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The Second Homology Group of the Mapping Class Group of an Orientable Surface

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Let F be an oriented surface of genus g with r boundary components and n distinguished points. The mapping class group $\Gamma=\Gamma(F)$ is $\pi_0(\mathrm{Diff}^+ F)$ where $\mathrm{Diff}^+(F)$ is the topological group of orientation preserving diffeomorphisms of F which fix the n points and restrict to the identity on ∂F. It is known [9] that $H_1(\Gamma)=0$ for $g\geqq 3$. In this paper we shall prove the

Theorem.

$$H_2(\Gamma)\cong\begin{cases}\mathbb{Z}^{n+1} & g\geqq 5,\ r+n>0\\ \mathbb{Z}\oplus\mathbb{Z}/(2g-2) & g\geqq 5,\ r=n=0.\end{cases}$$

As an immediate corollary we have the proof of a conjecture of Mumford [7]. To state this let $\mathfrak{M}_g^n$ be the moduli space for curves of genus g with n punctures ($r=0$). Γ acts properly discontinuously on the Teichmuller space $\mathcal{T}_g^n\cong\mathbb{C}^{3g-3+n}$ with quotient $\mathfrak{M}_g^n$. Furthermore, the stabilizer of any point is finite, so

$$H_i(\Gamma;\mathbb{Q})\cong H_i(\mathfrak{M}_g^n;\mathbb{Q}) \quad \text{for all } i.$$

The codimension of the subset of $\mathcal{T}_g^n$ on which Γ fails to act freely (the curves with automorphisms) increases with g, so in fact

$$H_i(\Gamma;\mathbb{Z})\cong H_i(\mathfrak{M}_g^n;\mathbb{Z}), \qquad g\gg i.$$

In [7] Mumford shows that the Picard group $\mathrm{Pic}(\mathcal{M})$ is isomorphic to $H^2(\Gamma;\mathbb{Z})$ and conjectures the latter is rank one, $g\geqq 3$. We prove this below for $g\geqq 5$.

Another interpretation of this theorem may be obtained by identifying $H_2(\Gamma)$ as bordism classes of fiber bundles $F\to W^4\to T$ where T is a closed oriented surface (Sect. 0). When F is closed every such bundle is bordant to $F\to W'\to T'$, a bundle admitting a section $s\colon T'\to W'$. The theorem then says that

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$$\begin{pmatrix} F \to W \\ \downarrow \\ T \end{pmatrix} = \partial \begin{pmatrix} F \to M^5 \\ \downarrow \\ Y^3 \end{pmatrix}$$

if and only if W has signature 0 and the self-intersection number of $s(T')$ is divisible by $2g-2$.

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Section 0. Outline of the Proof

When $g \geqq 3$ there are $n+1$ homomorphisms

$$S_0, \ldots, S_n \colon H_2(\Gamma) \to \mathbb{Z},$$

the components of a map φ to $\mathbb{Z}^{n+1}$ with image $\varphi = 4\mathbb{Z} \oplus \mathbb{Z}^n$. To describe these let Λ denote the topological group $\mathrm{Diff}^+(F)$, $B\Lambda$ its classifying space with $E\Lambda \to B\Lambda$ its universal covering. When $g \geqq 2$ each component of Λ is contractible [3] and $H_2(\Gamma) \cong H_2(B\Lambda)$. The latter is isomorphic to the bordism group $\Omega_2(B\Lambda)$. This means that every 2-cycle ξ on Γ may be represented by a map $T \xrightarrow{\xi} B\Lambda$, T some oriented closed surface, and $[\xi] = [\xi']$ in $H_2(\Gamma)$ iff $\xi \amalg -\xi'$ extends to $M^3 \to B\Lambda$, M^3 compact oriented with $\partial M = T \amalg -T'$.

The map ξ yields, by pulling back from $F \to E\Lambda \times_\Gamma F \to B\Lambda$, a fiber bundle $F \to W^4 \xrightarrow{\xi} T$. Set

$$S_0(\xi) = \mathrm{signature}(W^4).$$

The monodromy group of ξ lies in Λ, whose elements fix the n distinguished points on F, so ξ has n canonical sections $s_i \colon T \to W$. Set

$$S_i(\xi) = [s_i(T)]^2,$$

the self-intersection of $s_i(T)$ in W. These are well defined homomorphisms by the above discussion. Meyer [6] proves that $\mathrm{im}(S_0) = 4\mathbb{Z}$. We will show surjectivity of the other S_i's and independence of all $n+1$ S_i's in Sect. 4. We may then state our theorem more explicitly by saying that φ is an isomorphism for $r+n>0$ and has kernel $\mathbb{Z}/|X(F)|$ for $r=n=0$.

To compute $H_2(\Gamma)$ completely it will be necessary to construct a cellular action of Γ on a simply-connected 3-complex Y_3. This complex has its origins in [5].

A well known spectral sequence technique then allows us to find $H_2(\Gamma)$ in terms of $H_2(Y_3/\Gamma)$ and the lower homology groups of the stabilizers of the cells of Y_3.

In Sect. 1 we show $H_1(\Gamma) \cong 0$ for $g \geqq 3$ and $H_1(\Gamma; H_1 F) = 0$, $g \geqq 4$. These facts will be needed in Sects. 3 and 4. In Sect. 2 we describe the complex Y_3 and prove it is simply connected. Section 3 computes $H_2(\Gamma)$ with $r \geqq 1$, $g \geqq 5$, $n=0$ directly from the action on Y_3. In Sect. 4 we use various short exact sequences to deal with the general cases and finish the proof.

Section 1. $H_1(\Gamma)$ and $H_1(\Gamma; H_1(F))$

During the course of our computation of $H_2(\Gamma)$ it will be necessary to know the following:

Lemma 1.1. $H_1(\Gamma)=0$ *for* $g\geqq 3$; r, n *arbitrary.*

Lemma 1.2. $H_1(\Gamma; \widetilde{H_1(F)})=0$ *for* $g\geqq 4$; r, n *arbitrary.*

Lemma 1.1 was first proven by Powell [9] for $r=n=0$.

Proof of 1.1. Let $C\subset F$ be a simple closed curve. The Dehn twist on C, τ_C, is the mapping class obtained by splitting F open at C and reglueing by a 360° twist to the right. Dehn [2] proved that mappings of this form generate Γ.

Let F_0 be a sphere with four disks removed, Γ_0 its mapping class group (recall ∂F_0 must remain fixed). Label its boundary components $C_0, \ldots, C_3$ and write τ_i for the Dehn twist on a circle in $F_0-\partial F_0$ parallel to C_i. Also write C_{ij} for the circle enclosing C_i and C_j, shown in Fig. 1, τ_{ij} for the twist on C_{ij}. The following relation in Γ_0 is easily verified

$$\tau_0\tau_1\tau_2\tau_3=\tau_{12}\tau_{13}\tau_{23}. \qquad (*)$$

Since ∂F_0 remains fixed throughout, any embedding of F_0 in F will induce a relation in Γ.

As a first application of (*) we wish to prove Γ is generated by Dehn twists on nonseparating curves in F. If $C\subset F$ is parallel to a component of ∂F and $g\geqq 2$ choose two disjoint simple closed curves $\alpha_1, \alpha_2 \subset F$ with $F-\{\alpha_1, \alpha_2\}$ connected. It is easy to then find an embedding of F_0 in F with $C_1=\alpha_1$, $C_2=\alpha_2$ and $C_3=C$. All curves will be nonseparating in F except C, so τ_C is isotopic to a product of twists on nonseparating curves.

For arbitrary separating $C\subset F$, $g\geqq 3$, split F open on C and use the side of $F-C$ which has genus $\geqq 2$.

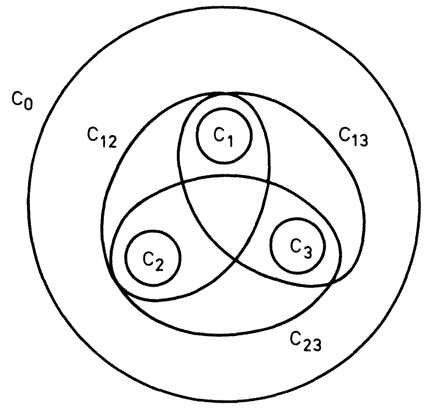

Fig. 1

Remark. For $g=2$, Γ is still generated by twists on nonseparating curves; the argument is only slightly more difficult.

We see immediately that $H_1(\Gamma)$ is cyclic because if C_2, C_1 are any two nonseparating curves, there exists $h \in \mathrm{Diff}^+(F)$ with $h(C_2)=C_1$. It follows that

$$\tau_{C_2}=h\tau_{C_1}h^{-1}$$

in Γ. To see $H_1(\Gamma)=0$ we notice that for $g \geqq 3$, there is an embedding of F_0 in F with all seven curves nonseparating. This completes the Proof of 1.1. $\square$

Proof of 1.2. Construct a free $\mathbb{Z}\Gamma$ resolution $L_* \to \mathbb{Z}$ with $L_0=\mathbb{Z}\Gamma$, L_1 the free $Z\Gamma$ module on symbols $\hat{\gamma}$ for $\gamma \subset F$ a nonseparating curve, L_2 the free $\mathbb{Z}\Gamma$ module on symbols for relations, etc.; boundary maps are defined in the usual way. For each curve $C \subset F$ we choose an orientation to obtain $h_C \in H_1(F)$. If γ is nonseparating, there is a basis $\{C_i\}$ of $H_1(F)$ with $C_1=\gamma$, $\gamma \cap C_2=$one point, and $\gamma \cap C_i=\emptyset$, $i \geqq 3$. Form F_i by splitting F along C_i, $i \neq 2$; since genus $(F_i) \geqq 3$, on F_i there is by Lemma 1.1 a relation R of the form

$$\tau_\gamma=\prod[\tau_{\gamma_j}, \tau_{\delta_j}], \qquad \gamma_j, \delta_j \subset \mathrm{F}_i.$$

Since τ_{γ_j}, τ_{δ_j} and τ_γ act trivially on h_{C_i}

$$\partial_2(\hat{R} \otimes h_{C_i})=\hat{\gamma} \otimes h_{C_i}, \qquad (i \neq 2).$$

To see that $\hat{\gamma} \otimes h_{C_2}$ bounds, choose orientations of C_1, C_2 so that $C_1 \cdot C_2=1$; then, setting $\tau_i=\tau_{C_i}$,

$$\tau_{1*}(\mathrm{h}_{C_2})=\mathrm{h}_{C_1}+\mathrm{h}_{C_2}=\tau_{2*}^{-1}(\mathrm{h}_{C_1}).$$

Therefore

$$\begin{aligned}\hat{C}_1 \otimes h_{C_2} &= \tau_1^{-1}(\hat{C}_1 \otimes \tau_{1*}(h_{C_2}))\\ &= \tau_1^{-1}\tau_2^{-1}(\hat{C}_1 \otimes h_{C_1}).\end{aligned}$$

But $\hat{C}_1 \otimes h_{C_1}=\partial_2(\hat{R} \otimes h_{C_1})$ by the above. $\square$

Section 2. The Cut System Complex

Based on work of [5] we now construct the complex Y_3 mentioned in Sect. 0. Throughout this section we shall assume $n=0$, $r \geqq 1$.

A *cut system* on F is $\langle C_i \rangle$, the isotopy class of a collection of disjoint simple closed curves $C_1, \ldots, C_g \subset F$ such that $F-(C_1 \cup \ldots \cup C_g)$ is connected and therefore planar. (We will often confuse a curve and its isotopy class.) There is no ordering of the circles and they are not oriented.

If C, C' are two isotopy classes set $I(C, C')$ equal to the minimum number of intersections (no signs) between any two representatives meeting transversely. When we replace $\langle C_i \rangle$ by $\langle C_i' \rangle$ where $I(C_i, C_i')=1$ and $C_j=C_j'$, $j \neq i$ we say $\langle C_i \rangle$ and $\langle C_i' \rangle$ differ by a *simple move*. We assume from now on that any circle omitted from the notation remains unchanged.

There are three main cycles of such moves

$$\begin{array}{ccc} \langle C_i \rangle & \text{———} & \langle C_i' \rangle \\ & \diagdown \quad \diagup & \\ & \langle C_i'' \rangle & \end{array} \qquad (\mathrm{R}_1)$$

$$\begin{array}{ccc} \langle C_i, C_j \rangle & \text{———} & \langle C_i, C_j' \rangle \\ | & & | \\ \langle C_i', C_j \rangle & \text{———} & \langle C_i', C_j' \rangle \end{array} \qquad (\mathrm{R}_2)$$

$$\begin{array}{cccc} & \langle C_i, C_j \rangle & \text{———} & \langle C_j, C_j' \rangle & \\ \langle C_i, C_i' \rangle & & & & \langle C_j, C \rangle. \\ & \langle C_i, C \rangle & \text{———} & \langle C, C' \rangle & \end{array} \qquad (\mathrm{R}_3)$$

Each edge corresponds to a simple move. An illustration is shown in Fig. 2 although there is no restriction on $I(C', C_i')$, $I(C', C_j')$, or $I(C_i', C_j')$. Note that R_3 is different from [5] but part 2 of Lemma 1.7 of that paper shows that use of R_3 is equivalent.

Let X_0 be the 0-complex with one vertex for each cut system on F; X_1 the 1-complex obtained by attaching a 1-cell for each simple move; $X_2 = X_1$ with 2-cells attached for the cycles R_1, R_2 and R_3.

Theorem 2.1 [5]. *X_2 is connected and simply connected.*

From this they outline a presentation of Γ.

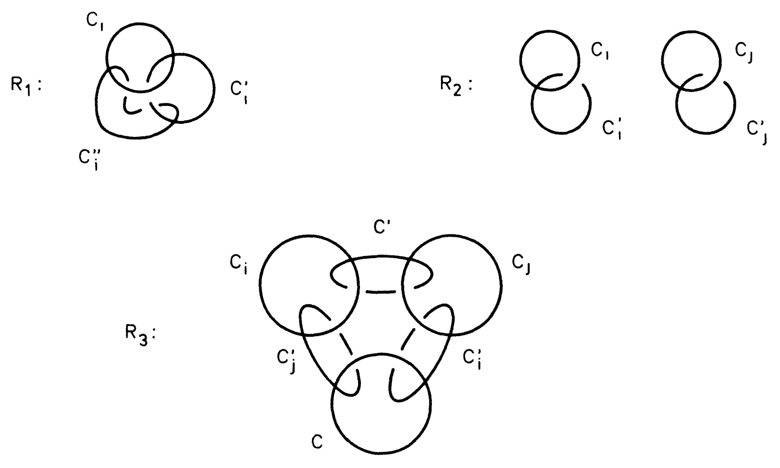

Fig. 2

Let $\varphi \cdot \langle C_i \rangle = \langle \varphi(C_i) \rangle$. Because all cells are determined by configurations of circles, extending linearly gives a natural action of Γ on X_2.

Our first task will be to describe a subcomplex $Y_2 \subset X_2$ with $Y_1 = X_1$, $\pi_1(Y_2) = 1$ and $\Gamma(Y_2) \subset Y_2$.

Consider type R_1 2-cells in X. Among such are those $\sigma(\gamma)$ corresponding to an R_1 cycle with $\{C_i, C_i', C_i''\} = \{\alpha_1, \beta_1, \gamma\}$ and $\{C_j, j \neq i\} = \{\alpha_j, j \neq 1\}$, where α_i and β_1 are the standard curves in Fig. 3. Under the action of Γ on X_2, every type R_1 2-cell of X_2 is identified with some $\sigma(\gamma)$, with γ ranging over a rather large but finite set of curves. For Y_2, include those 2-cells of X_2 in the Γ orbit of all $\sigma(\gamma)$ with γ among the $N = 2r + 2 + \binom{r-1}{2}$ curves of Fig. 4 (the parts of γ not drawn are straight arcs which do not link any handles).

Y_2 contains all 2-cells of X_2 of type R_2. For R_3, Y_2 contains all 2-cells in the Γ-orbit of a single 2-cell, the one corresponding to the R_3 cycle involving the standard cut system $\langle \alpha_i \rangle$ and the cycle of curves

$$(C_i, C', C_j, C_i', C, C_j') = (\alpha_1, \omega, \alpha_2, \beta_2, C_0, \beta_1)$$

shown in Fig. 3.

Clearly $\Gamma(Y_2) \subset Y_2$.

Theorem 2.2. *Y_2 is simply connected.*

Proof. Form X_3 by adding to X_2 the 3-cells of types Σ_1, Σ_2 and Σ_3 from Fig. 5.

(I) Consider first two 2-cells of type R_3 and the choice of the circle C_i'. $I(C_i', C) = I(C_i', C_j) = 1$, $I(C_i', C_k) = 0$, $k \neq j$.

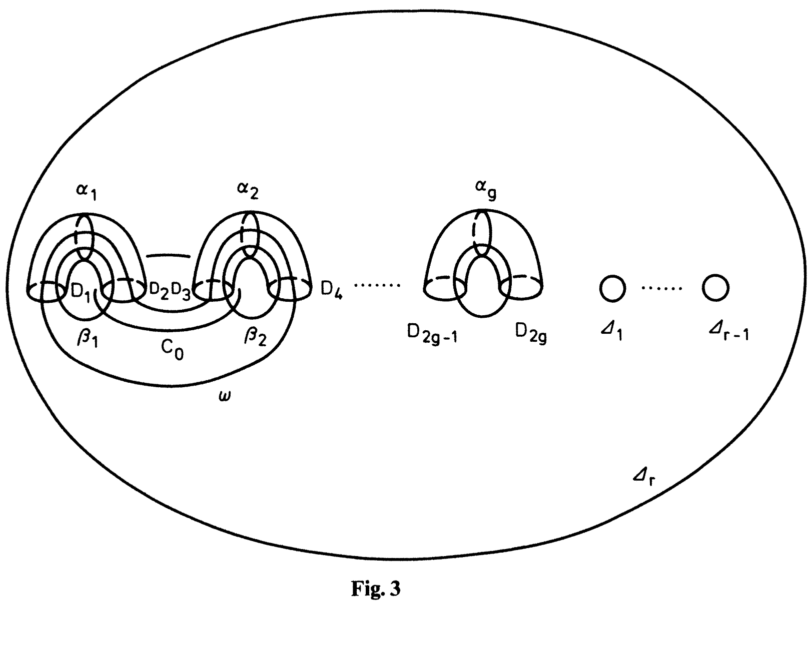

Fig. 3

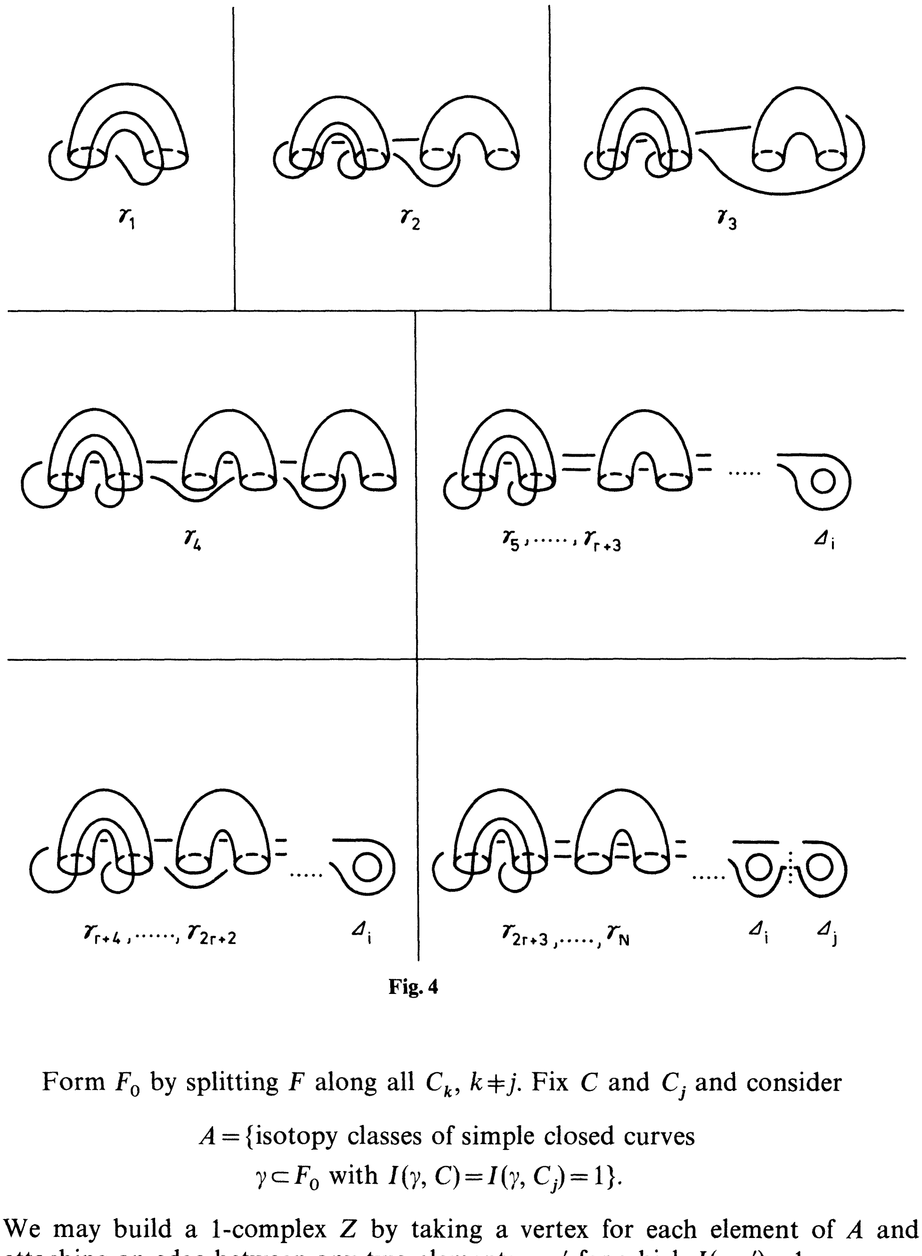

Fig. 4

Form F_0 by splitting F along all C_k, $k \neq j$. Fix C and C_j and consider

$$A = \{\text{isotopy classes of simple closed curves } \gamma \subset F_0 \text{ with } I(\gamma, C) = I(\gamma, C_j) = 1\}.$$

We may build a 1-complex Z by taking a vertex for each element of A and attaching an edge between any two elements γ, γ' for which $I(\gamma, \gamma') = 1$.

Lemma 2.3. *Z is connected.*

Proof. Fix a base point γ_0 in Z. F_0 is a torus with s holes. If s were 0 every element of A would be determined by its homology class. Furthermore, $C = C_j$ and each γ is $\gamma_0 + nC$, $n \in \mathbb{Z}$. Since $I(\gamma_0 + nC, \gamma_0 + mC) = |n - m|$ we see that Z may be identified with the real line, A the integer points.

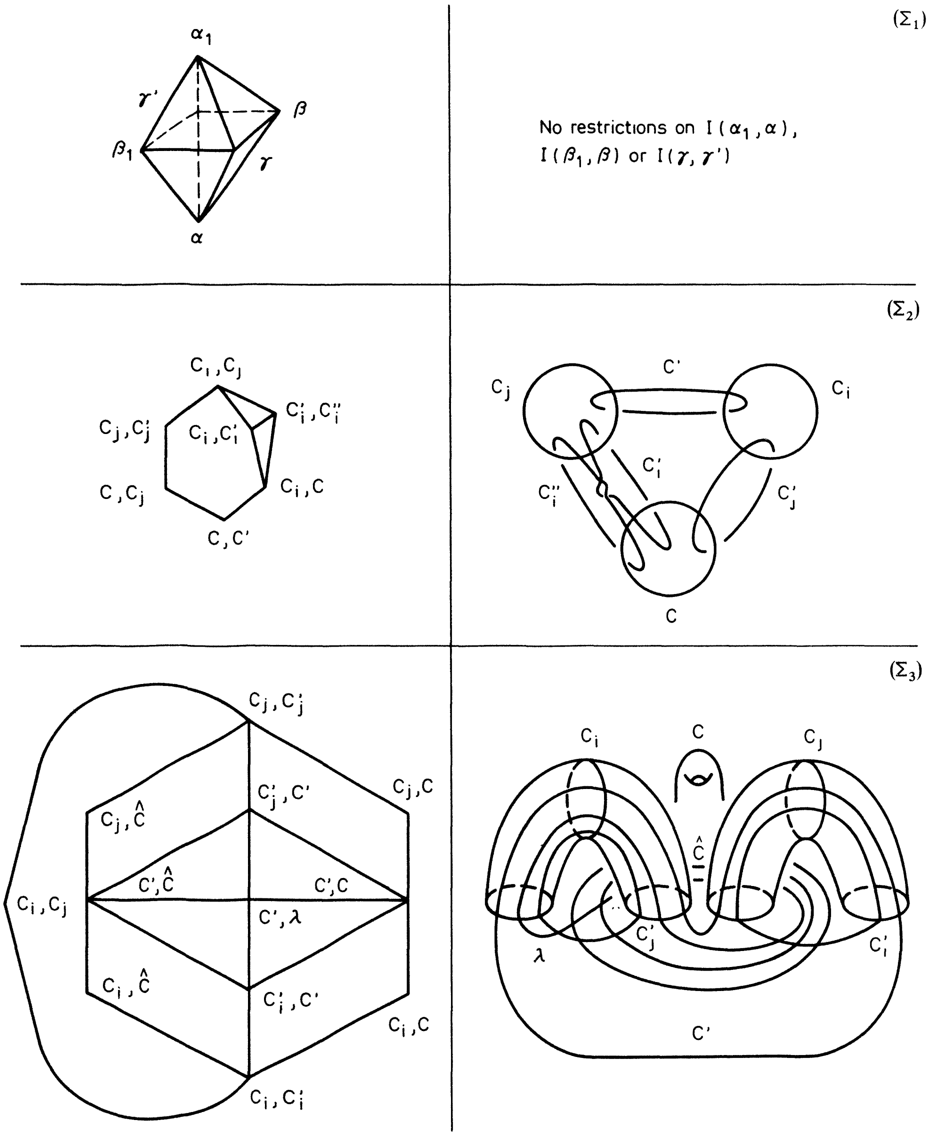

Fig. 5

For the general case proceed by induction on s. Attach a disk to one component of ∂F_0 to form F_1; $f: F_0 \to F_1$ the inclusion. Denote the set A for F_i by A_i; f induces a surjective map $f_*: A_0 \to A_1$. Suppose now that $\gamma_1 \in A_0$. By hypothesis there is a sequence $w_1, \ldots, w_r$ of elements of A_1 such that $I(w_i, w_{i+1}) = I(f_*(\gamma_0), w_1) = I(w_r, f_*(\gamma_1)) = 1$. We may choose $w_i^0, w_i^1 \in A_0$ with $f_*(w_i^j) = w_i$, $I(w_i^0, w_{i-1}^1) = 1$, $i > 1$, and $I(\gamma_0, w_1^0) = I(\gamma_1, w_r^1) = 1$.

Finally, an isotopy of $f(w_i^0)$ to $f(w_i^1)$ in F_1 gives rise to a sequence of curves $w_i^2, \ldots, w_i^j$ in F_0 each differing in F_1 by a single move across the disk $F_1 - F_0$. Choose $\eta_j \in A_0$ with $I(\eta_j, w_1^j) = I(\eta_j, w_1^{j+1}) = 1$ (for example by Dehn twisting a copy of w_i^j along C). These fill in the connection from γ_1 to γ_0 in Z. □

Lemma 2.3 also holds for C_j' and C'; therefore via type Σ_2 3-cells we see that we only need one 2-cell in Y_2 for each choice of $\langle C_k \rangle$ and C.

Consider C next. $F - \{C_k, C\}$ has two components. If either is a sphere with three holes, there is an element of Γ identifying $\{C_k\}$ with $\{\alpha_k\}$ and C with C_0 (Fig. 3). When both have more than 3 holes replace C by $\hat{C}$, a curve disjoint from C which cobounds with C_i and C_j a sphere with three holes (connect C_i to C_j by an arc disjoint from C and surger $C_i \amalg C_j$ to $\hat{C}$). It is not hard to find the extra curves C_i', C_j', C' and λ necessary to construct a 3-cell of type Σ_3. Therefore any two choices of C are equivalent and Y_2 has enough 2-cells of type R_3.

(II) No reduction is needed for type R_2. For R_1, we must reduce the collection of γ's needed for the 2-cells $\sigma(\gamma)$ to those of Fig. 4. Orient α_1, β_1, and γ so that $\alpha_1 \cdot \beta_1 = \alpha_1 \cdot \gamma = 1$. By switching α_1 and β_1 if necessary we may assume that $\beta_1 \cdot \gamma = -1$. Picture F as the boundary of a handlebody (with r 2-disks removed), label the attaching disks for the handles $D_1, \ldots, D_{2g}$ and write $\Delta_1, \ldots, \Delta_r$ for the curves of ∂F (Fig. 3). If necessary, isotope γ until the three points of intersection between α_1, β_1 and γ are distinct. Then $F - \{\alpha_i, \beta_1, \gamma\}$ has three components. If we orient α_1 once and for all, each γ is oriented by requiring $\alpha_1 \cdot \gamma = 1$; write F_1 for the region to the *right* of γ *after* it crosses α_1, F_2 for the region to the *left* of γ *before* it crosses α_1 and F_0 for the remaining region. Let ℓ_i be the number of ∂D_j ($j \geqq 3$) and Δ_j lying in F_i. The γ curves for Y_2 occur when (assuming γ is oriented to the right as it crosses α_1 in Fig. 4) $\ell_1 = 0$, $\ell_2 \leqq 2$ and F_2 contains:

$\ell_2 = 0$ nothing,

$\ell_2 = 1$ $D_3, \Delta_1, \ldots, \Delta_{r-2}$ or Δ_{r-1},

$\ell_2 = 2$ $D_3, D_4; D_3, D_5; D_3, \Delta_i$, $1 \leqq i \leqq r-1$ or Δ_i, Δ_j $1 \leqq i < j \leqq r-1$.

The index (ℓ_1, ℓ_2) is not an invariant of $\sigma(\gamma)$: If we rotate the first handle and switch orientations of α_1 and γ we obtain an equivalence between curves of type (ℓ_1, ℓ_2) and type (ℓ_2, ℓ_1). Furthermore, types $(0, \ell_2)$ and $(2g-2+r-\ell_2, 0)$ are equivalent by an isotopy of γ with F_0, F_1 and F_2 permuted cyclically ((012)).

For the reduction, first suppose $\ell_1 > 2$. Build an octahedron (type Σ_1) with vertices α_1, α, β_1, β, γ, γ' where α_1, β_1, γ are as above, α links $\ell_1 - 2$ holes in F_1 while β and γ' link these and one more (Fig. 6). Each face is translated by Γ to a $\sigma(\gamma)$; if (ℓ_1', ℓ_2') is the index of any of the seven new faces of the octahedron, it is straightforward to verify that $\ell_1' + \ell_2' < \ell_1 + \ell_2$. Using symmetry we are therefore reduced to the cases where ℓ_1, $\ell_2 \leqq 2$.

For (2, 2) and (2, 1) use the curves of Fig. 7, γ' links both holes of F_1 while α and β each link a different one. If γ is type $(2, \ell_2)$ with $\ell_2 = 1$ or 2, the other triangles have types $(1, \ell_2)$, $(1, 1)$, $(2, 0)$, $(0, \ell_2)$, $(0, 0)$ or $(1, 0)$.

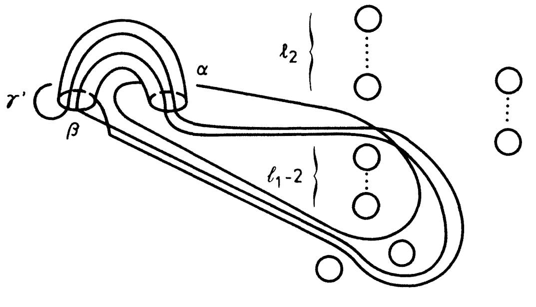

Fig. 6

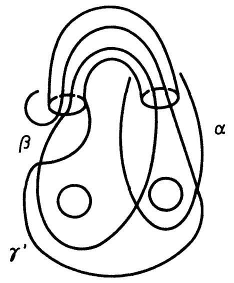

Fig. 7

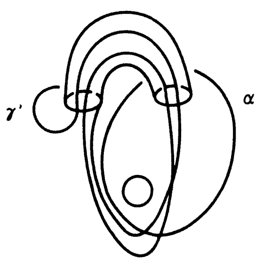

Fig. 8

With type (1, 1) use the curves of Fig. 8, where γ, α, β and γ' each link the hole of F_1. The other types are (0, 1), (0, 0). (1, 0) or (0, 2). Again by using symmetry we have reduced to types (0, 0), (0, 1) and (0, 2).

Finally, notice that in all three reductions any D_i or Δ_j lying in F_0 remains there. If Δ_r lies in F_2 for any of the γ of type $(0, \ell_2)$, $\ell_2 = 1, 2$ we use the equivalence $(0, \ell_2) \equiv (2g-2+r-\ell_2, 0)$ to put Δ_r in F_0. Then the reduction process is repeated to reach types $(0, \ell_2)$, $\ell = 0, 1, 2$ with $\Delta_r \subset F_0$. It is now easily verified that the action of Γ identifies the resulting $\sigma(\gamma)$ with those of Y_2. This finishes Theorem 2.2. □

Section 3. Computation of $H_2\Gamma$ for $g \geqq 5$, $r \geqq 1$, $n=0$

Let Y_2 be the 2-complex of Sect. 2. We must add two types of 3-cells to form Y_3. For the first type notice that the curve γ_1 of Fig. 4 is disjoint from β_2. Therefore we may construct a 3-cell Σ_4 from the fact that the cycle of simple moves $\alpha_1-\beta_1-\gamma_1-\alpha_1$ commutes with the moves $\alpha_2-\beta_2-\alpha_2$. It is a triangular prism and is pictured in Fig. 9. Y_3 has a 3-cell of type Σ_3 for each configuration identified by Γ to this cell.

For the second type we add to Y_3 any Σ_3 3-cell equivalent under Γ to the one with $(C_i, C_j, C_i', \hat{C}, C')=(\alpha_1, \alpha_2, \beta_2, \beta_1, C_0, \omega)$, with λ identified with any fixed curve λ_0 which meets the other curves properly (for example, $\lambda_0=\tau_{C_0}^{-1}(\beta_1)$, again τ_{C_0} is the right-handed Dehn twist on C_0; compare Figs. 3 and 5), and C the curve which encircles D_1 and D_4 and lies in front of D_2 and D_3, disjoint from ω ($\exists$ a map τ: $F \to F$ fixing ω', α_3, ..., α_g which interchanges α_1 with α_2, β_1 with β_2 and C with C_0). Γ acts on Y_3.

Suppose next that $B\Gamma$ is a CW complex and a $K(\Gamma, 1)$, $E\Gamma \to B\Gamma$ its universal covering. From the fiber product $\Delta = E\Gamma \times_\Gamma Y_3$ there is a natural projection $f\colon \Delta \to B\Gamma$, a fibration with fiber Y_3. This means $\pi_1(\Delta) \cong \Gamma$ so a $K(\Gamma, 1)$ may be constructed by attaching cells to Δ of dimension $\geqq 3$. There is therefore a well defined surjection $\varphi\colon H_2(\Delta) \to H_2(\Gamma)$.

Theorem 3.1. *Image* $(\varphi) \cong \mathbb{Z}$.

Corollary 3.2. $H_2(\Gamma) \cong \mathbb{Z}$ *for* $g \geqq 5$, $r \geqq 1$, $n=0$.

Proof of 3.1. Write (C_*, ∂_*^C) for the cellular chain complex of Y_3, (K_*, ∂_*^K) for that of $E\Gamma$. If $R=\mathbb{Z}\Gamma$,

$$M_k = \bigoplus_{i+j=k} C_i \otimes_R K_j$$

and

$$\partial_k^M = [\bigoplus(\partial_i^C \otimes_R 1_{K_j})] + [\bigoplus(1_{C_i} \otimes_R (-1)^i \partial_j^K)],$$

then (M_*, ∂_*^M) is the chain complex of Δ. Define a filtration of M_* by setting

$$F_p(M_k) = \bigoplus_{\substack{i+j=k \\ i \leqq p}} C_i \otimes_R K_j.$$

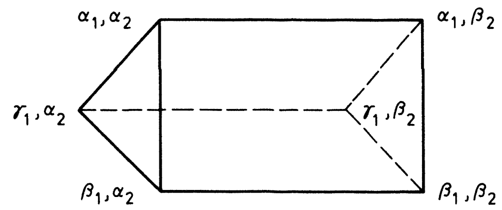

Fig. 9

The corresponding spectral sequence has $E^{\infty}_{p,q}=\dfrac{F_p(H_{p+q}(\Delta))}{F_{p-1}(H_{p+q}(\Delta))}$ where $F_p(H_{p+q}(\Delta))=\operatorname{Im}(H_{p+q}(\Delta_p)\to H_{p+q}(\Delta))$, $\Delta_p=E\Gamma\times_\Gamma Y_p$, $p\leqq 3$. Also

$$E^0_{p,q}=C_p\otimes_R K_q$$

and

$$d^0_{p,q}=1\otimes\partial^K_q.$$

Let σ_0 be the 0-cell of Y_2 corresponding to $\langle\alpha_i\rangle$, σ_1 the 1-cell for $\langle\alpha_1\rangle-\langle\beta_1\rangle$, $(\alpha_2,\ldots,\alpha_g$ understood), σ_2^i, $1\leqq i\leqq N$, the type R_1 2-cells for $\langle\alpha_1\rangle-\langle\beta_1\rangle-\langle\gamma_i\rangle-\langle\alpha_1\rangle$, σ_2^{N+1} the type R_2 2-cell for $\langle\alpha_1,\alpha_2\rangle-\langle\alpha_1,\beta_2\rangle-\langle\beta_1,\beta_2\rangle-\langle\beta_1,\alpha_2\rangle-\langle\alpha_1,\alpha_2\rangle$, σ_2^{N+2} the special type R_3 2-cell from the curves of Fig. 3, σ_3^1 the Σ_4 3-cell of Fig. 9 and σ_3^2 the Σ_3 3-cell described at the beginning of this section.

The action of Γ on C_p splits:

$$C_p=C_p^1\oplus\ldots\oplus C_p^{n_p}$$

with $\Gamma(C_p^i)\subset C_p^i$ and every generator of C_p^i identified by Γ with σ_p^i ($n_0=n_1=1$, $n_2=N+2$ and $n_3=2$). If Γ_p^i denotes the stabilizer of σ_p^i

$$C_p^i\cong R\otimes_{\mathbb{Z}[\Gamma_p^i]}\mathbb{Z}$$

via the correspondence $t\cdot g(\sigma_p^i)\leftrightarrow g\otimes t$. Hence

$$C_p\otimes_R K_q\cong\bigoplus_i(\langle\sigma_p^i\rangle\otimes_{\mathbb{Z}[\Gamma_p^i]}K_q).$$

Since $K_*\to\mathbb{Z}$ is a free $\mathbb{Z}[\Gamma_p^i]$ resolution

$$E^1_{p,q}\cong\bigoplus_i H_q(\Gamma_p^i;\langle\sigma_p^i\rangle).$$

Where $\langle\sigma_p^i\rangle\cong\mathbb{Z}\subset C_p$. Γ_0, Γ_2^i, $1\leqq i\leqq N$, Γ_2^{N+2} and Γ_3^1 act trivially on their respective $\langle\sigma_p^i\rangle$. Γ_1, Γ_2^{N+1} and Γ_3^2 however contain orientation reversing maps.

We begin the computations with

Lemma 3.3.

$$E^2_{p,0}\cong\begin{cases}\mathbb{Z} & p=0\\ 0 & p=1,3.\\ \mathbb{Z}^N\oplus\mathbb{Z}/2\mathbb{Z} & p=2\end{cases}$$

Proof. When Γ_p^i acts trivially on σ_p^i, $H_0(\Gamma_p^i;\langle\sigma_p^i\rangle)\cong\mathbb{Z}$. On the other hand for Γ_1, Γ_2^{N+1} and Γ_3^2 this group is $\mathbb{Z}/2\mathbb{Z}$. It is straight forward to then verify that

$$d^1: H_0(\Gamma_2^1;\langle\sigma_2^1\rangle)\to H_0(\Gamma_1;\langle\sigma_1\rangle)$$

is surjective, that

$$d^1: H_0(\Gamma_3^2;\langle\sigma_3^2\rangle)\to H_0(\Gamma_2^{N+1};\langle\sigma_2^{N+1}\rangle)\oplus H_0(\Gamma_2^1;\langle\sigma_2^1\rangle)$$

is injective and that

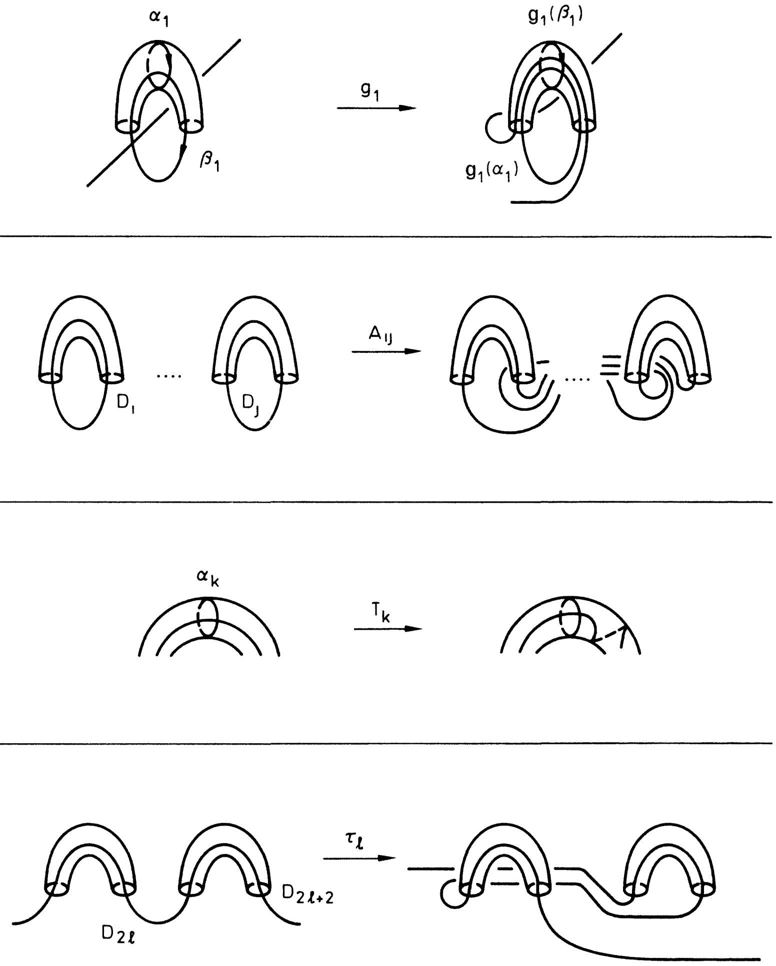

Fig. 10

$$d^1: H_0(\Gamma_3^1; \langle \sigma_3^1 \rangle) \to H_0(\Gamma_2^{N+2}; \langle \sigma_2^{N+2} \rangle) \oplus \ldots$$

is multiplication by 2 in the first factor.

The lemma follows. □

Next we must analyze the stabilizer of each cell. Write

$$\hat{\Gamma}_0 = \{f \in \Gamma_0 : f \text{ fixes the curves which determine the cell } \sigma_0 \text{ pointwise}\}$$

with a similar definition of $\hat{\Gamma}_1$, $\hat{\Gamma}_2^i$ and $\hat{\Gamma}_3^i$.

Lemma 3.4. $\hat{\Gamma}_0 \cong P_{2g+r-1} \times \mathbb{Z}^{g+r-1}$ *where P_n is the pure braid group on n strands. As generators we have $A_{i,j}$, $1 \leqq i < j \leqq 2g+r-1$ and T_k, $1 \leqq k \leqq g+r-1$ with $A_{i,j}$ the diffeomorphisms of F obtained by sliding D_i around D_j (where $D_{2g+j} = \Delta_j$, Figs. 3, 10) and T_k the Dehn twist on α_k, $1 \leqq k \leqq g$ or Δ_{k-g}, $g+1 \leqq k \leqq g+r-1$.*

The proof from [5] for closed surfaces is easily adapted. □

Let F_1 be obtained by splitting F along $\alpha_1 \cup \beta_1$; F_1 has genus $g-1$ with $r+1$ boundary components. One sees immediately that

$$\hat{\Gamma}_1 \cong \hat{\Gamma}_0(F_1)$$

with $\hat{\Gamma}_0(F_1) = \{f \in \Gamma(F_1): f | \{\alpha_2, \ldots, \alpha_g\} = 1\}$. We may likewise form F_2^i and F_3^i with

$$\hat{\Gamma}_2^i \cong \hat{\Gamma}_0(F_2^i)$$
$$\hat{\Gamma}_3^i \cong \hat{\Gamma}_0(F_3^i).$$

There are short exact sequences

$$1 \to \hat{\Gamma}_p^i \to \Gamma_p^i \to G_p^i \to 1 \qquad (*)$$

where G_p^i is the (finite) group of symmetries of σ_p^i. $G_0 \cong \pm \Sigma_g$, the group of signed permutations on g elements (the curves $\{\alpha_1, \ldots, \alpha_g\}$ may be permuted and have their orientations changed). Γ_0 is therefore generated by $\hat{\Gamma}_0$, x_1 and τ_ℓ, $1 \leqq \ell \leqq g-1$, where x_1 reverses α_1 (fixing $\alpha_2, \ldots, \alpha_g$) and τ_ℓ interchanges α_ℓ and $\alpha_{\ell+1}$ (Fig. 10; in Γ, $x_1 = g_1^2$, g_1 is the map σ from [5]). $G_1 \cong \mathbb{Z}/4\mathbb{Z} \times \pm \Sigma_{g-1}$ so Γ_1 is generated by $\hat{\Gamma}_1$, g_1, x_2 and τ_ℓ, $2 \leqq \ell \leqq g-1$. It will not be necessary (although it is not difficult) to compute G_2^i or G_3^i.

g_1 lies in the center of Γ_1 so $d^1: H_q(\Gamma_1; \langle \sigma_1 \rangle) \to H_q(\Gamma_0)$ is zero for all q. Combining this with Lemma 3.3 we see that the E^2 term of our spectral sequence is

$$\begin{array}{|llll}
H_2\Gamma_0 & & & \\
H_1\Gamma_0 & E_{1,1}^2 & & \\
\mathbb{Z} & 0 & \mathbb{Z}^N \oplus \mathbb{Z}/2\mathbb{Z} & 0. \\
\hline
\end{array}$$

To complete the argument we shall show

Lemma 3.5. *$H_1(\Gamma_0) \cong \mathbb{Z}^{N-1} \oplus \mathbb{Z}/2\mathbb{Z}$ and $E_{2,0}^3 \cong \mathbb{Z}$.*

Lemma 3.6. *If $F_p(H_2(\Gamma)) = \varphi(F_p(H_2(\Delta)))$ (φ the map of (3.1)),*

$$F_0(H_2(\Gamma)) = F_1(H_2(\Gamma)) = 0.$$

Since $H_2(\Delta) = F_2(H_2(\Delta))$ and $E_{2,0}^3 = E_{2,0}^\infty = F_2(H_2(\Delta))/F_1(H_2(\Delta))$ these lemmas complete the Proof of 3.1.

Proof of 3.5. P_n has the following presentation ([1], see the errata, the presentation in the book in incorrect):

Generators are A_{ij}, $1 \leqq i < j \leqq n$.

Relations are $[A_{ij}, B_{i,j,r,s}]$, $1 \leqq i < j \leqq n$, $1 \leqq r < s \leqq n$ where

$$B_{i,j,r,s} = \begin{cases} A_{rs} & r<s<i<j \quad \text{or} \quad i<r<s<j, \\ A_{rs}A_{rj} & s=i, \\ A_{rs}A_{ij}A_{sj} & i=r<s<j \\ A_{rs}[A_{rj}, A_{sj}] & r<i<s<j. \end{cases}$$

Clearly then

$$H_1(P_n\times\mathbb{Z}^m)\cong\mathbb{Z}^{\binom{n}{2}+m}$$

generated by the classes of A_{ij} and T_k. In particular $H_1(\hat{\Gamma}_p^i)$ is torsion free for all p, i. If we analyze the spectral sequence associated with (*) we find (since $H_2(G_p^i)$ is torsion)

$$0\to H_1(\hat{\Gamma}_p^i)/G_p^i\to H_1(\Gamma_p^i)\xrightarrow{\psi}H_1(G_p^i)\to 0.$$

For $p=0$, G_0 identifies

$$\begin{array}{lll} T_k & \text{with } T_1 & 1\leqq k\leqq g,\\ A_{ij} & \text{with } A_{23} & 1\leqq i<j\leqq 2g,\ (i,j)\neq(2s-1,2s),\\ & \text{with } A_{12} & (i,j)=(2s-1;2s),\ s\leqq g,\end{array}$$

or

$$\text{with } A_{1j} \quad 1\leqq i\leqq 2g,\ j>2g.$$

Thus $H_1(\hat{\Gamma}_0)/G_0\cong\mathbb{Z}^{N-1}$, generators $T_1, T_{g+1}, \ldots, T_{g+r-1}, A_{12}, A_{23}, A_{1i}, A_{ij}$, $2g<i<j\leqq 2g+r-1$.

$$H_1(G_0)\cong\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$$

with $\psi(x_1)$ and $\psi(\tau_1)$ as generators. The τ_1 of Fig. 10 satisfies

$$\tau_1^2=A_{23}^{-1}A_{13}^{-1}A_{34}^{-1}A_{24}^{-1}A_{14}^{-1}A_{34},$$

so $2(\tau_1+2A_{23})$ represents 0 in $H_1(\Gamma_0)$. x_1 on the other hand satisfies $x_1^2=A_{12}T_1^2$. Putting this all together shows $H_1(\Gamma_0)\cong\mathbb{Z}^{n-1}\oplus\mathbb{Z}/2\mathbb{Z}$.

$H_1(\Delta)=0$ means that

$$d^2\colon\mathbb{Z}^N\oplus\mathbb{Z}/2\mathbb{Z}\to\mathbb{Z}^{N-1}\oplus\mathbb{Z}/2\mathbb{Z}$$

must be surjective. Since $\alpha_2, \ldots, \alpha_g$ are fixed in type R_1 2-cells $d^2|\mathbb{Z}^N$ misses the $\mathbb{Z}/2\mathbb{Z}$ generated by τ_1+2A_{23}. Therefore $\operatorname{Ker}(d^2)\cong\mathbb{Z}$ and the lemma is proven. □

Proof of 3.6. Consider again the sequence (*). Combining a presentation of $G_0\cong\pm\Sigma_g$ with one for $\hat{\Gamma}_0$ gives us one for Γ_0. From this we may construct the 2-skeleton K_2 of a $K(\Gamma_0,1)$. Γ_0 and g_1 generate Γ and g_1 commutes with Γ_1. Form $\hat{K}_2$ from K_2 by adding a 1-cell for g_1 and 2-cells for the relations $g_1^2=x_1$ and $[g_1,\eta_i]$ with $\{\eta_i\}$ a generating set for Γ_1. $\hat{K}_2$ may be completed to K, a $K(\Gamma,1)$, by adding 2-cells (for σ_2^i), 3-cells, etc.

Part 1. $F_0(H_2(\Gamma))=\operatorname{Im}(H_2(\Gamma_0)\to H_2(\Gamma))$, for this we look at $K_2\hookrightarrow K$. The spectral sequence associated to (*) includes the terms $E_{0,2}^2\cong H_2(\hat{\Gamma}_0)/\pm\Sigma_g$, $E_{1,1}^2\cong H_1(\pm\Sigma_g; H_1(\hat{\Gamma}_0))$ and $E_{2,0}^2\cong H_2(\pm\Sigma_g)$. There is an exact sequence

$$1\to(\mathbb{Z}/2\mathbb{Z})^g\to\pm\Sigma_g\to\Sigma_g\to 1$$

with Σ_g generated by $\tau_1, \ldots, \tau_{g-1}$ and $(\mathbb{Z}/2\mathbb{Z})^g$ generated by $x_1, \ldots, x_g$ (where x_i reverses α_i, fixing $\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_g$).

$$\Sigma_g=\{\tau_1,\ldots,\tau_{g-1}\colon \tau_i\tau_{i+1}\tau_i=\tau_{i+1}\tau_i\tau_{i+1},\quad \tau_1^2=1,$$
$$[\tau_i,\tau_j]=1 \quad \text{if } |i-j|>1\}.$$

Also, Σ_g acts on $(\mathbb{Z}/2\mathbb{Z})^g$ by permutations of the x_i. All together $E^2_{2,0}$ contributes

$$[\tau_1, \tau_3],$$
$$[x_1, x_2]$$

and

$$[x_1, \tau_2]$$

to K_2.

$E^2_{0,2}$ contributes

$$[T_i, T_j],$$
$$[T_k, A_{ij}]$$

and

$$[A_{ij}, B_{ijrs}],$$

for various choices of i, j, k, r, and s.

Finally, after allowing for the identification of certain generators of $\hat{\Gamma}_0$ by $\pm\Sigma_g$ it is straightforward to check that $E^2_{1,1}$ contributes

$[\tau_2, T_k]$ $\quad k=1, \ k>g$
$[x_1, T_k]$ $\quad k=1, \ k>g$
$[A_{13}, \tau_1 \bar{A}_{34}]$
$[\tau_3, A_{ij}]$ $\quad (i, j)=(1, 2), (1, 3), (1, s), (s, t), \quad 2g<s<t$
$[x_3, A_{ij}]$ $\quad$ same (i, j).

What we finally see is that $H_2(\Gamma_0)$ is generated by commutators $[w_1, w_2]$ where either

a) w_1, w_2 are words in the generators which as elements of Γ are supported on disjoint subsurfaces one of which (say the one carrying w_2) is genus $\geqq 3$,

b) $(w_1, w_2)=(A_{13}, \tau_1 \bar{A}_{34})$ or

c) $(w_1, w_2)=(x_1, T_1)$.

For (a), Γ is perfect for $g \geqq 1$ so

$$w_2 = \prod_{i=1}^{r} [y_i, z_i] \tag{**}$$

with $[w_1, y_i]=[w_1, z_i]=1$ for all i with y_i, z_i words in the generators of Γ. In building a $K(\Gamma, 1)$ from K_2 we will therefore have a 3-cell of the form $P \times I$ where P is a $(2r+1)$-gon giving the relation (**) and the I factor is attached for w_1. Of course $\partial(P \times I) \equiv [w_1, w_2]$.

For (b) we again claim that $w_2 = \tau_1 \bar{A}_{34}$ has a commutator expansion (**) with $[A_{13}, y_i]=[A_{13}, z_i]=1$. Equivalently we may work with $[T_1 T_2 A_{13}, \tau_1 \bar{A}_{34}]$ since $[T_1 T_2, \tau_1]$ and $[T_1 T_2, \bar{A}_{34}]$ are known. Then $T_1 T_2 A_{13}$ is the twist T_0 on the curve C_0 which links D_1, D_3 in back of D_2. Furthermore $\tau_1 \bar{A}_{34}$ fixes C_0. We may therefore find the y_i and z_i in $F - C_0$, fixing C_0 guarantees $[T_0, y_i]=[T_0, z_i]=1$ back in Γ.

Finally, for (c) the situation is slightly different since x_1 reverses the orientation of α_1. Consider the trefoil knot $K \subset S^3$.

$$\pi_1(S^3 - K) \cong \{a, b \colon aba = bab\}.$$

The meridian m of K is a and the longitude ℓ is aba^2ba. Set $a=T_1$, $b=T_1^{-1}g_1T_1^{-1}$. Then $aba=bab$ is satisfied in Γ so we find $f: S^3 - \dot{N}(K) \to K(\Gamma, 1)$, $N(K)$ a tubular neighborhood of K, with $f(m)=T_1$ and $f(\ell) = aba^2ba = x_1$. $\mathrm{Im}(f)$ provides the nullhomology for $[T_1, x_1]$.

Part 2. $E^{\infty}_{1,1} \cong F_1(H_2(\Delta))/F_0(H_2(\Delta))$, $E^1_{1,1} = H_1(\Gamma_1; \langle \sigma_1 \rangle)$. Since $d^r | E^r_{1,1}$ is 0 for every $r \geqq 1$, we have $E^1_{1,1} \to E^{\infty}_{1,1} \to H_2(\Delta)/F_0(H_2(\Delta)) \to H_2(\Gamma)$; call the composition ψ. Image $(\psi) = F_1(H_2(\Gamma))$ is represented in K by the commutator 2-cells of $\hat{K}_2 - K_2$. The argument of Part 1 then applies to show $F_1(H_2(\Gamma)) = 0$. □

Section 4. Final Computations

Consider now the groups $\Gamma^n_{g,r}$ consisting of mapping classes of diffeomorphisms of F which fix $\Delta_1, \ldots, \Delta_r$, the boundary curves of F, as well as distinguished points $p_1, \ldots, p_n$. We will delete the indices r and/or n when equal to 0. Let π_g denote π_1 of the closed surface of genus g. We need two exact sequences:

$$1 \to \mathbb{Z} \xrightarrow{f_1} \Gamma^n_{g,r} \xrightarrow{f_2} \Gamma^{n+1}_{g,r-1} \to 1 \tag{A}$$

$$1 \to \pi_g \xrightarrow{f_3} \Gamma^1_g \xrightarrow{f_4} \Gamma_g \to 1. \tag{B}$$

Here f_2 is the map induced by adding a disk to Δ_r whose center becomes p_{n+1}, f_1 is the Dehn twist on a curve parallel to Δ_r, f_4 is obtained by forgetting p and f_3 comes by sliding p along a loop in F.

Analyzing the Hochschild-Serre-Lyndon spectral sequence for the *central extension* (A) gives E^2 term

$$\begin{array}{|llll} 0 & 0 & 0 & \\ \mathbb{Z} & H_1(\Gamma^{n+1}_{g,r-1}) & * & * \\ \mathbb{Z} & H_1(\Gamma^{n+1}_{g,r-1}) & H_2(\Gamma^{n+1}_{g,r-1}) & *. \\ \hline \end{array}$$

Inductively we may assume $H_2(\Gamma^n_{g,r}) \cong \mathbb{Z}^{n+1}$ and since $g>2$ $H_1(\Gamma^{n+1}_{g,r-1}) \cong H_1(\Gamma^n_{g,r}) \cong 0$. Therefore d^2: $H_2(\Gamma^{n+1}_{g,r-1}) \to H_0(\Gamma^{n+1}_{g,r-1})$ is surjective and $H_2(\Gamma^{n+1}_{g,r-1}) \cong \mathbb{Z}^{n+2}$ via $S_0, \ldots, S_{n+1}$ as required.

For (B) the E^2 term is

$$\begin{array}{|ccc} \mathbb{Z} & 0 & * \\ H_1(F_g)/\Gamma_g & H_1(\Gamma_g; H_1 F_g) & * \\ \mathbb{Z} & 0 & H_2(\Gamma_g). \\ \hline \end{array}$$

It is easy to check $H_1(F)/\Gamma = 0$. Also Lemma 1.2 says $H_1(\Gamma; H_1 F) = 0$ so there is an exact sequence

$$0 \to E^{\infty}_{0,2} \xrightarrow{\varphi} H_2(\Gamma^1_g) \xrightarrow{\psi} H_2(\Gamma_g) \to 0.$$

Recall

$$\left(\frac{S_0}{4}, S_1\right): H_2(\Gamma_g^1) \xrightarrow{\approx} \mathbb{Z} \oplus \mathbb{Z}.$$

Identifying $H_2(\Gamma_g^1)$ with $\Omega_2(B\,\mathrm{Diff}^+(F_g^1))$, image φ is generated by the class $[\eta]$ of the bundle over F induced by $f_3: \pi_g \to \Gamma_g^1$. Since S_0 factors through $H_2(\Gamma_g)$, $S_0[\eta]=0$. On the other hand $f_4 \circ f_3 = 0$ implies the total space of η is diffeomorphic to $F_g \times F_g$ with $s_1(F_g)$ the diagonal. Hence $S_1 \circ \varphi = 2-2g$ and the theorem follows. □

Section 5. Remarks

(1) Let f_1, f_2 be orientation preserving diffeomorphisms of F_g. Write $M^3_{f_i}$ for the mapping torus of f_i, $i=1,2$. Because the 3-dimensional bordism group is zero, $M_{f_i} = \partial W_i^4$ where W_i^4 is a compact oriented 4-manifold. By glueing W_1 and W_2 together along copies of $F \times I$ in M_{f_1} and M_{f_2} we obtain W^4 with $\partial W^4 = M_{f_2 f_1}$. Define

$$\Delta(f_1, f_2) = \mathrm{index}(W) - \mathrm{index}(W_1) - \mathrm{index}(W_2).$$

It is not difficult to check that Δ depends only on the isotopy classes of f_1 and f_2. Furthermore, Neumann [8] observes that Δ satisfies the cocycle condition

$$\Delta(f_2, f_3) - \Delta(f_1 f_2, f_3) + \Delta(f_1, f_2 f_3) - \Delta(f_1, f_2) = 0$$

for any $f_1, f_2, f_3 \in \Gamma_g$. Hence Δ represents an element of $H^2(\Gamma_g; \mathbb{Z})$. That Δ generates $H^2(\Gamma_g)$ may be seen directly from the isomorphism

$$\frac{S_0}{4}: H_2(M_g)/\mathrm{torsion} \to \mathbb{Z}.$$

(2) Γ is generated by Dehn twists $\tau(\gamma_i)$ on nonseparating circles γ_i (defined up to isotopy) in F. There is an easy relation among such Dehn twists, namely

$$\tau(\gamma_i)\tau(\gamma_j) = \tau(\gamma_j)\tau(\gamma_i^{\gamma_j}) \qquad (*)$$

where $\gamma_i^{\gamma_j}$ is the image of the curve γ_i under $\tau(\gamma_j)$. Let G be the group with generators $\tau(\gamma_i)$ taken over all nonseparating circles in F and relations of type (*) (for example when $g=1$, $G \cong \Gamma_{1,1}$). There is a natural surjection $G \xrightarrow{\varphi} \Gamma$. In [4] the kernel of φ is computed to be

$$H_2(\Gamma) \oplus \mathbb{Z}.$$

Because Γ is perfect $G' = [G, G]$ is also perfect and maps onto Γ. $H_1(G) = \mathbb{Z}$ so we have an exact sequence

$$0 \to H_2(\Gamma) \to G' \to \Gamma \to 1.$$

G' is the *universal central extension* of Γ. G' may also be obtained from G by adjoining the relation (*) from Sect. 1 for an embedding of F_0 in F with all curves nonseparating.

(3) Clearly Theorem 2.2 gives a simplification of Hatcher and Thurston's presentation of Γ. Wajnryb [10] has used this to give an incredibly simple presentation of Γ_g and $\Gamma_{g,1}$.

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Note Added in Proof

The author has been informed by Mumford that the computation of $H_2\Gamma$ gives a proof of the rational version of the Francetta conjecture.