PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

MAPPINGS BETWEEN ANRs THAT ARE FINE HOMOTOPY EQUIVALENCES

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It is shown in this note that every closed UV^∞ - map between separable ANRs is a fine homotopy equivalence.

We extend Lacher's result [6,7] that a closed UV^∞-map between locally compact, finite dimensional ANRs is a fine homotopy equivalence to the case of arbitrary separable ANRs. It is hoped that this theorem will be useful in studying manifolds modelled on the Hilbert Cube. (See [1], section PF3. Added in proof. See also [9]).

A set $A \subset X$ has property UV^∞ if for each open set U of X containing A, there is an open V, with $A \subset V \subset U$ such that V is null-homotopic in U. A mapping $f: X \to Y$ of X onto Y is a UV^∞-map if for each $y \in Y, f^{-1}(y)$ is a UV^∞ subset of X. The mapping f is said to be closed if the image of every closed set is closed and proper if the inverse image of every compact set is compact. An absolute neighborhood retract for metric spaces is denoted an ANR. If α is a cover of Y and g_1 and g_2 are maps of a space A into Y, g_1 is α-near g_2 if for each $a \in A$ there is a $U \in \alpha$ containing $g_1(a)$ and $g_2(a)$. The map g_1 is α-homotopic to $g_2, g_1 \overset{\alpha}{\simeq} g_2$, if there is a homotopy $\lambda: A \times I \to Y$ taking g_1 to g_2 with the property that for each $a \in A$ there exists $U \in \alpha$ containing $\lambda(\{a\} \times I)$. A map $f: X \to Y$ is a fine homotopy equivalence if for each open cover, α, of Y there exists a map $g: Y \to X$ such that $fg \overset{\alpha}{\simeq} id_Y$ and $gf \overset{f^{-1}(\alpha)}{\simeq} id_X$.

Various versions of Lemma 3 have been proven by Smale [8], Armentrout and Price [2], Kozlowski [5] and Lacher [6]. The difference in this lemma is that K is not required to be a finite dimensional complex.

Let K be a locally finite complex and j be a nonnegative integer. When there is no confusion we will not distinghish between the complex K and its underlying point set $|K|$. If σ is a simplex of K, then $N(\sigma, K) = \{\tau < K \mid \sigma \cap \tau \neq \phi\}$ and $\text{st}(\sigma, K) = \{\tau < K \mid \sigma < \tau\}$. Also K^j will denote the j-skeleton of K and ${}^jK = \{\sigma < K \mid |N(\sigma, K)| \subset |K^j|\}$. Let $\mathcal{U}$ be a covering of a space Y and B a subset of Y. The star of B with respect to $\mathcal{U}$, $\text{st}^1(B, \mathcal{U})$, is the set $\{U \in \mathcal{U} \mid B \cap U \neq \phi\}$. Inductively, $\text{st}^n(B, \mathcal{U})$ is defined to be $\text{st}(\text{st}^{n-1}(B, \mathcal{U}))$. A covering $\mathcal{V}$ is called a star^n refinement of $\mathcal{U}$ if the covering $\{\text{st}^n(V, \mathcal{V}) / V \in V\}$ refines $\mathcal{U}$. Every open covering of a metric space has an open star^n refinement for each positive integer n (c.f. [3]). We start by stating without proof two easily verified lemmas.

LEMMA 1. *Let K be a locally finite complex. Suppose $\phi: K \to Y$ is a map, $\mathcal{U}$ is an open cover of Y, and k is a nonnegative integer. Then there is a subdivision $\tilde{K}$ of K so that:*

(a) *if σ is a k-simplex of $\tilde{K}$, then $\phi(N(\sigma,\tilde{K})) \subset U$, for some $U \in \mathcal{U}$,*

(b) *if $\sigma < {}^{k-1}K$, then $\sigma < \tilde{K}$.*

We will call such a subdivision, $\tilde{K}$, a $(k,\mathcal{U})$-subdivision of K. We note that for any vertex, v, of $\tilde{K}$ with $v \not\in {}^{k-1}K$ it follows that $\phi(\mathrm{st}(v,\tilde{K})) \subset U$ for some $U \in \mathcal{U}$.

LEMMA 2. *Let $\mathcal{U}$ be an open cover of the paracompact space Y and $f: X \to Y$ a closed UV^∞-map. Then there is an open locally finite refinement $\mathcal{V}$ of $\mathcal{U}$ such that for each $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ satisfying*

(a) $\mathrm{st}(V,\mathcal{V}) \subset U$

(b) *if m is a positive integer and the map $\gamma: \partial B^m \to f^{-1}(\mathrm{st}(V,\mathcal{V}))$ is given, then γ can be extended to $\bar{\gamma}: B^m \to f^{-1}(U)$.*

We will call such a refinement, $\mathcal{V}$, a UV^∞ star refinement of $\mathcal{U}$.

LEMMA 3. *Let $f: X \to Y$ be a closed UV^∞-map of an arbitrary space, X, onto the paracompact space Y. Let K be a locally finite complex and J a subcomplex of K. Let $\phi: K \to Y$ and $\psi': J \to X$ be mappings such that $f\psi' = \phi | J$. Then given any open cover, α, of Y there exists a map $\psi: K \to X$ extending ψ' so that $f\psi$ is α-near ϕ.*

Proof. Let K_0 be a $(0,\alpha)$-subdivision of K and let $\alpha_0 = \alpha$. Define inductively a sequence of covers of Y, $\{\alpha_i\}_{i=0}^\infty$, and subdivisions of K_0, $\{K_i\}_{i=0}^\infty$, such that for each $i > 0, \alpha_i$ is a UV^∞ star refinement of α_{i-1} and K_i is an (i,α_i)-subdivision of K_{i-1}.

Define $\psi_0: K_0^0 \to X$ by letting $\psi_0(v) = \psi'(v)$ if $V \in J$ and otherwise an arbitrary element of $f^{-1}(\phi(v))$. Assume inductively that there exist maps $\{\psi_i: K_i^i \to X\}_{i=0}^n$ such that for $0 \leqq i \leqq n$:

(1) $\psi_i | J \cap K_i^i = \psi' | J \cap K_i^i$ and if $j < i$, $\psi_i | {}^jK_j = \psi_j | {}^jK_j$,

(2) if v is a vertex of K_i, $\psi_i(v) \in f^{-1}(\phi(v))$,

(3) if σ is a j-simplex of K_i^i and $k = \dim \mathrm{st}(\sigma, K_i^i)$, then $\phi(\mathrm{st}(\sigma,K_i)) \cup f\psi_i(\sigma) \subset U$, for some $U \in \alpha_{k-j}$.

[Note that $\psi_0: K_0^0 \to X$ satisfies these conditions since if σ is a 0-simplex of K_0^0 the dimension of $\mathrm{st}(\sigma, K_0^0)$ is 0 and the fact that K_0 is a $(0,\alpha_0)$-subdivision of K implies that $\phi(\mathrm{st}(\sigma,K_0)) \cup f\psi_0(\sigma) \subset U$ for some $U \in \alpha_0$.]

We wish now to define $\psi_{n+1}: K_{n+1}^{n+1} \to X$ satisfying conditions (1)-(3) for $i = n+1$. For each vertex v of K_{n+1}, let

$$\psi_{n+1}(v) = \begin{cases} \psi_n(v), \text{ if } v \text{ is a vertex of } {}^nK_n \\ \psi'(v), \text{ if } v \in J \\ \text{an arbitrary element of } f^{-1}(\phi(v)), \text{ otherwise} \end{cases}$$

Assume (subinductive statement) that $\psi_{n+1} | K^r_{n+1}$ has been defined so that

(1′) $\psi_{n+1} | J \cap K^r_{n+1} = \psi' | J \cap K^r_{n+1}$ and $\psi_{n+1} | {}^nK_n \cap K^r_{n+1} = \psi_n | {}^nK_n \cap K^r_{n+1}$,

(2′) if v is a vertex of K_{n+1}, $\psi_{n+1}(v) \in f^{-1}(\phi(v))$,

(3′) if σ is a j-simplex of K^r_{n+1} and $k = \dim \operatorname{st}(\sigma, K^{n+1}_{n+1})$, then $\phi(\operatorname{st}(\sigma, K_{n+1})) \cup f\psi_{n+1}(\sigma) \subset U$, for some $U \in \alpha_{k-j}$.

[Note that $\psi_{n+1} | K^0_{n+1}$ has been defined in such a manner that properties (1′)-(3′) are satisfied. Properties (1′) and (2′) follow immediately from the definition. Let v be a simplex of K^0_{n+1}. If v is a vertex of ${}^nK_n$, then property (3′) follows from the fact that ψ_n satisfies property (3) of the main inductive statement since in this case $\dim \operatorname{st}(v, K^{n+1}_{n+1}) = \dim \operatorname{st}(v, K^n_n)$. Suppose v is not a vertex of ${}^nK_n$. By the remark following Lemma 1, $\phi(\operatorname{st}(v, K_{n+1}))$ is contained in some element of α_{n+1} and hence property (3′) is again satisfied.]

Now let σ be an $(r+1)$-simplex of K_{n+1}. If σ is a subset of J, let $\psi_{n+1} | \sigma = \psi' | \sigma$. If $\sigma < {}^nK_n$, let $\psi_{n+1} | \sigma = \psi_n | \sigma$. Otherwise, let $k = \dim \operatorname{st}(\sigma, K^{n+1}_{n+1})$. For each r-simplex, τ, in $\partial\sigma$, there is a $u_\tau \in \alpha_{k-r}$ containing $\phi(\operatorname{st}(\tau, K_{n+1})) \cup f\psi_{n+1}(\tau)$. Let τ' be a fixed r-simplex in $\partial\sigma$ and note that $\psi_{n+1}(\partial\sigma) \subset f^{-1}(\operatorname{st}(u_{\tau'}, \alpha_{k-r}))$. Since α_{k-r} is a UV^∞ star refinement of α_{k-r-1}, there is a $U \in \alpha_{k-r-1} = \alpha_{k-(r+1)}$ containing $\operatorname{st}(U_{\tau'}, \alpha_{k-r})$ and an extension of $\psi_{n+1} | \partial\sigma$ which maps σ into $f^{-1}(U)$. We call this extension ψ_{n+1} and note that $\phi(\operatorname{st}(\sigma, K_{n+1})) \cup f\psi_{n+1}(\sigma) \subset U$. In this manner, extend ψ_{n+1} to K^{r+1}_{n+1} and note that conditions (1′)-(3′) are satisfied. This completes the subinductive argument and hence the main inductive argument.

We now define $\psi: K \to X$ by $\psi(x) = \lim_{n\to\infty} \psi_n(x)$. For any $x \in K$, the local finiteness of K assures that there exists an integer N so that $x \in {}^NK_N$. Hence for $n \geqq N$, $\psi_n(x) = \psi_N(x)$. Therefore ψ is well-defined and continuous. Let $x \in K$ and let σ be a simplex of maximal dimension containing x. Then there exists an integer N such that $|\sigma| \subset {}^NK_N$. Choose a simplex B in ${}^NK_N$ containing x and note that $\psi(x) = \psi_N(x)$. By inductive statement (3), there is an open set $U \in \alpha_i$, for some $i \geqq 0$, such that $\phi(\operatorname{st}(B, K_N)) \cup f\psi(B) \subset U$. Since α_i refines $\alpha_0 = \alpha$, there is a $V \in \alpha$ such that $\{\phi(x)\} \cup \{f\psi(x)\} \subset V$. Since ψ extends ψ', this completes the proof of Lemma 3.

REMARK. By a slightly more cumbersome process, ψ can be chosen so that $f\psi$ is a α-homotopic to ϕ.

THEOREM. *Let X and Y be separable ANRs and $f: X \to Y$ be a closed UV^{∞}-map. Then f is a fine homotopy equivalence.*

Proof. Let α be an open cover of Y, Let α_1 be a star[5] refinement of α and α_2 a star refinement of α_1. Let β be an open refinement of α_2 such that any two β-near maps from any space into Y are α_2-homotopic (such refinements exists since Y is an ANR, c.f. **[4]**).

By Hanner's characterization of separable ANRs (c.f. **[4]**), there exist a locally finite polyhedron Q and maps $c: Q \to Y$ and $s: Q \to Y$ with property that $sc \overset{\alpha_2}{\simeq} id_Y$. By Lemma 3, there is a map $v: Q \to X$ such that fv is β-near s. Define $g: Y \to X$ by $g = vc$. Note that fg is β-near sc and hence $fg \overset{\alpha_2}{\simeq} sc$. But $sc \overset{\alpha_2}{\simeq} id_Y$ and hence $fg \overset{\alpha_1}{\simeq} id_Y$. Denote this α_1-homotopy by h; then, $h: Y \times I \to Y$ is a α_1-homotopy with $h_0 = id_Y$ and $h_1 = fg$.

It remains to be shown that gf is $f^{-1}(\alpha)$ homotopic to id_X.

Choose a locally finite polyhedron, P, maps $b: \to P$ and $r: P \to X$ and a homotopy $W: X \times I \to X$ with the following properties:

(a) $W_0 = rb$ and $W_1 = id_X$

(b) W is limited by $f^{-1}(\alpha_1)$ and by $(gf)^{-1}\ (f^{-1}(\alpha_1))$.

Next, define $H: P \times I \to Y$ by $H(p,t) = h_t(fr(p))$ and note that $H(p,0) = fr(p)$ and $H(p,1) = fgfr(p)$. Define $G': P \times \{0,1\} \to X$ by $G'(p,0) = r(p)$, $G'(p,1) = gfr(p)$. Then by Lemma 3 there is a map $G: P \times I \to X$ extending G' with the property that fG is α_1-near H.

Define $\psi: X \times I \to X$ by $\psi(x,t) = G(b(x),t)$.

Note that: $\psi_0(x) = G(b(x),0) = G'(b(x),0) = rb(x)$ and $\psi_1(x) = G(b(x),1) = G'(b(x),1) = gfrb(x)$.

Now, W is a homotopy taking rb to id_X and is limited by $f^{-1}(\alpha_1)$. Also, since W is limited by $(gf)^{-1}\ (f^{-1}(\alpha_1))$, $gfW: X \times I \to X$, defined by $gfW(x,t) = gf(W(x,t))$, is a homotopy taking $gfrb$ to gf and is limited by $f^{-1}(\alpha_1)$.

Recall that α_1 is a star[5] refinement of α. Therefore, to show that $id_X \overset{f^{-1}(\alpha)}{\simeq} gf$, it suffices to show that $f\psi: X \times I \to Y$ is limited by star[3] (α_1). Fix $x \in X$. Since the homotopy h is limited by α_1, there exists $U \in \alpha_1$ with $h(f(x) \times I) \subset U$. we claim that $f(\psi(x \times I)) \subset \mathrm{st}^3(U)$.

Fix $t \in I$. Recall $f(\psi(x,t)) = f(G(b(x),t))$. Thus there exists $U' \in \alpha_1$ such that $f^{-1}(U')$ contains x and $rb(x)$. Hence $f(x)$ and $frb(x)$ are elements of U' and $U \cap U' \neq \phi$. Since h is limited by α_1, we can choose $U'' \in \alpha_1$ so that $h_t frb(x)$ and $frb(x)$ are elements of U''. Note that $U'' \cap U' \neq \phi$. Also, there exists $U''' \in \alpha_1$ containing $H(b(x),t)$ and $f(G(b(x),t))$, since fG is α_1-near H. But $H(b(x),t) =$

$h_t frb(x)$. Hence $U''' \cap U'' \neq \phi$ and we have completed the proof of the theorem by showing that $f\psi: X \times I \to Y$ is limited by $\text{star}^3(\alpha_1)$.

Added in proof. I would like to thank Bob Edwards for some suggestions concerning this paper and for pointing out that George Kozlowski [Images of ANR's, to appear] has shown that a UV^∞-map between ANR's is a homotopy equivalence.

REMARK. If in addition it is assumed that X and Y are locally compact and f is a *proper* map it follows immediately that f is a proper fine homotopy equivalence.

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Received March 12, 1974 and in revised form July 20, 1974.

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