgery obstruction.

For a fixed (X, w), let $I_{n+1}(X, w) \subseteq L_{n+1}(\mathbb{Z}\pi, w)$ denote the image of $\sigma_{n+1}(X)$. This is the subgroup of $L_{n+1}(\mathbb{Z}\pi, w)$ which acts trivially on the structure set $\mathscr{S}(X)$. The surgery exact sequence (and the s-cobordism theorem) shows that the elements of $I_{n+1}(X, w)$ are exactly the surgery obstructions of *relative* degree 1 normal maps

$$(f, b)\colon (W, \partial W) \to (X \times I, X \times \partial I),$$

where the f restricted to the boundary ∂W is a homeomorphism. By glueing a copy of $X \times I$ along the boundary components in domain and range, we obtain a closed manifold surgery problem $W \cup_{\partial W} (X \times I) \to X \times S^1$. By varying X over all closed manifolds with the same orientation character w, we define the *inertia subgroup*

$$I_{n+1}(\pi, w) \subseteq L_{n+1}(\mathbb{Z}\pi, w) \tag{1.2}$$

as the subgroup of the L-group generated by all of the inertia groups $I_{n+1}(X, w)$. By construction, $I_{n+1}(\pi, w) \subseteq C_{n+1}(\pi, w)$ for all fundamental group data (π, w), and $n \geq 5$. Here is our main result:

Theorem A. *Let π be a finitely-presented group and $w\colon \pi \to \mathbb{Z}/2$ an orientation character. The inertia subgroup $I_{n+1}(\pi, w)$ equals the group of closed manifold surgery obstructions $C_{n+1}(\pi, w) \subseteq L_{n+1}(\mathbb{Z}\pi), w)$, for all $n \geq 5$.*

As stated, this holds for the simple surgery obstructions in $L^s_{n+1}(\mathbb{Z}\pi, w)$, but the inertial or closed manifold subgroups of $L^h_{n+1}(\mathbb{Z}\pi, w)$ are just the images of $I_{n+1}(\pi, w)$ or $C_{n+1}(\pi, w)$ under the natural change of K-theory homomorphism $L^s \to L^h$ (or $L^h \to L^p$). It follows that the inertial and closed manifold subgroups are equal for all torsion decorations in $K_i(\mathbb{Z}\pi)$, $i \leq 1$. In [9] it was proved that the images of these two subgroups were equal in the *projective* surgery obstruction groups $L^p_{n+1}(\mathbb{Z}\pi, w)$, for π a finite group, and the question answered here was raised in [9, p. 107].

Remark 1.3. Fairly complete information is available about the closed manifold obstructions for finite fundamental groups [11, Theorem A], under the assumptions that the manifolds are oriented and surgery obstructions are measured up to *weakly simple* homotopy equivalence, with Whitehead torsion in $SK_1(\mathbb{Z}\pi)$. The outstanding open problems in this area are (i) to investigate the non-oriented case, (ii) to compute the *simple* closed manifold obstructions in L^s_*, and (iii) to decide whether the component $\kappa_4\colon H_4(\pi; \mathbb{Z}/2) \to L_6(\mathbb{Z}\pi)$ of the assembly map $A_*\colon H_*(B\pi^w; \mathbb{L}_\bullet) \to L_*(\mathbb{Z}\pi, w)$ is zero or non-zero (see Section 2 and [11, p. 352] for this notation).

For a finitely-presented group π of infinite order, the closed manifold subgroup $C_n(\pi, w)$ is contained in the image $A_n(\pi, w)$ of the assembly map, but they are not always equal (see Example 5.4). However, these subgroups do become equal after localizing at 2 (see

Theorem 4.2), or after stabilizing as follows. The periodicity isomorphism

$$L_n(\mathbb{Z}\pi, w) \xrightarrow{\times \mathbf{CP}^2} L_{n+4}(\mathbb{Z}\pi, w)$$

allows us to identity $L_n \cong L_{n+4k}$, for all $k \geq 0$. We define the *periodic* image of the assembly map $\mathfrak{A}_q(\pi, w)$, $0 \leq q \leq 3$, as the subgroup of $L_q(\mathbb{Z}\pi, w)$ generated by all of the images of the assembly maps $A_n(\pi.w)$, for $n \equiv q \pmod 4$.

Similarly, we define the *periodic* inertial subgroup $\mathfrak{I}_q(\pi, w)$ and the *periodic* closed manifold subgroup $\mathfrak{C}_q(\pi, w)$, $0 \leq q \leq 3$, as the subgroups of $L_q(\mathbb{Z}\pi, w)$ generated by all $I_n(\pi, w)$ and $C_n(\pi.w)$, respectively, for $n \equiv q \pmod 4$. After stabilization we obtain a result for all fundamental groups.

Theorem B. *Let π be a finitely-presented group and $w \colon \pi \to \mathbb{Z}/2$ an orientation character. The periodic inertial subgroup $\mathfrak{I}_q(\pi, w)$ and the periodic closed manifold subgroup $\mathfrak{C}_q(\pi, w)$ both equal the periodic image of the assembly map $\mathfrak{A}_q(\pi, w) \subseteq L_q(\mathbb{Z}\pi), w)$, for $0 \leq q \leq 3$.*

Remark 1.4. For infinite torsion-free groups, the L-theory assembly maps are conjectured to be isomorphisms [7], and this is currently an active area of research. For infinite groups π with torsion, conjecturally the contribution to $C_n(\pi)$ arising from finite subgroups is determined by the virtually cyclic subgroups of π. Theorem B is proved by showing that the periodic inertial subgroup $\mathfrak{I}_q(\pi, w)$ is equal to a periodic stabilization of the image of the assembly map (see Theorem 5.1).

Acknowledgement. The author would like to thank Matthias Kreck, Larry Smith, Andrew Ranicki and Larry Taylor for helpful conversations.

2. The surgery assembly map

We will need to use the relationship between the closed manifold subgroup and the image of the L-theory assembly map. Recall that there is a factorization due to Quinn [22] and Ranicki [23], [24, §18] (see also Nicas [20, §3]):

$$\begin{tikzcd}
{[X \times I, X \times \partial I; G/TOP, *]} \arrow[rrr, "\sigma_{n+1}(X)"] \arrow[d, "\cap [X, \partial X]_{\mathbb{L}^0}"] & & & L_{n+1}(\mathbb{Z}\pi, w) \\
H_{n+1}(X^w; \mathbb{L}_\bullet) \arrow[r, "i_\bullet"] & H_{n+1}(X^w; \mathbb{L}_0) \arrow[rr, "c_*"] & & H_{n+1}(B\pi^w; \mathbb{L}_0) \arrow[u, "A_{n+1}"']
\end{tikzcd}$$

of the surgery obstruction map $\sigma_{n+1}(X)$ through the assembly map A_{n+1}, where $\mathbb{L}_0$ denotes the (-1)-connective quadratic L-spectrum with $\mathbb{Z} \times G/TOP$ in dimension zero, and $[X, \partial X]_{\mathbb{L}^0}$ is the fundamental class for symmetric $\mathbb{L}^0$-theory. Cap product with this fundamental class gives a Poincaré duality isomorphism [24, 18.3] for $\mathbb{L}_0$, and for its 0-connective cover $\mathbb{L}_\bullet$ (which has G/TOP in dimension zero). In particular,

$$H^0(X, \partial X; \mathbb{L}_\bullet) = [X \times I, X \times \partial I; G/TOP, *] \;\cong\; H_n(X^w; \mathbb{L}_\bullet)\ .$$

The (co)fibration $i_\bullet \colon \mathbb{L}_\bullet \to \mathbb{L}_0$ of spectra induces a long exact sequence relating the two homology theories. Similarly, we have the formula

$$\sigma_n(X) = A_n \circ c_* \circ i_\bullet \circ (- \cap [X]_{\mathbb{L}^0}).$$

The notation X^w or $B\pi^w$ means the Thom spectrum of the line bundle over X or $B\pi$ induced by w, with Thom class in dimension zero, and the assembly maps A_* are induced by a spectrum-level composite

$$A_{\pi,w}\colon B\pi^w \wedge \mathbb{L}_0 \xrightarrow{a_{\pi,w}\wedge 1} \mathbb{L}^0(\mathbb{Z}\pi) \wedge \mathbb{L}_0 \longrightarrow \mathbb{L}_0(\mathbb{Z}\pi)$$

as described in [11, §1]. In particular, the homomorphisms A_n, $n \geq 0$, are just the maps induced on homotopy groups by $A_{\pi,w}$. We define the subgroup

$$A_n(\pi, w) = \operatorname{im}\Big(H_n(B\pi^w; \mathbb{L}_\bullet) \xrightarrow{i_\bullet} H_n(B\pi^w; \mathbb{L}_0) \xrightarrow{A_n} L_n(\mathbb{Z}\pi, w)\Big) \tag{2.1}$$

as the image of the assembly map restricted to $\mathbb{L}_\bullet$, for any dimension $n \geq 0$. Let $\mathfrak{A}_q(\pi, w)$ denote the image of the assembly map made *periodic*. We observe that the factorization of the surgery obstruction map implies that

$$C_n(\pi, w) \subseteq A_n(\pi, w) \subseteq L_n(\mathbb{Z}\pi, w),$$

and

$$I_{n+1}(\pi, w) \subseteq C_{n+1}(\pi, w) \subseteq A_{n+1}(\pi, w),$$

but the inertial subgroup and the closed manifold subgroup have purely geometric definitions independent of the assembly map.

Remark 2.2. In [24, 18.6(i)] it is stated without proof that $C_n(\pi, w) = A_n(\pi, w)$, for $n \geq 5$. This is not true in general (see Example 5.4), but we will verify this for π finite. For π any finitely-presented group, we show that $C_n(\pi, w) \otimes \mathbb{Z}_{(2)} = A_n(\pi, w) \otimes \mathbb{Z}_{(2)}$, for $n \geq 5$, and that $\mathfrak{C}_q(\pi, w) = \mathfrak{A}_q(\pi, w)$, for $0 \leq q \leq 3$.

3. The characteristic class formulas

We will use the characteristic class formulas for the surgery obstruction maps $\sigma_*(X)$ as presented by Taylor and Williams [28] (see also [31], and [11, §1] for the non-oriented case). Let $\mathbf{bo}(\Lambda)$ denote the connective KO-spectrum with coefficients in a group Λ. The Morgan-Sullivan characteristic class [19] is denoted $\mathcal{L} \in H^{4*}(BSTOP; \mathbb{Z}_{(2)})$, and $V \in H^{2i}(BSTOP; \mathbb{Z}/2)$ denotes the total Wu class.

(i) ([28, Theorem A]) The spectra $\mathbb{L}^0(\mathbb{Z}\pi, w)$ and $\mathbb{L}_0(\mathbb{Z}\pi, w)$ are generalized Eilenberg-MacLane spectra when localized at 2, and when localized away from 2 are both

$$\mathbf{bo}(\Lambda_0) \vee \mathbf{bo}(\Lambda_1) \vee \mathbf{bo}(\Lambda_2) \vee \mathbf{bo}(\Lambda_3),$$

where $\Lambda_i = \pi_i(\mathbb{L}^0(\mathbb{Z}\pi, w)) \otimes \mathbb{Z}[1/2]$. In particular, there is an equivalence of spectra $\mathbb{L}^0 \otimes \mathbb{Z}[1/2] \simeq \mathbf{bo}(\mathbb{Z}[1/2])$, defining a characteristic class

$$\Delta \in KO^0(\mathbb{L}_\bullet\, ; \mathbb{Z}[1/2])$$

whose associated map $\sigma\colon G/TOP[1/2] \xrightarrow{\simeq} BO[1/2]$ is the homotopy equivalence of infinite loop spaces due to Sullivan and Kirby-Siebenmann (see [27], [17], and the exposition in [18, 4.28]).

(ii) The splitting of $\mathbb{L}_0 \otimes \mathbb{Z}_{(2)}$ is given by universal cohomology classes $\ell \in H^{4*}(\mathbb{L}_0\,; \mathbb{Z}_{(2)})$ and $k \in H^{4*+2}(\mathbb{L}_0\,; \mathbb{Z}/2)$. The domain of the assembly map

$$H_n(B\pi^w; \mathbb{L}_0) \otimes \mathbb{Z}_{(2)} \xrightarrow{\approx} \bigoplus_{i \geq 0} H_{n-4i}(\pi; \mathbb{Z}^w_{(2)}) \oplus H_{n-4i-2}(\pi; \mathbb{Z}/2)$$

has a natural splitting induced by ℓ and k. The assembly map has component maps

$$\mathscr{I}_m\colon H_m(\pi; \mathbb{Z}^w_{(2)}) \to L_m(\mathbb{Z}\pi, w) \otimes \mathbb{Z}_{(2)},\ m \geq 0$$

and

$$\kappa_m\colon H_m(\pi; \mathbb{Z}/2) \to L_{m+2}(\mathbb{Z}\pi, w) \otimes \mathbb{Z}_{(2)},\ m \geq 0$$

which determine $A_n(\pi, w)$ completely (see [11, §1]).

(iii) ([28, Theorem C]) Let X be a closed n-manifold, with a reference map $c\colon X \to B\pi$ such that $c^*(w) = w_1(X)$. Let $u_X\colon X \to BSTOP$ classify the bundle ν_+ such that ν_+ plus the line bundle corresponding to w is the stable normal bundle ν_X. Let $f\colon X \to \mathbb{L}_\bullet$ determine a degree 1 normal map. Then

$$\sigma_X(f)_{(odd)} = A_* c_* \left(f^*(\Delta) \cap [X]_{\mathbf{bo}}\right)$$

gives the surgery obstruction localized away from 2, where $[X]_{\mathbf{bo}}$ denotes the **ko**-fundamental class of X. Furthermore, the 2-local surgery obstruction is given by

$$\sigma_X(f)_{(2)} = A_* c_* \left((u^*(\mathcal{L}) \cup f^*(\ell) + u^*(\mathcal{L}) \cup f^*(k) + \delta^*(u^*(VSq^1V) \cup f^*(k))) \cap [X]\right)$$

where δ^* denotes the integral Bockstein and A_* is the assembly map.

These formulas translate the given information about the manifold X and the surgery problem $f\colon X \to \mathbb{L}_\bullet$ into a collection of **ko**-homology classes (away from 2), or a collection of ordinary $\mathbb{Z}^w_{(2)}$ or $\mathbb{Z}/2$ homology classes for (π, w). The surgery obstruction $\sigma_X(f) \in L_n(\mathbb{Z}\pi, w)$ is then computed by applying the assembly map to these classes. There are similar formulas for the obstruction to a relative surgery problem defined by $f\colon \Sigma(X) \to \mathbb{L}_\bullet$, involving the relative fundamental class $[X \times I, X \times \partial I]$.

Remark 3.1. Note that the degree 0 component of the class $f^*(\ell)$ in the 2-local formula is zero. The class $f^*(\Delta)$ has a similar property which will be made precise in Lemma 5.2.

4. The proof of Theorem A (localized at 2)

We fix the fundamental group data (π, w). The idea of the proof (generalizing [9, §4]) is to construct enough inertial surgery problems to realize all possible elements of $C_{n+1}(\pi, w)$. The target manifolds for these surgery problems will have the form $X^n \times I$, where X^n is the total space of an S^{n-m}-bundle over Y^m, with structural group $\mathbb{Z}/2$, and the dimension of Y has the form $m = (n+1) - 4i$ or $m = (n+1) - 4i + 2$, for some $i > 0$. The surgery problems

$$(f, b)\colon (W, \partial W) \to (X \times I, X \times \partial I),$$

with $\deg f = 1$ and $b\colon \nu_W \to \nu_{X \times I}$ a bundle map covering f, will be constructed by glueing a simply-connected Milnor or Kervaire manifold surgery problem *fibrewise* into a tubular neigbourhood of $Y \subset X \times \{1/2\} \subset X \times I$. The details of this construction will be given below.

The basic input is the relation between bordism and homology or KO-theory.

(i) (localized at 2) The work of Thom [29] and Conner-Floyd [4] the Hurewicz map $\Omega_m^{SO}(X, A) \otimes \mathbb{Z}_{(2)} \to H_m(X, A; \mathbb{Z}_{(2)})$ for oriented bordism is surjective for every pair (X, A). Similarly, the Hurewicz map $\mathcal{N}_m(X, A) \to H_m(X, A; \mathbb{Z}/2)$ for unoriented bordism is surjective, $m \geq 0$.

(ii) (localized away from 2) There is an isomorphism

$$h_0 \colon \Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}(pt)} \mathbb{Z}[1/2] \xrightarrow{\approx} KO_*(X; \mathbb{Z}[1/2])$$

induced by the image of the $KO[1/2]$-fundamental class (see [18, 4.15]). In this tensor product, the action $\Omega_*^{SO}(pt) \to \mathbb{Z}[1/2]$ is given by the index homomorphism if $* = 4i$, and zero if $* \neq 4i$.

For finite groups, there is the following foundational result:

Theorem 4.1 (Wall [30, §7])**.** *For π a finite group, and w an orientation character, the localization map $L_n(\mathbb{Z}\pi, w) \to L_n(\mathbb{Z}\pi, w) \otimes \mathbb{Z}_{(2)}$ is injective.*

We now divide the argument into two cases, since it suffices to show that $I_{n+1}(\pi, w)$ and $C_{n+1}(\pi, w)$ are equal after tensoring with $\mathbb{Z}_{(2)}$ and $\mathbb{Z}[1/2]$ separately.

Theorem 4.2. *Let π be a finitely-presented group, $n \geq 5$, and $w \colon \pi \to \mathbb{Z}/2$ an orientation character. Then*

(i) $C_n(\pi, w) \otimes \mathbb{Z}_{(2)} = A_n(\pi, w) \otimes \mathbb{Z}_{(2)}$, *and*
(ii) $I_{n+1}(\pi, w) \otimes \mathbb{Z}_{(2)} = A_{n+1}(\pi, w) \otimes \mathbb{Z}_{(2)}$.

Corollary 4.3. *If π is a finite group , then $C_n(\pi, w) = A_n(\pi, w)$ and $I_{n+1}(\pi, w) = A_{n+1}(\pi, w)$, for all $n \geq 5$.*

Proof. For finite groups, the only elements of infinite order in $A_n(\pi, w)$ come from the trivial group (see [32, 13B.1]). Hence Theorem A for finite groups follows from the 2-local version and Theorem 4.1. □

The proof of Theorem 4.2. We first show that $I_{n+1}(\pi, w) \otimes \mathbb{Z}_{(2)} = A_{n+1}(\pi, w) \otimes \mathbb{Z}_{(2)}$, and note that $(ii) \Rightarrow (i)$ for $n + 1 \geq 6$. Alternately, a direct proof that $C_n(\pi, w) \otimes \mathbb{Z}_{(2)} = A_n(\pi, w) \otimes \mathbb{Z}_{(2)}$, for all $n \geq 5$, can be given along the same lines. The details are similar, but easier, and will be left to the reader.

We proceed as outlined above to construct enough inertial elements to generate the domain

$$H_{n+1}(B\pi^w; \mathbb{L}_\bullet) \otimes \mathbb{Z}_{(2)} \xrightarrow{\approx} \bigoplus_{i>0} H_{n+1-4i}(\pi; \mathbb{Z}^w_{(2)}) \oplus H_{n+1-4i+2}(\pi; \mathbb{Z}/2) \tag{4.4}$$

of the assembly map restricted to $\mathbb{L}_\bullet$.

Suppose that $\alpha \in H_m(\pi; \mathbb{Z}^w_{(2)})$ is a given homology class (with twisted coefficients given by the orientation character w). Let η denote the line bundle over $K(\pi, 1)$ with $w_1(\eta) = w$. By the Thom isomorphism,

$$\Phi \colon H_m(\pi; \mathbb{Z}^w_{(2)}) \cong H_{m+1}(E, \partial E; \mathbb{Z}_{(2)})$$

where $E = E(\eta)$ denotes the total space of the disk bundle of η. Let $h\colon (V^{m+1}, \partial V) \to (E, \partial E)$ be an oriented $(m+1)$-manifold, with reference map to $(E, \partial E)$, whose fundamental class $h_*[V, \partial V] = \Phi(\alpha)$. Now let $g\colon Y^m \to B\pi$ be the transverse pre-image of the zero section in $E(\eta)$, with $w_1(Y) = g^*(w)$. By construction, $g_*[Y] = \alpha$.

The model surgery problems with target $X \times I$ will be constructed from products $X = Y^m \times S^{n-m}$. A small tubular neighbourhood of $Y \subset X \times \{1/2\} \subset X \times I$ is homoemorphic to the product $U = Y \times D^{n+1-m}$. By our choice of dimensions, $n+1-m = 4i$ or $n+1-m = 4i-2$, for some $i > 0$. Let $\varphi\colon (M^{4i}, \partial M) \to (D^{4i}, \partial D^{4i})$ and $\psi\colon (K^{4i-2}, \partial K) \to (D^{4i-2}, \partial D^{4i-2})$ denote the simply-connected Milnor and Kervaire manifold surgery problems, whose surgery obstructions represent generators of $L_{4i}(\mathbb{Z})$ and $L_{4i-2}(\mathbb{Z})$ respectively. Recall that these are smooth surgery problems, with boundary manifolds ∂M^{4i} and ∂K^{4i-2} smooth homotopy spheres (at least if $4i > 4$), but homeomorphic to the standard sphere by the solution of the Poincaré conjecture [25]. In dimension 4, we need the E_8-manifold constructed by Freedman [8].

Now we define W by removing the interior of U from $X \times I$, and glueing in Y product with either the Milnor manifold M^{4i} or the Kervaire manifold K^{4i-2}. The degree 1 map $F\colon W \to X \times I$ is the identity outside of $U = Y \times D^{n+1-m}$, and inside U is given by $\mathrm{id} \times \varphi$ or $\mathrm{id} \times \psi$. Similarly, the bundle map $b\colon \nu_W \to \nu_{X\times I}$ is the identity over the complement of U and given by the simply-connected problem over U. We now have a degree 1 normal map $(f, b)\colon W \to X \times I$ which is the identity on the boundary, hence defines an element in $[\Sigma(X), G/TOP]$.

It follows from the characteristic class formula that the surgery obstruction

$$\sigma(f, b)_{(2)} = A_m(\alpha) + \text{ lower terms}$$

where $A_m = \mathscr{I}_m$ or $A_m = \kappa_m$, and the "lower terms" are the images under A_j for $j < m$ (in this formula we have identified $L_{n-4i} = L_n$ by periodicity).

The surgery problems constructed so far are enough to deal with degree $m = (n+1-4i)$ contributions from the first of the summands in formula (4.4) for $H_{n+1}(B\pi^w; \mathbb{L}_\bullet)$. To realize the $\mathbb{Z}/2$-homology classes $\beta \in H_m(\pi; \mathbb{Z}/2)$ arising from the second summand, we start with a possibly non-orientable manifold $g\colon Y^m \to B\pi$ with $g_*[Y] = \beta$.

In this case, $n+1-m = 4i-2$, for some $i > 0$, and we let ζ denote the line bundle over Y with $w_1(\zeta) = w_1(Y) + g^*(w)$. Let $\xi = (2i-1)\zeta \oplus (2i-1)\varepsilon$ be the Whitney sum of $(2i-1)$ copies of ζ, together with $(2i-1)$ copies of the trivial line bundle ε. Now let $p\colon X \to Y$ denote the total space of the associated sphere bundle $X = S(\xi)$, and observe that the class $w_1(X) = p^*(g^*(w))$. The fibre sphere has dimension $4i-3 = n-m$. Notice that the bundle ξ has structural group $\mathbb{Z}/2$, and the transition functions defining this bundle operate through the involution denoted $S^{4i-3}(2i-1)$, meaning the restriction to the unit sphere of the representation $\mathbb{R}^{2i-1}_+ \oplus \mathbb{R}^{2i-1}_-$ in which a generator of $\mathbb{Z}/2$ acts as $+1$ on the first subspace and as -1 on the second. Since $i > 0$ this bundle has non-zero sections, so we may choose an embedding of $Y \subset X$.

We will now show that the Kervaire sphere ∂K^{4i-2} admits an orientation-reversing involution which is $\mathbb{Z}/2$-equivariantly homeomorphic to $S^{4i-3}(2i-1)$. Recall that $W^{4i-3}(d)$ denotes the Brieskorn variety given by intersecting the solution set of the equation

$$z_0^d + z_1^2 + \cdots + z_{2i-1}^2 = 0$$

with the unit sphere in $\mathbb{C}^{2i}$, with $d \geq 0$ an odd integer. There is an involution T_d on $W^{4i-3}(d)$ given by complex conjugation $z_j \mapsto \bar{z}_j$ in each coordinate. It is known that $W^{4i-3}(d)$ is a homotopy sphere if d is odd, which is diffeomorphic to the standard sphere if $d \equiv \pm 1 \pmod 8$, and to the Kervaire sphere ∂K^{4i-2} if $d \equiv \pm 3 \pmod 8$ (see [12]). The complex conjugation involution extends to the perturbed zero set, which is diffeomorphic to the Kervaire manifold K^{4i-2} if $d \equiv \pm 3 \pmod 8$.

Lemma 4.5. *The involution $(W^{4i-3}(d), T_d)$, with d odd, is $\mathbb{Z}/2$-equivariantly homeomorphic to $S^{4i-3}(2i-1)$.*

Proof. These involutions were studied by Kitada [16], who gave necessary and sufficient conditions for $(W^{4i-3}(d), T_d)$ to be $\mathbb{Z}/2$-equivariantly diffeomorphic to $(W^{4i-3}(d'), T_{d'})$. We need only the easy part of his argument, namely that $(W^{4i-3}(d), T_d)$ is $\mathbb{Z}/2$-equivariantly normally cobordant to $S^{4i-3}(2i-1)$ by a normal cobordism which is the identity on a neighbourhood of the fixed set. The remaining surgery obstruction to obtaining an equivariant s-cobordism lies in the action of $L_{4i-2}(\mathbb{Z}[\mathbb{Z}/2], w) \cong \mathbb{Z}/2$ on the relative structure set of the complement of the fixed set. In the smooth category, this action is difficult to determine, but in the topological category the action is trivial (since this element is in the image of the assembly map). □

We can now glue in the simply-connected Kervaire manifold surgery problem in a tubular neighbourhood U of $Y \subset X \times I$. The boundary $\partial U = \widetilde{Y} \times_{\mathbb{Z}/2} S^{4i-3}$, where $\widetilde{Y}$ is the double covering of Y given by $w_1(\zeta)$ and the fibre sphere has the action $S^{4i-3}(2i-1)$. We have a homeomorphism

$$\widetilde{Y} \times_{\mathbb{Z}/2} S^{4i-3} \approx \widetilde{Y} \times_{\mathbb{Z}/2} \partial K^{4i-2}$$

given by Lemma 4.5, and this is used to glue in $\widetilde{Y} \times_{\mathbb{Z}/2} K^{4i-2}$ defined by the extension of the complex conjugation involution over K^{4i-2}. The characteristic class formula shows as before that the surgery obstruction

$$\sigma(f, b)_{(2)} = \kappa_m(\alpha) + \text{ lower terms}$$

where the "lower terms" are the images under A_j for $j < m$. □

5. The proof of Theorem A (at odd primes) and Theorem B

By Theorem 4.2, the inertial subgroup and the closed manifold subgroup are both equal to the image of the assembly map, after localization at 2. We now localize away from 2, and this is where we will need to stabilize to identify the image of the assembly map. As above, let $\mathfrak{A}_q(\pi, w)$, $0 \leq q \leq 3$, denote the periodic image of the assembly map, generated by all the $A_n(\pi, w)$ for $n \equiv q \pmod 4$. We will prove:

Theorem 5.1. *Let π be a finitely-presented group, and w an orientation character. Then*

(i) $I_{n+1}(\pi, w) \otimes \mathbb{Z}[1/2] = C_{n+1}(\pi, w) \otimes \mathbb{Z}[1/2]$, *and*
(ii) $\mathfrak{I}_q(\pi, w) = \mathfrak{C}_q(\pi, w) = \mathfrak{A}_q(\pi, w)$.

The procedure in this setting will be similar, but since the transfer map on L-theory

$$\mathrm{Res}\colon L_{n+1}(\mathbb{Z}\pi, w) \otimes \mathbb{Z}[1/2] \to L_{n+1}(\mathbb{Z}\pi^+) \otimes \mathbb{Z}[1/2]$$

to the orientation double covering ($\pi^+ = \ker w$) has the property that $\mathrm{Ind} \circ \mathrm{Res}$ is multiplication by 2, we may assume that w is trivial. The domain of the assembly map is now

$$H_{n+1}(B\pi; \mathbb{L}_0) \otimes \mathbb{Z}[1/2] \cong \mathbf{ko}_{n+1}(B\pi; \mathbb{Z}[1/2]),$$

and by the characteristic class formula we must consider the surgery obstructions of elements of the form

$$\alpha = (f^*(\Delta) \cap [V]_{\mathbf{bo}}) \in \mathbf{ko}_*(V; \mathbb{Z}[1/2]),$$

for some closed $(n+1)$-manifold V with reference map $c\colon V \to B\pi$. The map $f\colon V \to \mathbb{L}_\bullet$ classifies a given closed manifold surgery problem with range V.

We need more information about the class Δ. Recall that there is a Conner-Floyd isomorphism

$$h^0\colon \Omega^{4*}(X) \otimes_{\Omega^*(pt)} \mathbb{Z}[1/2] \xrightarrow{\approx} KO^0(X; \mathbb{Z}[1/2])$$

which gives the KO-theory for a finite complex X in terms of oriented cobordism away from 2.

Lemma 5.2. *The class $\Delta \in KO^0(G/TOP; \mathbb{Z}[1/2])$ is represented by a formal sum of classes $\widehat{\Delta}_k \in \Omega^{4k}(G/TOP) \otimes \mathbb{Z}[1/2]$ of positive degrees $k > 0$.*

Proof. We first recall the description of Δ given in [18, Chap. 4]. For each $k > 0$, let $S_k\colon \Omega_{4k}(G/TOP) \to \mathbb{Z}$ be the homomorphism which assigns to an element $f\colon X \to G/TOP$, the signature difference $(\operatorname{index} M - \operatorname{index} X)/8$ for the associated surgery problem $M \to X$. These are $\Omega_*(pt)$-module homomorphisms, where $\Omega_*(pt)$ acts on $\mathbb{Z}[1/2]$ via the signature in dimensions $\equiv 0 \pmod 4$, and zero otherwise.

By [18, Lemma 4.26], the collection $\{S_k\}$ induces a homomorphism

$$\sigma_0\colon KO_0(G/TOP; \mathbb{Z}[1/2]) \to \mathbb{Z}[1/2].$$

The proof uses Conner-Floyd and an inverse limit argument over finite skeleta of G/TOP. Now one applies the universal coefficient formula for KO-theory [33, (2.8)], and in particular the isomorphism

$$eval\colon KO^0(G/TOP; \mathbb{Z}[1/2]) \to \operatorname{Hom}_{\mathbb{Z}}(KO_0(G/TOP; \mathbb{Z}[1/2]), \mathbb{Z}[1/2])$$

to get the element

$$\Delta \in \widetilde{KO}^0(G/TOP; \mathbb{Z}[1/2]) = [G/TOP, BO[1/2]],$$

with $eval(\Delta) = \sigma_0$ (see [18, p. 86] and the proof of [18, (4.26)] for the assertion that $eval$ is an isomorphism). Note that the element Δ lies in reduced KO^0 since the homomorphisms S_k have positive degree. The associated map $\sigma\colon G/TOP \to BO[1/2]$ is the Sullivan homotopy equivalence (see [18, 4.28]).

By the Conner-Floyd isomorphism for cobordism, there is a unique element

$$\widehat{\Delta} \in \widetilde{\Omega}^{4*}(G/TOP) \otimes_{\Omega^*(pt)} \mathbb{Z}[1/2],$$

such that $h^0(\widehat{\Delta}) = \Delta \in \widetilde{KO}^0(G/TOP; \mathbb{Z}[1/2])$. We may consider this tensor product as a quotient of the corresponding direct product, and represent elements as infinite formal sums.

Quillen [21, Theorem 5.1] proved that the reduced cobordism group of a finite complex is generated by elements in strictly positive dimensions, modulo the action of $\Omega^*(pt)$. Therefore, $\widehat{\Delta}$ is represented in the tensor product by a formal sum of elements

$$\Delta_k \in \widetilde{\Omega}^{4k}(G/TOP; \mathbb{Z}[1/2])$$

with $k > 0$. □

We now consider an element $\alpha = (f^*(\Delta) \cap [V]_{\mathbf{bo}}) \in \mathbf{ko}_*(V; \mathbb{Z}[1/2])$ whose image under the assembly maps gives an element of $C_{n+1}(\pi, w) \otimes \mathbb{Z}[1/2]$. By Lemma 5.2 and Poincaré duality for bordism theory [2], we can express

$$\alpha = \sum_{k>0} a_k [Y^{n+1-4k}, g_k]$$

as a finite $\mathbb{Z}[1/2]$-linear combination of oriented manifolds $g_k \colon Y^{n+1-4k} \to V$, with $a_k \in \mathbb{Z}[1/2]$ and $k > 0$. Let $g \colon Y^{n+1-4k} \to B\pi$ be a manifold with reference map (induced by V), such that k is the smallest integer with $a_k \neq 0$. Hence $g_*([Y]_{\mathbf{bo}}) = \alpha +$ lower terms.

We will now construct an element in $I_{n+1}(\pi)$. We write $n + 1 - m = 4k$, and define $X = Y^m \times S^{4k-1}$. The surgery problem

$$(f, b) \colon (W^{n+1}, \partial W) \to (X \times I, X \times \partial I)$$

will be constructed as above, by gluing in the Milnor manifold surgery problem

$$(M^{4k}, \partial M) \to (D^{4k}, \partial D^{4k})$$

fibrewise along a tubular neighbourhood $U \subset X \times I$ of $Y \subset X \times \{1/2\}$ in the interior of $X \times I$. Let $f \colon \Sigma(X) \to \mathbb{L}_\bullet$ also denote the normal invariant of (f, b), which factors as the composite

$$f \colon \Sigma(X) \xrightarrow{project} Y \times D^{4k}/Y \times S^{4k-1} \xrightarrow{1\times\varphi} \mathbb{L}_\bullet,$$

where $\varphi \colon S^{4k} \to \mathbb{L}_\bullet$ is the normal invariant of the Milnor problem (i.e. the generator of $\pi_{4k}(\mathbb{L}_\bullet) = \mathbb{Z}$). The characteristic class formula

$$\sigma(f)_{(odd)} = A_*(g \times 1)_* \left(f^*(\Delta) \cap [Y \times S^{4i}]_{\mathbf{bo}}\right) = A_*(\alpha) + \text{ lower terms},$$

since $f^*(\Delta) \cap [Y \times S^{4i}]_{\mathbf{bo}} = [Y]_{\mathbf{bo}}$, and $g_*([Y]_{\mathbf{bo}}) = \alpha +$ lower terms. This completes the proof of part(i) of Theorem 5.1.

Remark 5.3. This formula is consistent with the rationalization of the calculation at 2. Note that the the Poincaré dual $\mathcal{L}(Y)$ of the $\mathcal{L}$-genus gives the rational part of the $\mathbb{L}^0$-theory fundamental class $[Y]_{\mathbb{Q}} \cap \mathcal{L}(Y) \in H_{m-4*}(Y; \mathbb{Q})$, by [24, 25.17]. Under the equivalence $\mathbb{L}^0 \otimes \mathbb{Z}[1/2] \simeq \mathbf{bo}(\mathbb{Z}[1/2])$, the fundamental class $[Y]_{\mathbb{L}^0} \in H_m(Y; \mathbb{L}^0 \otimes \mathbb{Z}[1/2])$ maps to $[Y]_{\mathbf{bo}} \in \mathbf{ko}_m(Y; \mathbb{Z}[1/2])$.

The proof of Theorem B. By Theorem 4.2 it is enough to show that $\mathfrak{I}_q(\pi, w) \otimes \mathbb{Z}[1/2] = \mathfrak{A}_q(\pi, w) \otimes \mathbb{Z}[1/2]$, for $0 \leq q \leq 3$. We start with an integer $m \equiv q \pmod 4$, $m \geq 5$, and an arbitrary element $A_m(\alpha) \in \mathfrak{A}_m(\pi, w) \otimes \mathbb{Z}[1/2]$, which is the image of an element $\alpha \in \mathbf{ko}_m(\pi; \mathbb{Z}[1/2])$ under the assembly map.

Since KO-homology satisfies the wedge axiom, the group $\mathbf{ko}_m(\pi; \mathbb{Z}[1/2])$ is the direct limit of the $\mathbf{ko}$-homology of the finite skeleta of the classifying space $B\pi$. By using the

Conner-Floyd theorem [18, 4.15] and the 4-fold periodicity $\mathbf{ko}_{m+4k} = \mathbf{ko}_m$, $k > 0$, we can express

$$\alpha = \sum_{k>0} a_k[Y^{m+4k}, g_k]$$

as a finite $\mathbb{Z}[1/2]$-linear combination of the images of fundamental classes $[Y]_{\mathbf{bo}}$ of oriented manifolds $g_k\colon Y^{m+4k} \to B\pi$, $a_k \in \mathbb{Z}[1/2]$. The surgery obstruction of a typical element $(g_k)_*[Y]_{\mathbf{bo}} \in \mathbf{ko}_{n+1}(B\pi; \mathbb{Z}[1/2])$ in this sum lies in $C_{n+1}(\pi, w) \otimes \mathbb{Z}[1/2]$, where $n+1 := m+4k \geq 9$. But $C_{n+1}(\pi, w) \otimes \mathbb{Z}[1/2] = I_{n+1}(\pi, w) \otimes \mathbb{Z}[1/2]$, by Theorem 5.1, part (i), so we conclude that $A_m(\alpha) \in \mathfrak{I}_m(\pi, w) \otimes \mathbb{Z}[1/2]$. □

Example 5.4. We give an example (based on work of Conner and Smith) to show that, in a given dimension n, the image of the assembly map $A_n(\pi, w)$ is not always equal to the closed manifold subgroup $C_n(\pi, w)$. In particular, this contradicts [24, 18.6(i)], and shows that for a suitable finite complex K the elements of $H_n(K; \mathbb{L}_\bullet)$ are not always represented by closed manifold surgery problems.

We will need [10, Prop. 2.6], which is a variation of the Kan-Thurston theorem [14], [3]. For any finite complex X, there exists a finitely-presented group Γ_X with $B\Gamma_X$ of dimension $\leq \dim X$, and an epimorphism $\varphi\colon \Gamma_X \to \pi_1(X)$ with perfect kernel. Moreover, there is a lifting $\widetilde{\alpha}_X\colon X \to (B\Gamma_X)^+_{\ker\varphi}$ of the classifying map $\alpha_X\colon X \to B\pi_1(X)$ which is a homotopy equivalence. In other words, X is obtained by applying the Quillen plus construction to $B\Gamma_X$. It follows (from the Atiyah-Hirzebruch spectral sequence) that $\mathbf{ko}_*(X) \cong \mathbf{ko}_*(B\Gamma_X)$. it is therefore enough to produce the following example:

Lemma 5.5 (Conner-Smith)**.** *There exists a finite complex X such that the natural map*

$$\Omega^{SO}_m(X) \otimes \mathbb{Z}[1/2] \to \mathbf{ko}_m(X; \mathbb{Z}[1/2])$$

is not surjective onto the odd torsion in some dimension $m \geq 5$.

Proof. In a series of papers Conner and Smith studied the relation of complex bordism to connective complex K-theory. Our interest is in real connective K-theory, but since $MU \simeq MSO \wedge \Sigma^2 MSO$ away from 2, their results apply to our situation.

According to a result of Johnson and Smith [13, Theorem 1], for a finite complex X the natural map $\Omega^U_*(X) \to \mathbf{k}_*(X)$ is onto if and only if the projective dimension of $\Omega^U_*(X)$ over $\Omega^U_*(pt)$ is ≤ 2. The right-hand side is connective complex K-homology theory. On the other hand, by a result of Conner and Smith [6, Theorem 5.1], a large N-skeleton X of $K(Z/p, n)$, p an odd prime, will have a large homological dimension over MU. We may pick one with $\operatorname{hom.dim}_{\Omega^U_*(pt)} \Omega^U_*(X) \geq 3$, and with p an odd prime, and both n and N fairly large (see also [26, p. 854] for an explicit example). Such a finite complex X gives the required example. □

We note that the reduced **ko**-theory of X will be p-torsion, since the $1/p$-type of X is a wedge of spheres. In Conner and Smith [5, Theorem 10.8], they show that any class in $\mathbf{ko}_m(X; \mathbb{Z}[1/2]$ can be realized by a closed manifold after stabilizing by powers of the periodicity element $t = [CP^4]$. The statement above shows that stabilizing is actually necessary.

References

[1] J. F. Adams, *Lectures on generalised cohomology*, Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three), Springer, Berlin, 1969, pp. 1–138.

[2] M. F. Atiyah, *Bordism and cobordism*, Proc. Cambridge Philos. Soc. **57** (1961), 200–208.

[3] G. Baumslag, E. Dyer, and A. Heller, *The topology of discrete groups*, J. Pure Appl. Algebra **16** (1980), 1–47.

[4] P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33, Academic Press Inc., Publishers, New York, 1964.

[5] P. E. Conner and L. Smith, *On the complex bordism of finite complexes*, Inst. Hautes Études Sci. Publ. Math. (1969), 117–221.

[6] ———, *Homological dimension of complex bordism modules*, Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969), Markham, Chicago, Ill., 1970, pp. 472–482.

[7] F. T. Farrell and L. E. Jones, *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc. **6** (1993), 249–297.

[8] M. H. Freedman, *The disk theorem for four-dimensional manifolds*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983) (Warsaw), PWN, 1984, pp. 647–663.

[9] I. Hambleton, *Projective surgery obstructions on closed manifolds*, Algebraic K-theory, Part II (Oberwolfach, 1980), Springer, Berlin, 1982, pp. 101–131.

[10] I. Hambleton and J.-C. Hausmann, *Acyclic maps and Poincaré spaces*, Algebraic topology, Aarhus 1982 (Aarhus, 1982), Springer, Berlin, 1984, pp. 222–245.

[11] I. Hambleton, R. J. Milgram, L. R. Taylor, and B. Williams, *Surgery with finite fundamental group*, Proc. London Math. Soc. (3) **56** (1988), 349–379.

[12] F. Hirzebruch and K. H. Mayer, *$O(n)$-Mannigfaltigkeiten, exotische Sphären und Singularitäten*, Springer-Verlag, Berlin, 1968.

[13] D. C. Johnson and L. Smith, *On the relation of complex cobordism to connective K-theory*, Proc. Cambridge Philos. Soc. **70** (1971), 19–22.

[14] D. M. Kan and W. P. Thurston, *Every connected space has the homology of a $K(\pi, 1)$*, Topology **15** (1976), 253–258.

[15] R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press, Princeton, N.J., 1977, With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[16] Y. Kitada, *Orientation reversing involutions on Brieskorn spheres*, Kodai Math. J. **9** (1986), 361–367.

[17] I. Madsen, V. Snaith, and J. Tornehave, *Infinite loop maps in geometric topology*, Math. Proc. Cambridge Philos. Soc. **81** (1977), 399–430.

[18] I. Madsen and R. J. Milgram, *The classifying spaces for surgery and cobordism of manifolds*, Annals of Mathematics Studies, vol. 92, Princeton University Press, Princeton, N.J., 1979.

[19] J. W. Morgan and D. P. Sullivan, *The transversality characteristic class and linking cycles in surgery theory*, Ann. of Math. (2) **99** (1974), 463–544.

[20] A. J. Nicas, *Induction theorems for groups of homotopy manifold structures*, Mem. Amer. Math. Soc. **39** (1982), vi+108.

[21] D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. **7** (1971), 29–56 (1971).

[22] F. Quinn, *A geometric formulation of surgery*, Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969), Markham, Chicago, 1970, pp. 500–511.

[23] A. A. Ranicki, *The total surgery obstruction*, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), Springer, Berlin, 1979, pp. 275–316.

[24] ———, *Algebraic L-theory and topological manifolds*, Cambridge University Press, Cambridge, 1992.

[25] S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. (2) **74** (1961), 391–406.

[26] L. Smith, *On realizing complex bordism modules. Applications to the stable homotopy of spheres*, Amer. J. Math. **92** (1970), 793–856.
[27] D. Sullivan, *Geometric periodicity and the invariants of manifolds*, Manifolds–Amsterdam 1970 (Proc. Nuffic Summer School), Lecture Notes in Mathematics, Vol. 197, Springer, Berlin, 1971, pp. 44–75.
[28] L. R. Taylor and E. B. Williams, *Surgery spaces: formulae and structure*, Algebraic topology, Waterloo, 1978 (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1978), Springer, Berlin, 1979, pp. 170–195.
[29] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.
[30] C. T. C. Wall, *On the classification of Hermitian forms. V. Global rings*, Invent. Math. **23** (1974), 261–288.
[31] ______, *Formulae for surgery obstructions*, Topology **25** (1976), 189–210.
[32] ______, *Surgery on compact manifolds*, second ed., American Mathematical Society, Providence, RI, 1999, Edited and with a foreword by A. A. Ranicki.
[33] Z.-i. Yosimura, *Universal coefficient sequences for cohomology theories of* CW*-spectra*, Osaka J. Math. **12** (1975), 305–323.

Department of Mathematics, McMaster University, Hamilton, Ontario L8S 4K1, Canada
E-mail address: hambleton@mcmaster.ca