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Existence of irreducible representations for knot complements with nonconstant equivariant signature

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1 Introduction

The purpose of this paper is to establish the existence of irreducible $SU(2)$ representations of knot complement fundamental groups near abelian representations where the equivariant knot signature changes. For knots in S^3, Frohman and Klassen showed the existence of irreducible representations near abelian representations corresponding to simple roots of the Alexander polynomial. They raised the question whether an analogous result holds for multiple roots (see [F-K]). This paper gives a more general existence result for knots in homology spheres. It will be shown that if the Tristam-Levine equivariant signature of a knot takes different values on opposite sides of a root of the Alexander polynomial, then there exist irreducible representations near the corresponding abelian representation. This condition holds, for example, for any knot whose Alexander polynomial has a root of odd multiplicity on the unit circle in $\mathbf{C}$.

Let Y be the complement of an open tubular neighborhood of a knot in a homology sphere. Fix an orientation on Y. Choose a longitude λ for Y; this is a simple closed curve in ∂Y representing a primitive element of $H_1(\partial Y;\mathbf{Z})$ which is null homologous in Y. Also choose a meridian μ, i.e., a simple closed curve in ∂Y representing a generator for $H_1(Y;\mathbf{Z})$. We require that $\mu\cdot\lambda = 1$ with the induced orientation on $T^2 = \partial Y$. (We use the convention that $T(\partial Y)\oplus\langle\text{outward normal}\rangle = TY$.)

Let F be a Seifert surface with boundary λ, and choose an orientation of the normal bundle of F in Y. If $\{x_i\}_{1\le i\le g}$ is a basis for $H_1(F;\mathbf{Z})$, let x_i^+ denote the pushoff of x_i in the positive normal direction. Define the linking matrix V by $V_{ij} = \ell k(x_i, x_j^+)$.

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The symmetrized Alexander matrix for Y is the matrix

$$A(t) = t^{\frac{1}{2}} V - t^{-\frac{1}{2}} V^T.$$

Consider $B(t) = (1-t)V + (1-t^{-1})V^T$. Note that $B(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})A(t)$, so the complex values $t \neq \pm 1$ for which $B(t)$ is singular are exactly the roots of the Alexander polynomial $\Delta(t) = \det A(t)$.

If t is a unit complex number, then $B(t)$ is a Hermitian matrix and hence has only real eigenvalues. The equivariant knot signature of Y, denoted by $\mathrm{Sign}B(t^2)$, is the function from $U(1)$ to $\mathbf{Z}$ taking t to the number of positive eigenvalues minus the number of negative eigenvalues for $B(t^2)$, counted with multiplicity. (See [K-K-R] or [H2] for details.) This function is independent of the choice of F, $\{x_i\}$, and normal bundle orientation, and it changes sign if the orientation on Y is reversed. The relationship between $B(t)$ and the Alexander matrix implies that $\mathrm{Sign}B(t^2)$ is continuous in $t \in U(1)$ except possibly at square roots of roots of the Alexander polynomial. Note that $\mathrm{Sign}B(1) = 0$.

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the standard orthonormal basis for $su(2)$ corresponding to the identification of $SU(2)$ with the space of unit quaternions. We shall consider $U(1) = \{\exp(\mathbf{i}\theta)\} \subset SU(2)$, and we make the identifications $\mathrm{span}(\mathbf{i}) = \mathbf{R}$ and $\mathrm{span}(\mathbf{j}, \mathbf{k}) = \mathbf{C}$.

We now state the main result in this paper. For each $0 < \alpha < \pi$, let $\rho_\alpha : \pi_1 Y \to SU(2)$ be the abelian representation taking μ to $\exp(\mathbf{i}\alpha)$.

Theorem 1. *For any unit root $e^{i2\alpha}$ of $\Delta(t)$ where the right and left hand limits $\lim_{\beta\to\alpha^\pm} \mathrm{Sign}B(e^{i2\beta})$ do not agree, there is a continuous family of irreducible $SU(2)$ representations of $\pi_1(Y)$ limiting to ρ_α.*

Corollary 2. *For any odd multiplicity root $e^{i2\alpha}$ of $\Delta(t)$, there is a continuous family of irreducible $SU(2)$ representations of $\pi_1(Y)$ limiting to ρ_α.*

Remark: There are examples of prime knots for which the Alexander polynomial has only roots of odd multiplicity (greater than 1) on the unit circle and for which the Casson invariant $\lambda'(\kappa) = 0$. An example can be constructed as follows. Let $\Delta(t) = (t - 1 + t^{-1})^3(-t + 3 - t^{-1})^3$ be the Alexander polynomial of the composite of 3 trefoils and 3 figure eight knots. Kondo's construction in [Ko] gives a prime knot with this as its Alexander polynomial. The Casson invariant is 0 and $\det B_\kappa(t^2)$ changes sign at $t = e^{\frac{i\pi}{6}}$ since $\Delta(t^2)$ does. (The reason we look for a prime knot for an interesting example is that if either of the knots in a composite has a family of irreducible representations limiting to an abelian representation, then, by a simple gluing argument, the composite does also.)

In the course of proving the main result we shall also prove the following fact.

Corollary 3. *Suppose for some $0 < \alpha < \pi$ the matrix $B(e^{i2\alpha})$ has nontrivial kernel, and suppose that, as $t \in U(1)$ moves through the value $e^{i\alpha}$, all eigenvalues of $B(t^2)$ touching zero cross zero transversely, and all do so in the same direction. Then all of the irreducible representations ρ near ρ_α send λ to $\exp(\mathbf{i}\sigma(\rho))$ for some* *small $\sigma(\rho) \neq 0$, where the sign of $\sigma(\rho)$ corresponds to the direction the eigenvalues go through 0.*

Corollary 4. *If κ is a knot in a homology sphere and there exists any value $0 < \alpha < \pi$ satisfying the hypotheses of Corollary 3, then for n sufficiently large, both the homology spheres obtained by $\frac{1}{n}$ and $-\frac{1}{n}$ surgery on κ have nontrivial $SU(2)$ representations.*

We now outline the proof of the main theorem. First identify the set of representations of $\pi_1(Y)$ into $SU(2)$ modulo conjugation with the moduli space of flat $SU(2)$ connections on $Y \times SU(2)$ modulo gauge equivalence. We show that for arbitrarily small perturbations of the flatness equation the perturbed flat moduli space contains irreducible connections. Using a limiting argument, we show that this property continues to hold for the unperturbed flat moduli space.

The paper is organized as follows. Section 2 contains basic results about perturbing the flatness equation and the perturbed flat moduli space for 3-manifolds with torus boundary. Subsection 3.1 contains a statement of the basic existence theorem for irreducible perturbed flat connections under certain assumptions of nondegeneracy. Subsection 3.2 provides a proof of this result. A proof of Corollary 3 is also given in this subsection. Section 4 then provides proofs of our main result along with Corollaries 2 and 4.

2 The structure of the flat moduli space

We begin by describing $SU(2)$ gauge theory on 3-manifolds with torus boundary, recalling results from [H1].

Let $\mathcal{A}$ denote the space of connections on $Y \times SU(2)$. Given a fixed trivialization of this principal bundle, we may identify $\mathcal{A}$ with the space of $su(2)$ valued 1-forms on Y, $\Omega^1(Y; su(2))$. We complete this space using the L^2_2 Sobolev norm. Let $\mathcal{G} = \mathrm{Aut}(Y \times SU(2))$ be the gauge group, with the L^2_3 completion. To each connection A is associated its curvature 2-form, $F(A) = dA + A \wedge A$, and A is said to be flat if $F(A) = 0$.

The flat moduli space is the quotient $\mathcal{M} = F^{-1}(0)/\mathcal{G}$. There is a standard method of perturbing the flatness equation in order to obtain a moduli space which is nondegenerate (nondegenerate will be given a precise definition below), used, for example, in [T], [F], and [H1]. We sketch it below; see [H1] for more details.

Let $\{\gamma_i : S^1 \times D^2 \to Y\}_{1 \leq i \leq n}$ be a collection of embeddings of the solid torus into Y whose images are disjoint. Let η be the product of a nonnegative bump function on D^2 with support in the interior and the standard volume form on D^2. Let $\{\overline{h}_i : \mathbf{R} \to \mathbf{R}\}_{1 \leq i \leq n}$ be a collection of C^2 functions. Let $\mathrm{tr}\ \mathrm{hol}_{\gamma_i}(x, A)$ be the trace of the holonomy of the connection A around the curve $\gamma_i(S^1 \times \{x\})$. We define a function $h : \mathcal{A} \to \mathbf{R}$ by the formula

$$h(A) = \sum_{i=1}^{n} \int_{D^2} \overline{h}_i(\mathrm{tr}\ \mathrm{hol}_{\gamma_i}(x, A))\eta(x).$$

Definition 5. *A function h constructed in this way is called an* **admissible perturbation function***.*

Now fix a Riemannian metric on Y, and let $*$ denote the Hodge star operator on $su(2)$ valued forms. Given an admissible perturbation function h, we define

$$\zeta_h(A) = - * \frac{1}{2\pi} F(A) + \nabla h(A),$$

where ∇h denotes the L^2 gradient of h.

Definition 6. *A connection A is called* **perturbed flat** *if it satisfies the equation $\zeta_h(A) = 0$. The* **perturbed flat moduli space** *of Y is*

$$\mathcal{M}_h = \zeta_h^{-1}(0)/\mathcal{G}.$$

We shall sometimes write $\mathcal{M}_h(Y)$ to avoid confusion with the perturbed flat moduli space of the zero surgery on Y, which is denoted by $\mathcal{M}_h(Y_0)$.

The structure of the perturbed flat moduli space for a 3-manifold with boundary was described in [H1]. We now summarize the results for the case when the boundary is a torus. Let $*d_{A,h} = *\frac{1}{2\pi} d_A - \text{Hess}h(A)$. Let $\mathcal{H}^1_{A,h}(Y; su(2))$ and $\mathcal{H}^1_{A,h}(Y, \partial Y; su(2))$ be (the harmonic spaces representing) the first and second cohomology groups of the following elliptic complex (where the grading goes 0,1,2,3):

$$0 \to \Omega^0(Y; su(2)) \xrightarrow{d_A} \Omega^1(Y; su(2)) \xrightarrow{*d_{A,h}} \Omega^1(Y; su(2)) \xrightarrow{d_A^*} \Omega^0(Y; su(2)) \to 0.$$

Let $\mathcal{M}_h^*, \mathcal{M}_h^{U(1)}$, and $\mathcal{M}_h^{SU(2)}$ denote the portions of $\mathcal{M}_h$ consisting of irreducible, abelian (noncentral), and central orbits, respectively.

Definition 7. *The perturbed flat moduli space $\mathcal{M}_h$ is* **nondegenerate** *if it satisfies the following 5 properties (and otherwise* **degenerate***):*

(a) There are no noncentral orbits in $\mathcal{M}_h$ which are central when restricted to ∂Y.
(b) For every $[A] \in \mathcal{M}_h^{SU(2)}$, $\mathcal{H}^1_{A,h}(Y, \partial Y; su(2)) = 0$.
(c) For all but finitely many orbits $[A] \in \mathcal{M}_h^{U(1)}$, $\dim \mathcal{H}^1_{A,h}(Y, \partial Y; su(2)) = 0$, and for the remaining abelian orbits $\dim \mathcal{H}^1_{A,h}(Y, \partial Y; su(2)) = 2$ and $\text{Stab}A$ acts nontrivially on $\mathcal{H}^1_{A,h}(Y, \partial Y; su(2))$.
(d) At each abelian orbit $[A] \in \mathcal{M}_h^{U(1)}$ with $\dim \mathcal{H}^1_{A,h}(Y, \partial Y; su(2)) = 2$, the family of Hermitian matrices H_t (defined below) has transverse spectral flow.
(e) For each $[A] \in \mathcal{M}_h^*$, dim $\mathcal{H}^1_{A,h}(Y, \partial Y; su(2)) = 1$.

Condition (c) implies that $\mathcal{M}_h^{U(1)}$ is a smooth 1-manifold. To define the matrix H_t in condition (d) we first choose a family of connections A_t with $[A_t]$ parameterizing an open set in $\mathcal{M}_h^{U(1)}$ with $\mathcal{H}^1_{A_0,h}(Y, \partial Y; su(2))$ nonzero. The orthogonal complement of $T_{[A_0]}\mathcal{M}_h^{U(1)}$ in $\mathcal{H}^1_{A_0,h}(Y; su(2))$ is isomorphic to $\mathcal{H}^1_{A_0,h}(Y, \partial Y; su(2))$. The action of $\text{Stab}A_0$ gives this space a complex structure, and we may view it as $\mathcal{H}^1_{A_0,h}(Y;\mathbf{C})$. Then we define a 1-parameter family of Hermitian forms on $\mathcal{H}^1_{A_0,h}(Y;\mathbf{C})$ by the formula

$$H_t(\alpha,\beta) = \langle *d_{A_t,h}\alpha,\beta\rangle.$$

H_t is a cohomological pairing on relative cohomology. When the complex dimension of this cohomology is 1, H_t is simply a real scalar, i.e.,

$$H_t(\alpha,\beta) = \lambda(t)\langle\alpha,\beta\rangle$$

for some real valued function $\lambda(t)$. Condition (d) requires that $\lambda'(0) \neq 0$.

In the unperturbed situation, a jump in $\mathcal{H}^1_A(Y;su(2))$ occurs (for abelian connections) exactly at connections with meridinal holonomy conjugate to $\mathrm{hol}_\mu A = \exp(\mathbf{i}\theta)$ where $B_\kappa(e^{i2\theta})$ has a zero eigenvalue. The spectral flow of $B_\kappa(e^{i2\theta})$ through such a point is the negative of the spectral flow of H_t (see [H2]).

Remark: The second part of condition (c) insures that $\mathcal{M}^{U(1)}$ is smooth even at the points where the cohomology jumps.

The flat moduli space for the torus is equal to $\mathcal{M}_{T^2} = T^2/\mathbf{Z}_2$, known as the pillowcase. It is topologically a 2-sphere, but has 4 "corners" corresponding to the central orbits, i.e., the fixed points of the involution. There is a restriction map $r : \mathcal{M}_h \to \mathcal{M}_{T^2}$.

Theorem 8. *(Theorem 15 and Corollary 24, [H1]) $\mathcal{M}_h$ is compact. If $\mathcal{M}_h$ is nondegenerate, then it has the following structure. $\mathcal{M}_h^{SU(2)}$ consists of 2 points. $\mathcal{M}_h^{U(1)}$ is a smooth 1-dimensional manifold, compact except for two open ends which limiting to the central orbits. $\mathcal{M}_h^*$ is a 1-dimensional manifold, compact except for open ends which limit to distinct points on $\mathcal{M}_h^{U(1)}$ where $\dim \mathcal{H}^1_{A,h}(Y,\partial Y;su(2)) = 2$. Each such abelian orbit where the relative cohomology jumps is the limit of exactly one such irreducible end. The restriction map r is an immersion on each stratum.*

Remark: The structure of the flat moduli space around the bifurcation points, where the irreducible stratum meets the abelian stratum, is the foundation of the existence result in this paper. There is a gap in the proof in [H1], so we state this claim as Theorem 12 and provide a complete proof in Section 3.

Given a flat abelian connection A, let Sym_A denote the set of symmetric bilinear forms on $\mathcal{H}^1_A(Y,\partial Y;su(2))$ which are $\mathrm{Stab}A$ invariant. Given a collection of n disjoint embedded loops $\{\ell_i\}_{1\le i\le n}$, let

$$D\ \mathrm{tr}\ \mathrm{hol}_{\ell_I} : \mathbf{R}^n \to \mathrm{Hom}(\mathcal{H}^1_A(Y;su(2)),\mathbf{R})$$

be the linear function which takes the vector $(b_1,\dots b_n)$ to the homomorphism

$$\alpha \mapsto \sum_{i=1}^n b_i \frac{\partial}{\partial s}(\mathrm{tr}\ \mathrm{hol}_{\ell_i}(A+s\alpha))|_{s=0}.$$

Similarly, let $D^2\ \mathrm{tr}\ \mathrm{hol}_{\ell_I} : \mathbf{R}^n \to \mathrm{Sym}_A$ be the linear function which takes the vector $(b_1,\dots b_n)$ to the bilinear form

$$(\beta_1, \beta_2) \mapsto \sum_{i=1}^{n} b_i \frac{\partial^2}{\partial s \partial t}(\mathrm{tr}\ \mathrm{hol}_{\ell_i}(A + s\beta_1 + t\beta_2))|_{s=t=0}.$$

Proposition 9. *(Lemma 38, Lemma 60, and Theorem 15, [H1]) There is a finite collection of disjoint embedded loops $\{\ell_i\}_{1\leq i\leq n}$ with the following properties:*

1. *For all irreducible flat connections A the map*

$$D\ \mathrm{tr}\ \mathrm{hol}_{\ell_I} : \mathbf{R}^n \to \mathrm{Hom}(\mathcal{H}^1_A(Y; su(2)), \mathbf{R})$$

is surjective.

2. *For all abelian flat connections A the map*

$$D\ \mathrm{tr}\ \mathrm{hol}_{\ell_I} \oplus D^2\ \mathrm{tr}\ \mathrm{hol}_{\ell_I} : \mathbf{R}^n \to \mathrm{Hom}\mathcal{H}^1_A(Y; \mathbf{R}) \oplus Sym_A$$

is surjective.

Choose a collection of loops $\{\ell_i\}$ as in the previous proposition and let $\{\gamma_i\}$ be a corresponding collection of embeddings of solid tori into disjoint tubular neighborhoods of the loops. Let $\overline{\mathcal{E}} = C^2(\mathbf{R}, \mathbf{R})$ and $\mathcal{E} = \overline{\mathcal{E}}^n$. Let $\mathcal{E}_1 \subset \mathcal{E}$ be the subset of n-tuples $(\overline{h}_1, \ldots, \overline{h}_n)$ for which the associated perturbed flat moduli space is degenerate.

Theorem 10. *(Theorem 15, [H1]) There is a neighborhood U of $(0, \ldots, 0) \in \overline{\mathcal{H}}^n$ such that $U_1 = \mathcal{E}_1 \cap U$ has codimension 1.*

For any path $h_t : [0, \epsilon] \to U$, define

$$\mathcal{M}_{\{h_t\}} = \{([A], t) \in \mathcal{A}/\mathcal{G} \times [0, \epsilon] |\ \zeta_{h_t}(A) = 0\}.$$

Proposition 11. *(Proposition 49, [H1]) $\mathcal{M}_{\{h_t\}}$ is compact.*

3 Existence of irreducible orbits in the nondegenerate case

3.1 Statement of the theorem and some comments

The existence theorem in the nondegenerate situation is the following.

Theorem 12. *Suppose that in some neighborhood of an abelian orbit $[A_0] \in \mathcal{M}_h^{U(1)}$ with nonzero $\mathcal{H}^1_{A_0,h}(Y, \partial Y; su(2))$ the nondegeneracy conditions (c) and (d) are satisfied. Then there is a neighborhood $U \subset \mathcal{A}/\mathcal{G}$ of $[A_0]$ such that $U \cap \mathcal{M}_h^*$ is a smooth arc with one open end limiting to $[A_0]$. The tangent space to the image of $\overline{U \cap \mathcal{M}_h^*}$ is transverse to that of $\mathcal{M}_h^{U(1)}$ in the pillowcase.*

This theorem will be proved in the next subsection by a somewhat indirect route. We discuss here what goes wrong with the more direct approach.

The perturbed flat moduli space near an abelian orbit $[A_0]$ is homeomorphic to the zero set of the Kuranishi map

$$\Phi : \mathcal{H}^1_{A_0,h}(Y; su(2)) \cong \mathbf{R} \oplus \mathbf{C} \to \mathcal{H}^1_{A_0,h}(Y, \partial Y; su(2)) \cong \mathbf{C}$$

where Φ is defined as follows (see [H1], Section 6.4, for details). The implicit function theorem gives a map $\psi : \mathcal{H}^1_{A_0,h}(Y; su(2)) \to *d_{A_0,h}\Omega^1_\tau(Y; su(2))$ (here τ denotes the Dirichlet boundary conditions) such that

$$\Pi_{\ker d^*_{A_0}} \zeta_h(A + \alpha + \psi(\alpha)) \in \mathcal{H}^1_{A_0,h}(Y, \partial Y; su(2)).$$

The map Φ is defined by $\Phi(\alpha) = \Pi_{\ker d^*_{A_0}} \circ \zeta_h(A + \alpha + \psi(\alpha))$. These maps are $\mathrm{Stab}A_0 \cong U(1)$ equivariant. The linearization of Φ at $(t, 0)$ in the $\mathbf{C}$ direction, composed with inclusion of relative cohomology into absolute, is equal to H_t. One would like to argue that Φ is a 1-parameter family of gradient vector fields on $\mathbf{C}$ (here we identify the relative and absolute $\mathbf{C}$ valued cohomology through the inclusion of relative into absolute) and hence must be, up to change of coordinates, $\Phi(t, z) = tz$.

The vector field $\Phi(t, z)$ may not be a gradient vector field on $\mathbf{C}$, however. Recall from [H1] that $-\frac{1}{2\pi} * F(A) + \nabla h(A)$ is not the L^2 gradient of a function on $\mathcal{A}$, but rather the gradient of a section of a $U(1)$ bundle defined with respect to a connection on that bundle. This connection, restricted to the graph of the function from $\mathcal{H}^1_{A_0,h}(Y; su(2))$ to $\mathcal{A}$ given by $\alpha \mapsto A_0 + \alpha + \psi(\alpha)$, may not be flat. Thus the gradient with respect to this connection may not in fact be a conservative vector field.

The difficulty is to rule out families of vector fields on $\mathbf{C}$ such as

$$(t, x, y) \mapsto (tx - y(x^2 + y^2)^n, ty - x(x^2 + y^2)^n),$$

which is $U(1)$ equivariant and has the same linearization along $\mathbf{R} \oplus \{0\}$, but has no zeros off $\mathbf{R} \oplus \{0\}$. The existence of such families was pointed out to the author by Eric Klassen.

To avoid this difficulty we propose a somewhat different argument. Consider the closed manifold Y_0 obtained by 0-surgery on Y. In this setting, $- * \frac{1}{2\pi} F(A) + \nabla h(A)$ is truly the L^2 gradient of $\mathrm{CS}(A) + h(A) : \mathcal{A} \to \mathbf{R}$, where CS denotes the Chern-Simons function, given by

$$\mathrm{CS}(A) = \frac{1}{4\pi} \int_{Y_0} \mathrm{tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A).$$

We establish the existence of a family of irreducible connections on Y_0 whose restrictions to Y are flat and which limit to the orbit $[A_0]$ as required. In addition, we can describe the position of the image of the nearby irreducible orbits in the pillowcase.

3.2 The picture on Y_0

In this subsection we consider connections on a closed manifold Y_0. We begin with a completely general description of the perturbed flat moduli space near a flat connection (with no assumptions of nondegeneracy), and then add the nondegeneracy assumptions 1–3 as needed.

As before for manifolds with boundary, let $\zeta_h(A)$ denote $-*\frac{1}{2\pi}F(A)+\nabla h(A)$ and $X_A = \{A_0 + a \mid d_{A_0}^* a = 0\}$. For any closed subspace $W \subset \Omega^1(Y_0; su(2))$, denote by Π_W the orthogonal projection onto W. The next lemma describes the Kuranishi picture for the perturbed flat moduli space near $[A_0]$. For a proof see [H1], Section 6.4, or [M-M-R], Section 12.1.

Lemma 13. *Let A_0 be a smooth perturbed flat connection. There exist:*

(a) a $\mathrm{Stab}(A_0)$ *equivariant neighborhood V_{A_0} of* 0 *in $\mathcal{H}^1_{A_0,h}(Y_0; su(2))$,*
(b) a $\mathcal{G}$ equivariant neighborhood U_{A_0} of A_0 in $\mathcal{A}$,
(c) a $\mathrm{Stab}(A_0)$ *equivariant real analytic embedding*

$$\phi : V_{A_0} \to X_A$$

whose differential at 0 *is just the inclusion of $\mathcal{H}^1_{A_0,h}(Y_0; su(2))$ into* $\ker d_{A_0}^* \cap \Omega^1(Y_0; su(2))$*, and*
(d) a $\mathrm{Stab}(A_0)$ *equivariant map*

$$\Phi : V_{A_0} \to \mathcal{H}^1_{A_0,h}(Y_0; su(2))$$

such that ϕ maps $\Phi^{-1}(0)$ homeomorphically onto the zero set of $\zeta_h|_{X_A \cap U_A}$.

The map ϕ is defined by $\phi(\alpha) = A_0+\alpha+\psi(\alpha)$ where $\psi(\alpha) \in *d_{A_0}\Omega^1(Y_0; su(2))$ solves

$$\Pi_{*d_{A_0}\Omega^1(Y_0;su(2))}(*d_{A_0}\psi(\alpha) + *\frac{1}{2}[\alpha + \psi(\alpha) \wedge \alpha + \psi(\alpha)]) = 0.$$

In other words, the graph of ψ has the property that for any $\alpha \in V_{A_0}$,

$$\Pi_{\ker d_{A_0}^*}\zeta_h(A_0 + \alpha + \psi(\alpha)) \perp *d_{A_0,h}\Omega^1(Y_0; su(2)).$$

The map Φ is given by

$$\Phi(\alpha) = \Pi_{\ker d_{A_0}^*}\zeta_h(A_0 + \alpha + \psi(\alpha)).$$

Assumption 1: Assume that Y_0 is the zero framed surgery on the knot complement Y. Let $[A_0] \in \mathcal{M}_h^{U(1)}(Y_0)$ and assume that $\mathcal{H}^1_{A_0,h}(Y; \mathbf{R}) \cong \mathbf{R}$, which guarantees that $\mathcal{M}_h^{U(1)}(Y)$ meets nondegeneracy condition (c) near $[A_0]$. Finally, assume that $\mathcal{H}^1_{A_0,h}(Y; \mathbf{C})$ is nonempty and the graphs of the eigenvalues of the family of bilinear forms H_t defined earlier are transverse to zero at $t = 0$ and all cross it in the same direction.

Proposition 14. *There is an additional perturbation which does not change the topological structure of $\mathcal{M}_h(Y)$ near $[A_0]$, but changes its image in the pillowcase by a diffeomorphism of the pillowcase (minus the corners) in such a way that the new abelian arc lines up with the flat connections on the Dehn filling in Y_0. After performing this perturbation, all the perturbed flat abelian connections on Y near $[A_0]$ extend over Y_0 as perturbed flat connections.*

Proof: The proof uses the description of perturbed flat connections in Lemma 61, [H1]. If the abelian stratum around $[A_0]$ maps into the pillowcase to a curve which is transverse to the circles $\{\mathrm{hol}_\mu = \text{constant}\}$, then by doing an additional perturbation using a curve in a tubular neighborhood of the boundary torus ∂Y we can make this piece of the abelian stratum lie on the $\{\mathrm{hol}_\lambda = \mathrm{id}\}$ arc in the pillow case. Specifically, choose the curve to be a meridian with framing a parallel meridian in the same $T^2 \subset T^2 \times [0,1]$ (implicit in the definition of an admissible function is a choice of framings of the images of the solid tori), and choose the function of trace appropriately.

If the tangent direction to the abelian stratum at $[A_0]$ is vertical, then first do a perturbation using a trivially framed longitude to tip it slightly so that it satisfies the former hypothesis. Then perturb as above. The key fact used to prove this proposition is that a perturbation using a trivially framed longitude and or meridian changes the picture of $r : \mathcal{M}_h \to \mathcal{M}_{T^2}$ by a diffeomorphism of $\mathcal{M}_{T^2} \setminus \{\text{centrals}\}$, so this doesn't affect any of the properties of $\mathcal{M}$ which concern us here. $\square$

Proposition 15. *The additional perturbation in the Proposition 14 does not alter the cohomology of Y at A_0, nor does it affect the transversality condition on the eigenvalues of H_t.*

Proof: Let h denote the "background" perturbation on Y and let h' be the perturbation function constructed in the previous proposition. We identify orbits of $(h + h')$-perturbed flat connections on Y with the pairs of orbits of $([A], [A']) \in \mathcal{M}_h(Y) \times \mathcal{M}_{h'}(T^2 \times [0,1])$ which agree on the torus $\partial Y = T^2 \times \{0\}$.

We sketch the proof, leaving the details as an exercise for the reader. A_0 extends uniquely (up to gauge) over $Y \cup (T^2 \times [0,1])$ to a perturbed flat connection. We shall use the same notation for this extension.

$\mathcal{H}^1_{A_0,h'}(T^2 \times [0,1]; su(2))$ is two dimensional, and the restriction map to the cohomology of either boundary component is a surjection. The way to see this (in the harder case, when h' consists of two perturbation curves) is to consider first $\mathcal{H}^1_{A_0,h'}(T^2 \times [0,1] \setminus \{\text{the two perturbation curves}\}; su(2))$, which equals the ordinary real cohomology of this space with $\mathbf{R}$ coefficients (4-dimensional). Then use a Mayer Vietoris argument to check that the subspace consisting of cohomology classes whose restrictions to the boundaries of the perturbation curves lie in the image of the (perturbed) cohomology on the solid tori has the required properties. (Note that the second claim does not contradict the fact that the image of $\mathcal{H}^1_{A_0,h'}(T^2 \times [0,1]; su(2))$ under restriction must be a Lagrangian subspace of the direct sum $\mathcal{H}^1_{A_0,h'}(T^2 \times \{0\} \cup T^2 \times \{1\}; su(2))$ with its symplectic structure. This symplectic structure is the difference of the pull backs of the two pillowcase symplectic structures because the orientations on the tori differ. The Lagrangian property is then easily verified.)

The Mayer Vietoris sequence applied to $Y \cup T^2 \times [0,1]$ now implies that $\mathcal{H}^1_{A_0,h+h'}(Y \cup T^2 \times [0,1]; su(2)) \cong \mathcal{H}^1_{A_0,h}(Y; su(2))$, and similarly for relative first cohomology. In addition, it implies that relative 1-dimensional classes on the union are represented by forms which are exact on $T^2 \times [0,1]$. The signs of the derivatives of the eigenvalues of H_t as they pass through 0 are detected by the cohomology pairing on $\mathcal{H}^1_{A_0,h+h'}(Y \cup T^2 \times [0,1])$, which agrees with the one on Y (see [H2], Section 4). $\square$

By Propositions 14 and 15, we can make the following assumption without any loss of generality with regard to irreducible orbits near $[A_0]$.

Assumption 2: The abelian perturbed flat connections on Y in the arc through $[A_0]$ extend over the 0-surgery Y_0.

Remark: This is not a generic situation; reducible and irreducible orbits on Y_0 are isolated for generic perturbations. We are deliberately putting ourselves in this degenerate situation. Also, there is nothing special about the 0-framing for the surgery. We choose this particular Dehn filling simply because in the unperturbed case there is no perturbation required for the abelians on Y to extend over this closed 3-manifold.

For the remainder of this subsection we shall work on Y_0, and the connections, Chern-Simons function, etc., are on this closed 3-manifold unless otherwise specified. We shall use the same notation to denote the perturbed flat connections on Y_0 as their restrictions to Y.

We can assume after gauge transformation that A_0 takes values in the fixed 1-dimensional subspace $\mathbf{R} \subset su(2)$, and that $\mathrm{hol}_\mu A_0 = \exp(\mathbf{i}\alpha)$ for $0 < \alpha < \pi$. The stabilizer $U(1)$ action on $su(2)$ valued forms is compatible with our decomposition $su(2) = \mathbf{R} \oplus \mathbf{C}$. In particular, the perturbed flat de Rham cohomology decomposes accordingly, $\mathcal{H}^1_{A_0,h}(Y_0; su(2)) = \mathcal{H}^1_{A_0,h}(Y_0;\mathbf{R}) \oplus \mathcal{H}^1_{A_0,h}(Y_0;\mathbf{C})$.

Proposition 16. *Any perturbed flat abelian connection on a 3-manifold is gauge equivalent to a smooth connection.*

Proof: So long as h is smooth, this follows from a standard argument using elliptic regularity. In case h is not smooth, then A is still gauge equivalent to a smooth connection off the perturbation solid tori. On the solid tori, A can be put into the canonical form described in Corollary 62, [H1], which is smooth. $\square$

Assume A_0 is smooth. Consider the Kuranishi picture near A_0. Let α_0 be a nonzero element of $\mathcal{H}^1_{A_0,h}(Y_0;\mathbf{R}) = T_{[A_0]}\mathcal{M}^{U(1)}$ which points away from the trivial connection. Then the abelian stratum near $[A_0]$ is parameterized by $A_t = A_0 + t\alpha_0 + \psi(t\alpha_0, 0)$.

Let $H_t(Y_0)$ denote the family of Hermitian bilinear forms on $\mathcal{H}^1_{A_0,h}(Y_0;\mathbf{C})$ given by

$$H_t(Y_0)(\beta_1,\beta_2) = \langle *d_{A_t,h}\beta_1, \beta_2\rangle_{L^2(Y_0)}.$$

The spectral flow of $H_t(Y_0)$ coincides with that of $H_t(Y)$.

Proposition 17. *The linearization of Φ at $(t\alpha_0, 0, \ldots, 0)$ in the $\mathbf{C}^n$ direction agrees with H_t to the order of t^2. In particular, these two families of symmetric bilinear forms on $\mathcal{H}^1_{A_0,h}(Y_0;\mathbf{C})$ have the same spectral flow. The transversality requirement on the eigenvalues of H_t implies the same for $\Phi_*(t\alpha_0, 0)$.*

Proof: Recall that

$$\Phi(\alpha,\beta)=\Pi_{\ker d^*_{A_0}}(*d_{A_0,h}\psi(\alpha,\beta)+*\frac{1}{2}[(\alpha,\beta)+\psi(\alpha,\beta)\wedge(\alpha,\beta)+\psi(\alpha,\beta)]).$$

The linearization of Φ at $(\alpha,0)$ takes a tangent vector β in the $\mathcal{H}^1_{A_0,h}(Y_0;\mathbf{C})$ direction to

$$\Phi_*(\alpha,0)(\beta)=\Pi_{\ker d^*_{A_0}}(*d_{A_0,h}\psi_*(\alpha,0)(\beta)+*[(\alpha,0)+\psi(\alpha,0)\wedge\beta+\psi_*(\alpha,0)(\beta)]).$$

Thus

$$\begin{aligned}\langle\Phi_*(t\alpha_0,0)(\beta_1),\beta_2\rangle &= \langle *[(t\alpha_0,0)+\psi(t\alpha_0,0)\wedge\beta_1],\beta_2\rangle+\\ &\quad\langle *[(t\alpha_0,0)+\psi(t\alpha_0,0)\wedge\psi_*(t\alpha_0,0)(\beta_1)],\beta_2\rangle.\end{aligned}$$

The first term is exactly $H_t(\alpha,\beta)$.

Since ψ is a real analytic map and $\psi(0,0)=0$ and $\psi_*(0,0)=0$, there is a constant C such that, whenever $\|t\alpha_0\|_{L^2_2}\le 1$,

$$\|\psi(t,\alpha_0,0)\|_{L^2_2}\le Ct^2$$

and

$$\|\psi_*(t\alpha_0,0)(\beta)\|_{L^2_2}\le Ct^2\|\beta\|_{L^2_2}.$$

By the Multiplication Theorem for Sobolev spaces,

$$\|*[t\alpha_0+\psi(t\alpha_0,0)\wedge\psi_*(t\alpha_0,0)(\beta)]\|_{L^2}\le C't^2\|\beta\|_{L^2_2}.$$

□

We now make our final assumption and prove Theorem 12.

Assumption 3: Suppose now, in addition, that $\mathcal{H}^1_{A_0,h}(Y,\partial Y;su(2))$ has complex dimension 1. By Theorem 8, for generic h, this is the case at each abelian orbit where this cohomology is nonempty.

Proof of Theorem 12: When the extra cohomology at A_0 is only of complex dimension 1, the StabA_0 invariance becomes a much stronger condition on the function $(\mathrm{CS}+h)\circ\phi$. Let β_0 be a nonzero element of $\mathcal{H}^1_{A_0,h}(Y_0;\mathbf{C})$. Then $(\mathrm{CS}+h)\circ\phi(t\alpha_0,re^{i\theta}\beta_0)$ depends only on t and $|r|$.

To complete the proof, we perturb once again, so that the abelian parts of $\mathcal{M}_h(Y)$ and $\mathcal{M}(S^1\times D^2)$ no longer match up. For simplicity, we leave the existing perturbation on Y alone and add a new function of trace of holonomy around the Dehn filling core to $\mathrm{CS}+h$. Basically, we want to gradually sweep $\mathcal{M}(S^1\times D^2)$ across the pillowcase to detect which irreducible orbits in $\mathcal{M}_h(Y)$ have images on either side of $r(\mathcal{M}_h^{U(1)}(Y))$.

Choose an admissible function $h':\mathcal{A}(Y_0)\to\mathbf{R}$ defined using the core of the Dehn filling in such a way that $\langle\nabla(h'\circ\phi)(t\alpha_0,0)),\alpha_0\rangle=1$ and $h'\circ\phi(0,0)=0$. Explicit computation of the gradient of h' shows that we can take h' to be an appropriate *decreasing* function of the trace of the holonomy around the core in the same direction as λ (an increasing function would force $\langle \nabla h', \alpha_0 \rangle < 0$). The crucial observation is that any connection on Y_0 which is $h + h'$ perturbed flat restricts to Y to give an h perturbed flat connection.

Consider the function $f_\epsilon(t, r) = (\mathrm{CS} + h + \epsilon h') \circ \phi : \mathbf{R} \oplus \mathbf{C} \to \mathbf{R}$. Then

$$f_\epsilon(t, r) = f_0(0, 0) + \epsilon t + \frac{\lambda(t)}{2} r^2 + O(t^2 r^2) + O(r^4).$$

The lower order terms depend not only on t and r but also on ϵ.

A local model for the flat moduli space of Y near $[A_0]$ is the quotient by the $\mathbf{Z}_2$ symmetry ($r \mapsto -r$) of the set

$$\{(t, r) |\ \frac{\partial f_\epsilon(t, r)}{\partial t} = \frac{\partial f_\epsilon(t, r)}{\partial r} = 0 \text{ for some } \epsilon\}.$$

This set is the union of $\{(t, 0)\}$ and the image under projection onto the (t, r) coordinate plane of

$$N = \{(t, r, \epsilon) |\ r \neq 0, \frac{\partial f_\epsilon(t, r)}{\partial t} = \frac{1}{r} \frac{\partial f_\epsilon(t, r)}{\partial r} = 0\} \cup \{(t, 0, 0)\}.$$

Let $P = (P_1, P_2) : \mathbf{R}^3 \to \mathbf{R}^2$ where $P_1(t, r, \epsilon) = \frac{\partial f_\epsilon(t,r)}{\partial t}$ and

$$P_2(t, r, \epsilon) = \begin{cases} \lambda(t) & r = 0 \\ \frac{1}{r} \frac{\partial f_\epsilon(t,r)}{\partial r} & r \neq 0 \end{cases}$$

The linearization of P at $(0, 0, 0)$ is

$$DP(0, 0, 0) = \begin{bmatrix} 0 & 0 & 1 \\ \lambda'(0) & * & 0 \end{bmatrix}.$$

The implicit function theorem now implies that there are smooth functions $\epsilon(r)$ and $t(r)$ such that for r small, $(t(r), r, \epsilon(r))$ parameterizes N near $(0, 0, 0)$. This shows that, up to gauge equivalence, there is a smooth 1-dimensional family of irreducible connections on Y limiting to $[A_0]$.

The calculation above implies slightly more. Since $\epsilon'(0) = t'(0)$ and $\epsilon''(0) \neq 0$ it follows that the family of irreducible orbits leaves the abelian stratum transversely (in the pillowcase). □

Proof of Corollary 3: We first prove the corollary under the additional assumption (3). Notice that the t component of $\nabla f_\epsilon(t, r)$ has no zeros when $\epsilon = 0$ or ϵ has the same sign as $\lambda'(0)$. It suffices then to determine the position of the perturbed flat connections on the Dehn filling in terms of the sign of ϵ.

As was noted above, in order for the perturbation function h' to satisfy $\langle \nabla h' \circ \phi, \alpha_0 \rangle > 0$, h' must be a decreasing function of trace of the holonomy around the Dehn filling core. This has the effect that $\epsilon h'$ perturbed flat connections on the Dehn filling are mapped to the front of the pillowcase, that is, the side where $(\mu, \lambda) \mapsto (\exp(\mathbf{i}\tau), \exp(\mathbf{i}\sigma))$ for $0 < \tau < \pi$ and $0 < \sigma < \pi$, when $\epsilon < 0$ and to the back when $\epsilon > 0$.

Without assumption (3), we can make a similar argument. Let $\lambda_i(t)$, $i = 1, \dots, n$, be the eigenvalues of $H_t(Y_0)$. Let $\beta_1(t), \dots, \beta_n(t)$ be a 1-parameter family of bases of corresponding eigenvectors for $\mathcal{H}^1_{A_0,h}(Y_0; \mathbf{C})$. and let

$$r(t\alpha_0, (x_1 + iy_1)\beta_1, \dots, (x_n + iy_n)\beta_n) = (x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2)^{\frac{1}{2}}.$$

This time the function $(\mathrm{CS} + h) \circ \phi(t\alpha_0, (x_1 + iy_1)\beta_1, \dots, (x_n + iy_n)\beta_n)$ has the form

$$\sum_{i=1}^{n} \frac{\lambda_i(t)}{2}(x_i^2 + y_i^2) + O(t^2 r^2) + O(r^3).$$

The t component of the gradient Φ is

$$\sum_{i=1}^{n} \frac{\lambda_i'(t)}{2}(x_i^2 + y_i^2) + O(tr^2) + O(r^3).$$

Consideration of the same family of perturbations now completes the proof. □

Remark: The assumption that the spectral flow is only in one direction is necessary to conclude that there are no irreducible representations which take the longitude to the identity near ρ_α. For example, if Y is the complement of the square knot, then there is a component of $\mathcal{M}^*(Y)$ which limit to orbits in $\mathcal{M}^{U(1)}(Y)$ and whose image in $\mathcal{M}_{T^2}$ coincides with that of (part of) $\mathcal{M}^{U(1)}(Y)$. In this example, the total spectral flow of H_t through these abelian limit points is zero. By taking a composite of two right handed and one left handed trefoils, however, we get an example of this behavior where the spectral flow is algebraically nonzero.

Proof of Corollary 4: Suppose there exists an α satisfying the hypotheses. Theorem 12 says that there is a continuous family of irreducible flat orbits limiting to the flat abelian orbit corresponding the representation $\mu \mapsto \exp(\mathbf{i}\alpha)$. By Corollary 3, the image of this irreducible family in the pillowcase is on one side of the abelian arc $\mathrm{hol}_\lambda = \mathrm{id}$. Before perturbation, $r(\mathcal{M}(Y))$ is symmetric under the involution $(\mathrm{hol}_\lambda, \mathrm{hol}_\mu) \mapsto (\mathrm{hol}_\lambda, -\mathrm{hol}_\mu)$ (as can be seen from the Wertinger presentation of the fundamental group). This symmetry implies that the irreducible moduli space limits to the abelian arc from both sides. Thus for $|n|$ large enough, this family of irreducible representations of $\pi_1 Y$ must intersect the curve of slope $\pm\frac{1}{n}$ in the pillowcase, which corresponds to the set of representations of $\pi_1 T^2$ which extend over the corresponding Dehn surgeries. □

4 General existence theorem

In this section we use Theorem 12 to prove Theorem 1.

Proof of Theorem 1: Find a collection of curves in Y satisfying the conclusion of Proposition 9, and let U and U_1 be as in Theorem 10. Choose a path $h_s : [-\epsilon, \epsilon] \to U$ with $h_0 = 0$ which is transverse to U_1. We can take ϵ small enough that $\mathcal{M}_{h_s}$ is nondegenerate when $0 < s \le \epsilon$.

Let $A_{0,0}$ be the flat abelian connection with $\mathrm{hol}_\mu A = \exp(\mathbf{i}\alpha)$. There is a 2-parameter family of abelian connections $A_{s,t}$ near the connection $A_{0,0}$ such that $[A_{s,t}] \in \mathcal{M}_{h_s}^{U(1)}$. Let

$$H_{s,t}(\beta_1, \beta_2) = \langle * d_{A_{s,t},h_s}\beta_1, \beta_2\rangle$$

be the corresponding 2-parameter family of bilinear forms on $\mathcal{H}^1_{A_{0,0}}(Y; \mathbf{C})$.

Let B be an arbitrarily small ball around $r[A_{0,0}]$ in $\mathcal{M}_{T^2}$. By shrinking B and ϵ if necessary, we can assume there is a $\delta > 0$ such that

1. for $(s, t) \in [0, \epsilon] \times [-\delta, \delta]$, $\det H_{s,t} = 0$ implies $r[A_{s,t}] \in B$.
2. for $s \in [0, \epsilon]$, $r[A_{s,\pm\delta}] \notin B$.
3. for $s \in [0, \epsilon]$, each curve $\{r[A_{s,t}] |\ t \in [-\delta, \delta]\}$ intersects B.

Loosely speaking, the family of perturbations separates the spectral flow points along the abelian stratum but still keeps their images in the pillowcase in the small ball B.

Let $\overline{\mathcal{M}}^*_{h_s}$ denote the closure of the irreducible stratum of $\mathcal{M}_{h_s}$, i.e., the irreducible stratum compactified by adding the abelian limit points. For all $0 < s \le \epsilon$, $r(\overline{\mathcal{M}}^*_{h_s})$ consists of an immersed compact 1-manifold with an odd number of endpoints in the interior of B. Therefore $r(\overline{\mathcal{M}}^*_{h_s}) \cap \partial B \ne \emptyset$ for all $0 < s \le \epsilon$.

By Proposition 11, $r(\overline{\mathcal{M}}^*_{\{h_s\}}) \cap \partial B$ (where s ranges over $[0, \epsilon]$) is compact, and hence $r(\overline{\mathcal{M}}^*_{h_0}) \cap \partial B = r(\overline{\mathcal{M}}^*) \cap \partial B \ne \emptyset$. Since the same is true for arbitrarily small B, $[A_{0,0}]$ is in $\overline{\mathcal{M}}^*$. If there were no continuous path in $r(\overline{\mathcal{M}}^*)$ connecting $[A_0]$ to ∂B, then we could separate $[A_0]$ and $r(\overline{\mathcal{M}}^*) \cap \partial B$ by a continuous loop $\gamma : S^1 \to (B \setminus r(\overline{\mathcal{M}}^*))$. The above argument showing that $r(\overline{\mathcal{M}}^*) \cap \partial B \ne \emptyset$ could then be applied to $r(\overline{\mathcal{M}}^*) \cap \gamma(S^1)$ to give a contradiction. $\square$

Proof of Corollary 2: Under the hypothesis of the corollary,

$$\Delta(t) = (t - t_0)^p g(t)$$

for some unit complex number t_0, odd integer p, and function $g(t)$ which is nonzero and holomorphic on some neighborhood of t_0. Parameterize the unit circle near $t_0 = e^{is_0}$ by

$$e^{is} = e^{is_0} + r(s)e^{i\theta(s)}$$

where $r(s)$ changes sign at $s = s_0$.

Since $\Delta(e^{is}) = r(s)^p e^{ip\theta(s)} g(t_0 + r(s)e^{i\theta(s)})$ is a real valued function, $e^{ip\theta(s)} g(t_0 + r(s)e^{i\theta(s)})$ is real valued (and nonzero). Thus $\Delta(e^{is})$ changes sign since $r(s)^p$ does.

To complete the proof, note that

$$\det B(t^2) = \det((t^{-1} - t)A(t^2)) = (t^{-1} - t)^{2g}\ \det(A(t^2))$$

is a nonzero real valued function multiplied by $\Delta(t^2)$, so if $\Delta(t^2)$ changes sign then an odd number of eigenvalues of $B(t^2)$ must also change sign. $\square$

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