Annals of Mathematics, **102** (1975), 101-137

# Higher simple homotopy theory

By A. E. HATCHER*

## 1. Introduction

In this paper we globalize J. H. C. Whitehead's simple homotopy theory [17] by constructing a homotopy functor Wh from polyhedra to simplicial $H$-spaces, such that Whitehead's theory amounts to the calculation of $\pi_0 \mathrm{Wh}(K)$, the arc-components of $\mathrm{Wh}(K)$. "Higher simple homotopy theory" is then concerned with the full homotopy type of $\mathrm{Wh}(K)$, for example, its higher homotopy groups.

Recall from Whitehead's simple homotopy theory the basic geometric operation of an elementary collapse, written $L_0 \searrow L_1$, where $L_0$ and $L_1$ are finite cell complexes such that $L_0$ is obtained from $L_1$ by attaching a ball along a face in its boundary. The equivalence relation generated by elementary collapses is called simple homotopy equivalence, and the main theorem is that a homotopy equivalence of finite complexes is simple if and only if a single algebraically defined obstruction (the torsion), lying in an abelian group which depends only on the fundamental group of the spaces involved, vanishes.

Simple homotopy equivalences are not hard to find in nature. A useful recognition criterion in the PL category, due to M. M. Cohen [7], is the following: A PL map $f\colon L_0 \to L_1$ is a simple homotopy equivalence if all the point inverses $f^{-1}(*)$ are non-empty and contractible. Cohen called such maps *contractible mappings*. For example, an elementary collapse $L_0 \searrow L_1$ can be realized by an evident contractible mapping. More recently, T. A. Chapman [3] has vastly generalized Cohen's theorem to the CW category (with "contractible" replaced by "cell-like"), thereby proving a conjecture of Whitehead that homeomorphisms are simple.

One nice property of PL contractible mappings not shared by elementary collapses is that they are closed under composition. Thus we can form the category $\mathcal{C}$ whose objects are finite polyhedra (say, finite subpolyhedra of $\mathbf{R}^\infty$ for definiteness) and whose morphisms are PL contractible mappings. We have also the full subcategory $\mathcal{C}_K$ of $\mathcal{C}$ whose objects are polyhedra homotopy equivalent to the fixed polyhedron $K$, and the subcategory $\mathcal{C}(K) \subset \mathcal{C}_K$ whose objects contain $K$ as a deformation retract and whose morphisms restrict to

* Supported in part by NSF grant GP 34324X.

the identity on $K$. One of several equivalent definitions of $\mathrm{Wh}(K)$ is the classifying space $B\mathcal{C}(K)$. This is (the geometric realization of) the simplicial space whose $k$-simplices are the compositions $L_0 \xrightarrow{f_1} L_1 \longrightarrow \cdots \xrightarrow{f_k} L_k$ in $\mathcal{C}(K)$. The various $(k-1)$-faces of such a $k$-simplex are obtained by deleting an $L_i$ and, if $0 < i < k$, composing $f_i$ with $f_{i+1}$. The classifying spaces $B\mathcal{C}_K$ and $B\mathcal{C}$ are defined similarly. The arc-components of $B\mathcal{C}$ are, by Cohen's theorem, exactly the simple homotopy types of finite polyhedra. $B\mathcal{C}_K$ is a union of components of $B\mathcal{C}$, those containing polyhedra homotopy equivalent to $K$. Also, $\pi_0 \mathrm{Wh}(K)$ is just the group called "$\mathrm{Wh}(K)$" in [8], where Whitehead's theorem is reformulated to say that $\pi_0 \mathrm{Wh}(K)$ is naturally isomorphic to $\mathrm{Wh}_1(\pi_1 K)$, the algebraic torsion group, quotient of $K_1\mathbf{Z}[\pi_1 K]$. (See also [10], [13], [15] for similar geometric definitions of Whitehead torsion.)

To breathe a little life into this categorical nonsense, we start by showing that $B\mathcal{C}$ actually classifies something: fibrations in the PL category, that is, PL maps which satisfy the covering homotopy property for polyhedra (Serre fibrations). Thus $\mathcal{C}$ is the "structure group" for PL fibrations. Intuitively, the idea is that PL contractible mappings are "local" homotopy equivalences, and the covering homotopy property is essentially a local condition. An immediate corollary is that, over a connected base, the fibers of a PL fibration all have the same simple homotopy type, not just the same homotopy type, as one might expect.

More usually in topology one works with homotopy fibrations, meaning maps of arbitrary spaces which are Serre fibrations, or equivalently, PL maps which are only quasi-fibrations (satisfying the weak covering homotopy property [9]). Homotopy fibrations with fibers homotopy equivalent to $K$ are classified by $BG(K)$, where $G(K)$ is the $H$-space of self-homotopy equivalences of $K$. Passing from PL fibrations to homotopy fibrations induces a map of classifying spaces, whose homotopy fiber turns out to be $\mathrm{Wh}(K) = B\mathcal{C}(K)$:

$$(*) \qquad \mathrm{Wh}(K) \longrightarrow B\mathcal{C}_K \longrightarrow BG(K)\ .$$

Thus $\mathrm{Wh}(K)$ measures the global difference between PL contractible mappings and general homotopy equivalences. Curiously, $\mathrm{Wh}(K)$ has much more structure than $B\mathcal{C}_K$ or $BG(K)$: it is a homotopy functor of $K$ and, functorially, an infinite loopspace.

A deeper justification for our definition of $\mathrm{Wh}(K)$ is that we can use it to prove a parametrized version of the PL $h$-cobordism theorem. Recall that the $h$-cobordism theorem says in effect that for a given compact connected PL manifold $M^n$, $n \geqq 5$, the various $h$-cobordisms $(W, M)$ are in one-to-one correspondence with the components of $\mathrm{Wh}(M)$, via their torsions. In particular, there exists a product structure $(W, M) \approx (M \times I, M)$ if and only if $W$ lies in the identity component of $\mathrm{Wh}(M)$. The parametrized version deals with the comparison between different product structures on $(M \times I, M)$. Let $\mathcal{P}(M)$ denote the (simplicial) space of PL homeomorphisms of $M \times I$ fixed on $M$ (i.e., pseudo-isotopies). Then we construct a natural map $\mathcal{P}(M) \to \Omega\,\mathrm{Wh}(M)$ which is $k$-connected whenever $n = \dim M$ is large with respect to $k$ ($n \geqq 3k + 8$ will do). This is the main theorem of the paper.

An immediate consequence is that there is a "stable" dimension range $n \gg i$ where $\pi_i\mathcal{P}(M^n)$ depends only on the homotopy type of $M$. In particular, the inclusion $\mathcal{P}(M) \hookrightarrow \mathcal{P}(M \times I)$, $f \mapsto f \times \mathrm{id}_I$, induces an isomorphism on $\pi_i$ if $n \gg i$. With results of Chapman, this leads to a neat reformation of higher simple homotopy theory in terms of compact Hilbert cube manifolds. Such manifolds have the form $K \times Q$ for $K$ a finite polyhedron and $Q$ the Hilbert cube. $K$ is determined only up to simple homotopy type, so we may as well take it to be a PL manifold $M$. Then $\mathcal{P}_{\mathrm{TOP}}(M \times Q) \simeq \bigcup_q \mathcal{P}(M \times I^q) \simeq \Omega\,\mathrm{Wh}(M)$, the first equivalence, by [5]. For the composite equivalence $\mathcal{P}_{\mathrm{TOP}}(M \times Q) \simeq \Omega\,\mathrm{Wh}(M)$ we can replace $M$ by $K$, and we have a diagram

$$\begin{array}{ccccccc}
\mathcal{P}_{\mathrm{TOP}}(K \times Q) & \to & \mathrm{Homeo}\,(K \times Q) & \to & G(K \times Q) & \to & G(K \times Q)/\mathrm{Homeo}\,(K \times Q) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \\
\Omega\,\mathrm{Wh}(K) & \longrightarrow & \Omega B\mathcal{C}_K & \longrightarrow & G(K) & \longrightarrow & \mathrm{Wh}(K)
\end{array}$$

where the upper row is a fibration sequence by [6] and the lower row continues (*). The map $G(K \times Q)/\mathrm{Homeo}(K \times Q) \to \mathrm{Wh}(K)$ is a homotopy equivalence on identity components, and the precise situation with $\pi_0$ is covered by Chapman's original proof of the topological invariance of Whitehead torsion: A homotopy equivalence $f\colon K \to K'$ is simple if and only if $f \times \mathrm{id}\colon K \times Q \to K' \times Q$ is homotopic to a homeomorphism [4]. The equivalence of $G(K \times Q)/\mathrm{Homeo}(K \times Q)$ with a union of components of $\mathrm{Wh}(K)$ therefore globalizes this result and gives the "topological invariance of higher torsions," viz., that the composition $\mathrm{Homeo}(K) \hookrightarrow G(K) \to \mathrm{Wh}(K)$ is null-homotopic. I should add that the equivalence of $G(K \times Q)/\mathrm{Homeo}(K \times Q)$ with a then-hypothetical higher simple homotopy theory was predicted to me a couple of years ago by F. S. Quinn.

A further application of the stable equivalence $\mathcal{P}(M) \simeq \Omega\,\mathrm{Wh}(M)$ is the calculation[1]

$$\pi_1\,\mathrm{Wh}(M) \approx \pi_0\mathcal{P}_{\mathrm{PL}}(M) \approx \pi_0\mathcal{P}_{\mathrm{Diff}}(M) \approx \mathrm{Wh}_2(\pi_1 M) \oplus \mathrm{Wh}_1^+(\pi_1 M; \mathbf{Z}_2 \times \pi_2 M)\ ,$$

[1] Assuming the first Postnikov invariant $k_1 \in H^3(\pi_1 M; \pi_2 M)$ of $M$ vanishes—see the footnote in § 10.

the second and third equivalences by [2] and [11], respectively. (One could also prove this directly.) Here $\mathrm{Wh}_2(\pi_1)$ is a certain quotient of $K_2\mathbf{Z}[\pi_1]$; $\mathrm{Wh}_1^+(\pi_1; \mathbf{Z}_2 \times \pi_2)$ is described in Section 10.

The calculation of $\pi_1 \mathrm{Wh}(M)$ shows that higher simple homotopy theory is not a functor of fundamental groups alone, as is the classical theory. Probably the best general statement about the dependence of $\mathrm{Wh}(K)$ on $K$ is that if $K \to K'$ is $k$-connected, $k > 1$, then the induced map $\mathrm{Wh}(K) \to \mathrm{Wh}(K')$ is $(k-1)$-connected. This is proved in Section 7 by a homotopy excision argument. In fact, we show that Wh satisfies an excision property formally analogous to excision in ordinary homotopy theory. Consequently there is a stable simple homotopy theory $s_*(K)$ which is a generalized homology theory. R. K. Lashof had previously constructed this theory in terms of pseudo-isotopy spaces (using our stability result on $\mathcal{P}(M) \to \mathcal{P}(M \times I)$). Also, using results of Morlet and Chenciner he calculated the coefficient groups: $s_i(S^0) \approx \pi_{i-2}\mathcal{P}_{\mathrm{Diff}}(D^n)$ for $n$ large. Since $\mathcal{P}_{\mathrm{PL}}(D^n)$ is contractible by the Alexander trick, the effect of this is that $s_*$ measures the difference between $\mathcal{P}_{\mathrm{Diff}}$ and $\mathcal{P}_{\mathrm{PL}}$, in the stable range. It is known (see [16], [18]) that the first non-vanishing $s_i(S^0)$ occurs for $i = 3$.

By way of example we give in the last section of the paper an easy construction of some non-trivial elements of $\pi_1 \mathrm{Wh}(K)$ whenever $\pi_1 K \neq 0$, together with a way of injecting these into $\pi_{n+1}\mathrm{Wh}(K \times T^n)$, $T^n$ the $n$-torus, for any $n \geqq 1$.

In a later paper we intend to clarify the relationship between $\mathrm{Wh}(K)$ and higher algebraic $K$-theory by defining higher Whitehead groups $\mathrm{Wh}_i(\pi_1 K)$ and natural maps $\pi_{i-1}\mathrm{Wh}(K) \to \mathrm{Wh}_i(\pi_1 K)$ and $K_i\mathbf{Z}[\pi_1 K] \to \mathrm{Wh}_i(\pi_1 K)$. The best one could hope would be for these two maps to be surjective (they are for $i = 1, 2$), but even this seems unlikely in general. Similar remarks apply to a second family of functors $\mathrm{Wh}_i^+(\pi_1 K; \mathbf{Z}_2 \times \pi_2 K)$ which extend the summand $\mathrm{Wh}_1^+(\pi_1 K; \mathbf{Z}_2 \times \pi_2 K)$ of $\pi_1 \mathrm{Wh}(K)$. And these two invariants are just the beginning.

I am indebted to T. A. Chapman and F. S. Quinn for some stimulating conversations about the material of this paper.

## 2. PL fibrations

We will be working in the PL category. All polyhedra will be subpolyhedra of $\mathbf{R}^\infty$ though we usually neglect to mention the specific embeddings in $\mathbf{R}^\infty$. For simplicity we will consider only finite polyhedra. The extension to the locally finite case (with proper maps) is straightforward; the result would be a "higher infinite simple homotopy theory," generalizing [13].

This section contains preliminary material on fibrations in the PL category, the main results being the local characterization given in 2.1 and its global form in 2.5.

We begin by describing a completely general way of decomposing any PL map $\pi: E \to B$ into elemental blocks over the simplices of some triangulation of $B$. Given a chain of PL maps

$$L_0 \xrightarrow{f_1} L_1 \longrightarrow \cdots \xrightarrow{f_k} L_k ,$$

the *iterated mapping cylinder* $M(f_1, \cdots, f_k)$ is defined inductively to be the ordinary mapping cylinder of the composition $M(f_1, \cdots, f_{k-1}) \to L_{k-1} \xrightarrow{f_k} L_k$, where the unmarked arrow is the obvious projection. Thus for $k = 1$ we have the usual mapping cylinder, for $k = 2$ the mapping cylinder of $M(f_1) \to L_1 \xrightarrow{f_2} L_2$, etc. (Note that mapping cylinders are well-defined PL objects by 9.5 of [7]; see also [1].) By an *iterated mapping cylinder decomposition* of $\pi: E \to B$ we mean: Over each simplex $\sigma$ of some triangulation of $B$, $\pi^{-1}(\sigma)$ is given as an iterated mapping cylinder $M(f_1^\sigma, \cdots, f_k^\sigma)$, where $k = \dim \sigma$, such that $\pi: \pi^{-1}(\sigma) \to \sigma$ is identified with the standard projection $M(f_1^\sigma, \cdots, f_k^\sigma) \to \Delta^k$. Moreover, these structures are to be compatible when we pass from $\sigma$ to simplices of $\partial\sigma$.

To obtain an iterated mapping cylinder decomposition of an arbitrary PL map $\pi: E \to B$, choose triangulations of $E$ and $B$ (which we still call $E$ and $B$) and barycentric subdivisions $E'$ and $B'$ such that $\pi: E \to B$ and $\pi: E' \to B'$ are simplicial. Let $b_0, \cdots, b_k$ be barycenters of simplices $\beta_0 > \cdots > \beta_k$ of $B$ and let $L_i = \pi^{-1}(b_i)$. Define PL maps $f_{i+1}: L_i \to L_{i+1}$ by sending a barycenter $e_i \in L_i$ of a simplex $\varepsilon_i$ of $\pi^{-1}(\beta_i)$ to the barycenter $e_{i+1}$ of $\varepsilon_{i+1} = \varepsilon_i \cap \pi^{-1}(\beta_{i+1})$, and extending linearly. Then $\pi^{-1}(b_0 \cdots b_k)$ is identified naturally with $M(f_1, \cdots, f_k)$, and $\pi: \pi^{-1}(b_0 \cdots b_k) \to b_0 \cdots b_k$ is the projection $M(f_1, \cdots, f_k) \to \Delta^k$. (To see this it suffices to consider the case that $E$ and $B$ are simplices and $\pi$ is simplicial.)

PROPOSITION 2.1. *Suppose $\pi: E \to B$ is a* PL *map which is a Serre fibration (briefly, a* PL *fibration). Then the maps $f_j^\sigma$ in any iterated mapping cylinder decomposition of $\pi$ are contractible mappings,* i.e., *all point-inverses $(f_j^\sigma)^{-1}(*)$ are non-empty and contractible. Conversely, if in some iterated mapping cylinder decomposition of $\pi$ all the maps $f_j^\sigma$ are contractible mappings, then $\pi$ is a* PL *fibration.*

*Proof.* We first show that the $f_j^\sigma$'s must be contractible mappings if $\pi$ is a PL fibration. For this it suffices to choose $E = M(f)$ for $f: L_0 \to L_1$. If $x \in L_1$, then $M(f \mid f^{-1}(x))$ is a cone $C$ on $f^{-1}(x)$, and we apply the covering homotopy property in the diagram

$$\begin{array}{ccc} & \nearrow & M(f) \\ & & \downarrow \pi \\ C & \xrightarrow{\pi} & [0, 1] \end{array}$$

to the homotopy $C \xrightarrow{\pi} [0, 1] \xrightarrow{h_s} [0, 1]$, $h_s(t) = \min(1 - s, t)$, where $\pi^{-1}(t) = L_0$ for $0 \leqq t < 1$ and $\pi^{-1}(1) = L_1$. The effect is to produce a continuous family of contractions of $f^{-1}(x)$ in each slice $L_0 = \pi^{-1}(t)$, $0 \leqq t < 1$. By continuity, the contraction must take place in a neighborhood of $f^{-1}(x) \subset L_0$ for $(s, t)$ near $(0, 1)$, so $f^{-1}(x)$ must actually be contractible in itself.

For the converse we start with an iterated mapping cylinder decomposition of $\pi$, with respect to some triangulation of $B$. Since being a PL fibration is a local property with respect to $B$, it will suffice to show that $\pi$ is a PL fibration over the star, in the barycentric subdivision of $B$, of each vertex $v \in B$. (These stars can be enlarged slightly by isotopy so that their interiors still cover $B$.) Thus we may assume $B = \text{star}(v) = C(A)$, the cone on $A = \text{link}(v)$, and, by induction on $\dim B$, that on $L = \pi^{-1}(A)$, $\pi$ is a PL fibration. Moreover, $E = \pi^{-1}(B)$ has the structure of a mapping cylinder $M(f)$, where $f: L \to K = \pi^{-1}(v)$ is such that its restriction to each fiber $\pi^{-1}(a)$ in $L$ is a contractible mapping (being one of the $f_j^{\sigma}$'s in the given iterated mapping cylinder decomposition of $\pi$ or else the identity). Thus the proposition is reduced to:

LEMMA 2.2. *Given a* PL *fibration* $\pi: L \to A$ *and a fiber-preserving* PL *contractible mapping* $(f, \pi): L \to K \times A$, *then the natural projection* $\bar{\pi}: M(f) \to C(A)$ *to the cone on* $A$ *is a* PL *fibration.*

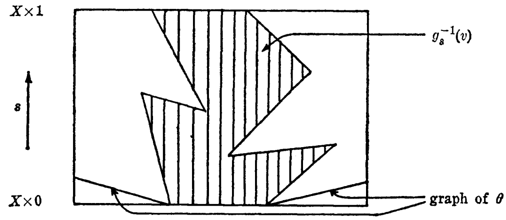
*Proof.* Let $F: M(f) \to K \times C(A)$ be the obvious map. We claim:

(H) $F$ has a homotopy inverse $G$ for which there is a homotopy $H_u$: $M(f) \to M(f)$ from the identity to $GF$ such that $\bar{\pi} H_u = \bar{\pi}$ and such that $H_u$ is fixed on $K = \bar{\pi}^{-1}(v)$, where $v$ is the cone point of $C(A)$.

Assuming this, we can proceed as follows. Given a lifting problem

$$\begin{array}{ccc} & \overset{\tilde{g}_0}{\nearrow} & M(f) \\ & & \downarrow \bar{\pi} \\ X & \xrightarrow[g_s]{} & C(A) \end{array}$$

let $\theta: X \to I$ be such that $\theta^{-1}(0) = g_0^{-1}(v)$ and $g_s(x) \neq v$ if $s \leqq \theta(x) \neq 0$.




Define $g'_s: X - g_0^{-1}(v) \longrightarrow C(A) - \{v\}$ to be $g_s$ if $s \leqq \theta$ and $g_\theta$ if $s \geqq \theta$. This lifts to $\tilde{g}'_s: X - g_0^{-1}(v) \longrightarrow M(f) - \tilde{\pi}^{-1}(v)$ since $\tilde{\pi} \mid M(f) - \tilde{\pi}^{-1}(v)$ is a PL fibration. Then define $\tilde{g}_s: X \longrightarrow M(f)$ for $s \leqq \theta$ by $\tilde{g}_s = H_{s/\theta}\tilde{g}'_s$, where if necessary $\theta$ is replaced by a smaller function so that each arc $g_s(x)$, $0 \leqq s \leqq \theta(x)$, approaches the constant arc $g_0(x_0)$ as $x$ approaches $x_0 \in g_0^{-1}(v)$. (This assures that $\tilde{g}_s$ is a continuous extension of $\tilde{g}_0$.) We are left with the problem of lifting $g_s$ for $\theta \leqq s \leqq 1$ with an initial position $\tilde{g}_\theta$ which factors through the trivial fibration $K \times C(A) \longrightarrow C(A)$. This can certainly be done. □

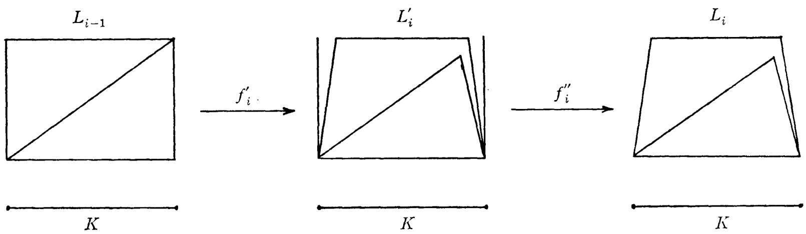
Towards proving (H) we first reprove a result of M. M. Cohen [7]:

PROPOSITION 2.3. *A contractible mapping $f: L \longrightarrow K$ is a simple homotopy equivalence.*

Note that this then holds also for the restrictions $f \mid f^{-1}(K')$, $K'$ a subpolyhedron of $K$. Hence contractible mappings are closed under composition.

*Proof of* 2.3. Choose triangulations of $K$ and $L$ such that $f$ is simplicial. Since $f$ is surjective, we can regard it as a kind of collapsing map: for each simplex $\sigma$ of $K$, $f$ collapses the subcomplex $f^{-1}(\sigma)$ of $L$ onto $\sigma$. Thinking of $f$ in this way, we can factor it as $L \xrightarrow{f_0} L_0 \xrightarrow{f_1} L_1 \longrightarrow \cdots \longrightarrow K$ where $f_i$ collapses the inverse image of the $i$-skeleton $K^i$ of $K$ to $K^i$. We will show that each $f_i$ is a simple homotopy equivalence, which implies that their composition $f$ is also.

Let $\sigma$ be an $i$-simplex of $K^i = L_i^i$. Since $f$ is simplicial we can choose an $i$-cell of $L_{i-1}$ mapped isomorphically onto $\sigma$ by $f_i$. Identify $\sigma$ with this $i$-cell. Then since $f_i$ is a contractible mapping, there is a deformation retraction of $f_i^{-1}(\sigma)$ onto $\sigma$, rel $f_i^{-1}(\partial\sigma)$, preserving the "fibers" $f_i^{-1}(x)$. Using these deformations for the various $\sigma$ in $K^i$ we can pass continuously from $L_{i-1}$ to $L'_i = L_i \cup f_i^{-1}(K^i)$ by a homotopy of the attaching maps of the cells in $f^{-1}(K - K^i)$, thinking of $L_{i-1}$ as a CW complex (with PL attaching maps). The resulting homotopy equivalence $f'_i: L_{i-1} \longrightarrow L'_i$ is therefore simple. The collapse $f''_i: L'_i \longrightarrow L_i$ is clearly a homotopy equivalence; it is simple since the collapsing splits into the disjoint collapses $f_i^{-1}(\sigma) \longrightarrow \sigma$ each of which takes place over a contractible, hence simply-connected, part of $L_i$. The composition $f''_i f'_i$ is homotopic to $f_i$ (in fact, by a homotopy which is arbitrarily small with respect to projection on $K$), so $f_i$ is a simple homotopy equivalence. □




Returning to assertion (H), consider first the unfibered case $A$ = point. Retracing the steps in the proof that $f$: $L \to K$ is a homotopy equivalence, we can construct $g$: $K \to L$ and a homotopy $h_u$: $L \to L$ from the identity to $gf$ such that $fh_u$ is arbitrarily close to $f$. Letting $fh_u$ approach $f$ as we slide down $M(f)$ to $K \subset M(f)$ we obtain $G$: $K \times I \to M(f)$ and a homotopy $H_u$: $M(f) \to M(f)$ from the identity to $GF$ (preserving the projection $\bar{\pi}$ to $I$) such that $H_u$ is the identity on $K \subset M(f)$.

In the fibered case when $f$: $L \to K$ is replaced by $(f, \pi)$: $L \to K \times A$, begin with $g$: $K \times A \to L$ and $h_u$: $L \to L$ as above. These two maps commute with projection to $A$ up to homotopy. Applying the covering homotopy property to this homotopy, we can deform $g$ and $h_u$ so that they commute with projection to $A$ exactly. Then construct $G$ and $H_u$ as in the unfibered case. □

As a consequence of 2.1, being a PL fibration is a purely local property; i.e., given $\pi$: $E \to B$, if every point of $E$ has a neighborhood $U$ such that $\pi \mid U$: $U \to \pi(U)$ is a PL fibration, then $\pi$ is a PL fibration. Also, $\pi$ is a PL fibration if over one-dimensional subpolyhedra of $B$ it is a PL fibration.

The next lemma, which follows from 2.1, will be useful later.

LEMMA 2.4. *Let* $K \to B$ *and* $L \to B$ *be* PL *fibrations,* $K' \subset K$ *a subfibration, and* $\phi$: $K' \to L$ *a fiber map. Then* $L \cup_\phi K \to B$ *is a* PL *fibration.*

*Proof.* A mapping cylinder in an iterated mapping cylinder decomposition of $L \cup_\phi K$ can be chosen of the form $M(g \cup_\phi f)$, coming from a commutative diagram

$$\begin{array}{ccc} & K_0 \xrightarrow{f} K_1 & \\ \nearrow & & \nearrow \\ K'_0 \xrightarrow{f'} K'_1 & & \\ \searrow \phi_0 & & \searrow \phi_1 \\ & L_0 \xrightarrow{g} L_1 & \end{array}$$

with $f$, $f'$, and $g$ contractible mappings. To check that $g \cup_\phi f$ is a contractible mapping, consider first a point $x \in L_1$, so that

$$(g \cup_\phi f)^{-1}(x) = g^{-1}(x) \cup_{\phi_0} f^{-1}(\phi_1^{-1}(x)) .$$

Now $f^{-1}(\phi_1^{-1}(x))$ deforms into $(f')^{-1}(\phi_1^{-1}(x))$ since both are homotopy equivalent to $\phi_1^{-1}(x)$ under $f$ and $f'$, respectively. So $(g \cup_\phi f)^{-1}(x)$ deforms into $g^{-1}(x)$ which is contractible by hypothesis. Hence $(g \cup_\phi f)^{-1}(x)$ is contractible if $x \in L_1$. In the opposite case $x \notin L_1$, $(g \cup_\phi f)^{-1}(x) = f^{-1}(x)$ is also contractible. □

We define now a classifying space for PL fibrations.

*Definition.* The simplicial space $\mathcal{S}$ has as a typical $k$-simplex a finite subpolyhedron $L \subset \mathbf{R}^\infty \times \Delta^k$ such that the projection $L \to \Delta^k$ is a PL fibration. The face and degeneracy maps are the obvious ones induced by restriction and projection of $\Delta^k$ to its $(k-1)$-faces.

Using the local characterization in 2.1 it is clear that $\mathcal{S}$ is a Kan complex. Any PL fibration $E \to B$ (with compact fibers) is induced from a map $B \to \mathcal{S}$. Homotopic maps correspond to "homotopic" fibrations, i.e., fibrations which are restrictions to $B \times \{0\}$ and $B \times \{1\}$ of a fibration over $B \times I$. In this sense $\mathcal{S}$ classifies PL fibrations.

Heuristically, $\mathcal{S}$ can be thought of as "the space of all finite polyhedra", or more precisely, as the (PL) singular complex of this "space". For if $\pi\colon E \to B$ is a PL fibration, then the covering homotopy property says somehow that the fibers $\pi^{-1}(x)$ are polyhedra which vary "continuously" with $x$. For example, one might ask, when can one finite polyhedron be deformed "continuously" into another, or in other words, what are the arc-components of $\mathcal{S}$? By 2.1 this amounts to asking when two polyhedra $L_0$ and $L_1$ can be joined by a chain of contractible mappings. By 2.3, $L_0$ and $L_1$ must have the same simple homotopy type. Conversely, since elementary collapses are contractible mappings, we see that the arc-components of $\mathcal{S}$ are exactly the simple homotopy types of finite polyhedra.

Recall the definition of the category $\mathcal{C}$ and its classifying space $B\mathcal{C}$ from Section 1. We can define a map $B\mathcal{C} \to \mathcal{S}$ by sending the $k$-simplex $L_0 \xrightarrow{f_1} L_1 \to \cdots \xrightarrow{f_k} L_k$ of $B\mathcal{C}$ to its iterated mapping cylinder $M(f_1, \cdots, f_k)$ embedded in $\mathbf{R}^\infty \times \Delta^k$ by general position, preserving the projection to $\Delta^k$.

PROPOSITION 2.5. *$B\mathcal{C} \to \mathcal{S}$ is a homotopy equivalence.*

*Proof.* We will be a little sketchy, since the result will not be used essentially in the rest of the paper. Let $L$ be a $k$-simplex of $\mathcal{S}$ with a chosen lifting $L^0$ to $B\mathcal{C}$ over $\partial\Delta^k$, representing an element of $\pi_k(\mathcal{S}, B\mathcal{C})$. As in the paragraph preceding 2.1 we can decompose $L$ into iterated mapping cylinders via triangulations of $L$ and $\Delta^k$ such that $L \to \Delta^k$ is simplicial, thereby obtaining a lift $L^1$ to $B\mathcal{C}$ over $\Delta^k$. The problem is to arrange things so that $L^1$ is homotopic in $B\mathcal{C}$ to $L^0$ over $\partial\Delta^k$.

$L^0$ is given as a union of iterated mapping cylinders $M(f_1^\sigma, \cdots, f_j^\sigma)$, one for each $j$-simplex $\sigma$ of a triangulation of $\partial\Delta^k$. Triangulate the vertices $L_i^\sigma$ of $M(f_1^\sigma, \cdots, f_j^\sigma)$ so that each $f_i^\sigma$ is simplicial. This triangulation depends on $\sigma$, but we can suppose that if $\tau$ is a face of $\sigma$, then the $\tau$-triangulations subdivide the $\sigma$-triangulations. Since the $f_i^\sigma$'s are simplicial, each $M(f_1^\sigma, \cdots, f_j^\sigma)$ decomposes into iterated mapping cylinders of the restrictions of the $f_i^\sigma$'s to simplices of the $L_i^\sigma$'s. Now triangulate $L$ so that all these iterated mapping subcylinders are subcomplexes and form $L^1$ from this triangulation.

To begin constructing a homotopy in $B\mathcal{C}$ from $L^0$ to $L^1$ we first subdivide $\partial\Delta^k$ so $L^0$ and $L^1$ have the same vertices $L_i^\sigma$ and differ only in the maps $L_i^\sigma \to L_{i+1}^\sigma$, say $f_i^\sigma$ for $L^0$ and $g_i^\sigma$ for $L^1$. The change in $L^0$ resulting from subdividing $\partial\Delta^k$ can clearly be realized by a homotopy of $L^0$ in $B\mathcal{C}$.

Next, we construct homotopies from the $g_i^\sigma$'s to the $f_i^\sigma$'s, inductively over the skeletons of our triangulation of $L_0^\sigma$. On the restriction $M(f_1^\sigma, \cdots, f_j^\sigma) \mid \Delta^l$ to a simplex $\Delta^l$ of $L_0^\sigma$, assuming $f_i^\sigma$'s and $g_i^\sigma$'s already agree on $M(f_i^\sigma, \cdots, f_i^\sigma) \mid \partial\Delta^l$, we can perform the well-known Alexander trick of radially coning off $g_i^\sigma$ to $f_i^\sigma$. That is, for successively smaller concentric simplices $\Delta_s^l \subset \Delta^l$, we use the maps $g_i^\sigma$ on $M(f_1^\sigma, \cdots, f_j^\sigma) \mid \Delta_s^l$ and the maps $f_i^\sigma$ on $M(f_1^\sigma, \cdots, f_j^\sigma) \mid \Delta^l - \Delta_s^l$. This deformation of the $g_i^\sigma$'s on $M(f_1^\sigma, \cdots, f_j^\sigma) \mid \Delta^l$ extends naturally via a regular neighborhood of $\Delta^l$ in $L_0^\sigma$ to a deformation on all of $M(f_1^\sigma, \cdots, f_j^\sigma)$, and then we continue with $(l+1)$-simplices of $L_0^\sigma$. In the end we get a deformation of $M(f_1^\sigma, \cdots, f_j^\sigma)$ to $M(g_1^\sigma, \cdots, g_j^\sigma)$ as simplices in $B\mathcal{C}$ (the underlying space $M(f_1^\sigma, \cdots, f_j^\sigma)$ is unchanged during the deformation).

It remains to piece together these deformations over the various simplices $\sigma$ of $\partial\Delta^k$, the trouble being that the deformation we have constructed over $\sigma$ depends on the triangulation of $L_0^\sigma$. But for faces $\tau$ of $\sigma$ we will have chosen the triangulation of $L_0^\tau$ to be a subdivision of the triangulation of the appropriate $L_i^\sigma$, so further applications of the Alexander trick will provide a way to glue everything together. Details are left to the diligent reader. $\square$

*Remark.* The results of this section have analogues for PL quasi-fibrations, in which "contractible mappings" are replaced by "homotopy equivalences" throughout.

### 3. The simple homotopy functor $\mathcal{S}(K)$

Recall that a $k$-simplex of $\mathcal{S}$ is a PL fibration $L \hookrightarrow \mathbf{R}^\infty \times \Delta^k \to \Delta^k$. It will be convenient to label such a simplex by its fibers $L_t = L \cap \mathbf{R}^\infty \times \{t\}$,

$t \in \Delta^k$, and to think of $L_t$ as a $k$-parameter family of polyhedra.

*Definition.* For a fixed polyhedron $K \subset \mathbf{R}^\infty$, let $\mathcal{S}_K$ be the subcomplex of $\mathcal{S}$ consisting of those $L_t$'s of the homotopy type of $K$, and let $\mathcal{S}(K) \subset \mathcal{S}_K$ be the subcomplex consisting of $L_t$'s which contain $K$ as a deformation retract.

$\mathcal{S}_K$ is just a union of components of $\mathcal{S}$. Restricting 2.5 to the subcategories $\mathcal{C}(K) \subset \mathcal{C}_K \subset \mathcal{C}$ (see Section 1) we obtain:

PROPOSITION 3.1. *The maps $B\mathcal{C}(K) \longrightarrow \mathcal{S}(K)$ and $B\mathcal{C}_K \longrightarrow \mathcal{S}_K$ are homotopy equivalences.* □

Thus $\mathcal{S}(K)$ is, up to homotopy, the space called $\mathrm{Wh}(K)$ in Section 1.

PL fibrations and homotopy fibrations with fibers of the homotopy type of $K$ are classified by $\mathcal{S}_K$ and $BG(K)$, respectively. Here $G(K)$ is the $H$-space of homotopy equivalences $K \longrightarrow K$. Since PL fibrations are homotopy fibrations, there is a forgetful map $\mathcal{S}_K \longrightarrow BG(K)$.

PROPOSITION 3.2. *The homotopy fiber of $\mathcal{S}_K \longrightarrow BG(K)$ is $\mathcal{S}(K)$.*

*Proof.* The fiber $\mathcal{F}(K)$ of $\mathcal{S}_K \longrightarrow BG(K)$ consists of pairs $(L_t, f_t)$, where $L_t$ is a simplex in $\mathcal{S}_K$ and $f_t\colon K \longrightarrow L_t$ is a family of homotopy equivalences. In fact, this datum is exactly a fiber-homotopy trivialization of the fibration $L_t \mapsto t$. If $(L_t, f_t) \in \mathcal{F}(K)$, the mapping cylinder $M(f_t)$ lies in $\mathcal{S}(K)$, since by 2.4, $M(f_t) \mapsto t$ is a PL fibration. The correspondence $(L_t, f_t) \mapsto M(f_t)$ gives a map $\mathcal{F}(K) \longrightarrow \mathcal{S}(K)$ which has as a homotopy inverse the map $L_t \mapsto (L_t, K \hookrightarrow L_t)$, as one can easily check. □

*Remark.* At the $\pi_0$ level the fibration sequence

$$G(K) \longrightarrow \mathcal{S}(K) \longrightarrow \mathcal{S}_K \longrightarrow BG(K)$$

is just the well-known representation of the set $\pi_0\mathcal{S}_K$ of simple homotopy types within the homotopy type of $K$ as the orbit space of $\pi_0\mathcal{S}(K) \approx \mathrm{Wh}_1(\pi_1 K)$ under the action of $\pi_0 G(K)$. (See [8, § 24].)

PROPOSITION 3.3. *$\mathcal{S}(K)$ is a covariant functor of $K$, from the homotopy category of finite polyhedra to the homotopy category of (simplicial) infinite loopspaces.*

COROLLARY 3.4. *If $K$ is contractible then so is $\mathcal{S}(K)$.*

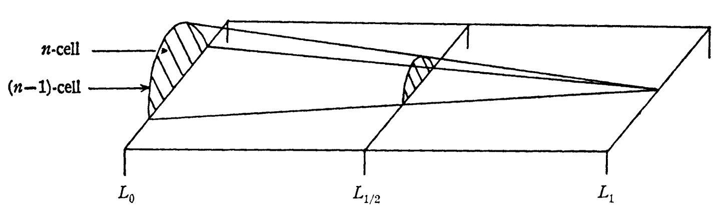
This is because if $K$ is a point we can just cone off all $L_t$ in $\mathcal{S}(K)$ uniformly.

*Proof of* 3.3. For a map $f: K \to K'$ and $L_t \in \mathcal{S}(K)$, set $f_*(L_t) = K' \cup_f L_t$. By 2.4 this lies in $\mathcal{S}(K')$. Also by 2.4 a homotopy of $f$ induces a homotopy of $f_*(L_t)$ in $\mathcal{S}(K')$.

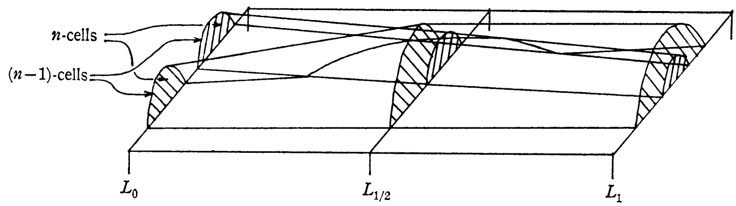
The composition operation "+" in $\mathcal{S}(K)$ is "disjoint union with the two copies of $K$ identified." To achieve the disjunction we will make essential use of the given embeddings in $\mathbf{R}^\infty$. Write $\mathbf{R}^\infty$ as $\mathbf{R}_1^\infty \times \mathbf{R} \times \mathbf{R}_2^\infty$. For each rectangle $R = (a_1, b_1) \times (a_2, b_2) \times \cdots$ in $\mathbf{R}_2^\infty$, with $0 \leqq a_i < b_i \leqq 1$ and $(a_i, b_i) = (0, 1)$ for sufficiently large $i$, let the cone $C(R)$ be the union of rays in $\mathbf{R} \times \mathbf{R}_2^\infty$ from the origin $0 \times 0$ through points in $1 \times R$. We can assume that all $L_t \in \mathcal{S}(K)$ lie in $\mathbf{R}_1^\infty \times C((0, 1) \times (0, 1) \times \cdots)$ and that $L_t \cap \mathbf{R}_1^\infty \times 0 \times 0 = K$. Now to form $L_t + L_t'$, first compress $L_t$ linearly into $\mathbf{R}_1^\infty \times C((0, 1/2) \times (0, 1) \times \cdots)$ and $L_t'$ linearly into $\mathbf{R}_1^\infty \times C((1/2, 1) \times (0, 1) \times \cdots)$. Then set $L_t + L_t'$ equal to the union of the shifted $L_t$ and $L_t'$, which now intersect only in $K$. This sum operation clearly makes $\mathcal{S}(K)$ a homotopy associative $H$-space. In fact, $\mathcal{S}(K)$ now has an obvious "little cubes" structure, making it into an infinite loopspace [12]. For example, homotopy commutativity follows by the familiar argument using the first two coordinates of $\mathbf{R}_2^\infty$ to slide around in. □

## 4. Families of PL cell complexes

For a more detailed study of $\mathcal{S}(K)$ we will need to replace it by a homotopy equivalent space, whose $k$-simplices are $k$-parameter families of polyhedra $L_t$ with chosen decompositions into PL cells. Each $L_t$ will be constructed from $K$ by successively attaching cells of various dimensions by PL attaching maps. The idea in defining a $k$-parameter family is to allow two kinds of operations: homotopies of attaching maps, and the "collapsing" of certain collections of cells by coning them off to a point. For example, an elementary collapse $L_0 \searrow L_1$ can be realized by a one-parameter family $L_t$, $0 \leqq t \leqq 1$, as in the following picture.




Another one-parameter family, demonstrating a homotopy of attaching maps, is the following.




As this example illustrates, in parametrized settings it is unreasonable to require attaching maps to be skeletal in each parameter slice. Indeed, it is impossible to go from $L_0$ to $L_1$ by a homotopy through skeletal attaching maps. Thus the PL cell complexes we permit may not be decomposed as CW complexes, though the underlying spaces are nice polyhedra.

The full definition of a $k$-parameter family of PL cell complexes is somewhat complicated:

(i) Set $L_t^{(0)} = K$, $t \in \Delta^k$.

(ii) Inductively, build $L_t^{(i)}$ from $L_t^{(i-1)}$ by

(a) attaching a PL cell $e_t^{n_i}$ via a PL $k$-parameter family of maps $\varphi_t^i$: $S^{n_i-1} \longrightarrow L_t^{(i-1)}$ and then

(b) collapsing $e_t^{n_i}$ to a point $p_t^i \in L_t^{(i-1)}$ over some subcomplex of $\{t \in \Delta^k \mid \varphi_t^i(S^{n_i-1}) = p_t^i\}$.

Globally, $L_t = \bigcup_i L_t^{(i)}$ is assumed to satisfy:

(iii) For each collapse point $p$, the cells of $L_t$ collapsing to $p$ are attached consecutively, as a block, without intervening non-collapsing cells.

(iv) The underlying polyhedra of the family $L_t$ form a $k$-simplex of $\mathcal{S}(K)$. That is,

(a) $L_t \mapsto t$ is a PL fibration.

(b) $K \hookrightarrow L_t$ is a homotopy equivalence.

A family of PL cell complexes constructed according to these rules we call a *basic* $k$-parameter family. A *general* family consists of basic families over the simplices of some subdivision of $\Delta^k$. Thus in a general $k$-parameter family the cells need not be attachable in the same order all over $\Delta^k$. (The actual order in which cells are attached is not part of the data of $L_t$, only the decomposition of $L_t - K$ into cells.)

*Definition.* For $0 \leqq i \leqq j \leqq \infty$, $\mathcal{S}_i^j(K)$ is the simplicial space whose $k$-simplices are general $k$-parameter families of PL cell complexes $L_t$ such that all cells of $L_t - K$ have dimensions in the range $[i, j]$. The face and degeneracy maps in $\mathcal{S}_i^j(K)$ are the obvious ones.

$\mathcal{S}_i^j(K)$ is clearly a Kan complex, since we include general $k$-parameter families.

*Remarks.* (1) If the normalization condition (iii) were not assumed, it could easily be achieved by a small homotopy of attaching maps.

(2) In the presence of (iii), condition (iv. a) is equivalent to:

(iv. a′) Near each collapse point $p$, the block $C_t(p)$ of cells which collapse to $p$ attaches to $L_t^{(i)}$ (for appropriate $i$) inside a small contractible neighborhood $N_t \subset L_t^{(i)}$, such that $N_t$ is a deformation retract of $N_t \cup C_t(p)$.
Thus $\mathcal{S}_i^j(K)$ can be defined without mention of PL fibrations.

(3) Any family $L_t \in \mathcal{S}_i^j(K)$ can be homotoped in $\mathcal{S}_i^j(K)$ to a family in which all collapsing is *split*, i.e., of the form $L_t \vee C_t \to L_t$, where $C_t$ is contractible. To achieve this, first use the deformation retractions $N_t \cup C_t(p) \to N_t$ of (iv. a′) to push all non-collapsing cells off $C_t(p)$, then use a contraction of $N_t$ to make $C_t(p)$ attach to $N_t$ at a point.

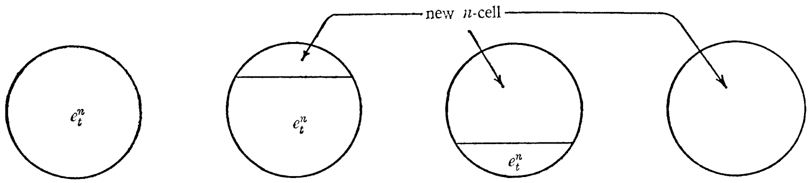
(4) One can always tell by inspection when a non-zero-dimensional cell of $L_t - K$ collapses: it shrinks to a point. But when two or more 0-cells merge into one, there is no intrinsic way to tell which 0-cells collapse and which is the survivor. So let us agree as a convention that collapsing 0-cells are distinguished. It is easy to achieve this by a homotopy of the family $L_t$: Let $e_0^t \in L_t$, $t \in \Delta^l \subset \Delta^k$, be a family of 0-cells. Extend $e_t^0$ to a point $p_t \in L_t$ for $t \in \Delta^k$, and attach a line segment $[0, 1]_t$ (with the usual cell decomposition) to $L_t$ by identifying $0_t$ with $p_t$. Then collapse $[0, 1]_t$ to $1_t$ over $\Delta^l$. The effect is to replace $e_t^0$ by $1_t$ over $\Delta^l$ (and $1_t$ collapses over any part of $\Delta^l$ where $e_t^0$ collapsed). Moreover, any 0-cells of the original $L_t$ which abut the new $1_t$ over $\Delta^l$ are now distinguished as collapsing 0-cells, since $1_t$ survives near $\Delta^l$.

PROPOSITION 4.1. *The natural map $\mathcal{S}_0^\infty(K) \to \mathcal{S}(K)$ obtained by ignoring cell decompositions is a homotopy equivalence.*

*Proof.* Represent an element of $\pi_k(\mathcal{S}(K), \mathcal{S}_0^\infty(K))$ by a family $L_t \in \mathcal{S}(K)$, $t \in \Delta^k$, having a given lift $L_t^0 \in \mathcal{S}_0^\infty(K)$ over $\partial\Delta^k$. Choose a triangulation $T$ of $L = \bigcup_t L_t$ such that the families of cells of $L_t^0$ are subcomplexes and such that the projection $\pi: L \to \Delta^k$ is linear on each simplex of $T$. Intersecting $T$ with $L_t$ gives each individual $L_t$ a PL cell complex structure, but the resulting family $L_t^1$ of PL cell complexes may not lie in $\mathcal{S}_0^\infty(K)$ because cells may not collapse to points but to cells of positive dimension. However, if we perturb $\pi$ slightly so that for each simplex $\Delta^l$ in $T$, $\pi(\Delta^l)$ has maximal dimension, namely min $(k, l)$, then the collapsing will be of the sort in $\mathcal{S}_0^\infty(K)$. By repeated application of 4.3 below, this perturbation can be done in $\mathcal{S}(K)$, and over $\partial\Delta^k$ it is clear that there is induced a deformation of $L_t^0$ in $\mathcal{S}_0^\infty(K)$. The final step is then given by:

LEMMA 4.2. *Let the family* $L_t^1 \in \mathcal{S}_0^\infty(K)$ *be a subdivision of the family* $L_t^0 \in \mathcal{S}_0^\infty(K)$. *Then* $L_t^0$ *and* $L_t^1$ *are homotopic in* $\mathcal{S}_0^\infty(K)$.

*Proof.* Proceeding inductively over simplices of the parameter domain, we need only to do a relative construction for $t \in \Delta^k$. Consider the following homotopy within a cell $e_t^n$ of $L_t$ ($n > 0$). In the first half of the homotopy we split $e_t^n$ into two $n$-cells and an $(n-1)$-cell by intersecting it with a family of parallel hyperplanes:




In the second half of the homotopy we reverse the process, but with $e_t^n$ replaced by its subdivision in $L_t^1$. Doing this simultaneously for all cells of $L_t^0$ gives a homotopy from $L_t^0$ to $L_t^1$ in $\mathcal{S}_0^\infty(K)$. In order to make this homotopy fixed over $\partial\Delta^k$ (where by assumption $L_t^0 = L_t^1$) we can first deform $L_t^0$ to be constant on the segments $t \times [0, \varepsilon]$ of a collar neighborhood $\partial\Delta^k \times [0, \varepsilon]$, then damp the homotopy down to zero along these segments. □

LEMMA 4.3. *Let* $\pi\colon L \to \Delta^k$ *be a* PL *fibration, and let* $\pi'$ *be obtained from* $\pi$ *by perturbing the image of one vertex of some triangulation of* $L$ *in which* $\pi$ *is linear on simplices, then extending linearly. If the vertex lies over the interior of* $\Delta^k$ *and the perturbation is sufficiently small, then* $\pi'$ *is also a* PL *fibration.*

*Proof.* Along a line in $\Delta^k$ the given triangulation of $L$ gives a decomposition of $\pi$, near a given $\pi$-slice $L_0$, into a mapping cylinder projection $M(f_{-1}) \cup M(f_1) \to [-1, 1]$ for contractible mappings $L_{-1} \xrightarrow{f_{-1}} L_0 \xleftarrow{f_1} L_1$. If the line in $\Delta^k$ is parallel to the direction of the perturbation, the fibers $L_t'$ of $\pi'$ near $L_0$ can be obtained, up to isotopy along the rays of $M(f_{-1})$ or $M(f_1)$, as follows. For some function $\phi_t\colon L_0 \to [-1, 1]$, $L_t'$ intersects $M(f_{-1} \mid f_{-1}^{-1}(x)) \cup M(f_1 \mid f_1^{-1}(x))$ in the slices $\pi^{-1}(\phi_t(x))$, $x \in L_0$. The projection $g_t\colon L_t' \to L_0$ is a contractible mapping, being the restriction of $M(f_{-1}) \cup M(f_1) \to L_0$. So at least the fibers of $\pi'$ are homotopy equivalent to the fibers of $\pi$.

We now check that $\pi'$ is a PL fibration along this line in $\Delta^k$ parallel to the direction of the perturbation. A given $x \in L_0$ lies in a minimal simplex of the triangulation of $L$, which intersects $L_0$ in a convex cell $\sigma$. We distinguish two cases:

(1) $\sigma$ does not lie in a fiber of $\pi'$. Then near $x$ the fibers $L_t'$ are inde-

pendent of $t$, up to isotopy, so $\pi'$ is a fibration near $x$.

(2) $\sigma$ lies in a fiber $L'_0$. Writing $\pi'$ as $M(f'_-) \cup M(f'_+) \to L'_0$ for maps $L'_- \xrightarrow{f'_-} L'_0 \xrightarrow{f'_+} L'_+$, then we can describe $(f'_+)^{-1}(x)$ as follows. Let $N$ be a transverse section of $\sigma$ in $L_0$ at $x$. Thus $N$ is a join $\partial N * x$, where $\partial N$ is the link of $\sigma$ in a neighborhood of $\sigma$ in $L_0$. Let $\partial N_+$ be spanned by the vertices of $\partial N$ for which $\phi_0 > 0$, and set $N_+ = \partial N_+ * x$. Then $(f'_+)^{-1}(x)$ is $g_0^{-1}(N_+)$, and so is contractible since $N_+$ is a cone. The same arguments apply to points of $L'_0$ near $x$, hence $\pi' \mid M(f'_+)$ is a PL fibration near $x$. And similarly with $f'_-$. Thus $\pi'$ is a PL fibration along lines in the direction of the perturbation.

Consider now a mapping cylinder $M(f')$ for $\pi'$ in a direction other than that of the perturbation, $f'\colon L'_0 \to L'_1$. If $y \in L'_1$ and $x \in (f')^{-1}(y)$, let $M(f)$, $f\colon L_0 \to L_1$, be a mapping cylinder for $\pi$ such that $x \in L_0$ and $y \in L_1$. Then $(f')^{-1}(y) = (g_0 f)^{-1}(y)$, where $g_0\colon L'_0 \to L_0$ is as above. Since $g_0$ and $f$ are contractible mappings, $(f')^{-1}(y)$ is contractible. □

Probably a more general statement than 4.3 is true, that *any* perturbation of $\pi$ sufficiently small with respect to the given triangulation of $L$ is still a PL fibration. (The proof just given applies in fact to the case that all vertices are perturbed in one direction.)

## 5. Suspension in $\mathbb{S}(K)$

In this section we define a suspension operation $\Sigma\colon \mathbb{S}(K) \to \mathbb{S}(K)$ and prove two facts about it: that it is a homotopy inverse for the $H$-space structure "+" on $\mathbb{S}(K)$, and that it satisfies a nice stability property. An "external" suspension $\mathbb{S}(K) \to \Omega\mathbb{S}(SK)$, apparently unrelated to $\Sigma$, will be defined in Section 7.

Let $r\colon L \to K$ be a retraction. Its suspension $\Sigma r\colon \Sigma L \to K$ is defined as follows: $\Sigma L$ is $L \times I$ with $L \times \partial I$ collapsed to $K \times \partial I$ via $r \times \mathrm{id}_{\partial I}$, then with $K \times I$ collapsed to $K$ via projection. And $\Sigma r$ is given by $\Sigma r(x, s) = r(x)$. For example, if $K$ is a point, $\Sigma L$ is just the usual reduced suspension.

Now let $L_t$ be a family in $\mathbb{S}(K)$. Then there is a family of deformation retractions $r_t\colon L_t \to K$ (in the strong sense that $K$ is fixed during the homotopy $r_t \simeq \mathrm{id}$), unique up to canonical homotopy. Consider the suspensions $\Sigma r_t\colon \Sigma L_t \to K$. By 2.4, the two collapses by means of which $\Sigma L_t$ is obtained from $L_t \times I$ preserve PL fibrations. So $\Sigma L_t$ is again a family in $\mathbb{S}(K)$. Thus we obtain $\Sigma\colon \mathbb{S}(K) \to \mathbb{S}(K)$, determined up to homotopy (the choice of $r_t$).

PROPOSITION 5.1. *The map* $\Sigma + \mathrm{id}\colon \mathbb{S}(K) \to \mathbb{S}(K)$ *is null-homotopic.*

*Proof.* Consider the cone operator $C$ on $\mathbb{S}(K)$ defined by $CL_t = L_t \times I/\{(L_t \times 0) \cup (K \times I) \sim K\}$ as in the definition of $\Sigma$. A deformation $r_t \simeq \mathrm{id}$

gives a homotopy $C \simeq \Sigma + \mathrm{id}$ in $\mathcal{S}(K)$, by 2.4. But also $C \simeq 0$ from the projection $L_t \times I \to L_t \times 0$ which is a contractible mapping. □

As a simple application we globalize the well-known product formula for Whitehead torsion. The sum formula can be treated in a similar fashion (see e.g., § 23 of [8] for the classical case).

PROPOSITION 5.2. *Let $C$ be a finite connected polyhedron. Then the product map $\mathcal{S}(K) \to \mathcal{S}(K \times C)$, $L_t \mapsto L_t \times C$, is homotopic to $\chi(C)\cdot i$, where $\chi(C)$ is the Euler characteristic of $C$ and $i$: $\mathcal{S}(K) \to \mathcal{S}(K \times C)$ is induced by inclusion of a factor.*

*Proof.* Choose a triangulation of $C$. Then if $L_t \in \mathcal{S}(K)$, deform $L_t \times C$ as follows. For $\Delta^n \in C$, push $L_t \times \mathring{\Delta}^n$ down into $K$ over $\partial\Delta^n$ via the deformation $r_t \simeq 1$. Then compose with a contraction of $K \times \Delta^n$ to $K \times 0$. This deforms $L_t \times \mathring{\Delta}^n$ to the $n$-fold suspension $\Sigma^n L_t$. Doing this for all simplices of $C$ by downward induction on $n$, $L_t \times C$ is deformed to $\sum_{\Delta^n \in C}(\Sigma^n L_t) \simeq \chi(C)\cdot i(L_t)$. □

It will be useful to have $\Sigma$ operate in $\mathcal{S}_0^\infty(K)$ as well as in $\mathcal{S}(K)$. *A priori* the suspension of a family $L_t \in \mathcal{S}_0^\infty(K)$ need not lie in $\mathcal{S}_0^\infty(K)$, since the suspension of a block $C_t(p)$ of cells which collapse to a point $p \in L_t - K$ collapses to a line $p \times I$, not a point. But there is an easy way to correct this, by contracting $\Sigma C_t(p)$ to the center of $p \times I$ as it collapses.

PROPOSITION 5.3. *The map $\Sigma\colon \mathcal{S}_i^j(K) \to \mathcal{S}_{i+1}^{j+1}(K)$ is $(2i - j)$-connected, provided $i > 1$.*

LEMMA 5.4. *Let the retraction $r\colon L \to K$ be $i$-connected. Then the suspension $\Sigma\colon \pi_k(r) \to \pi_{k+1}(\Sigma r)$ is an isomorphism for $k < 2i$ and an epimorphism for $k = 2i$.*

*Proof.* Let $C$ be the mapping cylinder of $r$. In $C \times [-1, 1]$ we have the subspaces $C = C \times 0$, $A = C \times -1 \cup L \times [-1, 0]$, and $B = C \times 1 \cup L \times [0, 1]$. Consider the following diagram:

$$\begin{array}{ccc}
\pi_k(C, A \cap B) & & \\
{\scriptstyle\partial}\uparrow{\scriptstyle\approx} & & \\
\pi_{k+1}(C \times [0, 1]; C, B) & \xrightarrow[\approx]{\partial} & \pi_k(B, A \cap B) \\
\downarrow & & \downarrow \\
\pi_{k+1}(C \times [-1, 1]; C \times [-1, 0], A \cup B) & \xrightarrow[\approx]{\partial} & \pi_k(A \cup B, A) \\
\uparrow{\scriptstyle\approx} & & \\
\pi_{k+1}(C \times [-1, 1], A \cup B)\,. & &
\end{array}$$

The four maps labelled isomorphisms come from the various long exact sequences of the triads $(C \times [0, 1]; C, B)$ and $(C \times [-1, 1]; C \times [-1, 0], A \cup B)$, while the two other vertical arrows are induced by inclusion. The composite $\pi_k(C, A \cap B) \to \pi_{k+1}(C \times [-1, 1], A \cup B)$ can be identified with $\Sigma\colon \pi_k(r) \to \pi_{k+1}(\Sigma r)$. Since the pairs $(A, A \cap B)$ and $(B, A \cap B)$ are $i$-connected by hypothesis, $(B, A \cap B) \to (A \cup B, A)$ is $2i$-connected by homotopy excision, and the result follows. □

*Proof of* 5.3. Consider first a single $L'_t \in \mathcal{S}^{j+1}_{i+1}(K)$ which we wish to desuspend. Suppose inductively that we have homotoped the attaching maps in a subcomplex of $L'_t$ to a suspension $\Sigma L^{(\cdot)}_t$, together with the deformation retraction $r'_t\colon L'_t \to K$ restricted to $\Sigma L^{(\cdot)}_t$, say $\Sigma r_t$. Then the attaching map $S^l \to \Sigma L^{(\cdot)}_t$ of a cell $e^{l+1}_t \in L'_t$, plus $r'_t \mid e^{l+1}_t$, give an element of $\pi_{l+1}(\Sigma r_t)$. Since $r_t\colon L^{(\cdot)}_t \to K$ is $i$-connected, the lemma says that $e^{l+1}_t$ and $r'_t \mid e^{l+1}_t$ can be desuspended provided $l \leqq j \leqq 2i$. This is the inductive step in showing that $\Sigma$ is 0-connected if $2i - j \geqq 0$.

Desuspending a family $L'_t$ representing an element of $\pi_*(\Sigma)$ is done inductively over basic $k$-simplices, and cell by cell within each basic $k$-simplex. Desuspending cells $e^{l+1}_t$ over $\Delta^k$, with the assumption of a desuspension over $\partial\Delta^k$, is possible if $l + k \leqq j + k \leqq 2i$ if one uses a straightforward fibered version of homotopy excision, which we leave to the reader. Collapses are desuspended to collapses by considering them as the case that $K = N_t$, in the notation of Remark (2) of Section 4.

The assumption $i > 1$ is needed to assure that the desuspension $L_t$ of $L'_t$ actually lies in $\mathcal{S}^j_i(K)$. For if $i > 1$ then $\pi_1 K \to \pi_1 L_t$ is an isomorphism and

$$H_*(L_t, K; \mathbf{Z}[\pi_1 K]) \approx H_*(\Sigma L_t, K; \mathbf{Z}[\pi_1 K]) = 0 ,$$

so $K$ is a deformation retract of $L_t$. □

## 6. Trading up cells into two dimensions

One of the main geometric steps in the proof of Whitehead's theorem on simple homotopy types is the assertion that any homotopy equivalence is, modulo elementary expansions and collapses, an inclusion, $K \hookrightarrow L$ such that the cells of $L - K$ all have dimension either $n$ or $n + 1$, for a fixed $n$. We prove now a parametrized version of this which will be the basis for all the deeper results about $\mathcal{S}(K)$ in the remainder of the paper.

THEOREM 6.1. *The inclusion* $\mathcal{S}^{n+1}_n(K) \hookrightarrow \mathcal{S}^\infty_0(K)$ *is* $(n-1)$*-connected if* $n > 1$.

*Proof.* Let the family $L_t$ represent an element of $\pi_k \mathcal{S}^\infty_0(K)$, say $L_t \in \mathcal{S}^j_0(K)$

for some $j \geqq n+1$. By 6.2 below, $L_t$ can be deformed to a family $L'_t \in \mathcal{S}^j_{j-1}(K)$. Then by 5.2 $L'_t$ is homotopic to an iterated suspension $\Sigma^{j-n-1}L''_t$ of a family $L''_t \in \mathcal{S}^{n+1}_n(K)$, provided $k \leqq n-1$. If $j$ is chosen so that $j-n-1$ is even, then by 5.1, $L''_t$ is homotopic to $L'_t$. This gives surjectivity of

$$\pi_k\mathcal{S}^{n+1}_n(K) \longrightarrow \pi_k\mathcal{S}^\infty_0(K) .$$

Injectivity is similar. If $L_t$, $t \in \Delta^{k+1}$, is a contraction in $\mathcal{S}^j_0(K)$ of an element of $\pi_k\mathcal{S}^{n+1}_n(K)$ for some $j$ with $j-n-1$ even, then adjoin to $\Delta^{k+1}$ an external collar supporting a homotopy of $L_t \mid \partial\Delta^{k+1}$ to $\Sigma^{j-n-1}L_t \mid \partial\Delta^{k+1}$. The resulting family $L'_t \in \mathcal{S}^j_0(K)$ can be deformed to $L''_t \in \mathcal{S}^j_{j-1}(K)$, rel boundary, by 6.2 below. If $k < n-1$, $L''_t$ desuspends to a contraction in $\mathcal{S}^{n+1}_n(K)$ of the original $L_t \mid \partial\Delta^{k+1}$. □

It remains to prove:

PROPOSITION 6.2. $\mathcal{S}^j_{i+1}(K) \hookrightarrow \mathcal{S}^j_i(K)$ *is a homotopy equivalence* $(i+1<j)$.

*Proof.* Let $L_t$ represent a class in $\pi_n(\mathcal{S}^j_i(K), \mathcal{S}^j_{i+1}(K))$, and let $e^i_t$ be one of the $i$-cells of $L_t$, attached by $\phi_t\colon S^{i-1} \to L^{(\cdot)}_t$. We will show how to cancel $e^i_t$ over a $k$-simplex $\Delta^k$, assuming a cancellation given over $\partial\Delta^k$, introducing only $i+1$- and $i+2$-cells in the process. Induction on $k$ will then allow $e^i_t$ to be cancelled everywhere, and the result will follow by iterating for other $i$-cells.

The actual elimination of $e^i_t$ over $\Delta^k$ proceeds just as in the unparametrized case:

LEMMA 6.3. *If $\phi_t$ extends to a map $\bar\phi_t\colon D^i \to L^{(\cdot)}_t$ over $\Delta^k$, which also collapses to a point over $\partial\Delta^k$, then $e^i_t$ can replaced by an $i+2$-cell over $\Delta^k$.*

*Proof.* We may use $\bar\phi_t$ to deform $\phi_t$ to the constant map. Then $\bigcup_{t\in\Delta^k} e^i_t$ is an $(i+k)$-sphere in $L = \bigcup_{t\in\Delta^k} L_t$. By rechoosing the extension $\bar\phi_t$ by an element of $\pi_{i+k}(L) = \pi_{i+k}(K)$ if necessary, we may assume the $i$-sphere $e^i_t$ (i.e., trivially attached $i$-cell) bounds a disc $D^{i+1}_t$ in $L_t$ which also collapses over $\partial\Delta^k$. Now introduce a trivial pair of cells $e^{i+1}_t$ and $e^{i+2}_t$ which together form a disc attached to $L_t$ at a point of $D^{i+1}_t$ and collapsing over $\partial\Delta^k$. The discs $D^{i+1}_t$ give a homotopy of the attaching map of $e^{i+1}_t$ so that $\partial e^{i+1}_t = e^i_t$. Next, push all higher cells off the disc $e^i_t \cup e^{i+1}_t$. Then cancel $e^i_t$ and $e^{i+1}_t$. □

It remains to find the extension $\bar\phi_t$. When $k=0$ this is trivial—cells of lowest dimension must be attached by null-homotopic maps. But for $k>0$, $\bigcup_{t\in\Delta^k} \phi_t\colon S^{i+k-1} \to \bigcup_{t\in\Delta^k} L^{(\cdot)}_t$ may well be non-trivial. Our aim will be to kill $\phi_t$ simultaneously for all $t \in \Delta^k$ by attaching new $i+1$-cells to $L^{(\cdot)}_t$.

First we specify more closely the subcomplexes $L^{(\cdot)}_t$, in which of course the index $(\cdot)$ varies with $t$, since the order of attaching $e^i_t$ may vary with $t$.

We want to choose a subdivision $T$ of $\Delta^k$ such that each cell of $L_t$ is contained in $L_t^{(\cdot)}$ over an open subset of $\Delta^k$ which is a union of open simplices of $T$. To achieve this we begin with a subdivision $T'$ of $\Delta^k$ such that $L_t$ consists of basic simplices in $\mathfrak{S}_i^j(K)$ over simplices of $T'$. We can assume each cell $e_t$ of $L_t$ lies in $L_t^{(\cdot)}$ over a subcomplex $S(e_t)$ of $T'$. Then we can make $e_t \in L_t^{(\cdot)}$ attach only to cells of $L_t^{(\cdot)}$ over a regular neighborhood $N(e_t)$ of $S(e_t)$ by a small homotopy of attaching maps. For if $e_t$ is disjoint from a cell $e_t'$ over $S(e_t)$, then it is disjoint from the center of $e_t'$ over a neighborhood of $S(e_t)$ and so it can be pushed off $e_t'$ near $S(e_t)$. Finally, we enlarge $L_t^{(\cdot)}$ by including $e_t$ over the interior of $N(e_t)$. After doing this for all cells $e_t$ we subdivide $T'$ to a triangulation $T$ in which the $N(e_t)$ are subcomplexes. (In this construction we regard collapse points as extensions of the cells which collapse to them. Thus if a collapse point $p$ belongs to $L_t^{(\cdot)}$, then in nearby $t$-slices all the cells collapsing to $p$ also lie in $L_t^{(\cdot)}$.)

Now we show how to construct $\bar{\phi}_t$ inductively over skeletons of the triangulation $T$ of $\Delta^k$. We have already remarked that $\bar{\phi}_t$ exists over vertices of $T$; in fact, we can first deform $\phi_t$ into $K$ by general position and then choose for $\bar{\phi}_t$ the image of $e_t^i$ under the retraction $r_t\colon L_t \to K$. Assume inductively that $\bar{\phi}_t$ has been constructed over the $(l-1)$-skeleton of $T$ such that $\bar{\varphi}_t \simeq e_t^i$ (rel $\varphi_t$) in $L_t$, and consider the problem of extending $\bar{\varphi}_t$ over an $l$-simplex $\Delta^l$. By the construction of $T$ in the first place, and by induction thereafter, we can assume:

(∗) $\bar{\varphi}_t$ extends from $\partial\Delta^l$ up to a neighborhood $N \subset \Delta^l$ of the center of $\Delta^l$ such that $L_t^{(\cdot)} \mapsto t$ is a fibration over $N$.

The obstruction to finishing the extension is the homotopy class of a map $\alpha\colon S^{i-1+l} \to L_t^{(\cdot)}$, $t \in N$. In $L_t$, the cells $e_t^i$ themselves provide a contraction $\bar{\alpha}$ of $\alpha$. Since $L_t^{(\cdot)} \mapsto t$ is a fibration over $N$, we can assume $\bar{\alpha}$ is spread as an $(l-1)$-parameter family $(D_t^{i+1}, \partial D_t^{i+1}) \to (L_t, L_t^{(\cdot)})$, $t \in D^{l-1}$, where $N = D^{l-1} \times D^1$, such that $D_t^{i+1}$ reduces to a point for $t \in C$, a collar neighborhood of $\partial D^{l-1}$ in $D^{l-1}$. Extend $(D_t^{i+1}, \partial D_t^{i+1}) \to (L_t, L_t^{(\cdot)})$ over $N \times D^{k-l}$, a neighborhood in $\Delta^k$. Now adjoin a trivial pair of cells $(e_t^{i+1}, e_t^{i+2})$ which together form a ball attached to $L_t^{(\cdot)}$ at a point in $D_t^{i+1}$. Using $D_t^{i+1}$, deform $e_t^{i+1}$ so that it attaches to $L_t^{(\cdot)}$ along $\partial D_t^{i+1}$. If we let $L_t^{(\cdot)}$ include the cell $e_t^{i+1}$ over $N \times D^{k-l}$ and the cell $e_t^{i+2}$ over $C \times D^1 \times D^{k-l}$, then we have killed the obstruction to extending $\bar{\varphi}_t$ over $\Delta^l$.

To guarantee that (∗) is preserved we need onlych oose $N \times D^{k-l}$ to include the centers of all simplices of $T$ which it meets and to be radially "starlike" in these simplices. This completes the induction step in the construction of

$\bar{\phi}_t$, and hence the proof of 6.2. □

## 7. Excision in $\mathcal{S}(K)$

Suppose our base space "$K$" is the union of two subpolyhedra $A$ and $B$. Then we have inclusions

$$\begin{array}{ccc} \mathcal{S}(A \cap B) & \subset & \mathcal{S}(B) \\ \cap & & \cap \\ \mathcal{S}(A) & \subset & \mathcal{S}(A \cup B) \end{array}$$

THEOREM 7.1. *If $(A, A \cap B)$ is $m$-connected and $(B, A \cap B)$ is $n$-connected, with $m > 1$, then the map*

$$\pi_k(\mathcal{S}(B), \mathcal{S}(A \cap B)) \longrightarrow \pi_k(\mathcal{S}(A \cup B), \mathcal{S}(A))$$

*induced by inclusions is surjective if $k \leqq m + n - 2$ and injective if $k < m + n - 2$.*

As in ordinary homotopy theory, excision implies a suspension theorem. Consider the (ordinary) suspension $SK$ of a polyhedron $K$ as the union of two cones $A$ and $B$ on $K$, with $A \cap B = K$. Since $\mathcal{S}(A)$ and $\mathcal{S}(B)$ are contractible, the homotopy fibers of $\mathcal{S}(A \cap B) \hookrightarrow \mathcal{S}(B)$ and $\mathcal{S}(A) \hookrightarrow \mathcal{S}(A \cup B)$ are $\mathcal{S}(K)$ and $\Omega\mathcal{S}(SK)$, respectively. The map of homotopy fibers $\mathcal{S}(K) \rightarrow \Omega\mathcal{S}(SK)$ induced by the inclusion $(\mathcal{S}(B), \mathcal{S}(A \cap B)) \subset (\mathcal{S}(A \cup B), \mathcal{S}(A))$ sends $L_t \in \mathcal{S}(K)$ to the loop $L_t^s$ in $\mathcal{S}(SK)$ defined by

$$L_t^s = SK \cup (L_t \times s) \subset SL_t\,, \qquad 0 \leqq s \leqq 1\,,$$

where $SL_t$ is, as usual, $L_t \times I$ with $L_t \times 0$ and $L_t \times 1$ collapsed to points.

COROLLARY 7.2. *If $K$ is $m$-connected, the suspension map $\mathcal{S}(K) \rightarrow \Omega\mathcal{S}(SK)$ is $(2m - 1)$-connected.* □

Thus the iteration

$$\mathcal{S}(K) \longrightarrow \Omega\mathcal{S}(SK) \longrightarrow \Omega^2\mathcal{S}(S^2K) \longrightarrow \cdots$$

eventually becomes highly connected, and we can speak of the stable simple homotopy functor $s(K) = \lim \Omega^n\mathcal{S}(S^nK)$. The stable simple homotopy groups $s_i(K) = \pi_i s(K) = \lim \pi_{i+n}\mathcal{S}(S^nK)$ satisfy excision by 7.1, hence form a generalized homology theory $s_*(K)$.

In the course of the proof of 7.1 we will obtain also the following:

PROPOSITION 7.3. *If $K \rightarrow K'$ is $m$-connected, $m > 1$, then the induced map $\mathcal{S}(K) \rightarrow \mathcal{S}(K')$ is $(m - 1)$-connected.*

For example, taking $K'$ to be an Eilenberg-MacLane space $K(\pi_1 K, 1)$, this is the well-known fact that $\pi_0\mathcal{S}(K)$ depends only on $\pi_1 K$. (If $K' = K(\pi_1 K, 1)$ is not a finite complex, take "$\mathcal{S}(K')$" here to be $\bigcup_\alpha \mathcal{S}(K'_\alpha)$, the union over finite subpolyhedra $K'_\alpha \subset K'$.) The proposition does not extend to $m = 1$ since a surjection $\pi_1 K \rightarrow \pi_1 K'$ may not induce a surjection $\mathrm{Wh}_1(\pi_1 K) \rightarrow \mathrm{Wh}_1(\pi_1 K')$, e.g., if $\pi_1 K$ is a free group, in which case $\mathrm{Wh}_1(\pi_1 K)$ is zero.

LEMMA 7.4. *Given $K' = K \cup e^{m+1}$ and $L_t \in \mathfrak{S}_i^{i+1}(K')$, $t \in \Delta^k$, we can deform the attaching maps of cells in $L_t - K'$ so that they attach to $e^{m+1}$ over a neighborhood of a $(k - m)$-dimensional subcomplex of $\Delta^k$, staying in $\mathfrak{S}_i^{i+1}(K)$ over any part of $\Delta^k$ where $L_t \in \mathfrak{S}_i^{i+1}(K)$.*

*Proof.* Consider for example a cell $e_t^{i+1} \in L_t - K'$ and suppose inductively that lower cells have already been fixed up. Let $\phi_t \colon S^i \rightarrow L_t^{(\cdot)}$ be the attaching map of $e_t^{i+1}$. In general position $\phi_t$ will be transverse to the center $x_t$ of the top cell of $L_t^{(\cdot)}$ (as a $k$-parameter family), and the pullback $\bigcup_t \phi_t^{-1}(x_t)$ will be a manifold $I_1 \subset S^i \times \Delta^k$ of dimension $\leqq k$. Outside a small neighborhood $N_1$ of $I_1$ we can push $\phi_t$ off this top cell of $L_t^{(\cdot)}$. Then repeat the process for the next highest cell of $L_t^{(\cdot)}$ to get a manifold $I_2 \subset S^i \times \Delta^k - N_1$ with neighborhood $N_2$, etc. So we can assume that outside $N = \bigcup_j N_j$ the family $\phi_t$ maps into $K'$. Extend $\phi_t$ to $\bar{\phi}_t \colon D^{i+1} \rightarrow L_t^{(\cdot)} \cup e_t^{i+1}$ by identifying the interior of $D^{i+1}$ with $e_t^{i+1}$. We can choose a family of $(i + 1)$-balls $D_t \subset D^{i+1} \times \{t\}$ such that $\bigcup_t D_t$ is a small neighborhood of the fiberwise join of $N$ with $0 \times \Delta^k$ in $D^{i+1} \times \Delta^k$, as follows. Write $D^{i+1}$ as the union of a small concentric disc $D_0^{i+1}$ with an annulus $S^i \times [0, 1]$, $S^i \times 1 = \partial D^{i+1}$. Let $f \colon S^i \times \Delta^k \rightarrow [0, 1]$ be 1 on $N$ and 0 away from $N$. Then set

$$D_t = D_0^{i+1} \times \{t\} \cup \{(x, s, t) \in S^i \times [0, 1] \times \{t\} \mid s \leqq F(x, t)\} ,$$

i.e., the "shadow" of the graph of $f$. Note that $\bigcup_t D_t$ has a $(k + 1)$-dimensional spine, since $N$ has a $k$-dimensional spine.

Now let $r_t \colon L_t \rightarrow K'$ be a family of retractions. Since $\phi_t = r_t \phi_t$ outside $N$, $r_t \bar{\phi}_t \mid D^{i+1} - \mathring{D}_t$ defines a homotopy of $\phi_t$ to a new attaching map $\phi'_t$ which equals $\phi_t$ on $N$ and $r_t \bar{\phi}_t \mid \partial D_t$ outside $N$. In general position the image of the spine of $\bigcup_t D_t$ under $r_t \bar{\phi}_t$ will meet the center of $e^{m+1}$ in a $(k - m)$-dimensional complex, so outside a neighborhood of this we may assume $\phi'_t$ attaches $e_t^{i+1}$ to $K' - e^{m+1} = K$. This is the inductive step. Note that if for certain $t \in \Delta^k$ the original $L_t$ lies in $\mathfrak{S}_i^{i+1}(K) \subset \mathfrak{S}_i^{i+1}(K')$, then choosing $r_t$ to retract $L_t - K$ onto $K$ guarantees that the modified $L_t$ is still in $\mathfrak{S}_i^{i+1}(K)$. □

*Proof of* 7.3. Since $\mathfrak{S}$ is a homotopy functor, we can take $K \rightarrow K'$ to be an inclusion such that $K' - K$ consists of cells of dimension $\geqq m + 1$. For each of these cells in turn apply 7.4 to a family $L_t$ representing an element of $\pi_k(\mathfrak{S}_i^{i+1}(K'), \mathfrak{S}_i^{i+1}(K))$, $k < m$. The hypothesis $m > 1$ is only necessary to assure that $\pi_1 K \approx \pi_1 K'$, so that the new $L_t$ contains $K$ as well as $K'$ as a deformation retract. □

*Proof of* 7.1. By induction it suffices to take $A = K \cup e^{m+1}$, $B = K \cup e^{n+1}$. Let $L_t \in \mathcal{S}_i^{i+1}(A \cup B)$, $t \in D^k$, be such that, writing $\partial D^k = D_+^{k-1} \cup D_-^{k-1}$, we have $L_t \in \mathcal{S}_i^{i+1}(A)$ for $t \in D_-^{k-1}$, $L_t \in \mathcal{S}_i^{i+1}(B)$ for $t \in D_+^{k-1}$, and $L_t \in \mathcal{S}_i^{i+1}(A \cap B)$ for $t \in D_+^{k-1} \cap D_-^{k-1}$. Applying 7.4, we obtain a set $S_A \subset D^k - D_+^{k-1}$ over which cells of $L_t - (A \cup B)$ attach to $A$, and $S_A$ has a spine of dimension $k-m$. Similarly we obtain $S_B \subset D^k - D_-^{k-1}$ with a $(k-n)$-dimensional spine. In general position we can assume that $S_B$ is disjoint from $\operatorname{sh}(S_A)$, the "shadow" of $S_A$ under projection to $D_-^{k-1}$, provided $(k-m)+(k-n) < k-1$, or $k \leqq m + n - 2$. Then there is an evident homotopy of the family $L_t$, fixed over $D_+^{k-1}$, which excises $\operatorname{sh}(S_A)$ from its parameter domain. Again, the hypothesis $m > 1$ implies that $L_t$ actually is in $\mathcal{S}_i^{i+1}(B)$ over the new $D^k$ and similarly that $L_t$ is in $\mathcal{S}_i^{i+1}(A \cap B)$ over the new $D_-^{k-1}$. This shows that the inclusion $(\mathcal{S}_i^{i+1}(B), \mathcal{S}_i^{i+1}(A \cap B)) \subset (\mathcal{S}_i^{i+1}(A \cup B), \mathcal{S}_i^{i+1}(A))$ is surjective on $\pi_k$ and injective on $\pi_{k-1}$. □

## 8. Families of PL handlebodies

We continue to work entirely within the PL category. Let $W^{n+1}$ be a compact connected manifold and let $M^n$ be a codimension zero submanifold of $\partial W$. By a *handlebody structure h on* $(W, M)$ we mean:

(1) There exists a filtration of $W$ by codimension zero submanifolds $W = W^{(N)} \supset W^{(N-1)} \supset \cdots \supset W^{(0)}$ where $W^{(0)} \approx M \times I$ is a collar on $M \times 0 = M \subset W$ (with $\partial M \times I \subset \partial W$) and $W^{(i)}$ is obtained from $W^{(i-1)}$ by attaching a handle $D^{n_i} \times D^{n+1-n_i}$ via an embedding $\varphi^i\colon S^{n_i-1} \times D^{n+1-n_i} \to \partial W^{(i-1)}$.

(2) A product structure on the collar $W^{(0)} \approx M \times I$ and on each handle $D^{n_i} \times D^{n+1-n_i}$ is given. That is, the homeomorphisms $W^{(0)} \approx M \times I$ and $W^{(i)} - W^{(i-1)} \approx \mathring{D}^{n_i} \times D^{n+1-n_i}$ are specified only up to product homeomorphism $f_1 \times f_2\colon M \times I \circlearrowleft$ or $f_1 \times f_2\colon D^{n_i} \times D^{n+1-n_i} \circlearrowleft$. (Thus in a product structure on $X_1 \times X_2$ the collections of slices $\{X_1 \times \{x_2\} \mid x_2 \in X_2\}$ and $\{\{x_1\} \times X_2 \mid x_1 \in X_1\}$ are well-defined.)

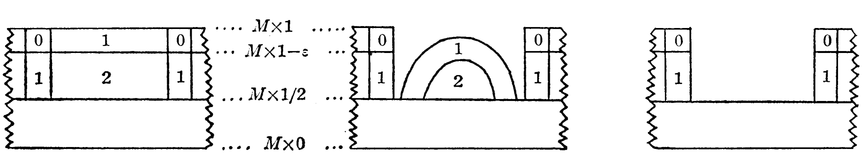
Now we define the notion of a $k$-parameter family of handlebody structures $h_t$ on $(W, M)$, $t \in \Delta^k$. We first allow the submanifolds $W^{(i)}$ to vary through a $k$-isotopy $W_t^{(i)}$, as follows. For $i = 0$ the collar $W_t^{(0)}$ moves by $k$-isotopy fixed on $M = M \times 0$. For $i > 0$ the handle $D^{n_i} \times D^{n+1-n_i}$ moves by a $k$-isotopy $D_t^{n_i} \times D_t^{n+1-n_i}$. In particular, the attaching maps $\varphi^i$ can vary by a $k$-isotopy $\varphi_t^i$.

But also we want to allow certain collections of handles to be coned off to a point. The prescription for this goes inductively on $k$, as follows. For

$t_0 \in \Delta^k$, suppose that over the boundary of a small neighborhood of $t_0$ in $\Delta^k$ the $(k-1)$-parameter family $h_t$ has a filtration with layers $W_t^{(i)} \subset W_t^{(j)}$ such that the handles of $W_t^{(j)} - W_t^{(i)}$ attach to $\partial W_t^{(i)}$ inside a disc $D_t^n$ in such a way that $W_t^{(i)}$ is homeomorphic to $W_t^{(j)}$ by a family of homeomorphisms fixed outside a small neighborhood of $D_t^n$ in $W_t^{(i)}$. In this situation we can obtain a $k$-parameter family $W_t^{(j)}$ by simply shrinking $W_t^{(j)} - W_t^{(i)}$ radially to a point $p_t \in D_t^n$ as $t$ goes radially from the boundary of the neighborhood of $t_0$ to $t_0$ (the Alexander trick). Thus the submanifolds $W_t^{(j)}$ change to $W_{t_0}^{(j)} = W_{t_0}^{(i)}$ by isotopy. More generally, we allow simultaneously a finite number of such collapsing operations to be going on independently at distinct points $p_t \in W$.

A $k$-parameter family of handlebody structures $h_t$ on $(W, M)$ obtained by the above process we call a *basic* family. A *general* family $h_t$ is one which, over the simplices of some subdivision of $\Delta^k$, consists of basic families. The distinction between basic and general is that in a basic family all the handles can be attached in one order independent of $t$ (though the choice of such an order is not part of the data of $h_t$). The collection of all $k$-parameter families, $k = 0, 1, 2, \cdots$, forms a simplicial space which we denote $\mathcal{G}(W, M)$. The face and degeneracy maps are the obvious ones, and $\mathcal{G}(W, M)$ is clearly a Kan complex.

*Example.* We can consider as one-parameter families according to the above definition all of the handle operations used in the proof of the PL $h$-cobordism theorem, namely isotopies of attaching maps of various sorts or cancellations of complementary pairs of handles. As a very special case, let $(W, M) = (M \times I, M \times 0)$ be given first the handlebody structure $h_0$ with no handles and just the collar $W_0^{(0)} = M \times I$ (in the given product structure). A second handlebody structure $h_1$ on $(M \times I, M \times 0)$ can be obtained from a handlebody structure on $M$ as follows. The collar $W_1^{(0)}$ of $h_1$ is $M \times [0, 1/2]$, and for each $i$-handle $D^i \times D^{n-i}$ of $M$ we can consider $D^i \times (D^{n-i} \times [1-\varepsilon, 1])$ as an $i$-handle in $M \times I$ and $(D^i \times [1/2, 1-\varepsilon]) \times D^{n-i}$ as an $(i+1)$-handle. See figure (a). We can isotope these two handles in $M \times I$ so they attach only to $M \times 1/2$, as in figure (b). The resulting complementary pair can then be coned off, producing figure (c). Doing this successively for all handles of $h_1$, we obtain a one-parameter family $h_t$ connecting $h_1$ with $h_0$.




(a) (b) (c)

The reason for permitting collapses in $\mathcal{G}(W, M)$ more general than the collapse of a complementary pair of handles is that with the more general collapses we can prove the following key result.

THEOREM 8.1. *$\mathcal{G}(W, M)$ is contractible.*

*Proof.* Let $h_t$, $t \in S^k$, represent an element of $\pi_k \mathcal{G}(W, M)$. We will show how to homotope the family $h_t$ to a constant family. To begin, we deform $h_t$ so that the collars $W_t^{(0)} \approx M \times I$ all agree with a chosen standard collar. The collars $W_t^{(0)}$ are all standard at $M = M \times 0$, so by an isotopy within $W_t^{(0)}$ we can assume they are standard near $M \times 0$, say on $M \times [0, 1/2]$. Then using the one-parameter family in the preceding example, we can decompose the remaining non-standard half of $W_t^{(0)}$ into handles by a homotopy of the family $h_t$, leaving us with only standard collars.

The idea for the rest of the proof is to show that $h_t = h_t^0$ and a constant family $h_t^1$ have a common subdivision $h_t^{1/2}$, in the obvious sense that $h_t^{1/2}$ intersects each family of handles $D_t^i \times D_t^{n+1-i}$ of $h_t^0$ or $h_t^1$ in a union of handles of $h_t^{1/2}$, and in fact, in a family in $\mathcal{G}(D^i \times D^{n+1-i}, \partial D^i \times D^{n+1-i})$. Then by the obvious handlebody version of 4.2, a family is homotopic to a subdivision of itself, so $h_t^0 \simeq h_t^{1/2} \simeq h_t^1$.

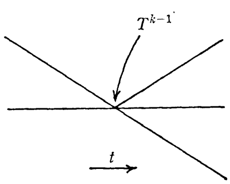
Let $T$ be a triangulation of $W \times S^k$ such that the $k$-parameter families of handles of $h_t^0$ and $h_t^1$ are subcomplexes. We can assume $T$ is transverse to the slices $W_t = W \times \{t\}$, so that a simplex $\Delta^l$ of $T$ intersects each slice $W_t$ in a ball $B_t^{l-k}$, a point $p_t$, or not at all. For a slice $W_t$ this ball decomposition leads to a handlebody structure $h_t^{1/2}$ in the usual way: small neighborhoods of the 0-balls are 0-handles, small neighborhoods of the 1-balls, minus the previously constructed 0-handles, are 1-handles, etc. We shall call such handles *handle neighborhoods* $N_t(B_t^{l-k})$ or $N_t(p_t)$ (though they are not really neighborhoods). Unfortunately, this construction fails to give a $k$-parameter family in $\mathcal{G}(W, M)$. The trouble comes in a neighborhood $N$ of the $k-1$ skeleton $T^{k-1}$ of $T$, where a ball $B_t^{l-k}$ can shrink to a point $p_t \in T^{k-1}$, although its handle neighborhood $N_t(B_t^{l-k})$ cannot change to the 0-handle $N_t(p_t)$ "continuously," i.e., by any natural path in $\mathcal{G}(W, M)$. So we will have to splice things together somehow near $T^{k-1}$.

We can choose the neighborhood $N$ of $T^{k-1}$ to have the following properties:

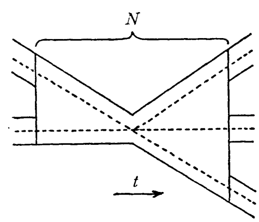
(1) $N = N^{k-1} \supset \cdots \supset N^0$, where $N^i$ is a neighborhood of $T^i$ obtained from $N^{i-1}$ by adding handle neighborhoods $N(\Delta^i)$ of the $i$-simplices of $T^i$, such that $N_t(\Delta^i) = N(\Delta^i) \cap W_t$ is a $k$-parameter family of 0-handles over the $i$-handle $\pi(N(\Delta^i))$ in $S^k$, where $\pi\colon W \times S^k \to S^k$ is projection.

(2) If the $i$-handle $\pi(N(\Delta^i))$ is written as $D^i \times D^{k-i}$, then over $D^i \times \partial D^{k-i}$ the 0-handle $N_t(\Delta^i)$ is the union of 0-handles $N_t(\Delta^j)$ for $j > i$ and handles $N_t(B_t^{l-k})$ of $W_t - N$.

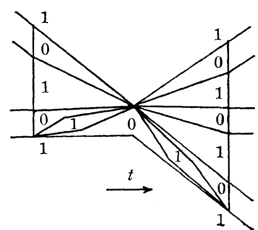
Now we can extend the family of handlebody structures $h_t^{1/2}$ on $W \times S^k - N$ to all of $W \times S^k$, one simplex $\Delta^i$ of $T^{k-1}$ at a time by downward induction on $i$, as follows. Over $D^i \times \partial D^{k-i}$ we have inductively a family of handlebody structures on the $(n+1)$-disc $N_t(\Delta^i)$. Since $\mathcal{G}(D^{n+1}, \phi) \simeq *$ by 8.2 below, this extends to a family over $D^i \times D^{k-i}$. See figures (d), (e), and (f) for an example when $k = 1$. Note also that any two such extensions of $h_t^{1/2}$ over $N$ are homotopic, by the relative form of the same argument.



(d) Triangulation $T$



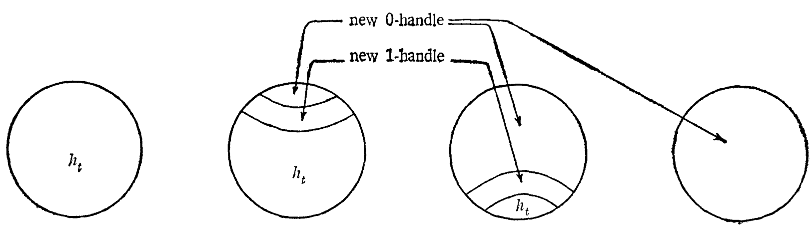
(e) Handle Nbds $N_t(B_t^{l-k})$



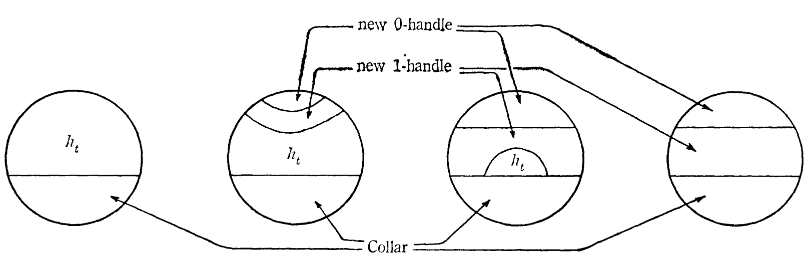
(f) Splicing over $N$

LEMMA 8.2. *$\mathcal{G}(D^{n+1}, \phi)$ and $\mathcal{G}(D^{n+1}, D^n)$ are contractible.*

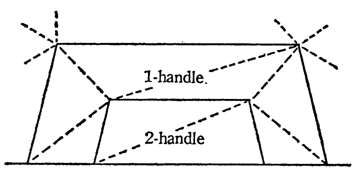
*Proof.* We can uniformly cone off any family $h_t$ in $\mathcal{G}(D^{n+1}, \phi)$ to a single 0-handle, as in the following sequence:



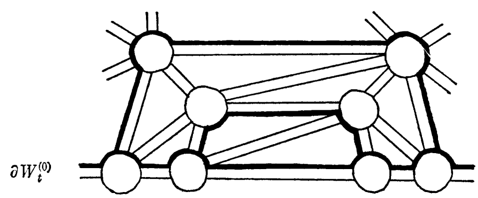

For $\mathcal{G}(D^{n+1}, D^n)$ we can take all collars to be standard and then deform $h_t \in \mathcal{G}(D^{n+1}, D^n)$ to a fixed (1, 0)-handle pair: □




Returning to the main argument, the family $h_t^{1/2}$ as constructed is not yet a subdivision of $h_t^0$ (or $h_t^1$). Even in the unparametrized case a handle neighborhood $N_t(B_t^{l-k})$ of a ball $B_t^{l-k}$ in the intersection of two handles of $h_t^0$ will not lie inside either handle. But such a ball $B_t^{l-k}$ lies in $\mathring{D}_t^i \times \partial D_t^{n+1-i}$ for only one handle $D_t^i \times D_t^{n+1-i}$ of $h_t^0$, so let us enlarge this handle by a small collar on $\mathring{D}_t^i \times \partial D_t^{n+1-i}$ consisting of all these $N_t(B_t^{l-k})$ for which $B_t^{l-k} \subset \mathring{D}_t^i \times \partial D_t^{n+1-i}$. In other words we are moving $h_t^0$ to $h_t^{1/4}$ by a small isotopy so that $h_t^{1/2}$ becomes a subdivision of $h_t^{1/4}$. A similar modification is required for our fixed collar $W_t^{(0)} \approx M \times I$: we expand it slightly to include all $N_t(B_t^{l-k})$ with $B_t^{l-k} \subset W_t^{(0)}$. Of course we do not want to subdivide $W_t^{(0)}$, so we just let $h_t^{1/2} = h_t^{1/4}$ on $W_t^{(0)}$. See figures (g) and (h).




(g) $h_t^0$




(h) $h_t^{1/4}$ (heavy lines)

In the parametrized case we also have to take extra care in extending $h_t^{1/2}$ to $N$ so that it restricts to a family of handlebody structures on the handles of $h_t^{1/4}$. This can be done by building the extension of $h_t^{1/2}$ to $N$, one handle of $h_t^{1/4}$ at a time, using the second half of 8.2. Also, where a handle of $h_t^{1/4}$ collapses to a point, choose the extension of $h_t^{1/2}$ to $N$ to collapse simultaneously to a point. The result of these modifications of $h_t^{1/2}$ is a family $h_t^{1/3}$, homotopic to $h_t^{1/2}$ since it differs only on $N$, and homotopic to $h_t^{1/4}$ since it is a subdivision of $h_t^{1/4}$. Thus $h_t^0 \simeq h_t^{1/2}$, and similarly $h_t^{1/2} \simeq h_t^1$. □

It would be interesting to know if the analogue of 8.1 is true in the topological category (for manifolds admitting a topological handlebody structure).

### 9. The parametrized PL $h$-cobordism theorem

A well-known representation for the classifying space of the simplicial group $\mathrm{PL}(W)$ of (PL) homeomorphisms of a manifold $W$ is the space of all submanifolds of $\mathbf{R}^\infty$ homeomorphic to $W$. To make this more precise and to give a relative form for $\mathrm{PL}(W, M)$, the homeomorphisms restricting to the identity on a codimension zero submanifold $M$ of $\partial W$, first let $\mathfrak{E}(W, M)$ denote the simplicial space of embeddings of $W$ in $\mathbf{R}^\infty$ agreeing with a given fixed embedding on $M$. Then $\mathrm{PL}(W, M)$ operates freely on $\mathfrak{E}(W, M)$ by composition: $f \in \mathrm{PL}(W, M)$ sends $g \in \mathfrak{E}(W, M)$ to $g \circ f^{-1} \in \mathfrak{E}(W, M)$. The principal simplicial fibration

$$\mathrm{PL}(W, M) \longrightarrow \mathcal{E}(W, M) \longrightarrow \bar{\mathcal{E}}(W, M) = \mathcal{E}(W, M)/\mathrm{PL}(W, M)$$

is universal for $\mathrm{PL}(W, M)$ since $\mathcal{E}(W, M)$ is clearly contractible. The space $\bar{\mathcal{E}}(W, M)$ is the representation of $B\mathrm{PL}(W, M)$ described at the beginning of this paragraph.

Recall that a $k$-simplex of the simple homotopy space $\mathcal{S}(M)$ is a subpolyhedron $L \subset \mathbf{R}^\infty \times \Delta^k$ such that the projection $\pi: L \to \Delta^k$ is a PL fibration, and such that a fixed copy of $M = M \times \{t\}$ is a deformation retract of each fiber $\pi^{-1}(t)$. In particular, if $(W, M)$ is an $h$-cobordism, simplices of $\bar{\mathcal{E}}(W, M)$ are simplices of $\mathcal{S}(M)$. Thus we have an inclusion of $\bar{\mathcal{E}}(W, M)$ in $\mathcal{S}(M)$, in fact, in the component of $\mathcal{S}(M)$ having torsion equal to that of the $h$-cobordism $(W, M)$. (Note that $\bar{\mathcal{E}}(W, M)$ is always connected.)

THEOREM 9.1. *The inclusion* $\bar{\mathcal{E}}(M \times I, M) \hookrightarrow \mathcal{S}(M)$ *is $k$-connected (onto the identity component of* $\mathcal{S}(M)$*) provided* $\dim M = n \geqq 3k+5$. *Consequently the looping* $\mathrm{PL}(M \times I, M) \to \Omega\mathcal{S}(M)$ *is $k$-connected if* $n \geqq 3k + 8$.

Of course $\mathrm{PL}(M \times I, M)$ is just the PL pseudo-isotopy space, denoted by $\mathcal{P}(M)$ in the introduction.

COROLLARY 9.2. *The stabilization* $\sigma: \mathcal{P}(M) \hookrightarrow \mathcal{P}(M \times I)$, $\sigma(f) = f \times \mathrm{id}_I$, *induces an isomorphism on* $\pi_k$ *if* $n \geqq 3k + 11$ *and an epimorphism if* $n = 3k + 10$.

*Proof.* Consider the diagram

$$\begin{array}{ccc} \bar{\mathcal{E}}(M \times I, M) & \hookrightarrow & \mathcal{S}(M) \\ \sigma\downarrow & & \sigma\downarrow \\ \bar{\mathcal{E}}(M \times I \times I, M \times I) & \hookrightarrow & \mathcal{S}(M \times I) \end{array}$$

where $\sigma$ takes $X \subset \mathbf{R}^\infty$ to $X \times I \subset \mathbf{R}^\infty \times I \subset \mathbf{R}^\infty \times \mathbf{R} = \mathbf{R}^\infty$. On $\mathcal{S}$, $\sigma$ is clearly a homotopy equivalence since the projection $X \times I \to X$ is a contractible mapping. □

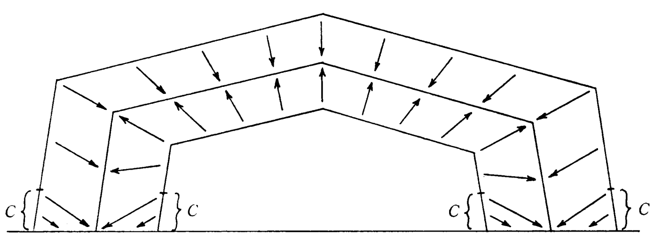
We approach the proof of the theorem by giving another version of $B\mathrm{PL}(W, M)$ in terms of handlebodies. $\mathrm{PL}(W, M)$ acts by composition on the space $\mathcal{G}(W, M)$ of handlebody structures on $(W, M)$, $f \in \mathrm{PL}(W, M)$ carrying $h \in \mathcal{G}(W, M)$ to its image $f(h)$ under $f$. Then using the diagonal action we have a principal fibration

$$\mathrm{PL}(W, M) \longrightarrow \mathcal{E}(W, M) \times \mathcal{G}(W, M) \longrightarrow \overline{\mathcal{E} \times \mathcal{G}}(W, M) .$$

By 8.1, this is also universal for $\mathrm{PL}(W, M)$. For each $(g, h) \in \mathcal{E}(W, M) \times \mathcal{G}(W, M)$ there is the induced handlebody structure $g(h)$ on $(g(W), g(M))$. This is invariant under the diagonal action of $\mathrm{PL}(W, M)$, so we can interpret

$\overline{\mathcal{E} \times \mathcal{G}}(W, M)$ as the space of handlebodies in $\mathbf{R}^\infty$ homeomorphic to $W$, collared on the fixed $M = g(M)$. Now let $W = M \times I$ and denote by $\mathcal{H}(M)$ the subspace of $\overline{\mathcal{E} \times \mathcal{G}}(M \times I, M)$ consisting of handlebodies whose collars agree with a fixed collar. Note that the inclusion $\mathcal{H}(M) \hookrightarrow \overline{\mathcal{E} \times \mathcal{G}}(M \times I, M)$ is a homotopy equivalence. (A collar on $M$ in $W \subset \mathbf{R}^\infty$ is essentially a path of embeddings $M \to \mathbf{R}^\infty$ having a fixed initial point, and the space of such paths is clearly contractible.)

Given a handlebody structure $h$ on $(W, M)$ there is a well-known way of associating to it a PL cell complex, in the sense of Section 4, by collapsing each handle to its core and collapsing the collar $M \times I$ to $M$. More precisely, a handle $D^i \times D^{n+1-i}$ collapses to $D^i \times \{*\} \cup \partial D^i \times D^{n+1-i}$, so there are some choices involved: first the point $* \in \mathring{D}^{n+1-i}$ and then the actual collapse $D^i \times D^{n+1-i} \to D^i \times \{*\} \cup \partial D^i \times D^{n+1-i}$, which depends on choosing a small collar $C$ on $\partial D^i \times \partial D^{n+1-i}$ in $D^i \times \partial D^{n+1-i}$ to collapse to $\partial D^i \times D^{n+1-i}$ (see the figure below). However, the space of such choices is contractible.




Moreover, collapsing to their cores the handles of a $k$-parameter family of handlebodies gives rise to a $k$-parameter family of cell complexes $L_t$ as in Section 4. Thus we have a map $c\colon \mathcal{H}(M) \to \mathcal{S}_0^\infty(M)$, determined up to homotopy—the aforementioned choices made in collapsing a handle to its core. Letting $\mathcal{H}_i^j(M) \subset \mathcal{H}(M)$ denote the subspace of handlebodies whose handles have indices in the range $[i, j]$, we have a diagram

$$\begin{array}{ccc} \bar{\mathcal{E}}(M \times I, M) & \hookrightarrow & \mathcal{S}(M) \\ \simeq\uparrow & & \simeq\uparrow \\ \mathcal{H}(M) & \xrightarrow{\;c\;} & \mathcal{S}_0^\infty(M) \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ \mathcal{H}_i^{i+1}(M) & \xrightarrow{\;c\;} & \mathcal{S}_i^{i+1}(M) . \end{array}$$

Since collapsing handles to their cores is a contractible mapping, the diagram is homotopy commutative.

PROPOSITION 9.3. *$\mathcal{H}_i^{i+1}(M) \hookrightarrow \mathcal{H}(M)$ is $k$-connected if $k + 2 \leqq i \leqq n - k - 2$.*

PROPOSITION 9.4. *$\mathcal{H}_i^{i+1}(M) \xrightarrow{c} \mathcal{S}_i^{i+1}(M)$ is $k$-connected (onto the identity component of $\mathcal{S}_i^{i+1}(M)$) if $2i < n - k$ and $n \geqq 5$.*

These two propositions prove the theorem: choose $i = k + 2$ and use the fact that $\mathcal{S}_i^{i+1}(M) \hookrightarrow \mathcal{S}_0^{\infty}(M)$ is $(i - 1)$-connected by 6.1.

The analogue of 9.3 in the smooth category, with handlebodies replaced by $\mathcal{C}^\infty$ functions and gradient-like vector fields, is known only for small values of $k$ (see § 5.3 of [11] for the cases $k = 1, 2$). This is because the behavior of singularities of $\mathcal{C}^\infty$ functions of higher codimension is not well understood. In the PL category there is no such problem since, roughly speaking, one can always just cone off by the Alexander trick—PL singularities are very "flabby". Perhaps there is an alternate approach in the smooth category which avoids the mysteries of $\mathcal{C}^\infty$ singularities.

To prove 9.3 it will suffice to show:

$$(*) \qquad \mathcal{H}_{i+1}^j(M) \hookrightarrow \mathcal{H}_i^j(M) \text{ is } k\text{-connected if } i \leqq n - k - 3$$

since by working with dual handlebodies the same argument shows that $\mathcal{H}_i^j(M) \hookrightarrow \mathcal{H}_i^{j+1}(M)$ is highly connected. The proof of $(*)$ consists of translating the proof of 6.2 into handlebody terms. The main difficulty is to deform attaching maps of cells into embeddings, which are needed in order to attach handles. The technique for doing this is contained in the following handlebody analogue of 6.3.

LEMMA 9.5. *Suppose the handle $D_t^i \times D_t^{n+1-i}$ in the family $W_t \in \mathcal{H}_i^j(M)$, $t \in \Delta^k$, collapses over $\partial\Delta^k$. If its attaching map $\varphi_t$: $S^{i-1} \times D^{n+1-i} \to \partial W_t^{(*)}$ is null-homotopic in $W_t^{(*)}$ by a homotopy which also collapses to a point over $\partial\Delta^k$ and if $i \leqq n - k - 3$, then the given $i$-handle can be replaced by an $(i + 2)$-handle over $\Delta^k$.*

*Proof*. The hypothesis on $\varphi_t$ gives a family $\psi_t$: $D^{i+1} \to W_t$ such that, if we write $\partial D^{i+1} = D_+^i \cup D_-^i$,

(i) $\psi_t \mid D_+^i$ is a slice $D_t^i \times \{x\}$ for some $x \in \partial D_t^{n+1-i}$;

(ii) $\psi_t(D_-^i) \subset W_t^{(*)}$;

(iii) $\psi_t(D^{i+1})$ is a point over $\partial\Delta^k$.

We claim that $\psi_t$ can be improved to satisfy also

(iv) $\psi_t$ is an embedding into a level $\partial W_t^{(\cdot)}$ in each $t$-slice, $t \in \mathring{\Delta}^k$ (with the superscript $(\cdot)$, like $(*)$, depending on $t$);

(v) $\psi_t(\partial D^{i+1}) \cap (D_t^i \times D_t^{n+1-i}) = D_t^i \times \{x\}$.

If we assume this, the given $i$-handle can be traded for an $(i + 1)$-handle over $\Delta^k$ just as in the unparametrized case: Introduce a trivial $(i + 1, i + 2)$-handle pair at the level $\partial W_t^{(\cdot)}$, use $\psi_t$ to isotope the attaching map of the

$(i+1)$-handle to be $\psi_t \mid \partial D^{i+1}$, then cancel the given $i$-handle with the new $(i+1)$-handle, using $\psi_t \mid D^i_-$ to isotope the $i$-handle to a trivially attached handle.

Now we show how to achieve (iv) and (v). In general position $\psi_t(D^{i+1})$ will be disjoint from the center points of all handles provided $i < n - k$. Then we can push $\psi_t(D^{i+1})$ into $\bigcup_l \partial W_t^{(l)}$. Next, working one handle $D_t^j \times D_t^{n+1-j}$ at a time, in the reverse of the order of attaching, put $\psi_t$ in general position with respect to the sphere $\{*\} \times \partial D_t^{n+1-j}$ so that $\psi_t$ is an embedding there (the singularity set of $\psi_t$, $\Sigma_t$, is of dimension $2(i+k+1)-(n+k) < i \leqq j = \operatorname{codim}(\{*\} \times \partial D_t^{n+1-j})$ assuming $i \leqq n-k-3$). Then $\Sigma_t$ can be pushed off $D_t^j \times D_t^{n+1-j}$, and eventually down into $M \times \{1\}$. Let $I_t$ be the union of the transverse pullbacks $\psi_t^{-1}(\{*\} \times \partial D_t^{n+1-j})$ for the various handles $D_t^j \times D_t^{n+1-j}$. This has dimension $\leqq k+1$. We can assume that $\psi_t$ maps the complement of a neighborhood $N_t$ of the cone $cI_t \subset D^{i+1}$ into $M \times \{1\}$. Now replace $\psi_t$ by its restriction to the $(i+1)$-disc $N_t$. (Even though $cI_t$ may not vary continuously in $t$, the neighborhoods $N_t$ can be taken to be a nice PL family of discs in $D^{i+1}$.) This restriction to $N_t$ will be an embedding if we make $\Sigma_t \cap cI_t = \varnothing$, which requires $\operatorname{codim} \Sigma_t = (n+k)-(i+k+1) > \dim(cI_t) = k+2$, or $i < n-k-3$. We can improve this to $i \leqq n-k-3$ since condition (iv) can be weakened to require only that $\psi_t$ be an embedding on each concentric $S^i$ in $D^{i+1}$.

To push $\psi_t$ into a single level $\partial W_t^{(\cdot)}$ we must avoid situations where $\psi_t$ maps $D^{i+1}$ across the top of one handle, which attaches across a second handle, which attaches across a third handle, etc., the last handle in this chain attaching to the image of $\psi_t$ again. The coincidences of $\psi_t(D^{i+1})$ across a $j_1$-handle across . . . a $j_l$-handle are, in general position, of dimension $\leqq i + k - j_l + 2 - l$ in the parameter domain $\Delta^k$. Over this part of $\Delta^k$ the intersections of $\psi_t(D^{i+1})$ with the attaching sphere of the $j_l$-handle are of dimension $\leqq 2i + k - n + 2 - l$. So we can make $cI_t$ disjoint from these chains of coincidences (and hence excise them) provided $(k+2)+(2i+k-n+2-l) < i+k+1$, or $i < n-k-3+l$. This finishes 9.5. □

To complete the proof of 9.3 we must kill the obstructions to the attaching map of an $i$-handle being (slicewise) homotopically trivial. It was shown in the proof of 6.2 how this can be done homotopically by adding $(i+1)$- and $(i+2)$-cells. The technique in the proof of 9.5 for producing embeddings suitable for attaching handles works just as well here, to show that the obstructions can also be killed by introducing $(i+1)$- and $(i+2)$-handles. □

*Proof of* 9.4. Let $L_t \in \mathcal{S}_i^{i+1}(M)$ for $t \in \Delta^k$ have a lifting $W_t \in \mathcal{H}_i^{i+1}(M)$ for $t \in \partial\Delta^k$, which we wish to extend over $\Delta^k$. We can assume to start that collapsing handles of $W_t$ are attached after all other handles, since in general position the collapse points of $W_t$, of dimension $\leqq k-2$ over $\partial\Delta^k$, will be disjoint from (the core spheres of) attaching maps of non-collapsing handles if $(k-2)+(i+k-1)<n+k-1$, or $i<n-k+2$. Also, over $\Delta^k$ we can deform collapsing cells of $L_t$ to attach last. Now suppose inductively that we have thickened into handles all non-collapsing cells of $L_t$ up to a level $L_t^{(\cdot)}$, to form $W_t^{(\cdot)}$ over $\Delta^k$, and let $\varphi_t\colon S^i \to L_t^{(\cdot)} \simeq W_t^{(\cdot)}$ be the attaching map of the next $(i+1)$-cell (the case of an $i$-cell is similar). In general position $\varphi_t(S^i)$ will be disjoint from the cores of handles in $W_t^{(\cdot)}$ if $(i+k)+(i+1+k)<n+k+1$, or $2i<n-k$. Then $\varphi_t$ can be deformed into $\partial W_t^{(\cdot)}$, where handles are supposed to attach. Approximate $\varphi_t$ by an embedding (if $2(i+k)<n+k$, or $2i<n-k$ again). In order to use $\varphi_t$ to attach an $(i+1)$-handle we need a trivialization of the normal bundle $\nu(\varphi_t(S^i), \partial W_t^{(\cdot)})$. (Note that we are in the stable range.) Let $r_t\colon L_t \to M$ be a family of deformation retractions and consider $r_t^*(\tau(M))$ where $\tau(M)$ is the stable tangent bundle of $M$. Assume inductively that we have an isomorphism $r_t^*(\tau(M)) \approx \tau(W_t^{(\cdot)})$ on $W_t^{(\cdot)}$. The cell $e_t^{i+1}$ attached via $\varphi_t$ gives a trivialization of $r_t^*(\tau(M))$ on $\varphi_t(S^i)$ and of $\tau(\varphi_t(S^i))$, hence also of $\nu(\varphi_t(S^i), \partial W_t^{(\cdot)})$. So $e_t^{i+1}$ can be thickened to a handle, at least away from where it collapses.

To turn collapsing cells into collapsing handles, choose a triangulation of the set of collapse points and proceed by downward induction on the simplices of this collapse set as follows. For an $l$-simplex $\Delta^l$ (which we can identify with its projection to the parameter domain $\Delta^k$) one has given over the boundary of its dual cell $D^{k-l}$ in $\Delta^k$ layers $W_t^{(p)} \subset W_t^{(q)}$, with the handles of $W_t^{(q)} - W_t^{(p)}$ to be coned off at $\Delta^l \cap D^{k-l}$. Excising all but a neighborhood of these handles from $W_t^{(p)}$ leaves us with an $h$-cobordism $(V_t, D_t^n)$ in each $t$-slice of $\partial D^{k-l}$ since the corresponding cells of $L_t$ do collapse at $\Delta^l \cap D^{k-l}$. By the (simply-connected) $h$-cobordism theorem $(V_t, D_t^n) \approx (D_t^{n+1}, D_t^n)$ if $n \geqq 5$. Moreover, since $\mathrm{PL}(D^{n+1}, D^n)$ is contractible by the Alexander trick, we can choose trivializations of $(V_t, D_t^n)$ consistently for all $t \in \partial D^{k-l}$. This is just what is required for the coning off of the handles of $W_t^{(q)} - W_t^{(p)}$ over $D^{k-l}$. This completes the induction step and the proof of 9.4, hence of 9.1. $\square$

Compared with the proof of 9.3, the proof of 9.4 is rather crude, using only general position to embed below the middle dimension. It seems that the more delicate excision technique of 9.3 for deforming attaching maps into embeddings in levels $\partial W_t^{(\cdot)}$ does not work in the relative case when embeddings are given over $\partial\Delta^k$. Using the absolute case one can only show that

$$\pi_k\mathcal{H}_i^{i+1}(M) \longrightarrow \pi_k\mathcal{S}_i^{i+1}(M)$$

is onto (and presumably split) if $i \leqq n - k - 2$, hence that $\pi_{k-1}\mathcal{P}(M) \approx \pi_k\mathcal{H}(M)$ maps onto $\pi_k\mathcal{S}(M)$ if $n \geqq 2k + 4$.

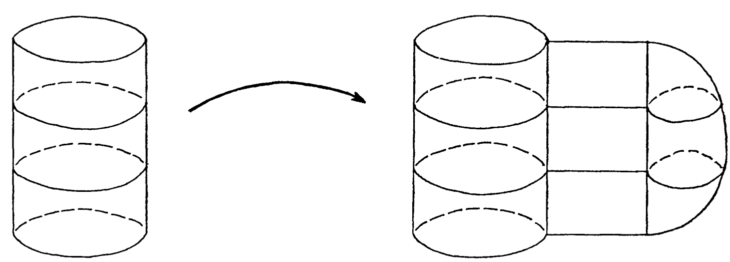
## 10. A few examples

From the pseudo-isotopy theorem ([11], [16]) we have[2]

$$\pi_1\mathcal{S}(K) \approx \mathrm{Wh}_2(\pi_1 K) \oplus \mathrm{Wh}_1^+(\pi_1 K; \mathbf{Z}_2 \times \pi_2 K) .$$

Let us consider the second summand $\mathrm{Wh}_1^+(\pi_1; \mathbf{Z}_2 \times \pi_2)$. Denote by $(\mathbf{Z}_2 \times \pi_2)[\pi_1]$ the additive abelian group of finite linear combinations $\sum \alpha_i\sigma_i$ for $\alpha_i \in \mathbf{Z}_2 \times \pi_2$ and $\sigma_i \in \pi_1$. Then $\mathrm{Wh}_1^+(\pi_1; \mathbf{Z}_2 \times \pi_2)$ is just the quotient group of $(\mathbf{Z}_2 \times \pi_2)[\pi_1]$ modulo the subgroup generated by elements of the form $\alpha\sigma - \alpha^\tau\tau\sigma\tau^{-1}$ and $\beta\cdot 1$, where $\alpha^\tau$ denotes the usual action of $\tau \in \pi_1$ on the $\pi_2$ component of $\alpha$ and the trivial action on the $\mathbf{Z}_2$ component.

A one-parameter family $L_t \in \mathcal{S}(K)$, $t \in [0, 1]$, representing a generator $\alpha\sigma$ of $\mathrm{Wh}_1^+(\pi_1 K; \mathbf{Z}_2 \times \pi_2 K)$ has the form $L_t = (K \vee S^i) \cup_{\varphi_t} e_t^{i+1}$, $i > 2$, the cell $e_t^{i+1}$ being attached by the level slices $\varphi_t$ of a map "proj $+\ \alpha\sigma$": $S^i \times I \to K \vee S^i$ which is the result of first pinching $S^i \times I$ as in the following picture,




then, on $S^i \times I$ projecting to $S^i \subset K \vee S^i$, on $I \times I$ wrapping each level segment around a loop representing $\sigma \in \pi_1 K$, and on $S^{i+1}$ taking the sum $\alpha_1 + [\alpha_2, \mathrm{id}_{S^i}]$, where $\alpha_1 \in \pi_{i+1}(S^i) \subset \pi_{i+1}(K \vee S^i)$ is the $\mathbf{Z}_2$ component of $\alpha$ and $[\alpha_2, \mathrm{id}_{S^i}]$ denotes Whitehead product with $\alpha_2$, the $\pi_2 K$ component of $\alpha$. Thus $L_0 = L_1$ and the family $L_t$ is a loop in $\mathcal{S}(K)$. If a loop based at the natural basepoint $K \in \mathcal{S}(K)$ is desired, just connect $L_0$ to $K$ by first pulling in the

[2] It appears now (May, 1975) the proof in Part II of [11] (Part II is not joint with Wagoner) that the $\pi_2$ part of the second obstruction is well-defined makes implicit use of an additional hypothesis on $K$, namely, that the first Postnikov invariant $k_1 \in H^3(\pi_1 K; \pi_2 K)$ of $K$ is zero. When $k_1$ is non-zero all that the arguments of [11] prove is that $\pi_1\mathcal{S}(K)$ maps onto $\mathrm{Wh}_2(\pi_1 K) \oplus \mathrm{Wh}_1^+(\pi_1 K; \mathbf{Z}_2)$ with kernel a quotient of $\mathrm{Wh}_1^+(\pi_1 K; \pi_2 K)$.

"tail" where $\varphi_0$ wraps around $\sigma$, then collapsing the resulting $K \vee D^{i+1}$ to $K$.

It is not too hard to figure out why loops $L_t$ constructed according to the above scheme for elements $\alpha\sigma - \alpha^{\tau}\tau\sigma\tau^{-1}$ and $\beta \cdot 1$ in $(\mathbf{Z}_2 \times \pi_2 K)[\pi_1 K]$ should be homotopically trivial in $\mathcal{S}(K)$. But to prove that this is exactly the kernel of the resulting map $(\mathbf{Z}_2 \times \pi_2 K)[\pi_1 K] \to \pi_1\mathcal{S}(K)$ is a real task; this is essentially the content of Part II of [11]. Computing the kernel of the $k$-parameter analogue $\pi_k^s(\Omega K) \to \pi_k\mathcal{S}(K)$ looks like a very tough problem.

Now let us compute the terms in the exact sequence of 3.2:

$$(1) \qquad \longrightarrow \pi_1 G(K) \longrightarrow \pi_1\mathcal{S}(K) \longrightarrow \pi_1\mathcal{S}_k \longrightarrow \pi_0 G(K) \longrightarrow \pi_0\mathcal{S}(K) \longrightarrow$$

in the case $K = S^1$. Since $S^1$ is a $K(\mathbf{Z}, 1)$ it is easy to check that $G(S^1) \simeq O(2)$, the orthogonal group. Thus $\mathcal{S}_{S^1} \to BG(S^1)$ has a section. Also, $\mathrm{Wh}_1(\mathbf{Z})$ and $\mathrm{Wh}_2(\mathbf{Z})$ are known to vanish. So (1) becomes, for $K = S^1$,

$$(2) \qquad \longrightarrow \mathbf{Z} \xrightarrow{\ 0\ } \frac{\mathbf{Z}_2[t, t^{-1}]}{\mathbf{Z}_2[1]} \longrightarrow \frac{\mathbf{Z}_2[t, t^{-1}]}{\mathbf{Z}_2[1]} \times_T \mathbf{Z}_2 \rightleftarrows \mathbf{Z}_2 \longrightarrow 0$$

where $T$ denotes the twisted product in which conjugation by the generator of $\mathbf{Z}_2$ interchanges $t$ and $t^{-1}$.

It is instructive to compare this with the case $K = S^1 \vee S^i$ with $i > 2$. Clearly $\pi_0 G(S^1 \vee S^i) \approx \mathbf{Z}_2 \times \mathbf{Z}_2$, generated by the degree $-1$ maps on $S^1$ and $S^i$. Also, one can check that $\pi_1 G(S^1 \vee S^i) \approx \mathbf{Z}_2[t, t^{-1}]$, generated by the loops $\psi_t = \mathrm{id}_{S^1} \vee \varphi_t$, where $\varphi_t$ is the family of maps $S^i \to S^1 \vee S^i$ described above. Lemma 10.1 below says that $\pi_1 G(S^1 \vee S^i) \to \pi_1\mathcal{S}(S^1 \vee S^i)$ is surjective, so the sequence (1) becomes now

$$(3) \qquad \longrightarrow \mathbf{Z}_2[t, t^{-1}] \xrightarrow{\text{proj}} \frac{\mathbf{Z}_2[t, t^{-1}]}{\mathbf{Z}_2[1]} \xrightarrow{\ 0\ } \mathbf{Z}_2 \times \mathbf{Z}_2 \xrightarrow{\ \approx\ } \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow 0\,.$$

Thus, although $\pi_*\mathcal{S}(K)$ depends only on the low dimensional skeletons of $K$, the maps in (1) depend on all of $K$.

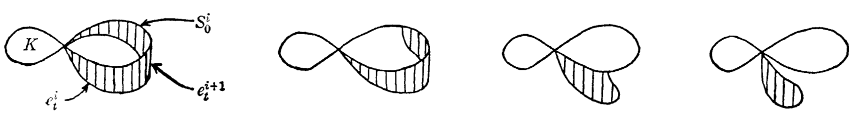
The interpretation of (2) and (3) in terms of homeomorphisms of Hilbert cube manifolds (see the introduction) is somewhat curious. On $S^1 \times Q$ there are many, many homeomorphisms which are homotopic but not isotopic, but on $(S^1 \vee S^i) \times Q$ for $i > 2$ there are none.

LEMMA 10.1. *The image of* $\pi_1 G(K \vee S^i) \to \pi_1\mathcal{S}(K \vee S^i)$ *contains the summand* $\mathrm{Wh}_1^+(\pi_1; \mathbf{Z}_2 \times \pi_2)$ *if* $i > 2$.

*Proof*. The map $G(K) \to \mathcal{S}(K)$ sends $f\colon K \to K$ to its mapping cylinder $M(f) = \big(K \times [0, 1] \cup K\big)/\big(x \times 1 \sim f(x)\big)$, considered as lying in $\mathcal{S}(K)$ by identifying $K$ with $K \times 0 \subset M(f)$. In the case at hand $\psi_t\colon K \vee S^i \to K \vee S^i$ is the identity on $K$, so we may collapse $K \times [0, 1] \subset M(\psi_t)$ to $K \times 0$ by a contractible mapping. The resulting $L_t$ has the form $L_t = (K \vee S_0^i \vee S_1^i) \cup_{\gamma_t} e_t^{i+1}$, where $S_0^i$ and $S_1^i$ correspond to $S^i \times 0$ and $S^i \times 1$ in $M(\psi_t)$ and $\gamma_t$ is a level slice of the map

$$\text{proj}_0 + \text{proj}_1 + (\alpha\sigma)_1 \colon S^i \times I \longrightarrow K \vee S_0^i \vee S_1^i \,.$$

Now we consider $S_1^i$ as a trivially attached $i$-cell $e_t^i$, and, simultaneously for each $t \in [0, 1]$, we deform its attaching map across $S_0^i$, producing again a trivially attached $e_t^i$. But on examining what has happened to $\gamma_t$ we find that $\text{proj}_0 + \text{proj}_1 + (\alpha\sigma)_1$ has changed to $(\alpha\sigma)_0 + \text{proj}_1 + (\alpha\sigma)_1$. The following diagrams depict this change at $t = 0$ (or 1), when $L_0$ changes to $K \vee S_0^i \vee D^{i+1}$.




We can assume that $(\alpha\sigma)_0$ is added to $\text{proj}_1$ in the $t$-interval $[0, 1/2]$ and that $(\alpha\sigma)_1$ is added in $[1/2, 1]$. Then for $t \in [0, 1/2]$ the pair $(e_t^{i+1}, e_t^i)$ can be collapsed by an elementary collapse. The remaining $L_t$ for $t \in [1/2, 1]$ is just the family called $L_t$ at the beginning of this section, with $K$ replaced now by $K \vee S^i$. $\square$

Finally, we show that $\pi_i \mathbb{S}(K)$ is a direct summand of $\pi_{i+1}\mathbb{S}(K \times S^1)$, giving a cheap way to mass produce elements of $\pi_* \mathbb{S}(-)$ from the known models in $\pi_0$ and $\pi_1$. (The technique is also known, independently, to Hsiang-Sharpe and Burghelea-Lashof-Rothenberg.)

PROPOSITION 10.2. *For a compact manifold $M$, $\mathcal{P}(M \times I)$ is a homotopy retract of $\Omega\mathcal{P}(M \times S^1)$. Hence $\mathbb{S}(K)$ is a homotopy retract of $\Omega\mathbb{S}(K \times S^1)$.*

*Proof.* Let us change notation slightly and let $\mathcal{P}(M)$ denote pseudo-isotopies $M \times I \to M \times I$ fixed on $M \times 0 \cup \partial M \times I$, rather than just on $M \times 0$. This new $\mathcal{P}(M)$ is homeomorphic to the old one since $(M \times I, M \times 0) \approx (M \times I, M \times 0 \cup \partial M \times I)$. Also, for $M$ a proper submanifold of some $V$ (i.e., $M \cap \partial V = \partial M$) denote by $\mathcal{P}(M, V)$ the space of proper embeddings $M \times I \to V \times I$ agreeing with the given $M \to V$ on $M \times 0 \cup \partial M \times I$, and such that $M \times 1 \to V \times 1$. We will be interested in the following diagram:

$$\begin{array}{ccccc}
\Omega\mathcal{P}(M \times S^1) & & \xrightarrow{\Omega\lambda_1} & & \Omega\mathcal{P}(M \times \mathbf{R}) \\
 & \nwarrow^{\tau} & & \nearrow_{\simeq} & \\
\Omega\rho_1 \Big\downarrow & & \mathcal{P}(M \times I) & & \simeq \Big\downarrow \Omega\rho_2 \\
 & \swarrow & & \searrow^{\simeq}_{\sigma} & \\
\Omega\mathcal{P}(M, M \times S^1) & & \xrightarrow{\Omega\lambda_2} & & \Omega\mathcal{P}(M, M \times \mathbf{R}) \,.
\end{array}$$

The maps $\lambda_1$ and $\lambda_2$ are obtained by lifting to the cover $M \times \mathbf{R} \to M \times S^1$, while the fibrations $\rho_1$ and $\rho_2$ come from restriction to $M \times 1 \subset M \times S^1$ and $M \times 0 \subset M \times \mathbf{R}$. The fiber of $\rho_2$ is clearly contractible, so $\Omega\rho_2$ is a homotopy equivalence. The fiber of $\rho_1$ can be identified with $\mathcal{P}(M \times I)$, and via this identification, $\tau$ is defined by taking $f \in \mathcal{P}(M \times I) \subset \mathcal{P}(M \times S^1)$, forming the commutator $\theta f \theta^{-1} f^{-1}$ where $\theta: M \times S^1 \times I \circlearrowleft$ is rotation through the angle $\theta \in S^1$, then considering $\theta f \theta^{-1} f^{-1}$ as a loop by letting $\theta$ run around $S^1$. The two unlabelled maps and $\sigma$ are defined to make the diagram commute.

For example, $\sigma$ is restriction of a pseudo-isotopy $(M \times I) \times I \circlearrowleft$ to the family of slices $(M \times s) \times I \subset (M \times I) \times I \subset (M \times \mathbf{R}) \times I$ for $0 \leqq s \leqq 1$, followed by the translation $x \to x - s$ in the $\mathbf{R}$ direction. With this description we recognize $\sigma$ as a section for the map $\partial$ in the following fibration sequence (identifying $(\mathbf{R}, 0)$ with $((0, \infty), 1)$):

$$\Omega\mathcal{P}(M, M \times (0, \infty)) \xrightarrow{\partial} \mathcal{P}(M \times I) \longrightarrow E \xrightarrow{\rho} \mathcal{P}(M, M \times (0, \infty)) .$$

Here $E$ is the space of proper embeddings $M \times [0, 1] \times I \to M \times [0, \infty) \times I$ which are the identity on $M \times 0 \times I \cup M \times [0, 1] \times 0 \cup \partial M \times [0, 1] \times I$ and carry $M \times [0, 1] \times 1$ to $M \times [0, \infty) \times 1$. The map $\rho$ is restriction to $M \times 1 \times I$; that $\rho$ is a fibration depends crucially on the compactness of $M$.

Since $E$ is evidently contractible (essentially by restriction to the segments $M \times [0, s] \times I$), $\partial$ and $\sigma$ are homotopy inverses, and we have shown that $\mathcal{P}(M \times I) \simeq \Omega\mathcal{P}(M \times \mathbf{R})$ is a homotopy retract of $\Omega\mathcal{P}(M \times S^1)$. These constructions respect the stabilization $M \to M \times I \to M \times I^2 \to \cdots$, so passing to the limit we get $\Omega\mathcal{S}(M)$ a homotopy retract of $\Omega^2\mathcal{S}(M \times S^1)$. To deloop this we appeal to [14] (which inspired the present proposition) for the assertion that $\pi_0\mathcal{S}(M) \approx \mathrm{Wh}_1(\pi_1 M)$ is a direct summand of $\pi_1\mathcal{S}(M \times S^1) \approx \pi_0\mathcal{P}(M \times S^1)$. □

*Remarks.* 1. The proposition holds also in the smooth and topological categories, with the same proof.

2. The pseudo-isotopy spaces $\mathcal{P}(M)$ and $\mathcal{P}(M, V)$ can be replaced by the spaces of homeomorphisms $M \to M$ and embeddings $M \to V$, both rel $\partial M$.

3. The map $\lambda_2$ has a section induced by embedding $\mathbf{R}$ in $S^1$. Burghelea-Lashof-Rothenberg have shown that $\rho_1$ also splits. This is clear if $M = N \times I$ for some $N$—an inclusion $I \times S^1 \subset I \times I$ gives $N \times I \times S^1 \subset N \times I \times I$, hence a projection of $\mathcal{P}(N \times I \times S^1)$ onto the fiber of $\rho_1$, $\mathcal{P}(N \times I \times I)$. Thus we get a formula

$$\Omega\mathcal{S}(M \times S^1) \simeq \Omega\mathcal{S}(M) \times \mathcal{S}(M) \times \Omega\mathfrak{N}(M)$$

where $\mathfrak{N}(M)$ is the fiber of the stabilized $\lambda_2$. Presumably the relationship of this to a well-known formula in algebraic $K$-theory for $K_*A[t, t^{-1}]$ is more than coincidental.

PRINCETON UNIVERSITY

REFERENCES

[1] E. AKIN, Transverse cellular mappings of polyhedra, Trans. A.M.S. **169** (1972), 401–438.

[2] D. BURGHELEA and R. LASHOF, The homotopy type of the space of diffeomorphisms, Trans. A.M.S. **196** (1974), 1–50.

[3] T. A. CHAPMAN, Cell-like mappings of Hilbert cube manifolds: applications to simple homotopy theory, Bull. A.M.S. **79** (1973), 1286–1291.

[4] ———, Topological invariance of Whitehead torsion, Am. J. of Math. **96** (1974), 488–497.

[5] ———, Concordances of Hilbert cube manifolds, preprint.

[6] ———, Homotopic homeomorphisms of Hilbert cube manifolds, Proc. Park City (Utah) Topology Conference 1974, to appear.

[7] M. M. COHEN, Simplicial structures and transverse cellularity, Ann. of Math. **85** (1967), 218–245.

[8] ———, *A course in simple-homotopy theory*, Springer-Verlag, 1973.

[9] A. DOLD, Partitions of unity in the theory of fibrations, Ann. of Math. **78** (1963), 223–255.

[10] B. ECKMANN and S. MAUMARY, Le groupe des types simples d'homotopie sur un polyèdre, in: *Essays on Topology and Related Topics*, ed. A. Haefliger and R. Narasimhan, Springer-Verlag, 1970, 173–187.

[11] A. HATCHER and J. WAGONER, *Pseudo-isotopies of compact manifolds*, Astérisque **6** (1973).

[12] J. P. MAY, The geometry of iterated loopspaces, Springer Lecture Notes 271, 1972.

[13] L. C. SIEBENMANN, Infinite simple homotopy types, Indag. Math. **32** (1970), 479–495.

[14] ———, Torsion invariants for pseudo-isotopies of closed manifolds, Notices A.M.S. **14** (1967), 942.

[15] R. STÖCKER, Whiteheadgruppe topologischer Räume, Invent. Math. **9** (1970), 271–278.

[16] I. A. VOLODIN, Generalized Whitehead groups and pseudo-isotopies, Uspekhi Math. Nauk **27** #5 (1972), 229–230.

[17] J. H. C. WHITEHEAD, Simple homotopy types, Amer. J. of Math. **72** (1950), 1–57.

[18] K. IGUSA, Princeton Ph. D. Thesis, 1975.

(Received March 19, 1975)