LOCALIZATION IN HERMITIAN K-THEORY OF RINGS

JENS HORNBOSTEL AND MARCO SCHLICHTING

ABSTRACT. We prove localization and *dévissage* theorems for the hermitian K-theory of rings analogous to well-known theorems in algebraic K-theory. Our proofs rely among others on a study of derived categories, on a generalization of a theorem of Pedersen-Weibel to the hermitian setting and on a cofinality result for triangular Witt groups. Applications include a proof of a conjecture of Karoubi and algebraic Bott periodicity.

INTRODUCTION

This article is about localization and *dévissage* in algebraic hermitian K-theory of rings. Recall from [Kar73, p. 308] that the hermitian K-theory space ${}_\epsilon K^h(A)$ of a ring A with anti-involution $A \to A^{op}$: $a \mapsto \bar{a}$ and a chosen central element $\epsilon \in A$ with $\epsilon\bar{\epsilon} = 1$ has the homotopy type of ${}_\epsilon K_0^h(A) \times B_\epsilon O(A)^+$ where ${}_\epsilon O(A)$ is the infinite ϵ-orthogonal group of A. Here ${}_\epsilon K_0^h(A)$ is the ϵ-Grothendieck-Witt group of A, *i.e.,* the Grothendieck group of the abelian monoid of isometry classes of finitely generated projective A-modules equipped with a non-degenerate ϵ-hermitian form. The monoid addition is given by the orthogonal sum of hermitian forms.

In the special case of the ring $A \times A^{op}$ with anti-involution $(a, b) \mapsto (b, a)$, the hermitian K-theory space ${}_\epsilon K^h(A \times A^{op})$ coincides with the algebraic K-theory space $K(A)$ of the ring A. In this sense, algebraic hermitian K-theory is a generalization of algebraic K-theory. Moreover, it is a way of studying the homology of more general groups than the general linear group, namely the orthogonal or symplectic group. For Karoubi, hermitian K-theory was the right framework for proving Bott periodicity theorems, in the topological as well as in the algebraic setting [Kar73],[Kar80].

In [Gra76, p.229-232], the following theorem of Quillen was proved. Let A be a ring, $\Sigma \subset A$ a multiplicative subset of central non-zero divisors, and let $\mathcal{T}_\Sigma$ be the exact category of finitely generated Σ-torsion A-modules of projective dimension at most 1. Then there is a homotopy fibration of K-theory spaces

$$K(\mathcal{T}_\Sigma) \to K(A) \to K(\Sigma^{-1}A).$$

Our Localization Theorem 1.15 provides the analogous statement for hermitian K-theory. It states that if 2 is invertible in A, there is a homotopy fibration

$${}_\epsilon U(\mathcal{T}_\Sigma) \to {}_\epsilon K^h(A) \to {}_\epsilon K^h(\Sigma^{-1}A).$$

Here ${}_\epsilon U(\mathcal{T}_\Sigma)$ is the loop space of the classifying space of an explicit category ${}_\epsilon \mathcal{W}(\mathcal{T}_\Sigma)$ associated to the exact category $\mathcal{T}_\Sigma$ equipped with the duality $Ext^1_A(\ \ , A) = Hom_A(\ \ , \Sigma^{-1}A/A) : \mathcal{T}_\Sigma \to \mathcal{T}_\Sigma^{op}$. The category ${}_\epsilon \mathcal{W}(\mathcal{E})$, defined by Giffen and Karoubi, is the hermitian analogue of Quillen's Q-construction and is defined for any exact category with duality ([CL86], [Uri90], Remark 1.11). Applied to the category of finitely generated projective A-modules equipped with the duality $Hom_A(\ \ , A)$, it yields a delooping of the homotopy fiber of the hyperbolic map $K(A) \to {}_\epsilon K^h(A)$ (Remark 1.12). So in this case our U-theory space coincides with Karoubi's U-theory space ${}_\epsilon U(A)$ of the ring A as defined in [Kar80].

In case A is commutative regular and $f \in A$ such that A/fA is regular as well, $\Sigma = \{f^n \mid n \in \mathbb{N}\}$, we have a homotopy equivalence ${}_\epsilon U(\mathcal{T}_\Sigma) \simeq {}_\epsilon U(A/fA)$ (theorem 1.19). In case A is a Dedekind domain, $\Sigma =$

Date: September 11, 2003.
1991 *Mathematics Subject Classification.* Primary 19G38; Secondary 18F25, 19G12.

$A - \{0\}$ the multiplicative set of non-zero elements, there is a homotopy equivalence ${}_\epsilon U(\mathcal{T}_\Sigma) \simeq \oplus_\epsilon U(A/\wp)$ ([Hor02], Theorem 1.22). Together with our localization theorem, this proves a conjecture of Karoubi from 1974. Namely, he conjectured the existence of a long exact sequence

$$... \to \oplus_\epsilon U_n(A/\wp) \to {}_\epsilon K^h_n(A) \to {}_\epsilon K^h_n(\Sigma^{-1}A) \to \oplus_\epsilon U_{n-1}(A/\wp) \to ...$$

where the sum is taken over all prime ideals $\wp \subset A$ different from 0 [Kar74, p.393]. The sequence was conjectured but not proved contrary to the footnote of *loco citato*. It can be used to explicitly compare the higher hermitian K-groups of rings of integers and their number fields in certain cases [Hor02].

Localization and *dévissage* theorems are well known for K-theory [Gra76] and Witt-theory [Bal00]. Heuristically, a theorem which is true for K and Witt theory ought to be true for hermitian K-theory as well. The link is provided by the "Karoubi induction principle" (5.20). The induction principle is based on Karoubi's *théoreme fondamental* [Kar80] (5.14) which is, to date, only known for additive categories with duality. However, the category $\mathcal{T}_\Sigma$ is an honest exact category with duality, *i.e.,* it has sequences which do not split. So in some sense, the article introduces methods that allow us to use the induction principle in spite of these restrictions.

There are three new tools which allow us to do this. First, there is a hermitian analogue (1.8) of Waldhausen's $S.$ construction [Wal85] which is homotopy equivalent to the Giffen-Karoubi category.

Second, there is a hermitian analogue (Theorem 3.6, 3.16) of a theorem of Pedersen and Weibel [PW89, Theorem 5.3]. It associates a homotopy fibration of hermitian K-theory spaces or spectra to certain sequences of additive categories with duality. Our proof applied to the K-theory version yields a simpler proof than the original one of [PW89] (which uses Thomason's double mapping cylinder [Tho82]) and the proof in [CP97] (which uses Waldhausen's machinery of [Wal85]).

Finally, we prove a cofinality theorem for Balmer's triangular Witt groups (Theorem A.2) which might be of independent interest. Let $\mathcal{B}$ be a triangulated category with duality in which 2 is invertible, and let $\mathcal{A} \subset \mathcal{B}$ be a full triangulated subcategory invariant under the duality. Suppose that every object in $\mathcal{B}$ is a direct summand of an object of $\mathcal{A}$. Then there is a 12-term periodic exact sequence whose terms are the higher triangular Witt groups of $\mathcal{A}$ and of $\mathcal{B}$ and Tate cohomology of $\mathbb{Z}/2\mathbb{Z}$ with coefficients in the $\mathbb{Z}/2\mathbb{Z}$-module $K_0(\mathcal{B})/K_0(\mathcal{A})$ where the action is induced by the duality on $\mathcal{B}$. At this point, we would like to thank Bruce Williams for having drawn our attention to a similar statement in Ranicki's work on L-theory [Ran81].

The strategies of proof of our main theorems - which are outlined at the beginning of sections 6 and 7, respectively - considerably differ from the known strategies used for proving the analogous statements in algebraic K-theory. This is partly due to the fact that no generalization of Karoubi's fundamental theorem [Kar80] from additive to exact categories with duality is known. In part it is due to the fact that many exact categories considered in ordinary algebraic K-theory don't have duality functors, *e.g.,* the category of finitely generated A-modules, A a Dedekind domain. Although the algebraic K-theory localization sequence is a consequence of our main result, our proof does not yield a new proof of it, the result on K-theory being an ingredient of our proof.

Methods developed here are used by the first author in [Hor] to show unstable and stable $\mathbf{A}^1$-representability of hermitian K-theory and Witt groups as well as Bott periodicity in the stable homotopy category of schemes of Morel-Voevodsky [MV99]. For affine schemes, a more elementary version of Bott periodicity is explained in 1.25. The representability results provide a proof that Witt groups are strictly $\mathbf{A}^1$-homotopy invariant, which is an ingredient in Morel's calculation of certain $\mathbf{A}^1$-homotopy groups of spheres.

This article generalizes and thus replaces "Localization in Hermitian K-theory for regular rings" (Preprint no. 179 of the "Preprintreihe SFB 478 - Geometrische Strukturen in der Mathematik", Münster 2001). We thank the SFB 478 Münster and the University of Illinois at Urbana-Champaign for financial support.

CONTENTS

1. Statement of the Main Results

We start with definitions in order to state our main theorems 1.15 and 1.19. At the end of the section we give two applications, namely Karoubi's conjecture 1.24 and algebraic Bott periodicity 1.25.

1.1. Let A be a ring. We write $A-$free for the category of finitely generated free right A-modules. We write $F(A)$ for the category with objects the natural numbers $\mathbb{N}$ and maps from n to m the abelian group of $m \times n$ matrices $(a_{i,j})$ with entries in A. Composition is matrix multiplication. Of course, $F(A)$ and $A-$free are equivalent categories.

We write $A-$proj for the category of finitely generated projective right A-modules. Recall that the *idempotent completion* (Karoubianisation, pseudo-abelianisation) of a category $\mathcal{C}$ is the category $\tilde{\mathcal{C}}$ whose objects are pairs (C,p) with $p : C \to C$ an idempotent in $\mathcal{C}$, *i.e.,* a map such that $p^2 = p$. Maps in $\tilde{\mathcal{C}}$ from (C,p) to (D,q) are the maps $f : C \to D$ such that $f \circ p = f = q \circ f$. Composition of maps in $\tilde{\mathcal{C}}$ is composition in $\mathcal{C}$. A category is called *idempotent complete* if the functor $C \to \tilde{\mathcal{C}} : C \mapsto (C, 1)$ is an equivalence.

We write $P(A)$ for the idempotent completion of $F(A)$. Of course, $P(A)$ is a small additive idempotent complete category equivalent to $A-$proj.

1.2 *Remark.* For most of the categories occurring in this article (see for example 1.1, 1.3), we will give explicit functorial constructions, because this makes it easier to check the existence of dualities and of duality preserving functors.

1.3. *The category $\mathcal{T}_\Sigma$.* Let $\Sigma \subset A$ be a multiplicative subset of central non-zero divisors. Let $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$ be the full subcategory of those right A-modules M for which there is an exact sequence of right A-modules

$$0 \to P_1 \stackrel{i}{\to} P_0 \to M \to 0 \tag{1.4}$$

with P_0 and P_1 in $A-$proj and $\Sigma^{-1}i : \Sigma^{-1}P_1 \to \Sigma^{-1}P_0$ an isomorphism (equivalently $\Sigma^{-1}M = 0$). The category $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$ is closed under extensions in the category of right A-modules. Declaring a sequence in $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$ exact if it is exact as a sequence of A-modules makes $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$ into an exact category [Kel96, Qui73, TT90].

We let $\mathcal{T}_\Sigma$ be the following functorial version of $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$. Objects are monomorphisms $i : P_1 \to P_0$ of A-modules $P_0, P_1 \in P(A)$ as above such that $\Sigma^{-1}i$ is an isomorphism. The group of morphisms from i to $j : Q_1 \to Q_0$ is the quotient of the abelian group of pairs (f_1, f_0) of morphisms $f_i : P_i \to Q_i$, $i = 0, 1$ with $f_0 i = j f_1$ modulo the pairs of the form (hi, jh) for some map $h : P_0 \to Q_1$. Objects for which i is the identity map of some $P \in P(A)$ are zero objects in $\mathcal{T}_\Sigma$, we identify them and call the resulting zero object base point. Simple homological algebra shows that the functor coker : $\mathcal{T}_\Sigma \to \mathcal{H}^1_{\Sigma,\mathrm{proj}} : i \mapsto \mathrm{coker}(i)$ is an equivalence of categories. Via this equivalence we define a structure of an exact category on $\mathcal{T}_\Sigma$.

1.5. A *category with duality* is a triple $(\mathcal{C}, \sharp, \eta)$ with $\mathcal{C}$ a category, $\sharp : \mathcal{C} \to \mathcal{C}^{op}$ a functor and $\eta : id_{\mathcal{C}} => \sharp\sharp$ a natural equivalence such that for all objects A of $\mathcal{C}$ we have $1_{A^\sharp} = \eta_A^\sharp \circ \eta_{A^\sharp}$. Given two categories with duality $(\mathcal{A}, \sharp, \eta)$, $(\mathcal{B}, \sharp, \tau)$, a functor $f : \mathcal{A} \to \mathcal{B}$ is called *duality preserving* if $\sharp \circ f = f^{op} \circ \sharp$ and $f(\eta_A) = \tau_{f(A)}$ for every object A of $\mathcal{A}$.

An *exact (resp. preadditive) category with duality* is a category with duality $(\mathcal{E}, \sharp, \eta)$ with $\mathcal{E}$ an exact (resp. preadditive, [Mac71]) category such that the duality functor $\sharp : \mathcal{E} \to \mathcal{E}^{op}$ is exact (resp. additive).

Given a category with duality $(\mathcal{C}, \sharp, \eta)$, its associated *hermitian category* $\mathcal{C}_h$ is defined as follows. An object is a pair (M, ϕ) with M an object of $\mathcal{C}$ and $\phi : M \overset{\cong}{\to} M^\sharp$ an isomorphism such that $\phi = \phi^\sharp \eta$. A morphism $\alpha : (M, \phi) \to (N, \psi)$ is a morphism $\alpha : M \to N$ in $\mathcal{C}$ such that $\alpha^\sharp \psi \alpha = \phi$.

Let $\epsilon \in \{+1, -1\}$ and $(\mathcal{C}, \sharp, \eta)$ be a a preadditive category with duality. Then we write ${}_\epsilon\mathcal{C}_h$ for the hermitian category associated with the category with duality $(\mathcal{C}, \sharp, \epsilon\eta)$.

1.6 *Example.* Let A be a ring with involution, *i.e.,* a ring equipped with a map $^- : A \to A^{op}$ satisfying $\overline{a+b} = \bar{a} + \bar{b}$, $\overline{ba} = \bar{a}\bar{b}$ and $\bar{\bar{a}} = a$. In other words $(A, ^-, id)$ is a preadditive category with duality which has exactly one (non base point, *cf.* 2.1) object. The duality $^-$ extends to $F(A)$ with $n^\sharp = n$, $(a_{i,j})^\sharp = (\bar{a}_{j,i})$, and to $P(A)$ with $(C, p)^\sharp = (C^\sharp, p^\sharp)$. So $(F(A), \sharp, id)$ and $(P(A), \sharp, id)$ are additive categories with duality.

On $A-$proj the duality $\sharp$ on $P(A)$ can be described in the more familiar way as follows. For M a right A-module and N an A-bimodule, let $\hom_{skew\ A}(M, N)$ be the right $\mathcal{A}$-module

$$\hom_{skew\ A}(M, N) := \{f \in \hom_{\mathbb{Z}}(M, N) | f(ma) = \bar{a} f(m)\}$$

which is an A-module via $fa(m) = f(m)a$. Then the duality on $A-$proj is $M^\sharp = \hom_{skew\ A}(M, A)$. The identification $id \to \sharp\sharp$ is $M \to M^{\sharp\sharp} : m \mapsto \overline{ev_m}$ where ev_m is evaluation at m.

If A is commutative with trivial involution, then $\sharp = \hom_A(\ \ , A)$ is the usual involution.

1.7. *The duality on $\mathcal{T}_\Sigma$.* Let $(A, \sharp)$ be a ring with involution, and let $\Sigma \subset A$ a multiplicative subset of central non-zero divisors closed under the involution. Then the ring of fractions $\Sigma^{-1}A$ is defined and $\sharp$ induces an involution on $\Sigma^{-1}A$ by $(s^{-1}a)^\sharp = (s^\sharp)^{-1}a^\sharp$.

The functor $\mathrm{Ext}^1_{skew\ A}(\ \ , A) = R^1 \hom_{skew\ A}(\ \ , A)$ induces an exact duality on $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$ and a natural isomorphism $\eta : Id \Rightarrow (Ext^1)^2$ (exercise, compare [Kar74]). By our assumptions on Σ, the localization map $A \to \Sigma^{-1}A$ is injective and a map of rings with involution. This implies that $i^\sharp : P_0^\sharp \to P_1^\sharp$ is injective for $i \in \mathcal{T}_\Sigma$, and the cokernel of $i^\sharp$ is Σ-torsion. Thus $i \mapsto i^\sharp$ defines a duality on $\mathcal{T}_\Sigma$ making $(\mathcal{T}_\Sigma, \sharp, id)$ into an additive category with duality. Since $\sharp$ is an explicit version of $\mathrm{Ext}^1_{skew\ A}(\ \ , A)$, it is exact, and thus $(\mathcal{T}_\Sigma, \sharp, id)$ is an exact category with duality.

1.8. *The simplicial category with duality $\mathcal{R}_*\mathcal{E}$.* Recall [Wal85] that for any exact category $\mathcal{E}$, Waldhausen constructs a simplicial exact category $S_*\mathcal{E}$ such that the classifying space of $iS_*\mathcal{E}$ is homotopy equivalent to $Q\mathcal{E}$, Quillen's Q-construction. For $\mathcal{E}$ an exact category with duality $\sharp$, point-wise application of $\sharp$ makes $S_n\mathcal{E}$ into an exact category with duality. We observe that $n \mapsto S_n\mathcal{E}$ is *not* a simplicial exact category with duality since the simplicial structure maps do not commute with dualities. But its edge-wise subdivision $n \mapsto S_{2n+1}\mathcal{E}$ *is* a simplicial exact category with duality.

More precisely, for $n \geq 0$ an integer, let $\mathbf{n}$ be the totally ordered set $\{n' < (n-1)' < ... < 0' < 0 < ... < (n-1) < n\}$ considered as a category with duality by declaring $l^\sharp = l'$, $(l')^\sharp = l$ for $0 \leq l \leq n$. We will write $'$ for this duality $\sharp$. The assignment $(\theta : [n] \to [m]) \mapsto (\underline{\theta} : \mathbf{n} \to \mathbf{m})$ with $\underline{\theta}(l) = \theta(l)$ and $\underline{\theta}(l') = \theta(l)'$ makes $[n] \mapsto \mathbf{n}$ into a cosimplicial category with duality.

Denote by $\mathcal{I}(n)$ the category of arrows in $\mathbf{n}$, *i.e.,* its objects are pairs $(p, q) \in \mathbf{n} \times \mathbf{n}$ with $p \leq q$ and the morphisms in $\mathcal{I}(n)$ are commutative squares in $\mathbf{n}$. The duality on $\mathbf{n}$ induces a duality on $\mathcal{I}(n)$. The cosimplicial structure $[n] \mapsto \mathbf{n}$ makes $[n] \mapsto \mathcal{I}(n)$ into a cosimplicial category with duality.

Let $(\mathcal{E}, \sharp, \eta)$ be an exact category with duality. We choose a zero object with $0^\sharp = 0$ and call it base point. Then $(\mathcal{R}_*\mathcal{E}, \sharp, \eta)$ is the following simplicial exact category with duality. Objects of $\mathcal{R}_n\mathcal{E}$ are functors $A : \mathcal{I}(n) \to \mathcal{E}$ where all the sequences $A_{pq} \to A_{pr} \to A_{qr}$ are admissible short exact sequences in $\mathcal{E}$ whenever $p \leq q \leq r \in \mathbf{n}$ and such that $A_{pp} = 0$ the base point zero object of $\mathcal{E}$. Morphisms are natural transformations of functors. The dual of an object is given by $(A^\sharp)_{p,q} := (A_{q',p'})^\sharp$. The dual of a morphism is also given by taking the point-wise dual and re-indexing. We set $(\eta_A)_{p,q} = \eta_{A_{p,q}}$. The exact structure on $\mathcal{R}_n\mathcal{E}$ we will use throughout this article is given point-wise by the additive split exact structure on $\mathcal{E}$. (The exact structure given pointwise by the exact structure on $\mathcal{E}$ will not appear in this article.) We also write $\mathcal{T}_\Sigma^\oplus$ for the category $\mathcal{T}_\Sigma$ of 1.7equipped with the split exact structure, which is therefore equivalent as an exact category to $\mathcal{R}_0\mathcal{T}_\Sigma$. The simplicial structure on $(\mathcal{R}_*\mathcal{E}, \sharp, \eta)$ is induced by the cosimplicial structure on $\mathcal{I}(*)$. We further write $\mathcal{R}_*^h\mathcal{E}$ for $(\mathcal{R}_*\mathcal{E})_h$.

It is straightforward to check that $(\mathcal{R}_*\mathcal{E}, \sharp, \eta)$ is a simplicial exact category with duality. Forgetting the duality, we see that $\mathcal{R}_n\mathcal{E}$ equals Waldhausen's $S_{2n+1}\mathcal{E}$ [Wal85], and $\mathcal{R}_*\mathcal{E}$ is just the edgewise subdivision [Wal85, p.375] of $S_*\mathcal{E}$.

1.9 *Notation.* Given a category $\mathcal{C}$, we write $i\mathcal{C}$ for the category with the same objects as $\mathcal{C}$ and morphisms the isomorphisms of $\mathcal{C}$.

1.10 Definition. For an exact category with duality $(\mathcal{E}, \sharp, \eta)$ we define a topological space ${}_\epsilon\mathcal{W}(\mathcal{E})$ as the realization of a bisimplicial set

$$ {}_\epsilon\mathcal{W}(\mathcal{E}) = |(p,q) \mapsto N_p i\ {}_\epsilon\mathcal{R}_q^h\mathcal{E}| $$

where N_* stands for the nerve of a category. We may write ${}_\epsilon\mathcal{W}(A)$ for the $\mathcal{W}$-theory space associated with $(P(A), \sharp, id)$ for A a ring with involution. The U-theory space of an exact category with duality $\mathcal{E}$ is

$$ {}_\epsilon U(\mathcal{E}) = \Omega\, {}_\epsilon\mathcal{W}(\mathcal{E}). $$

1.11 *Remark.* There is a category associated to $(\mathcal{E}, \sharp, \eta)$ due to Giffen and Karoubi [CL86, Uri90, Hor02, Schb] such that its classifying space is homotopy equivalent to $\mathcal{W}(\mathcal{E}) = |N_* i_\epsilon \mathcal{R}_*^h\mathcal{E}|$ [Schb]. The category is a hermitian analogue of Quillen's Q-construction.

1.12 *Remark.* For A a ring with involution in which 2 is invertible, there is a homotopy fibration

$$ K(A) \overset{H}{\to} {}_\epsilon K^h(A) \to {}_\epsilon\mathcal{W}(A). $$

Therefore, the U-theory space ${}_\epsilon U(A)$ defined here is homotopy equivalent to the U-theory space ${}_\epsilon U(A)$ defined by Karoubi [Kar73] (see also 5.8, 5.9). This result in its $\mathcal{W}$-category-guise is stated in [CL86]. But the article contains a crucial error as on p. 177 of [CL86]: the functor σ^* doesn't act as an inner automorphism as claimed. A proof which avoids this argument is given by the second author in [Schb].

1.13 *Remark.* The simplicial category $i_\epsilon\mathcal{R}_*^h\mathcal{E}$ is equivalent to $i\ {}_\epsilon S_*^e\mathcal{E}$ of [SY96] and the simplicial set $Ob({}_\epsilon\mathcal{R}_*^h\mathcal{E})$ is isomorphic to the simplicial set ${}_\epsilon s_*^e\mathcal{E}$ of [SY96]. However, it is important to consider $\mathcal{R}_*\mathcal{E}$ as a simplicial additive (exact) category with duality as we will see in the proof of our Localization Theorem 1.15.

1.14 *Remark.* Let $\mathcal{E}$ be an exact category with duality. The map from $Ob(\mathcal{R}_0^h\mathcal{E}) = Ob(\mathcal{E}_h)$ into the usual Witt group [Bal01, 1.6] $W(\mathcal{E})$ of $\mathcal{E}$ which sends a hermitian object to its class in $W(\mathcal{E})$, induces an isomorphism (exercise or [Uri90], [Hor02]) $\pi_0\mathcal{W}(\mathcal{E}) \overset{\sim}{\to} W(\mathcal{E})$.

We now state the main results 1.15 and 1.19 which will be proved in sections 6 and 7.

1.15 Theorem. (Localization) *Let $(A, \sharp)$ be a ring with involution in which 2 is a unit. Let $\Sigma \subset A$ be a multiplicative subset of central non-zero divisors closed under the involution. Then there is a homotopy fibration*

$$ {}_\epsilon U(\mathcal{T}_\Sigma) \to {}_\epsilon K^h(A) \to {}_\epsilon K^h(\Sigma^{-1}A) $$

where the map ${}_\epsilon K^h(A) \to {}_\epsilon K^h(\Sigma^{-1}A)$ is the result of applying the hermitian K-theory functor to the localization map $(A,\sharp) \to (\Sigma^{-1}A,\sharp)$ of rings with involution (1.7).

1.16 Corollary. *Under the hypothesis of 1.15 we have a long exact sequence*

$$\cdots \to {}_\epsilon U_n(\mathcal{T}_\Sigma) \to {}_\epsilon K^h_n(A) \to {}_\epsilon K^h_n(\Sigma^{-1}A) \to {}_\epsilon U_{n-1}(\mathcal{T}_\Sigma) \to \cdots \to {}_\epsilon U_0(\mathcal{T}_\Sigma) \to {}_\epsilon K^h_0(A) \to {}_\epsilon K^h_0(\Sigma^{-1}A).$$

Proof. This is the long exact sequence of homotopy groups associated to the homotopy fibration of 1.15. □

1.17 *Remark.* The map ${}_\epsilon K^h_0(A) \to {}_\epsilon K^h_0(\Sigma^{-1}A)$ is not surjective, in general, even if A is regular. This is because the map on classical Witt groups $W(A) \to W(\Sigma^{-1}A)$ is not surjective, in general, even for regular A. For A a Dedekind domain, the map on Witt groups is injective (see for example [MH73]), yet rarely an isomorphism.

The homotopy fibration of Theorem 1.15 extends to a homotopy fibration of spectra (7.3). If A is regular, then the negative homotopy groups of these spectra can be identified with the triangular Witt groups of Balmer (7.5). This yields a natural extension of the long exact sequence of Corollary 1.16 to the right (see 6.16 for a more detailed discussion).

1.18. Let $\sharp : A \to A^{op}$ be a ring with involution. Let $f \in A$ be a central non-zero divisor with $f^\sharp = f$ and let $\Sigma = \{f^n \mid n \in \mathbb{N}\}$. There is a functor $A \to \mathcal{T}_\Sigma$ of categories which sends A to $f : A \to A$ and a map $a : A \to A$ to the map of arrows $(a,a) : f \to f$ (f is central). As $f^\sharp = f$, the functor preserves dualities. Since the map $f : A \to A$ is sent to 0 in $\mathcal{T}_\Sigma$ we obtain a functor of categories with dualities $A/fA \to \mathcal{T}_\Sigma$ by passage to the quotient. The latter category is idempotent complete, so the functor extends to a duality preserving functor $P(A/fA) \to \mathcal{T}_\Sigma$. More precisely, we have duality preserving functors $P(A/fA) \to P(\mathcal{T}_\Sigma) \leftarrow \mathcal{T}_\Sigma$ where the last arrow is an equivalence because $\mathcal{T}_\Sigma$ is idempotent complete (see 2.3 for the definition of $P(\mathcal{T}_\Sigma)$).

1.19 Theorem. (Dévissage) *Let $\sharp : A \to A^{op}$ be a commutative ring with involution in which 2 is a unit, and let $f \in A$ be a non-zero divisor with $f^\sharp = f$. Assume that A and A/fA are regular. Write $\Sigma = \{f^n \mid n \in \mathbb{N}\}$. Then the inclusion of exact categories with duality $P(A/fA) \to \mathcal{T}_\Sigma$ (1.18) induces homotopy equivalences*

$${}_\epsilon\mathcal{W}(A/fA) \xrightarrow{\sim} {}_\epsilon\mathcal{W}(\mathcal{T}_\Sigma) \quad \text{and} \quad {}_\epsilon U(A/fA) \xrightarrow{\sim} {}_\epsilon U(\mathcal{T}_\Sigma).$$

1.20 *Remark.* The homotopy equivalences of Theorem 1.19 extend to homotopy equivalences of non-connective **W**- and **U**-spectra (see the proof of *dévissage* in section 7).

1.21. Let R be a (commutative) Dedekind domain with trivial involution and $\Sigma = R - 0$. For $0 \neq \wp$ a prime ideal in R, let $\Sigma_\wp = R_\wp - 0$. The localizations $R \to R_\wp$ induce by functoriality maps of categories with duality $\mathcal{T}_\Sigma \to \mathcal{T}_{\Sigma_\wp}$ which assemble to a duality preserving functor

$$\mathcal{T}_\Sigma \to \bigoplus_{(0)\neq\wp\subset R} \mathcal{T}_{\Sigma_\wp}$$

as the support of a finitely generated torsion module is a finite set of primes different from (0). This functor is easily checked to be an equivalence [Bas68, p. 509]. Choosing a local parameter $\pi_\wp$ for the dvr $R_\wp$ we have $R/\wp = R_\wp/\pi_\wp$ and $\Sigma_\wp^{-1}R_\wp = R[\pi_\wp^{-1}]$. Applying 1.19 with $A = R_\wp$ and $f = \pi_\wp$ yields a homotopy equivalence ${}_\epsilon W(R/\wp) \to {}_\epsilon W(\mathcal{T}_{\Sigma_\wp})$. Note that this homotopy equivalence depends on the choice of the local parameter $\pi_\wp$. Thus we have shown the following corollary.

1.22 Corollary. *Let R be a Dedekind domain with trivial involution and $\Sigma = R - 0$. Suppose $\frac{1}{2} \in R$. Then the inclusions $P(R/\wp) \to \mathcal{T}_\Sigma$ (1.21) of exact categories with duality induce isomorphisms*

$$\bigoplus_{(0)\neq\wp\subset R} {}_\epsilon U_n(R/\wp) \to {}_\epsilon U_n(\mathcal{T}_\Sigma).$$

□

1.23 *Remark.* This result was already established in [Hor02], but there a proof of the fact that the map of the corollary 1.22 and the similar map for V-theory are compatible with Karoubi's fundamental theorem was missing. Using arguments of section 7, this gap can be closed.

1.24. *Karoubi's conjecture.* Localization 1.15 and *Dévissage* 1.22 for Dedekind domains together yield a long exact sequence

$$\ldots \to \oplus_\epsilon U_n(A/\wp) \to {}_\epsilon K^h_n(A) \to {}_\epsilon K^h_n(\Sigma^{-1}A) \to \oplus_\epsilon U_{n-1}(A/\wp) \to \ldots$$

where the sum is taken over all prime ideals $\wp \subset A$ different from 0. As explained in the introduction, this was a conjecture of Karoubi.

1.25. *Application: Algebraic Bott periodicity.* First some notation. Let k be a field of characteristic $\neq 2$, and let R be a smooth k-algebra. We write X for $\operatorname{Spec} R$. Let $x : R \to k$ be a k-rational point of X. For a functor F from k-algebras to spectra, we write $F(X)$ for $F(R)$, and $\tilde{F}(X)$ for the reduced functor which is the fiber of $F(x)$. Given another pointed affine k-scheme (Y, y), we write $\tilde{F}(X \wedge Y)$ for the cofiber of $\tilde{F}(X \times y) \oplus \tilde{F}(x \times Y) \to \tilde{F}(X \times Y)$. The k-schemes $\mathbf{G}_m$ and $\mathbf{A}^1$ are pointed at 1.

Localization 1.15 and *Dévissage* 1.19 yield a homotopy fibration of spectra (see remarks 1.17 and 1.20)

$${}_\epsilon\mathbf{U}(R) \to {}_\epsilon\mathbf{K}^h(R[T]) \to {}_\epsilon\mathbf{K}^h(R[T, T^{-1}]).$$

The ring homomorphism $R \to R[T]$ induces a homotopy equivalence ${}_\epsilon\mathbf{K}^h(R) \stackrel{\sim}{\to} {}_\epsilon\mathbf{K}^h(R[T])$ as this is true for K-theory and Balmer's Witt groups (use Karoubi induction 5.20). Evaluating T at 1 yields a retraction of ${}_\epsilon\mathbf{K}^h(R) \to {}_\epsilon\mathbf{K}^h(R[T, T^{-1}])$ and thus a homotopy equivalence

$$\Omega_\epsilon\mathbf{K}^h(X \times \mathbf{G}_m) \simeq \Omega_\epsilon\mathbf{K}^h(X) \oplus {}_\epsilon\mathbf{U}(X).$$

In terms of reduced functors, this yields a homotopy equivalence

$$\Omega_\epsilon\tilde{\mathbf{K}}^h(X \wedge \mathbf{G}_m) \simeq {}_\epsilon\tilde{\mathbf{U}}(X). \tag{1.26}$$

Replacing R by UR (5.6) and using the homotopy equivalence 5.15 from Karboubi's fundamental theorem ${}_\epsilon\mathbf{K}^h(U^2R) \stackrel{\sim}{\to} {}_{-\epsilon}\mathbf{K}^h(V^2U^2R) \simeq {}_{-\epsilon}\mathbf{K}^h(S^2R)$ we find homotopy equivalences

$$\Omega_\epsilon\mathbf{U}(R[T, T^{-1}]) \simeq \Omega_\epsilon\mathbf{U}(R) \oplus {}_{-\epsilon}\mathbf{K}^h(R).$$

Again, in terms of reduced functors, this yields a homotopy equivalence

$$\Omega_\epsilon\tilde{\mathbf{U}}(X \wedge \mathbf{G}_m) \simeq {}_{-\epsilon}\tilde{\mathbf{K}}(X). \tag{1.27}$$

For a functor F as above, we further write $\tilde{F}(X \wedge \mathbb{P}^1)$ for the homotopy colimit of the diagram

$$\tilde{F}(X \wedge (\mathbb{P}^1 - \{0\})) \longleftarrow \tilde{F}(X \wedge \mathbf{G}_m) \longrightarrow \tilde{F}(X \wedge (\mathbb{P}^1 - \{\infty\})).$$

If $F = {}_\epsilon\mathbf{K}$ or $F = {}_\epsilon\mathbf{U}$, then this definition implies homotopy equivalences $\tilde{F}(X \wedge \mathbb{P}^1) \simeq \Omega\tilde{F}(X \wedge \mathbf{G}_m)$ as in these cases $\tilde{F}(X \wedge \mathbf{A}^1) \simeq *$. Now (1.26) and (1.27) become ${}_\epsilon\tilde{\mathbf{K}}^h(X \wedge \mathbb{P}^1) \simeq {}_\epsilon\tilde{\mathbf{U}}(X)$ and ${}_\epsilon\tilde{\mathbf{U}}^h(X \wedge \mathbb{P}^1) \simeq {}_\epsilon\tilde{\mathbf{K}}(X)$. In particular, we have homotopy equivalences

$${}_\epsilon\tilde{\mathbf{K}}^h(X \wedge (\mathbb{P}^1)^{\wedge 4}) \simeq {}_\epsilon\tilde{\mathbf{U}}(X \wedge (\mathbb{P}^1)^{\wedge 3}) \simeq {}_{-\epsilon}\tilde{\mathbf{K}}^h(X \wedge (\mathbb{P}^1)^{\wedge 2}) \simeq {}_{-\epsilon}\tilde{\mathbf{U}}(X \wedge \mathbb{P}^1) \simeq {}_\epsilon\tilde{\mathbf{K}}^h(X)$$

for X a smooth affine k-scheme.

Since in topology $(\mathbb{P}^1)^{\wedge 4}(\mathbf{C}) = S^8$, the previous homotopy equivalence is thought to be the algebraic analogue of the real Bott periodicity theorem

$$K^{top}_{\mathbb{R}}(X \wedge S^8) \simeq K^{top}_{\mathbb{R}}(X).$$

For more details and a generalization to non-affine smooth k-schemes in the framework of $\mathbf{A}^1$-homotopy category of schemes, we refer the reader to [Hor].

2. Filtering Categories

This section provides some formal properties of filtering maps of preadditive categories most of which are known. The observation 2.16 seems to be new. We need this abstract formalism to construct the maps ϕ_k in 6.5.

2.1. Recall [Mac71] that a *preadditive category* is a category whose hom-sets are abelian groups and whose composition is bilinear. In order for 2.17 to be true, we also demand that our preadditive categories in this article come equipped with a chosen 0 object called base point. Additive functors between preadditive categories are to preserve base points.

For instance, a ring is a preadditive category with exactly one object different from the base point. Denote by Pac the category of preadditive categories. It is complete, cocomplete and has a symmetric monoidal tensor product $\otimes$ defined as follows. The category $\mathcal{A} \otimes \mathcal{B}$ has objects pairs $A \wedge B$ with A an object of $\mathcal{A}$ and B an object of $\mathcal{B}$. The objects $A \wedge B$ with A or B a base point are identified with the base point of $\mathcal{A} \otimes \mathcal{B}$. The Hom-sets are defined by

$$Hom_{\mathcal{A}\otimes\mathcal{B}}(A \wedge B, A' \wedge B') = Hom_{\mathcal{A}}(A, A') \otimes Hom_{\mathcal{B}}(B, B').$$

Composition is defined by $(a \otimes b) \circ (a' \otimes b') = (a \circ a') \otimes (b \circ b')$.

2.2. A *preadditive category with duality* is a category with duality $(\mathcal{A}, \sharp, \eta)$ such that $\sharp$ is a morphism in Pac. Denote by Pad the category of small, preadditive categories with duality and duality preserving functors. It is complete, cocomplete and has a symmetric monoidal tensor product. Limits, colimits and tensor products are formed in Pac and one observes that the resulting category inherits a duality. For instance, $(\mathcal{A}, \sharp, \eta) \otimes (\mathcal{B}, \sharp, \tau) = (\mathcal{A} \otimes \mathcal{B}, \sharp \otimes \sharp, \eta \otimes \tau)$.

2.3. For $\mathcal{A}$ in Pad, an $\mathcal{A}$-module is an additive functor from $\mathcal{A}^{op}$ to the category of abelian groups. The category of $\mathcal{A}$-modules is denoted by $\mathcal{A}-$Mod. Recall that the Yoneda embedding $\mathcal{A} \to \mathcal{A}-\text{Mod} : A \mapsto Hom_{\mathcal{A}}(\ \ , A)$ is fully faithful, and we may write A for the representable functor $Hom_{\mathcal{A}}(\ \ , A)$. Let $\mathcal{A}-$free be the category of finitely generated free $\mathcal{A}$-modules, *i.e.,* the full subcategory of $\mathcal{A}-$Mod of those modules which are finite direct sums of representable modules. We write $F(\mathcal{A})$ for the following functorial version of $\mathcal{A}-$free. Objects are sequences $(A_1, ..., A_n)$ of objects of A and maps are matrices of maps $A_i \to B_j$, $i = 1, ..., n$, $j = 1, ..., m$ where $(B_1, ..., B_m)$ is another object of $F(\mathcal{A})$. Composition is matrix multiplication. The empty sequence is declared to be the base point zero object and is identified with the objects $(0, ..., 0)$. So $F(\mathcal{A})$ is in Pac. Moreover, $F(\mathcal{A})$ has a symmetric strict monoidal direct sum operation $\oplus : (A_1, ..., A_n) \times (B_1, ..., B_m) \mapsto (A_1, ..., A_n, B_1, ...B_m)$. If $\mathcal{A}$ is in Pad, then so is $F(\mathcal{A})$ by applying dualities component-wise. Of course, $F(\mathcal{A})$ is a small preadditive category equivalent to $\mathcal{A}-$free.

Let $\mathcal{A}-$proj be the category of finitely generated projective $\mathcal{A}$-modules, *i.e.,* the full subcategory of $\mathcal{A}-$Mod of those modules which are direct factors of finitely generated free $\mathcal{A}$-modules. Let $P(\mathcal{A})$ be the idempotent completion (1.1) of $F(\mathcal{A})$. It is equivalent to $\mathcal{A}-$proj. Any duality on $\mathcal{A}$ induces a duality on $P(\mathcal{A})$ by $(F, p)^{\sharp} = (F^{\sharp}, p^{\sharp})$, $F \in F(\mathcal{A})$.

2.4 Definition. A full inclusion $\mathcal{A} \to \mathcal{U}$ of preadditive categories is called *right filtering* if every morphism $U \to A$ from an object of $\mathcal{U}$ to an object of $\mathcal{A}-$free factors through a direct factor of U belonging to $\mathcal{A}-$proj, *i.e.,* there is an idempotent p of U such that $\text{Im}(p)$ is in $\mathcal{A}-$proj and such that the map is the composition of the canonical projection $p : U \to \mathrm{Im}(p)$ and a map $\mathrm{Im}(p) \to A$. Note that we do not require the complement of $\mathrm{Im}(p)$ in U to be in $\mathcal{U}$.

A full inclusion $\mathcal{A} \to \mathcal{U}$ is called *left filtering* if $\mathcal{A}^{op} \to \mathcal{U}^{op}$ is right filtering, it is called *filtering* if it is both left and right filtering. We say that $\mathcal{U}$ *is* $\mathcal{A}$*-filtered* if $\mathcal{A} \subset \mathcal{U}$ is filtering.

2.5 *Remark.* Definition 2.4 is a slightly modified version of [PW89, p.355] and [Kar70, Définition 1.5] which is most convenient for our purposes.

2.6 Definition. A map $\mathcal{U} \to \mathcal{B}$ in *Pac* is called *cofiltering* if the kernel category $\mathcal{A}$, *i.e.,* the full subcategory of $\mathcal{U}$ of objects mapped to the base point of $\mathcal{B}$, is filtering in $\mathcal{U}$ and if the induced map $\mathcal{U}/\mathcal{A} \to \mathcal{B}$ is an equivalence of categories which is surjective on objects. The quotient $\mathcal{U}/\mathcal{A}$ in *Pac* is the category with set of objects the quotient set $Ob\mathcal{U}/Ob\mathcal{A}$ (all objects of $\mathcal{A}$ are identified with the base point zero object). Morphisms from U to V in $\mathcal{U}/\mathcal{A}$ are the morphisms of $\mathcal{U}$ modulo those which are sums of maps factoring through $\mathcal{A}$.

A sequence $\mathcal{A} \to \mathcal{U} \to \mathcal{B}$ in *Pad* is called *exact* if $\mathcal{A} \to \mathcal{U}$ is filtering, the composition maps $\mathcal{A}$ to the base point of $\mathcal{B}$ and the induced map $\mathcal{U}/\mathcal{A} \to \mathcal{B}$ is an equivalence of categories, surjective on objects.

2.7 Lemma. *Let $\mathcal{A}$ be a full subcategory of a preadditive category $\mathcal{U}$. Write $\hat{F}(\mathcal{U})$ for the full subcategory of $P(\mathcal{U})$ of objects U such that there is a A of $P(\mathcal{A})$ with $U \oplus A$ isomorphic to an object of $F(\mathcal{U})$. Then $\mathcal{A} \to \mathcal{U}$ is filtering iff $P(\mathcal{A}) \to \hat{F}(\mathcal{U})$ is filtering.*

Proof. The "if" part is clear by the definitinon of the filtering condition and the equivalence $\mathcal{A}-\mathrm{proj} \to P(\mathcal{A})-\mathrm{proj}$. For the "only if" part, let $\varphi : A \to U$ be a map with A in $P(\mathcal{A})$ and U in $\hat{F}(\mathcal{U})$. Choose $B, B' \in \mathcal{A}-\mathrm{proj}$ with $U \oplus B$ a free $\mathcal{U}$-module and $A \oplus B \oplus B'$ a free $\mathcal{A}$-module. The map $\phi = \varphi \oplus id_B \oplus 0 : A \oplus B \oplus B' \to U \oplus B$ factors through a direct factor $\mathrm{Im}(q)$ of $U \oplus B$ for $q = (q_{i,j})$ an idempotent of $U \oplus B$ with image in $\mathcal{A}-\mathrm{proj}$ (2.4). This means that $q \circ \phi = \phi$ which implies $q_{11}\varphi = \varphi$, $q_{12} = 0$ and $q_{22} = 1$. Idempotency of q implies $q_{11}^2 = q_{11}$ and $q_{21}q_{11} = 0$. Then the diagonal matrix $diag(q_{11}, 0)$ is an idempotent of $\mathrm{Im}(q)$ and thus has image $\mathrm{Im}(q_{11})$ in $\mathcal{A}-\mathrm{proj}$. Since $q_{11}\varphi = \varphi$, the map φ factors through $\mathrm{Im}(q_{11})$ which is a direct factor of U. The right filtering condition is similar. □

2.8 *Remark.* Note that in $\hat{F}(\mathcal{U})$ all direct factors lying in $P(\mathcal{A})$ have complements in $\hat{F}(\mathcal{U})$. More precisely, let $A \overset{j}{\to} U \overset{q}{\to} A$ be two maps with $qj = 1$, $A \in P(\mathcal{A})$ and $U \in \hat{F}(\mathcal{U})$. Then the cokernel of i exists in $\hat{F}(\mathcal{U})$. Note also that the functor $F(\mathcal{U}/A) \to \hat{F}(\mathcal{U})/P(\mathcal{A})$ is an equivalence.

2.9. *Calculus of fractions.* Recall that a set S of morphisms in a preadditive category $\mathcal{C}$ satisfies a "calculus of right fractions" if i) S is closed under composition and all identity maps belong to S, ii) given maps $f : X \to Y$, $s : Z \to Y$ with $s \in S$, there exist maps $g : W \to Z$, $t : W \to X$ with $t \in S$ and $ft = sg$ and iii) given $f : X \to Y$, $s : Y \to Z$ with $s \in S$ and $sf = 0$, there is a map $t : W \to X$ in S with $ft = 0$ [GZ67, I.2.2]. In this situation, the preadditive category $\mathcal{C}[S^{-1}]$, obtained from $\mathcal{C}$ by formally inverting the maps in S, has a simple description in terms of right fractions, *cf. loco citato.*

Let $\mathcal{A} \to \mathcal{U}$ is a full inclusion of preadditive categories, and let S be the set of split monomorphisms of $\hat{F}(\mathcal{U})$ with cokernels in $P(\mathcal{A})$. Then the full inclusion is right filtering if and only if S satisfies a calculus of right fractions. Conditions i) and iii) are obvious and ii) is equivalent to the right filtering condition (2.7, 2.8). By the respective universal properties, we get functors $\hat{F}(\mathcal{U})/P(\mathcal{A}) \to \hat{F}(\mathcal{U})[S^{-1}]$ and $\hat{F}(\mathcal{U})[S^{-1}] \to \hat{F}(\mathcal{U})/P(\mathcal{A})$ which are inverse equivalences.

2.10. *The cone ring C.* We now give the main example of a filtering map. The *cone ring* C is the ring of infinite matrices $(a_{i,j})_{i,j\in\mathbb{N}}$ with entries $a_{i,j} \in \mathbb{Z}$ such that in each row and in each column all but finitely many entries are zero. We have $C = \mathrm{colim}_{S\in I}\, C_S$ where I runs over the sets $S \subset \mathbb{N}\times\mathbb{N}$ consisting of pairs of integers (i, j) such that for every $m, n \in \mathbb{N}$ the set $S_{m,n} = \{(i, j) \in S |\ i = m \text{ or } j = n\}$ is finite. The module $C_S \subset C$ is the $\mathbb{Z}$-submodule of those matrices whose entries satisfy $a_{i,j} = 0$ whenever $(i, j) \notin S$.

As $\mathbb{Z}$-module, C_S is a countable product of copies of $\mathbb{Z}$. So C_S is torsion-free, hence flat over $\mathbb{Z}$. Since C is a filtered union of the C_S's, the ring C is a flat $\mathbb{Z}$-algebra. The map which sends a matrix $(a_{i,j})$ to its transpose ${}^t(a_{i,j}) = (a_{j,i})$ defines an involution $C \to C^{op}$. In this way, C is a ring with involution.

In C we fix the symmetric idempotent $p = {}^tp = (a_{i,j}) \in C$ with $a_{0,0} = 1$ and $a_{i,j} = 0$ otherwise. Let $\mathcal{C} = C \cup Im(p)$ be the full subcategory of C-Mod with the two non-base point objects C and $Im(p) = (C, p)$ (2.3). Remark that the endomorphism ring of $Im(p)$ is $\mathbb{Z}$, so that the functor $Im(p) \otimes \quad : \mathbb{Z} \to \mathcal{C}$ is fully faithful. In fact, it is *filtering* as we see by the following argument. A map $C \to Im(p)$ is given by an element $(a_{i,j}) \in C$ such that $(a_{i,j}) = p \circ (a_{i,j})$, *i.e.*, such that $a_{i,j} = 0$ for $i > 0$. The requirement that $(a_{i,j})$ be column finite implies that there is a $d \in \mathbb{N}$ such that $a_{0,j} = 0$ for $j > d$ as well. Let $t = (t_{i,j}) \in C$ be the matrix with entries $t_{i,j} = 1$ for $i = j+1$ and $t_{i,j} = 0$ otherwise, and let $\hat{t} = (t_{i,j}) \in C$ be the matrix with entries $t_{i,j} = 1$ for $j = i+1$ and $t_{i,j} = 0$ otherwise. Now we write the given map as the composition of the isomorphism $(\hat{t}^{d+1}\ p\hat{t}^d\ \ldots\ p\hat{t}\ p) : C \to C \oplus Im(p)^{d+1}$ (with inverse $(t^{d+1}\ t^dp\ \ldots\ tp\ p)$) and $(0\ a_{0,d}\ \ldots\ a_{0,0}) : C \oplus Im(p)^{d+1} \to Im(p)$. Similarly for maps $Im(p) \to C$. The filtering condition for other maps between free $\mathbb{Z}$-modules and objects of $\mathcal{C}$ follows formally from the above situation.

2.11. *Eilenberg swindle.* We will construct a ring map $f : C \to C$ such that $f(\mathrm{Im}(p)) \cong C$ as C-modules. Since $C \cong C \oplus \mathrm{Im}(p)$ we have a C-module isomorphism $e : f(C) \cong f(C \oplus \mathrm{Im}(p)) \cong f(C) \oplus f(\mathrm{Im}(p)) \cong f(C) \oplus C$. Then f extends to a functor $f : P(C) \to P(C)$, and e extends to a natural isomorphism $e : f \cong f \oplus id$. This is called "Eilenberg swindle" for C.

Let $\bar{C}$ be the ring of infinite, row and column finite, matrices indexed over $\mathbb{N} \times \mathbb{N}$ with entrees in k. Transposition of matrices makes $\bar{C}$ into a ring with involution. A choice of bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ yields an isomorphism $\bar{C} \to C$ of rings with involution. So it suffices to construct a map $f : C \to \bar{C}$ with $f(\mathrm{Im}(p)) \cong \bar{C}$ as $\bar{C}$-modules. The map f sends the matrix $(a_{i,j})$ to the row and column finite matrix $(b_{(r,m),(s,n)})$ with $b_{(r,m),(s,n)} = a_{r,s}$ if $m = n$ and $b_{(r,m),(s,n)} = 0$ if $m \neq n$. One checks that f is a map of rings with involution. Choose a bijection $\alpha : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ with inverse β. The right $\bar{C}$-module isomorphism $e : (\bar{C}, f(p)) = f(\mathrm{Im}(p)) \to \bar{C}$ is given by a matrix $a \in \bar{C}$ which sends the $(0,n)$-th standard vector to the $\alpha(n)$-th one, the others to 0. Its inverse $b \in \bar{C}$ sends the (i,j)-th standard vector to the $(0, \beta(i,j))$-th one. It is clear that $af(p) = a$, $f(p)b = b$, $ba = f(p)$ and $ab = 1_{\bar{C}}$, so a and b do define inverse isomorphisms between $f(\mathrm{Im}(p))$ and $\bar{C}$.

2.12. *The suspension ring S.* We now give the main example of a cofiltering map. Let M_∞ be the two-sided ideal of C consisting of those matrices which only have finitely many non-zero entries. The *suspension ring* S is the quotient ring C/M_∞. We give a description of S as a ring of fractions of C. In particular, S will then be a flat C-algebra, and *a fortiori* a flat $\mathbb{Z}$-algebra. Recall the matrix t (2.10). The set $\{t^n \mid n \in \mathbb{N}\}$ satisfies the axioms for a calculus of right fractions (2.9). As t is invertible in S (with inverse $\hat{t}$), the map $C \to S$ factors through $C[t^{-1}] \to S$ which, using the explicit description of $C[t^{-1}]$, is seen to be an isomorphism. The involution on C induces an involution on S. Henceforth, S will be considered a ring with involution.

Note that the "Eilenberg swindle" map $f : C \to C$ does not induce a map $S \to S$ since $f(t)$ is not invertible in S.

2.13 Lemma. *Let $\mathcal{A} \to \mathcal{U}$ be a filtering map of preadditive categories (2.4), and let $\mathcal{B}$ be any preadditive category. Then the map $\mathcal{A} \otimes \mathcal{B} \to \mathcal{U} \otimes \mathcal{B}$ is filtering. In particular, the functor $\_ \otimes B : Pac \to Pac$ preserves exact sequences.*

Proof. Let $\alpha_j = \sum_i a_{i,j} \otimes b_{i,j} : A_j \wedge B_j \to U \wedge B'$ be a finite set of maps with A_j, B_j and B', U in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{U}$, respectively. By the filtering assumption, $(a_{i,j}) : \bigoplus_{i,j} A_j \to U$ factors as the composition of $(\alpha_{i,j})$ and the canonical inclusion $\mathrm{Im}(p) \to U$ with p an idempotent of U and $\mathrm{Im}(p)$ in $\mathcal{A}-$proj. Clearly, $\mathrm{Im}(p) \wedge B'$ is a direct factor of $U \wedge B'$ lying in $\mathcal{A} \otimes \mathcal{B}-$proj. The α_j factor as $\sum_i \alpha_{i,j} \otimes b_{i,j} : A_j \wedge B_j \to \mathrm{Im}(p) \wedge B'$ followed by the canonical inclusion $\mathrm{Im}(p) \wedge B' \to U \wedge B'$. The right filtering condition is similar. □

2.14. *Coproducts and coequalizers.* Let $\mathcal{A}$, $\mathcal{B}$ be two preadditive categories. Their coproduct $\mathcal{A} \vee \mathcal{B}$ has as set of objects the coproduct in the category of pointed sets $Ob\mathcal{A} \vee Ob\mathcal{B}$. Maps in $\mathcal{A} \vee \mathcal{B}$ between objects of $\mathcal{A}$ and of $\mathcal{B}$ are zero. The natural inclusions of $\mathcal{A}$, $\mathcal{B}$ into $\mathcal{A} \vee \mathcal{B}$ are fully faithful. Similarly for infinite coproducts. Given two maps $f, g : \mathcal{A} \to \mathcal{U}$ between preadditive categories, the coequalizer $\mathcal{C}$ of f and g has as set of objects the coequalizer of $f, g : Ob\mathcal{A} \to Ob\mathcal{U}$ in the category of pointed sets. Write π for the map $Ob\mathcal{U} \to Ob\mathcal{C}$. Let X, Y be two objects of $\mathcal{C}$. The group $Hom_{\mathcal{C}}(X, Y)$ of morphisms from X to Y in $\mathcal{C}$ is the abelian group

$$\bigoplus_{(X_0,Y_0,X_1,Y_1,\dots,X_n,Y_n)} \mathrm{Hom}_{\mathcal{U}}(X_0, Y_0) \otimes \mathrm{Hom}_{\mathcal{U}}(X_1, Y_1) \otimes \mathrm{Hom}_{\mathcal{U}}(X_2, Y_2) \otimes \dots \otimes \mathrm{Hom}_{\mathcal{U}}(X_n, Y_n)/\sim$$

where $(X_0, Y_0, X_1, Y_1, \dots, X_n, Y_n)$ runs over all finite sequences of objects of $\mathcal{U}$ such that $\pi(X_0) = X$, $\pi(Y_i) = \pi(X_{i+1})$ for $0 \le i \le n-1$ and $\pi(Y_n) = Y$. The equivalence relation $\sim$ is generated by $\dots \otimes f(\alpha_i) \otimes \dots \sim \dots \otimes g(\alpha_i) \otimes \dots$, $\dots \otimes \alpha_i \otimes \alpha_{i+1} \otimes \dots \sim \dots \otimes \alpha_{i+1} \circ \alpha_i \otimes \dots$ if $Y_i = X_{i+1}$ and $\dots \otimes \alpha_{i-1} \otimes id_{X_i} \otimes \alpha_{i+1} \otimes \dots \sim \dots \otimes \alpha_{i-1} \otimes \alpha_{i+1} \otimes \dots$. Composition is concatenation of tensor products. It is easy to see that this defines a category $\mathcal{C}$ and that this category satisfies the universal property of a coequalizer of f and g.

2.15 Lemma. *The pushout of a full inclusion of preadditive categories along an arbitrary map is a full inclusion.*

Proof. Let $\mathcal{B} \overset{f}{\leftarrow} \mathcal{A} \overset{g}{\to} \mathcal{U}$ be a diagram of preadditive categories with g a full inclusion. Let $\mathcal{V}$ be the pushout of the diagram. It is also the coequalizer of $f, g : \mathcal{A} \to \mathcal{B} \vee \mathcal{U}$. Write as above $\pi = \bar{g} \vee \bar{f}$ for the map $\mathcal{B} \vee \mathcal{U} \to \mathcal{V}$. We have to show that the map $\bar{g} : \mathcal{B} \to \mathcal{V}$ is a full inclusion. It is certainly injective on objects and it is easy to check that it is full. Let B, B' be two objects of $\mathcal{B}$. We construct an inverse to the map $\bar{g} : \mathrm{Hom}_{\mathcal{B}}(B, B') \to \mathrm{Hom}_{\mathcal{V}}(B, B')$. Call a sequence of objects $(X_0, Y_0, X_1, Y_1, \dots, X_n, Y_n)$ as in 2.14 belonging to a non-zero summand non-degenerate if $Y_i \neq X_{i+1}$ for all $0 \le i \le n-1$. Remark that if $\pi(X_0) = B$ and $\pi(Y_n) = B'$ then all the X_i, Y_i of a non-degenerate sequence lie in $\mathcal{A}$ or $\mathcal{B}$. In the following diagram, the first map is composition of morphisms and the second is identity on $\mathrm{Hom}_{\mathcal{B}}$-factors and f on the $\mathrm{Hom}_{\mathcal{A}}$-factors and composing the remaining morphisms

$$\begin{array}{c} \bigoplus_{(X_0,Y_0,\dots,X_n,Y_n)} \mathrm{Hom}_{\mathcal{B}\vee\mathcal{U}}(X_0, Y_0) \otimes \dots \otimes \mathrm{Hom}_{\mathcal{B}\vee\mathcal{U}}(X_n, Y_n) \\ \downarrow \\ \bigoplus_{(X_0,Y_0,\dots,X_n,Y_n)} \text{ nondeg.} \, \mathrm{Hom}_{\mathcal{B}\vee\mathcal{U}}(X_0, Y_0) \otimes \dots \otimes \mathrm{Hom}_{\mathcal{B}\vee\mathcal{U}}(X_n, Y_n) \\ \downarrow \\ \mathrm{Hom}_{\mathcal{B}}(B, B'). \end{array}$$

The composition induces a map $\rho : \mathrm{Hom}_{\mathcal{V}}(B, B') \to \mathrm{Hom}_{\mathcal{B}}(B, B')$ which is inverse to $\bar{g}$ (check $\rho \circ \bar{g} = id$ and the surjectivity of $\bar{g}$). □

2.16 Lemma. *The pushout of a filtering map of preadditive categories along an arbitrary map is filtering.*

Proof. Keep the notations of 2.15, and suppose further that g is filtering. By 2.15 we already know that $\bar{g}$ is a full inclusion. Given a finite set of maps $b_j : B_j \to U$ in $\mathcal{V}$ with $B_j \in \mathcal{B}$ and $U \in \mathcal{U} \backslash \mathcal{A} = \mathcal{V} \backslash \mathcal{B}$, the maps b_j can be represented as finite sums of tensor products as in 2.14. Note that in our case each tensor product summand is equivalent to one of length 2, *i.e.*, $b_j = \sum_i \beta_{i,j} \otimes u_{i,j}$ with $u_{i,j} : A_{i,j} \to U$ a map in $\mathcal{U}$ and $\beta_{i,j} : B_j \to f(A_{i,j})$ a map in $\mathcal{B}$. By the left filtering property, we can factor $(u_{i,j})$ as $(a_{i,j}) : \bigoplus A_{i,j} \to \mathrm{Im}(p)$, p being an idempotent of U with image in $\mathcal{A}$−proj, followed by the canonical inclusion $\iota : \mathrm{Im}(p) \to U$. Then $b_j = \iota \circ (\sum_i \beta_{i,j} \otimes a_{i,j})$ is the required factorization. The right filtering property is similar. □

2.17 Lemma. *The pull-back of a cofiltering map is cofiltering*

Proof. Let $f : \mathcal{U} \to \bar{\mathcal{U}}$ be cofiltering with kernel category $\mathcal{A}$ (2.6). Let $g : \bar{\mathcal{B}} \to \bar{\mathcal{U}}$ be any map of preadditive categories. Write $\mathcal{B}$ for the pull-back of f along g. Its objects are pairs (B, U) with $B \in \bar{\mathcal{B}}$ and $U \in \mathcal{U}$ such that $g(B) = f(U)$. Maps are pairs of maps sent to the same map in $\bar{\mathcal{U}}$. An object (B, U) is in the kernel category of $\mathcal{B} \to \bar{\mathcal{B}}$ iff $B = 0$, *i.e.*, iff it is $(0, A)$ for some $A \in \mathcal{A}$. Then $A \mapsto (0, A)$ identifies $\mathcal{A}$ with the kernel category of $\mathcal{B} \to \bar{\mathcal{B}}$. Given a finite set of maps $(0, a_j) : (0, A_j) \to (B, U)$, the map $\bigoplus A_j \to U$ factors as $(\alpha_j) : \bigoplus A_j \to \mathrm{Im}(p)$ followed by $\iota : \mathrm{Im}(p) \to U$ for some idempotent p of U with image in $\mathcal{A}$−proj. Then $(0, a_j) = (0, \iota) \circ (0, \alpha_j)$. So $\mathcal{B}$ is left $\mathcal{A}$-filtered, and it is also right $\mathcal{A}$-filtered by the dual argument. As $\mathcal{U} \to \bar{\mathcal{U}}$ is full so are $\mathcal{B} \to \bar{\mathcal{B}}$ and $\mathcal{B}/\mathcal{A} \to \bar{\mathcal{B}}$. The latter map is faithful because a map $(b, u) \in \mathcal{B}$ is zero in $\bar{\mathcal{B}}$ iff $b = 0$. But then $u = 0$ in $\mathcal{U}/\mathcal{A}$ and $(0, u) = 0$ in $\mathcal{B}/\mathcal{A}$. Surjectivity on objects is obvious. □

2.18 Remark/Definition. The important property of cone and suspension is that the sequence

$$\mathbb{Z} \stackrel{Im(p)\otimes}{\longrightarrow} \mathcal{C} \longrightarrow S$$

is exact where the latter map is given by the map $C \to S$ and by sending $Im(p)$ to the base point. This follows from 2.10 and the explicit description of quotient categories (2.6). For $\mathcal{A}$ a preadditive category, write $S^n\mathcal{A} = \mathcal{A} \otimes S^{\otimes n}$ and call it the *n-th suspension of* $\mathcal{A}$. The functor S^n preserves exact sequences (2.13). Likewise, $\mathcal{C}\mathcal{A} = \mathcal{A} \otimes \mathcal{C}$ is called the *cone of* $\mathcal{A}$. By 2.13, we have an exact sequence

$$\mathcal{A} \stackrel{Im(p)\otimes}{\longrightarrow} \mathcal{C}\mathcal{A} \longrightarrow S\mathcal{A}. \tag{2.19}$$

Note that the inclusions $C \to \mathcal{C}$ and thus $C\mathcal{A} \to \mathcal{C}\mathcal{A}$ induce equivalences on projective module categories $P(C) \to P(\mathcal{C})$ and $P(C\mathcal{A}) \to P(\mathcal{C}\mathcal{A})$. So $C\mathcal{A}$ and $\mathcal{C}\mathcal{A}$ will have the same (hermitian) K-theories.

2.20 Lemma. [PW89, 5.2] *Let $\mathcal{U}$ be an $\mathcal{A}$-filtered additive category with $\mathcal{A}$ idempotent complete. Assume that direct factors of objects in $\mathcal{U}$ which lye in $\mathcal{A}$ have complements in $\mathcal{U}$ (2.8). Then for every morphism f in $\mathcal{U}$ which becomes an isomorphism in $\mathcal{U}/\mathcal{A}$ there is a split monomorphism i with cokernel in $\mathcal{A}$ such that $f \circ i$ is a split monomorphism with cokernel in $\mathcal{A}$. In particular, the inclusion of $\mathcal{A}$ into the kernel category of $\mathcal{U} \to \mathcal{U}/A$ is an equivalence.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{U}$ which is an isomorphism in $\mathcal{U}/\mathcal{A}$. Represent its inverse by a map $g : Y \to X$ in $\mathcal{U}$. Recall (2.9) that $\mathcal{U}/\mathcal{A} \cong \mathcal{U}[\Sigma^{-1}]$ with Σ the set of split monomorphisms with cokernel in $\mathcal{A}$. As Σ satisfies a calculus of right fractions, there is a split monomorphism $i : Z \to X$ with cokernel A in $\mathcal{A}$ such that $i = gfi$. Replacing X with the isomorphic $Z \oplus A$ such that i becomes the standard inclusion ${}^t(1\ 0)$, the map $p_Z g$, $p_Z = (1\ 0)$, becomes a retraction of fi. The map fi is a split monomorphism with cokernel in $\mathcal{A}$ iff the idempotent $p = 1_Y - fip_Z g$ has image in $\mathcal{A}$ because then the cokernel of fi exists by the assumption of the lemma. Since $p = 0$ in $\mathcal{U}/\mathcal{A}$ it factors as a map in $\mathcal{U}$ through an object of $\mathcal{A}$. Using the right filtering property, we can replace Y by the isomorphic $W \oplus B$ with B in $\mathcal{A}$ such that p becomes

$$\begin{pmatrix} 0 & w \\ 0 & b \end{pmatrix}.$$

Idempotency of the matrix means $b^2 = b$ and $wb = w$, so $p = {}^t(w\ 1)b(0\ 1)$. The image of p is isomorphic to the image of the projector b since ${}^t(w\ 1)$ is split injective. The image of b exists in $\mathcal{A}$ because $\mathcal{A}$ is idempotent complete. □

3. The Hermitian Pedersen-Weibel Theorem

In this section, we prove the hermitian analogue of a Theorem of Pedersen-Weibel [PW89] which allows us to construct non-connective hermitian K-theory spectra.

3.1. For a category $\mathcal{C}$, we write $B\mathcal{C} := |N_*\mathcal{C}|$ for the topological space given by the geometric realization of the nerve of $\mathcal{C}$. We will often drop the letter B to simplify notation. For $\mathcal{C}$ a category write $i\mathcal{C}$ for the category which has the same objects as $\mathcal{C}$ and whose morphisms are the isomorphisms of $\mathcal{C}$.

Let $(\mathcal{C}, \oplus)$ be a symmetric monoidal category such that for every object $C \in \mathcal{C}$ the translation functor $\oplus C : \mathcal{C} \to \mathcal{C}$ is faithful (compare [Gra76, p. 220, 2)]). For such a category, Quillen [Gra76] constructs a new category $i\mathcal{C}^{-1}\mathcal{C}$, which we abbreviate by $\mathcal{C}^+$, and a functor $\mathcal{C} \to \mathcal{C}^+$ such that $B\mathcal{C} \to B\mathcal{C}^+$ is a group completion [Gra76, Th p. 221].

For an additive category with duality $(\mathcal{A}, \sharp)$, we observe that the orthogonal sum $(A, \alpha) \oplus (B, \beta) := (A \oplus B, \alpha \oplus \beta)$ makes $(\mathcal{A}_h, \oplus)$ into a symmetric monoidal category meeting the above faithfulness condition.

3.2 Definition. Let $(\mathcal{A}, \sharp, \eta)$ be a preadditive category with duality. Then for $\epsilon \in \{+1, -1\}$, its *ϵ-hermitian K-theory space* is defined by

$$_\epsilon K^h(\mathcal{A}) = B(i\,_\epsilon P(\mathcal{A})_h)^+.$$

The n-th hermitian K-group of $\mathcal{A}$ is the n-th homotopy group of this space $_\epsilon K^h_n(\mathcal{A}) = \pi_n\, _\epsilon K^h(\mathcal{A})$, $n \geq 0$.

If $\epsilon = 1$, then we often drop the ϵ and write K^h and K^h_n instead of $_\epsilon K^h$ and $_\epsilon K^h_n$.

3.3 *Remark.* For a preadditive category $\mathcal{A}$ there is a *canonical involution* on $\mathcal{A} \times \mathcal{A}^{op}$ interchanging the two factors. Note that $i_\epsilon(\mathcal{A} \times \mathcal{A}^{op})_h$ is equivalent to $i\mathcal{A}$ so that $K^h(\mathcal{A}) \simeq K(\mathcal{A})$ where $K(\mathcal{A})$ denotes Quillen K-theory of $P(\mathcal{A})$. In this sense, hermitian K-theory generalizes algebraic K-theory. We remark however, that the definition of K_0 differs from the standard definition of Quillen and Thomason when considering additive categories which are not idempotent complete.

3.4 Definition. An object (M, ϕ) of $\mathcal{A}_h$ is called *hyperbolic* if there is an object L in $\mathcal{A}$ together with an isomorphism of hermitian objects

$$(H(L), \mu_L) := (L \oplus L^\sharp, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \cong (M, \phi)$$

For any additive category with duality $(\mathcal{A}, \sharp, \eta)$, we define the hyperbolic functor

$$H : i\mathcal{A} \to i\mathcal{A}_h$$

by $H(M) = (H(M), \mu_M)$ and $H(f) = f \oplus (f^{-1})^\sharp$. We write $\mathcal{A}_H$ for the full subcategory of $\mathcal{A}_h$ consisting of the hyperbolic objects.

3.5 *Remark.* If 2 is invertible in an additive category with duality $\mathcal{A}$, then there is an isometry

$$\begin{pmatrix} 1 & -1 \\ \frac{1}{2}\phi & \frac{1}{2}\phi \end{pmatrix} : (M, \phi) \oplus (M, -\phi) \overset{\cong}{\to} (H(M), \mu_M)$$

for any (M, ϕ) in $\mathcal{A}_h$. This implies that the full subcategory of hyperbolic objects is cofinal in $\mathcal{A}_h$. It follows that $i(\mathcal{A})^+_H \to i(\mathcal{A})^+_h$ induces an isomorphism on π_n for $n > 0$ and a monomorphism for $n = 0$.

If $\mathcal{A} = P(A)$ with A an algebra with involution and if $\frac{1}{2} \in A$, then the free hyperbolic modules are cofinal in $P(A)_h$ and hence the connected component of 0 of $K^h(A)$ is homotopy equivalent to Quillen's plus construction applied to $BO(A) = B colim_n \operatorname{Aut} H(A^n)$ [Kar80, Théorème 1.6].

Now we can state the main result of this section.

3.6 Theorem. *Let $\mathcal{U}$ be a preadditive category with duality in which 2 is invertible, and $\mathcal{A}$ a full subcategory with duality. Suppose that $\mathcal{U}$ is $\mathcal{A}$-filtered (2.4). Then the exact sequence of preadditive categories with duality $\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$ induces a homotopy fibration of spaces*

$$K^h(\mathcal{A}) \to K^h(\mathcal{U}) \to K^h(\mathcal{U}/\mathcal{A}).$$

3.7. Let $\mathcal{S}$ be a symmetric monoidal category acting on a category $\mathcal{X}$. Recall from [Gra76, p. 219] that the category $< \mathcal{S}, \mathcal{X} >$ has the same objects as $\mathcal{X}$ and that a morphism $X \to Y$ is an equivalence class of data (A, x) with A an object of S and $x : A \oplus X \to Y$ a morphism in $\mathcal{X}$. The data (A, x) is equivalent to (A', x') if there is an isomorphism $a : A \to A'$ such that $x' \circ (a \oplus id_X) = x$. Composition is given by the monoidal operation in $\mathcal{S}$. Remark that there is a natural inclusion of categories $\mathcal{X} \to < \mathcal{S}, \mathcal{X} >$.

We say that a map of symmetric monoidal categories is a *homotopy equivalence after group completion*, or a commutative square is *homotopy cartesian after group completion*, if the corresponding statement is true after applying the group completion functor $(\ )^+$.

3.8. Now let $\mathcal{T}$ be a symmetric monoidal category such that translations $\oplus T : \mathcal{T} \to \mathcal{T}$ are faithful and such that every morphism in $\mathcal{T}$ is an isomorphism.

Assume further that any morphism in $\mathcal{T}^{-1}\mathcal{T}$ is monic. Let $\mathcal{S}$ be a full symmetric monoidal subcategory of $\mathcal{T}$. The inclusion $\mathcal{S} \subset \mathcal{T}$ induces an action of $\mathcal{S}$ on $\mathcal{T}$.

3.9 Lemma. *In the situation of 3.8, the commutative square*

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow \\ < \mathcal{S}, \mathcal{S} > & \longrightarrow & < \mathcal{S}, \mathcal{T} > \end{array}$$

is homotopy cartesian after group completion and the lower left corner is contractible.

Proof. The lower left corner is contractible since it has an initial object, namely the unit object with respect to the monoidal structure of $\mathcal{S}$. Consider the map of symmetric monoidal categories $\mathcal{S} \to \mathcal{T}^{-1}\mathcal{T} : A \mapsto (0, A)$ inducing an action of the first category on the second. Remark that the translations $\mathcal{S} \to \mathcal{T}^{-1}\mathcal{T} : A \mapsto (X, A \oplus Y)$ are faithful for all objects (X, Y) in $\mathcal{T}^{-1}\mathcal{T}$. By [Gra76, p. 223] (choose $\mathcal{X} = \mathcal{T}^{-1}\mathcal{T}$), the commutative diagram

$$\begin{array}{ccc} \mathcal{S}^{-1}\mathcal{S} & \longrightarrow & \mathcal{S}^{-1}\mathcal{T}^{-1}\mathcal{T} \\ \downarrow & & \downarrow \\ < \mathcal{S}, \mathcal{S} > & \longrightarrow & < \mathcal{S}, \mathcal{T}^{-1}\mathcal{T} > \end{array}$$

is homotopy cartesian. Remark that all categories in the diagram are group complete. There is a map of commutative squares from the diagram in the statement of Lemma 3.9 to the above diagram which on the upper left corner is $A \mapsto (0, A)$, in the upper right corner is $X \mapsto (0, 0, X)$, on the lower left corner is the identity and on the lower right corner is $X \mapsto (0, X)$. The map of commutative squares is point-wise a homotopy equivalence after group completion. It follows that the diagram in 3.9 is homotopy cartesian after group completion. $\square$

3.10. Let $(\mathcal{U}, \sharp, \eta)$ be an additive category with duality and $\mathcal{A}$ an idempotent complete full additive subcategory closed under the duality. Suppose that $\mathcal{U}$ is $\mathcal{A}$-filtered (2.4) and that direct factors lying in $\mathcal{A}$ of objects of $\mathcal{U}$ have complements in $\mathcal{U}$ (compare 2.8). Write $\tilde{\mathcal{U}}_H$ for the full subcategory of $\mathcal{U}_h$ consisting of objects (U, λ) for which there is an (A, α) in $\mathcal{A}_h$ with $(U, \lambda) \oplus (A, \alpha)$ isomorphic to an object of $\mathcal{U}_H$. In the following proposition, we don't assume that 2 is invertible in $\mathcal{U}$.

3.11 Lemma. *Under the hypothesis of 3.10, for any split monomorphism $Y \to H(X)$ with cokernel in $\mathcal{A}$, there is a split monomorphism $H(Z) \to Y$ with cokernel in $\mathcal{A}$ such that the composition $H(Z) \to H(X)$ restricts the standard hyperbolic form on $H(X)$ to the standard hyperbolic form of $H(Z)$.*

Proof. Let the object A of $\mathcal{A}$ be a complement of Y in $H(X)$. So there is an isomorphism

$$\begin{pmatrix} x & y \\ a & b \end{pmatrix} : X \oplus X^\sharp \overset{\cong}{\to} Y \oplus A.$$

Since $\mathcal{U}$ is $\mathcal{A}$-filtered, there is an object B of $\mathcal{A}$, an isomorphism $X \cong Z \oplus B$ and a commutative diagram

$$\begin{array}{ccccc} A^\sharp & \xrightarrow{b^\sharp} & X & \xrightarrow{a} & A \\ & \searrow^{\begin{pmatrix} 0 \\ \beta^\sharp \end{pmatrix}} & \downarrow \cong & \nearrow_{(0 \ \ \alpha)} & \\ & & Z \oplus B & & \end{array}$$

One can see this as follows. Apply the definition of $\mathcal{U}$ being $\mathcal{A}$-filtered (2.4) to the map $X \to A$ to obtain a temporary Z and apply the definition again to the map $A^\sharp \to Z$ to obtain the decomposition in the diagram. The inclusion of $Z \oplus Z^\sharp$ into $(Z \oplus B) \oplus (Z^\sharp \oplus B^\sharp) \cong X \oplus X^\sharp \cong Y \oplus A$ followed by the canonical projection $Y \oplus A \to A$ is the zero morphism. It follows that the map $H(Z) \to H(X)$ induced by $Z \to Z \oplus B \cong X$ factors through Y. By construction, the hyperbolic form on $H(X)$ restricts to the hyperbolic form on $H(Z)$. □

3.12 Proposition. *In the situation of 3.10, the sequence of symmetric monoidal categories*

$$i\mathcal{A}_h \to i\tilde{\mathcal{U}}_H \to i(\mathcal{U}/\mathcal{A})_H$$

is a homotopy fibration after group completion.

Proof. The last map of the sequence factors as $i\tilde{\mathcal{U}}_H \to < i\mathcal{A}_h, i\tilde{\mathcal{U}}_H > \stackrel{\sigma}{\to} i(\mathcal{U}/\mathcal{A})_H$. We claim that the functor σ is a homotopy equivalence. Using lemma 3.9 and the fact that group completion preserves homotopy equivalences the proposition will follow. To simplify notation, we will assume $\sharp\sharp = id$, $\eta = id$.

First note that morphisms $(U, \phi) \to (V, \psi)$ in $< i\mathcal{A}_h, i\tilde{\mathcal{U}}_H >$ are in bijection with split monomorphisms $i : U \to V$ with cokernel in $\mathcal{A}$ such that $\phi = \psi_{|U} := i^\sharp \circ \psi \circ i$. To see this, let i be such a split monomorphism. The map $p = i \circ \phi^{-1} \circ i^\sharp \circ \psi$ is an idempotent of V whose image is isomorphic to U via i. Its cokernel $A = \mathrm{Im}(1-p)$ is in $\mathcal{A}$. We have $\psi \circ p = p^\sharp \circ \psi$ and $\psi \circ (1-p) = (1-p^\sharp) \circ \psi$. Hence, $\psi = \psi_{|\,\mathrm{Im}(p)} \oplus \psi_{|\,\mathrm{Im}(1-p)} = \phi \oplus \psi_{|A}$ and $[(A, \psi_{|A}), (1-p) \oplus i]$ defines a map $(U, \phi) \to (V, \psi)$. The other direction is obvious.

According to Theorem A of [Qui73, p.92] it suffices to show that the categories $(\sigma \downarrow x)$ are filtering, hence contractible, for all objects x of $i(\mathcal{U}/\mathcal{A})_H$. Since every object of $i(\mathcal{U}/\mathcal{A})_H$ is isomorphic to a hyperbolic object we can assume $x = H(X)$, the image of an hyperbolic object of $\mathcal{U}$, and we see that $(\sigma \downarrow H(X))$ is non-empty.

Let (U, ϕ) and (V, ψ) be two objects of $\tilde{\mathcal{U}}_H$ equipped with isomorphisms $u : (U, \phi) \to H(X)$ and $v : (V, \psi) \to H(X)$ in $(\mathcal{U}/\mathcal{A})_H$. The isomorphism $v^{-1}u$ in $\mathcal{U}/\mathcal{A}$ can be written as a fraction ji^{-1} with $i : W \to U$ and $j : W \to V$ split monomorphisms with cokernel in $\mathcal{A}$ (2.9, 2.20). Since $i^\sharp \circ \phi \circ i = j^\sharp \circ \psi \circ j$ in $\mathcal{U}/\mathcal{A}$, we can replace W by a "smaller" direct factor, still called W, of U and V with cokernels in $\mathcal{A}$ such that $\phi|_W = \psi|_W$ (2.9). As the object (U, ϕ) is in $\tilde{\mathcal{U}}_H$, U is a direct factor of a hyperbolic object $H(Y)$ with quotient in $\mathcal{A}$ and such that the form ϕ is the restriction of the hyperbolic form on $H(Y)$. According to lemma 3.11, we can then find a hyperbolic direct factor $H(Z)$ of W with quotient $W/H(Z)$ in $\mathcal{A}$ and such that the hyperbolic form on $H(Z)$ is the restriction of the one on $H(Y)$. It follows that the form on $H(Z)$ is the restriction of the form on W which was a common subform of (U, ϕ) and (V, ψ). It follows that for any two objects of $(\sigma \downarrow x)$ there is an object which is "smaller" than both.

Let $a, b : (U, \phi) \to (V, \psi)$ be two morphisms in $< i\mathcal{A}_h, i\tilde{\mathcal{U}}_H >$ and let $y : (V, \psi) \to H(X)$ be a map in $i(\mathcal{U}/\mathcal{A})_H$ such that $y \circ a = y \circ b$ in $i(\mathcal{U}/\mathcal{A})_H$. By the calculus of fractions (2.9), there is a split monomorphism $i : W \to U$ with cokernel in $\mathcal{A}$ such that $a \circ i = b \circ i$. Using the fact that $(W, \phi_{|W})$ is a (possibly degenerated) subform of an hyperbolic object with quotient in $\mathcal{A}$, as (U, ϕ) is, and lemma 3.11, it follows that there is a hyperbolic subobject $H(Z)$ of W and a morphism $c : H(Z) \to (U, \phi)$ such that $a \circ c = b \circ c$. □

3.13 Corollary. *For $\mathcal{A}$ a preadditive category in which 2 is invertible, there is a homotopy equivalence*

$$K^h(\mathcal{A}) \simeq \Omega K^h(S\mathcal{A}).$$

Proof. Applying Proposition 3.12 to the exact sequence $P(\mathcal{A}) \to \hat{F}(\mathcal{CA}) \to F(S\mathcal{A})$ (2.18, 2.7, 2.8) and cofinality for the total and base space (3.5), it remains to show that $K^h(\mathcal{CA})$ is contractible. But this follows from the Eilenberg swindle (2.11) which extends to a natural isometry $f \oplus id \cong f$ in $P(\mathcal{CA})$. This yields $K_n^h(f) + K_n^h(id) = K_n^h(f)$, hence $id_{K_n^h(\mathcal{CA})} = 0$. □

Proof of Theorem 3.6. By Lemma 2.13, the sequence $S\mathcal{A} \to S\mathcal{U} \to S\mathcal{U}/\mathcal{A}$ is also exact. Because of 2.8 we can apply Proposition 3.12 to the sequence $P(S\mathcal{A}) \to \hat{F}(S\mathcal{U}) \to F(S\mathcal{U}/\mathcal{A})$ and we obtain a homotopy fibration whose loop space is the homotopy fibration of Theorem 3.6 (use 3.13 and cofinality for total and base space). □

3.14 Definition. Let $\mathcal{A}$ be a preadditive category with duality in which 2 is invertible. Its *non-connective hermitian K-theory spectrum* is the spectrum $\mathbf{K}^h(\mathcal{A})$ whose n-th space is $K^h(S^n\mathcal{A})$. The structure maps are given by the homotopy equivalences of 3.13. Define its n-th hermitian K-groups as $K_n^h(\mathcal{A}) := \pi_n(\mathbf{K}^h(\mathcal{A}))$. For a more precise and functorial definition, we refer the reader to appendix B.

3.15 *Remark.* As $\mathbf{K}^h(\mathcal{A})$ is an Ω-spectrum, we have $\pi_n\mathbf{K}^h(\mathcal{A}) = \pi_n K^h(\mathcal{A})$ for $n \geq 0$ (3.2), so Definition 3.14 is compatible with Definition 3.2. For $n < 0$ the groups coincide with Karoubi's negative hermitian K-groups [KV73, Paragraphe 3].

3.16 Theorem. *If the preadditive category with duality $\mathcal{U}$ is $\mathcal{A}$-filtered and if 2 is invertible in $\mathcal{U}$, then the sequence $\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$ induces a homotopy fibration of non-connective hermitian K-theory spectra*

$$\mathbf{K}^h(\mathcal{A}) \to \mathbf{K}^h(\mathcal{U}) \to \mathbf{K}^h(\mathcal{U}/\mathcal{A}).$$

Proof. This is Theorem 3.6 applied to the exact sequences $S^n\mathcal{A} \to S^n\mathcal{U} \to S^n(\mathcal{U}/\mathcal{A})$ (2.13), $n \in \mathbb{N}$. □

3.17. *Mapping cones.* For $f : \mathcal{A} \to \mathcal{B}$ a map of preadditive categories, the pushout $\mathcal{C}(f)$ of the diagram $\mathcal{CA} \leftarrow \mathcal{A} \to \mathcal{B}$ is called the *mapping cone* (or simply the *cone*) of f.

Applying the functor $\mathbf{K}^h$ to a diagram obtained by a pushout in *Pad* of a map along a filtering map yields a homotopy cartesian square of spectra (2.16, 3.16). In particular, as $\mathbf{K}^h(\mathcal{CA})$ is contractible (see proof of 3.13), for any map $f : \mathcal{A} \to \mathcal{B}$ of preadditive categories, the sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}(f)$ of induces a homotopy fibration

$$\mathbf{K}^h(\mathcal{A}) \to \mathbf{K}^h(\mathcal{B}) \to \mathbf{K}^h(\mathcal{C}(f)).$$

In the literature, one uses a different cone construction, *e.g.,* [Kar80]. Let $\bar{\mathcal{C}}(f)$ be the pull-back of the diagram $\mathcal{CB} \to S\mathcal{B} \leftarrow S\mathcal{A}$ and call it the *limit cone* of f. Then there is an exact sequence $\mathcal{B} \to \bar{\mathcal{C}}(f) \to S\mathcal{A}$ (2.17) and a homotopy fibration (3.16, 3.13)

$$\mathbf{K}^h(\mathcal{A}) \to \mathbf{K}^h(\mathcal{B}) \to \mathbf{K}^h(\bar{\mathcal{C}}(f)).$$

There is a natural map $\mathcal{C}(f) \to \bar{\mathcal{C}}(f)$ given by the universal properties of limit and mapping cone. It follows from the above discussion that it induces a homotopy equivalence $\mathbf{K}^h(\mathcal{C}(f)) \to \mathbf{K}^h(\bar{\mathcal{C}}(f))$.

4. Background on Triangulated Categories

In this section, we review localization sequences for exact and triangulated categories and discuss how this applies to K-theory and Witt groups. For notation and conventions regarding triangulated categories and the bounded derived category $D_b(\mathcal{E})$ of an exact category $\mathcal{E}$, we refer the reader to [Kel96]. Complexes will be written homologically, *i.e.,* differentials lower degree.

4.1. We recall some classical facts about K_0 of triangulated categories, where K_0 in this paragraph 4.1 means usual K_0, *i.e.,* before taking idempotent completion. For an exact category $\mathcal{E}$, there is a natural isomorphism $K_0(\mathcal{E}) \cong K_0(D_b(\mathcal{E}))$ (exercise!). Let $\mathcal{S} \to \mathcal{T}$ be a *cofinal* map of triangulated categories, *i.e.,* fully faithful, and every object of $\mathcal{T}$ is a direct factor of an object of $\mathcal{S}$. If $K_0(\mathcal{S}) \to K_0(\mathcal{T})$ is an isomorphism, then $\mathcal{S} \to \mathcal{T}$ is an equivalence. This follows from Thomason's classification of dense subcategories [Tho97].

4.2 Definition. A sequence of triangulated categories $\mathcal{R} \to \mathcal{S} \to \mathcal{T}$ is called *exact up to direct factors* if $\mathcal{R} \to \mathcal{S}$ is fully faithful, and the functor from the Verdier quotient $\mathcal{S}/\mathcal{R}$ to $\mathcal{T}$ is *cofinal, i.e.,* fully faithful, and every object of the latter category is a direct factor of an object of the Verdier quotient. The sequence is called *exact*, if it is exact up to direct factors and if moreover $\mathcal{R}$ is closed under direct factors in $\mathcal{S}$ and $\mathcal{S} \to \mathcal{T}$ is, up to isomorphism, surjective on objects.

4.3. Let A be a ring and let $\Sigma \subset A$ be a multiplicative subset of central non-zero-divisors. Let $D_b^{\Sigma}(A\text{−proj}) \subset D_b(A\text{−proj})$ be the full subcategory of those complexes which are acyclic after localization at Σ. Then the sequence of triangulated categories

$$D_b^{\Sigma}(A\text{−proj}) \to D_b(A\text{−proj}) \to D_b(\Sigma^{-1}A\text{−proj}) \tag{4.4}$$

is *exact up to direct factors.* This is almost by definition, we only have to observe that the localization with calculus of fractions $A\text{−free} \to \Sigma^{-1}A\text{−free}$ induces a localization functor $Ch_b(A\text{−free}) \to Ch_b(\Sigma^{-1}A\text{−free})$ and *a fortiori* a localization $D_b(A\text{−free}) \to D_b(\Sigma^{-1}A\text{−free})$. Moreover, for any ring R, the functor $D_b(R\text{−free}) \to D_b(R\text{−proj})$ is cofinal.

We will give a description of $D_b^{\Sigma}(A\text{−proj})$ as the derived category of certain exact categories below. First, we notice the following.

4.5 Lemma. *The triangulated category $D_b^{\Sigma}(A\text{−proj})$ is generated, up to direct factors, by complexes of the form*

$$\cdots \to 0 \to A \xrightarrow{s} A \to 0 \to \cdots \tag{4.6}$$

with $s \in \Sigma$ in the sense that it is the smallest idempotent complete triangulated subcategory of $D_b(A\text{−proj})$ which is closed under isomorphisms and which contains the complexes (4.6).

Proof. Let $\mathcal{T}$ be the triangulated category generated by the complexes (4.6). It is clear that $\mathcal{T} \subset D_b^{\Sigma}(A\text{−free})$. Let (A_*, d_*) be a complex in $D_b^{\Sigma}(A\text{−free})$ which is acyclic (and hence contractible) after localization at Σ. Suppose $A_i = 0$ for $i > 1$. The map $\Sigma^{-1}d_1$ is split injective because of the acyclicity assumption. By the calculus of fractions, there is an A-module map $e : A_0 \to A_1$ and an $s \in \Sigma$ such that $ed_1 = s$. The maps e and id_{A_1} define a map of complexes from A_* to the complex $A_1 \xrightarrow{s} A_1$ concentrated in degree 0 and 1 which is a direct sum of complexes of the form (4.6). The contractible complex $A_1 \xrightarrow{id} A_1$ concentrated in degree 1 and 2 is a direct factor of the cone of the map of complexes. If $A_i \neq 0$ for some $i \leq -1$, then the complement of the contractible complex is shorter (*i.e.,* has fewer non-zero terms) than A_*. By construction, the shorter complex is in $\mathcal{T}$ if and only if A_* is, so we may reduce the number of non-zero objects of A_* inductively. If in the complex A_* we have $A_i = 0$ for $i \neq 0, 1$, then $\Sigma^{-1}d_1$ is an isomorphism. By the calculus of fractions, there are A-module maps $e : A_0 \to A_1$, $f : A_1 \to A_0$, $s, t \in \Sigma$ with $s = ed_1$ and $t = fe$. This implies that d_1 (and e, f) is an isomorphism modulo $\mathcal{T}$. Thus, the cone of A_* is a direct factor of an object of $\mathcal{T}$. □

Let $\mathcal{H}^1_{\Sigma,\text{free}}$ be the full subcategory of right A-modules of those modules M for which there is an exact sequence of right A-modules

$$0 \to A^n \xrightarrow{\sigma} A^n \to M \to 0 \tag{4.7}$$

with σ an upper triangular matrix whose diagonal has entries in Σ. The category $\mathcal{H}^1_{\Sigma,\text{free}}$ is extension closed in the category of right A-modules, and thus is declared to be a fully exact subcategory of this category. Let $\mathcal{P}^1_{\Sigma,\text{free}}$ be the full subcategory of right A-modules of those modules E for which there is an exact sequence of right A-modules

$$0 \to P \to E \to M \to 0 \tag{4.8}$$

with P finitely generated free and M in $\mathcal{H}^1_{\Sigma,\mathrm{free}}$. The category $\mathcal{P}^1_{\Sigma,\mathrm{free}}$ is extension closed in the category of right A-modules, and thus is a fully exact subcategory. The inclusion $\mathcal{H}^1_{\Sigma,\mathrm{free}} \subset \mathcal{P}^1_{\Sigma,\mathrm{free}}$ induces a triangle functor $D_b(\mathcal{H}^1_{\Sigma,\mathrm{free}}) \to D_b(\mathcal{P}^1_{\Sigma,\mathrm{free}})$ which by [Kel96, 12.1 b), 11.7] is fully faithful. Moreover, the inclusion $A-\mathrm{free} \subset \mathcal{P}^1_{\Sigma,\mathrm{free}}$ induces an equivalence $D_b(A-\mathrm{free}) \cong D_b(\mathcal{P}^1_{\Sigma,\mathrm{free}})$ by resolution. Since the exact category $\mathcal{H}^1_{\Sigma,\mathrm{free}}$ is generated by the modules of the form A/sA with $s \in \Sigma$, and since $D_b(\mathcal{H}^1_{\Sigma,\mathrm{free}}) \subset D^\Sigma_b(A-\mathrm{proj})$, it follows from Lemma 4.5 that we have an equivalence of triangulated categories

$$D^\Sigma_b(A-\mathrm{proj}) \cong D_b(\mathcal{H}^1_{\Sigma,\mathrm{free}})^\sim \tag{4.9}$$

where $^\sim$ denotes idempotent completion as before (1.1).

Recall that the exact category $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$ has already been introduced (1.3). By similar arguments as above, we obtain a triangle equivalence

$$D^\Sigma_b(A-\mathrm{proj}) \cong D_b(\mathcal{H}^1_{\Sigma,\mathrm{proj}}). \tag{4.10}$$

Note that the right hand term is already idempotent complete because $\mathcal{H}^1_{\Sigma,\mathrm{proj}}$ is [BS01].

4.11. *Triangulated and preadditive exactness.* Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be an exact sequence of preadditive categories in the sense of 2.6. Then the sequence of triangulated categories

$$D_b(P(\mathcal{A})) \to D_b(\hat{F}(\mathcal{B})) \to D_b(F(\mathcal{C}))$$

is exact (see 2.7 for the definition of $\hat{F}$). This is implicit in [CP97]. For an explicit proof and a generalization to exact categories, see [Scha, 2.6]. It follows that for any preadditive category $\mathcal{A}$, the sequence (2.19) induces an exact sequence up to direct factors.

$$D_b(\mathcal{A}-\mathrm{proj}) \to D_b(C\mathcal{A}-\mathrm{proj}) \to D_b(S\mathcal{A}-\mathrm{proj}). \tag{4.12}$$

Since $C\mathcal{A}-\mathrm{free} \to S\mathcal{A}-\mathrm{free}$ is surjective on objects, it follows that not only $K_0(C\mathcal{A}-\mathrm{proj}) = 0$ but also $K_0(S\mathcal{A}-\mathrm{free}) = 0$ (for the usual K_0, *i.e.,* before taking idempotent completions). Using 4.1, we see that the following sequence is exact

$$D_b(\mathcal{A}-\mathrm{proj}) \to D_b(C\mathcal{A}-\mathrm{proj}) \to D_b(S\mathcal{A}-\mathrm{free}). \tag{4.13}$$

4.14. Let A be a ring and let $\Sigma \subset A$ be a multiplicative subset of central non-zero divisors. Let B be a flat $\mathbb{Z}$-algebra. Then the set $\Sigma \otimes 1_B = \{f \otimes 1_B \mid f \in \Sigma\}$ is a central multiplicative subset of $A \otimes B$ which consists of non-zero divisors because of the flatness assumption. We have $(\Sigma \otimes 1_B)^{-1}(A \otimes B) = \Sigma^{-1}A \otimes B$. Recall that cone and suspension rings are flat $\mathbb{Z}$-algebras (2.10, 2.12). The localization map from the sequence (4.12) for $R = A$ to the sequence (4.12) for $R = \Sigma^{-1}A$ induces a sequence of "kernel categories"

$$D^\Sigma_b(A-\mathrm{proj}) \to D^{\Sigma\otimes 1_C}_b(\mathcal{C}A-\mathrm{proj}) \to D^{\Sigma\otimes 1_S}_b(SA-\mathrm{proj}). \tag{4.15}$$

4.16 Lemma. *The sequence of triangulated categories (4.15) is exact up to direct factors.*

Proof. It is clear that $D^\Sigma_b(A-\mathrm{proj})$ is the "kernel category" of $D^{\Sigma\otimes 1_C}_b(\mathcal{C}A-\mathrm{proj}) \to D^{\Sigma\otimes 1_S}_b(SA-\mathrm{proj})$. Therefore, it remains to show that $D^{\Sigma\otimes 1_C}_b(\mathcal{C}A-\mathrm{proj}) \to D^{\Sigma\otimes 1_S}_b(SA-\mathrm{proj})$ is a localization up to direct factors. We will show that $\otimes_C S : \mathcal{H}^1_{\Sigma\otimes 1_C,\mathrm{free}} \to \mathcal{H}^1_{\Sigma\otimes 1_S,\mathrm{free}}$ is a localization with calculus of left fractions. This implies that $D_b(\mathcal{H}^1_{\Sigma\otimes 1_C,\mathrm{free}}) \to D_b(\mathcal{H}^1_{\Sigma\otimes 1_S,\mathrm{free}})$ is a localization. By (4.9), this yields the claim.

Recall (2.12) that the suspension ring S is obtained from the cone ring C by a calculus of right fractions with respect to the multiplicative set $\{t^n \mid n \in \mathbb{N}\}$. Thus, for M a right $A\otimes C$-module, right multiplication with t^n, $n \in \mathbb{N}$, satisfies the axioms of a calculus of left fraction. Note that $S = \mathrm{colim}(C \stackrel{\times t}{\to} C \stackrel{\times t}{\to} C \stackrel{\times t}{\to} \cdots)$ and $M \otimes_C S = \mathrm{colim}(M \stackrel{\times t}{\to} M \stackrel{\times t}{\to} M \stackrel{\times t}{\to} \cdots)$. If M and N are finitely presented right CA-modules, then

$$\begin{aligned}\hom_{SA}(N \otimes_{CA} SA, M \otimes_{CA} SA) &= \hom_{CA}(N, M \otimes_{CA} SA) = \hom_{CA}(N, \mathrm{colim}(M \stackrel{\times t}{\to} M \stackrel{\times t}{\to} \cdots)) \\ &= \mathrm{colim}(\hom_{CA}(N, M) \stackrel{\hom(N,\times t)}{\longrightarrow} \cdots).\end{aligned}$$

This shows that any subcategory of finitely presented right CA-modules defines a localization with calculus of left fractions onto its image in the category of right SA-modules. Thus, for $\otimes_C S : \mathcal{H}^1_{\Sigma\otimes 1_C,\mathrm{free}} \to \mathcal{H}^1_{\Sigma\otimes 1_S,\mathrm{free}}$ to be a localization with calculus of left fractions is the same as to be surjective on objects, up to isomorphisms. Surjectivity on objects follows from the fact that any object of $\mathcal{H}^1_{\Sigma\otimes 1_S,\mathrm{free}}$ has a presentation (4.7) with A replaced by SA. The map σ can be lifted to give a presentation (4.7) with A replaced by CA because $CA \to SA$ is surjective. □

4.17 Lemma. *Let $\mathcal{A}$ be a full additive subcategory of an additive category $\mathcal{U}$. Suppose the inclusion $L : \mathcal{A} \subset \mathcal{U}$ admits a right adjoint $R : \mathcal{U} \to \mathcal{A}$ such that the counit $\epsilon_U : LRU \to U$ is monic for all U in $\mathcal{U}$. Then the quotient map $\mathcal{U} \to \mathcal{U}/\mathcal{A}$ induces an equivalence of triangulated categories*

$$D_b(\mathcal{U})/D_b(\mathcal{A}) \to D_b(\mathcal{U}/\mathcal{A}).$$

Proof. Recall that $D_b(\mathcal{A})$ is a full triangulated subcategory of $D_b(\mathcal{U})$ as both categories are homotopy categories of chain complexes. Thus the quotient category $D_b(\mathcal{U})/D_b(\mathcal{A})$ and the induced map to $D_b(\mathcal{U}/\mathcal{A})$ do exist. Both categories are generated as triangulated categories by the objects $U \in \mathcal{U}$ considered as complexes concentrated in degree zero. By the five lemma for homomorphism groups in triangulated categories, it suffices to show that the map

$$\hom_{D_b(\mathcal{U})/D_b(\mathcal{A})}(U, V[i]) \to \hom_{D_b(\mathcal{U}/\mathcal{A})}(U, V[i])$$

is an isomorphism for all objects $U, V \in \mathcal{U}$.

Write $\eta : id_{\mathcal{A}} \to RL$ for the unit of the adjunction. As L is a full inclusion, η is an isomorphism. Degree-wise application of L, R, η, ϵ on chain complexes yields an adjoint pair of functors between the derived categories and their unit and counit maps, still called L, R, η, ϵ. In this situation, the cone $c(\epsilon)$ of ϵ defines a triangle functor $D_b(\mathcal{U}) \to D_b(\mathcal{U})$ which is trivial on $D_b(\mathcal{A})$ and thus defines a triangle functor $D_b(\mathcal{U})/D_b(\mathcal{A}) \to D_b(U)$ which is known to be fully faithful (exercise, or see [Nee01, Proposition 9.1.18]). Thus we have to show that

$$\hom_{D_b(\mathcal{U})}(c(\epsilon_U), c(\epsilon_V)[i]) \to \hom_{D_b(\mathcal{U}/\mathcal{A})}(U, V[i])$$

is an isomorphism. For $i \neq -1, 0, 1$ both sides are trivial on the chain complex level. For $i = -1, 1$ the right hand side is trivial on the chain complex level. For $i = -1$, the left hand side is trivial on the chain complex level because ϵ is monic. For $i = 1$, chain maps on the left hand side are homotopic to zero as any map from an object of $\mathcal{A}$ to an object W of $\mathcal{U}$ factors through $RL(W)$. For $i = 0$, the right hand side is the set of maps from U to V modulo those which factor through an object of $\mathcal{A}$, thus modulo those which factor through $RL(V)$, by adjointness. This is exactly the description of the left hand side. □

Exact sequences of triangulated categories are useful because of the following theorem and because of Theorem 4.20 below.

4.18 Theorem. *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a sequence of exact categories such that $D_b(\mathcal{A}) \to D_b(\mathcal{B}) \to D_b(\mathcal{C})$ is exact up to direct factors. Then the induced sequence of non-connective K-theory spectra*

$$\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B}) \to \mathbf{K}(\mathcal{C})$$

is a homotopy fibration.

Proof. This is a special case of [Schc, 11.10], see also [Schc, 5.5, 11.13]. □

4.19. *Triangular Witt groups.* Let $(\mathcal{T}, \sharp, \eta)$ be a triangulated category with duality, Balmer defines its triangular Witt groups $W^n_B(\mathcal{T})$ as the monoid of isomorphism classes of symmetric spaces (*i.e.,* of hermitian objects in $\mathcal{T}$) relative to the duality functor $T^n \circ \sharp$, modulo "neutral spaces" [Bal00, Definition 2.12]. Balmer's triangular Witt groups are 4-periodic [Bal00, 2.14]. Let $\epsilon = 1$ or $\epsilon = -1$, we write ${}_\epsilon W^n_B(\mathcal{T})$ for $W^n_B(\mathcal{T}, \sharp, \epsilon\eta)$. By [Bal00, Remark 2.16], there is a natural isomorphism ${}_{-\epsilon}W^n_B(\mathcal{T}) \cong {}_\epsilon W^{n+2}_B(\mathcal{T})$.

Moreover, there is a natural isomorphism $W(\mathcal{E}) \to W^0_B(D_b(\mathcal{E}))$ for $\mathcal{E}$ an exact category with duality [Bal01, 4.3], [BW02, 1.4]. For $\mathcal{E}$ an exact category, we may write ${}_\epsilon W^n_B(\mathcal{E})$ for ${}_\epsilon W^n_B(D_b(\mathcal{E}))$.

4.20 Theorem. [Bal00, Theorem 6.2] *Let $\mathcal{R} \to \mathcal{S} \to \mathcal{T}$ be an exact sequence of triangulated categories (4.2) with duality in which 2 is invertible. Then there is a natural long exact sequence*

$$... \to W^n_B(\mathcal{R}) \to W^n_B(\mathcal{S}) \to W^n_B(\mathcal{T}) \to W^{n+1}_B(\mathcal{R}) \to ... \tag{4.21}$$

4.22 Lemma. *Let $\mathcal{D}$ be a triangulated category with duality and let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be triangulated subcategories with duality closed under direct factors in $\mathcal{D}$ and such that $\mathcal{A} \subset \mathcal{B}, \mathcal{C}$. Assume that the induced map on quotients $f : \mathcal{C}/\mathcal{A} \to \mathcal{D}/\mathcal{B}$ is an equivalence. Assume further that there is a triangle map $\mathcal{D} \to \mathcal{B}/\mathcal{A}$ which is the quotient map $\mathcal{B} \to \mathcal{B}/\mathcal{A}$ when restricted to $\mathcal{B}$ and which is trivial when restricted to $\mathcal{C}$. Then the two compositions $W^*(\mathcal{D}) \stackrel{b}{\to} W^*(\mathcal{B}/\mathcal{A}) \stackrel{\delta_{\mathcal{B}}}{\to} W^{*+1}(\mathcal{A})$ and $W^*(\mathcal{D}) \stackrel{a}{\to} W^*(\mathcal{D}/\mathcal{B}) \cong W^*(\mathcal{C}/\mathcal{A}) \stackrel{\delta_{\mathcal{C}}}{\to} W^{*+1}(\mathcal{A})$ differ by a sign.*

Proof. We have induced exact sequences of triangulated categories $\mathcal{C}/\mathcal{A} \stackrel{i}{\to} \mathcal{D}/\mathcal{A} \stackrel{\alpha}{\to} \mathcal{D}/\mathcal{C}$ and $\mathcal{B}/\mathcal{A} \stackrel{j}{\to} \mathcal{D}/\mathcal{A} \stackrel{\beta}{\to} \mathcal{D}/\mathcal{B}$ which induce split short exact sequences of Witt groups since $\beta i = f$ is an equivalence. This implies that the map $e = \alpha j$ induces an isomorphism on Witt groups and the split exact sequences

$$0 \longrightarrow W^*(\mathcal{B}/\mathcal{A}) \underset{e^{-1}\alpha}{\overset{j}{\rightleftarrows}} W^*(\mathcal{D}/\mathcal{A}) \underset{i}{\overset{f^{-1}\beta}{\rightleftarrows}} W^*(\mathcal{C}/\mathcal{A}) \longrightarrow 0$$

represent the middle term as the sum of the two outer terms. Thus we have $1 = if^{-1}\beta + je^{-1}\alpha$. By Balmer's localization sequence, the composition $W^*(\mathcal{D}) \stackrel{c}{\to} W^*(\mathcal{D}/\mathcal{A}) \stackrel{\delta_{\mathcal{A}}}{\to} W^{*+1}(\mathcal{A})$ is zero, hence we have $0 = \delta_{\mathcal{A}}c = \delta_{\mathcal{A}}if^{-1}\beta c + \delta_{\mathcal{A}}je^{-1}\alpha c = \delta_{\mathcal{C}}f^{-1}a + \delta_{\mathcal{B}}b$. □

5. Karoubi Induction

The purpose of this section is to review Karoubi's "Fundamental Theorem" and the induction principle that can be derived from it both in the language and the generality that turn out to be necessary for our proofs of localization and *dévissage*.

5.1. *The ring V.* Following [Kar80] we introduce a ring V. The ring homomorphism $F : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}^{op}$: $a \mapsto (a, a^\sharp)$, where $\mathbb{Z} \times \mathbb{Z}^{op}$ is equipped with the canonical involution (3.3), is duality preserving. The map F is called *forgetful map*. Let $\mathcal{V}$ be the limit cone (3.17) of $F : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}^{op}$. It is a preadditive category with duality which has as objects, the ring with involution V defined by the pull-back square

$$\begin{array}{ccc} V & \longrightarrow & S \\ \downarrow & & \downarrow F \\ C \times C^{op} & \longrightarrow & S \times S^{op} \end{array} \tag{5.2}$$

and some objects of $V-$proj. Note that in this description of V, we used the isomorphism of rings with involution $(\mathbb{Z} \times \mathbb{Z}^{op}) \otimes A = A \times A \to A \times A^{op} : (a, b) \mapsto (a, \bar{b})$.

5.3 Definition. For $\mathcal{A}$ a preadditive category with duality, we write $V\mathcal{A}$ for $\mathcal{A} \otimes V$, ${}_\epsilon V(\mathcal{A})$ for the loop space of ${}_\epsilon K^h(V\mathcal{A})$ and ${}_\epsilon \mathbf{V}(\mathcal{A})$ for the loop spectrum of ${}_\epsilon \mathbf{K}^h(V\mathcal{A})$.

5.4 *Remark.* The inclusion $V\mathcal{A}-$proj $\to \mathcal{A} \otimes \mathcal{V}-$proj is an equivalence of categories with duality. By 2.13 and 2.17, $\mathcal{A} \otimes \mathcal{V}$ is the limit cone of the forgetful map $F_{\mathcal{A}} := \mathcal{A} \otimes F : \mathcal{A} \to \mathcal{A} \times \mathcal{A}^{op}$. So there is a homotopy fibration (3.17, 3.3)

$${}_\epsilon V(\mathcal{A}) \to {}_\epsilon K^h(\mathcal{A}) \stackrel{F}{\to} K(\mathcal{A}).$$

5.5 *Remark.* Note that, as a kernel of a surjective map, $S \times C \times C^{op} \to S \times S^{op}$ of flat $\mathbb{Z}$-modules, V is a flat $\mathbb{Z}$-algebra. Moreover, lemma 5.11 below shows that $V \to S$ and $V \to C \times C^{op}$ are flat as well.

5.6. *The ring U.* We define an involution on the ring M_2 of 2×2 matrices with entries in $\mathbb{Z}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

The ring homomorphism $H : \mathbb{Z} \times \mathbb{Z}^{op} \to M_2 : (a, b) \mapsto diag(a, b^\sharp)$ is duality preserving, where $\mathbb{Z} \times \mathbb{Z}^{op}$ is equipped with the canonical involution. The map H is called *hyperbolic map.* Let $\mathcal{U}$ be the limit cone (3.17) of $H : \mathbb{Z} \times \mathbb{Z}^{op} \to M_2$. It is a preadditive category with duality which has as objects the ring with involution U defined by the pull-back square

$$\begin{array}{ccc} U & \longrightarrow & S \times S^{op} \\ \downarrow & & \downarrow H \\ M_2(C) & \longrightarrow & M_2(S) \end{array} \tag{5.7}$$

and some objects of $U-$proj.

5.8 Definition. For $\mathcal{A}$ a preadditive category with duality, we write $U\mathcal{A}$ for $\mathcal{A} \otimes U$, ${}_\epsilon U(\mathcal{A})$ for the loop space of ${}_\epsilon K^h(U\mathcal{A})$ and ${}_\epsilon \mathbf{U}(\mathcal{A})$ for the loop spectrum of ${}_\epsilon \mathbf{K}^h(U\mathcal{A})$.

5.9 *Remark.* As in 5.4 there is a homotopy fibration

$${}_\epsilon U(\mathcal{A}) \to K(\mathcal{A}) \overset{H}{\to} {}_\epsilon K^h(\mathcal{A})$$

since by Morita equivalence $M_2 \otimes \mathcal{A}-$proj and $\mathcal{A}-$proj are equivalent categories with duality.

5.10 *Remark.* Note that, as a kernel of a surjective map, $M_2(C) \times S \times S^{op} \to M_2(S)$ of flat $\mathbb{Z}$-modules, U is a flat $\mathbb{Z}$-algebra. Moreover, $S \times S^{op} \to M_2(S)$ and $M_2(C) \to M_2(S)$ are flat, the latter map is surjective, so $U \to S \times S^{op}$ and $U \to M_2(C)$ are flat by 5.11 below.

5.11 Lemma. *Let $f : B \to D$ and $g : C \to D$ be ring homomorphisms. Let A be the pull-back ring of f along g. If f and g are flat and g surjective, then the induced maps $\bar{g} : A \to B$ and $\bar{f} : A \to C$ are also flat.*

Proof. Let $\mathcal{E}$ be the following category: objects are triples (M, N, ϕ) where M is an object of B-Mod, N is an object of C-Mod and $\phi : N \otimes_C D \overset{\cong}{\to} M \otimes_B D$ is an isomorphism in D-Mod. A morphism from (M, N, ϕ) to (M', N', ϕ') is a pair of morphisms $\mu : M \to M'$ and $\nu : N \to N'$ such that $\phi' \circ (\nu \otimes_C 1_D) = (\mu \otimes_B 1_D) \circ \phi$. The category $\mathcal{E}$ is a Grothendieck abelian category with small projective generator $P = (B, C, id)$. Kernels, cokernels and sums are formed component-wise (use f, g flat). The projectivity claim follows from flatness and surjectivity of g. The fact that P is a generator is a consequence of the surjectivity of g. It follows that $\mathcal{E}$ is equivalent to $End(P)$-Mod [Pop73, 3.7, exercise 4]. With $A = End(P)$, by construction, the equivalence A-$Mod \to \mathcal{E}$ sends an A-module M to $(M \otimes_A B, M \otimes_A C, id)$. As an equivalence it is exact and so $A \to B$, $A \to C$ are flat ring homomorphisms. □

5.12 *Notation.* For A a ring with involution and $\mathcal{B}$ a preadditive category with involution we write $A^n\mathcal{B}$ for the preadditive category with involution $\mathcal{B} \otimes A^{\otimes n}$.

5.13. Karoubi defined a map ${}_{-1}K_0^h(V^{\otimes 2} \otimes \mathbb{Z}[\frac{1}{2}]) \to {}_1K_0^h(\mathbb{R})$ and an element z in ${}_{-1}K_0^h(V^{\otimes 2} \otimes \mathbb{Z}[\frac{1}{2}])$ corresponding to $1 \in {}_1K_0^h(\mathbb{R})$ under this map [Kar80, 3.3]. Let z be represented by $(P_1, \lambda_1) - (P_2, \lambda_2)$. For any preadditive category $\mathcal{A}$, tensoring with P_i defines a map $\otimes P_i \; : \mathcal{A} \to P(V^2\mathcal{A}) \leftarrow V^2\mathcal{A}$, tensoring with (P_i, λ_i) defines a map $\otimes(P_i, \lambda_i) \; : {}_\epsilon P(\mathcal{A})_h \to {}_{-\epsilon}P^2(V^2\mathcal{A})_h \leftarrow {}_{-\epsilon}P(V^2\mathcal{A})_h$, the left arrow map being an equivalence. Group completing yields maps $f_i : {}_\epsilon K^h(\mathcal{A}) \to {}_{-\epsilon}K^h(V^2\mathcal{A})$. By abuse of notation, we will write $\cup z = f_1 - f_2 : {}_\epsilon K^h(\mathcal{A}) \to {}_{-\epsilon}K^h(V^2\mathcal{A})$ as a map of spaces. Saying that cup product with z is a homotopy equivalence means by definition that on homotopy groups the difference $\pi_i(f_1) - \pi_i(f_2)$ induces an isomorphism for all i. The proof of the following theorem can be found in [Kar80].

5.14 Theorem. (Karoubi's "théorème fondamental") *For any preadditive category with involution $\mathcal{A}$ in which 2 is invertible, cup product with z induces a homotopy equivalence*

$$\cup\, z : {}_\epsilon K^h(\mathcal{A}) \xrightarrow{\sim} {}_{-\epsilon}K^h(V^2\mathcal{A}).$$

Proof. Every idempotent complete additive category with duality is the filtered colimit of categories with dualities of the form $P(A)$ for some rings with involutions A [Schb, appendix]. The theorem follows from Karoubi's *théorème fondamental* of [Kar80] where it has been proved for rings with involutions. □

5.15 *Remark.* The maps $\otimes(P_i, \lambda_i)$ induce maps of spectra $f_i : {}_\epsilon\mathbf{K}^h(\mathcal{A}) \xrightarrow{\sim} {}_{-\epsilon}\mathbf{K}^h(V^2\mathcal{A})$. Using the isomorphism $S^nV^2\mathcal{A} \cong V^2S^n\mathcal{A}$, we can apply Karoubi's fundamental theorem 5.14 with $S^n\mathcal{A}$ in place of $\mathcal{A}$ to obtain a homotopy equivalence of spectra

$$f_1 - f_2 : {}_\epsilon\mathbf{K}^h(\mathcal{A}) \xrightarrow{\sim} {}_{-\epsilon}\mathbf{K}^h(V^2\mathcal{A}).$$

5.16 *Remark.* [Kar80, 1.4] In this paragraph, all maps of preadditive categories have to be tensored with $\mathbb{Z}[\frac{1}{2}]$ which we omit to avoid cumbersome notations. By definition of the preadditive category with duality $\mathcal{V}$ (5.1), there is a filtering inclusion $\iota : \mathbb{Z} \times \mathbb{Z}^{op} \to \mathcal{V}$ with quotient S (2.17). Tensoring the hyperbolic map (5.6) with ι yields a commutative square in *Pad*

$$\begin{array}{ccc} (\mathbb{Z} \times \mathbb{Z}^{op}) \times (\mathbb{Z} \times \mathbb{Z}^{op})^{op} & \xrightarrow{\iota \times \iota^{op}} & \mathcal{V} \times \mathcal{V}^{op} \\ \downarrow{\scriptstyle H_{\mathbb{Z}\times\mathbb{Z}^{op}}} & & \downarrow{\scriptstyle H_{\mathcal{V}}} \\ M_2(\mathbb{Z} \times \mathbb{Z}^{op}) & \xrightarrow[M_2(\iota)]{} & M_2\mathcal{V}. \end{array}$$

Let j be the full inclusion $\mathbb{Z} \to P(\mathbb{Z} \times \mathbb{Z}^{op})$ sending $\mathbb{Z}$ to the image of the idempotent $(1,0) \in \mathbb{Z} \times \mathbb{Z}^{op}$. Then $P(H_{\mathbb{Z}\times\mathbb{Z}^{op}}) \circ (j \times j^{op})$ is a hermitian K-theory equivalence as it induces an equivalence on finitely generated projective modules by Morita equivalence. The map $P(\iota) \circ j$ is a K-theory equivalence because

$$\mathbf{K}(\mathbb{Z}) \overset{F=\Delta}{\to} \mathbf{K}(\mathbb{Z} \times \mathbb{Z}^{op}) \to \mathbf{K}(\mathcal{V})$$

is a homotopy fibration (3.17). Hence, the map $P(\iota \times \iota^{op}) \circ (j \times j^{op})$ is a hermitian K-theory equivalence (3.3). It follows that the homotopy cofibers in hermitian K-theory of $M_2(\iota)$ and $H_{\mathcal{V}}$ are homotopy equivalent, in other words, there is a hermitian K-theory equivalence of UV with S. The argument is also valid after tensoring with any preadditive category $\mathcal{A}$ with duality (2.13). In particular, there is a natural homotopy equivalence ${}_\epsilon\mathbf{K}^h(UV\mathcal{A}) \simeq {}_\epsilon\mathbf{K}^h(S\mathcal{A})$.

5.17 *Remark.* Applying Karoubi's *théorème fondamental* to the preadditive category with involution $U\mathcal{A}$ we find the more familiar version of this theorem, namely a homotopy equivalence [Kar80]

$$\Omega_\epsilon U(\mathcal{A}) \xrightarrow{\sim} {}_{-\epsilon}V(\mathcal{A}).$$

This follows from the above discussion.

5.18. *Karoubi Witt groups.* Let $\mathcal{A}$ be a preadditive category with duality. The hyperbolic functor (3.4) induces a morphism of spectra $H : \mathbf{K}(\mathcal{A}) \to_\epsilon \mathbf{K}^h(\mathcal{A})$. Following Karoubi [Kar80] one defines

$${}_\epsilon W^K_n(\mathcal{A}) := \operatorname{coker}(K_n(\mathcal{A}) \xrightarrow{H} {}_\epsilon K^h_n(\mathcal{A})), \quad n \in \mathbb{Z}\ .$$

These groups are 4-periodic up to 2-torsion [Kar80]. Note that $W^K_0(\mathcal{A}) = W(\mathcal{A}{-}\mathrm{proj})$, the usual Witt group.

5.19 *Remark.* We show in A.4 that $W_n^K(\mathcal{A}) \otimes \mathbb{Z}[\frac{1}{2}] = W_B^{-n}(\mathcal{A}-\mathrm{proj}) \otimes \mathbb{Z}[\frac{1}{2}]$. Nevertheless, there are interesting elements in the 2-torsion of Karoubi's higher Witt groups which do not exist in Balmer's Witt groups. For example, let F be a field of characteristic different from 2, the determinant of an automorphism of a hermitian object defines a surjective map $W_1^K(F) \to {}_2F^\times = \mathbb{Z}/2\mathbb{Z}$. However, the Balmer Witt group $W_B^{-1}(F)$ is trivial [Bal01, 5.6].

5.20 Lemma. (Karoubi induction) *Let $f : \mathcal{A} \to \mathcal{B}$ be a map of preadditive categories with duality such that 2 is invertible in $\mathcal{A}$ and $\mathcal{B}$. Suppose f induces isomorphisms $K_*(\mathcal{A}) \to K_*(\mathcal{B})$, $* \in \mathbb{Z}$ and either a) ${}_\epsilon K_i^h(\mathcal{A}) \to {}_\epsilon K_i^h(\mathcal{B})$ are isomorphisms for $i = N, N+1$ and $\epsilon = 1, -1$, or b) ${}_\epsilon W_B^n(P(\mathcal{A})) \to {}_\epsilon W_B^n(P(\mathcal{B}))$ are isomorphisms for $n \in \mathbb{Z}$ and $\epsilon = 1, -1$. Then the induced maps ${}_\epsilon K_i^h(\mathcal{A}) \to {}_\epsilon K_i^h(\mathcal{A})$ are isomorphisms for $\epsilon = 1, -1$ and a) for all $i \geq N$, b) for all $i \in \mathbb{Z}$, respectively.*

Proof. To prove part a), observe that our hypothesis has the following implications: $\pi_N({}_\epsilon V(f))$ iso (five lemma) $\Rightarrow$ ${}_{-\epsilon}U_{N+1}(f)$ iso (5.17) $\Rightarrow$ ${}_{-\epsilon}K^h_{N+2}(f)$ epi (five lemma) $\Rightarrow$ $\pi_{N+1}({}_{-\epsilon}V(f))$ epi (five lemma) $\Rightarrow$ ${}_\epsilon K^h_{N+2}(f)$ iso (5.17 and five lemma). The claim follows by induction. To establish part b), let $C(f)$ be the mapping cone (3.17) associated with f. By 2.16, we have exact sequences up to direct factors (4.12) $D_b(\mathcal{A}-\mathrm{proj}) \to D_b(C\mathcal{A}-\mathrm{proj}) \to D_b(S\mathcal{A}-\mathrm{proj})$ and $D_b(\mathcal{B}-\mathrm{proj}) \to D_b(C(f)-\mathrm{proj}) \to D_b(S\mathcal{A}-\mathrm{proj})$, and f induces a map between them which is the identity on $D_b(S\mathcal{A}-\mathrm{proj})$. By assumption, Theorem 4.18 and $\mathbf{K}(C\mathcal{A}) \simeq 0$, it follows that $\mathbf{K}(C(f)) \simeq 0$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the image categories of $D_b(C\mathcal{A}-\mathrm{proj}) \to D_b(S\mathcal{A}-\mathrm{proj})$ and $D_b(C(f)-\mathrm{proj}) \to D_b(S\mathcal{A}-\mathrm{proj})$, respectively. The cofinal inclusion $\mathcal{T}_1 \subset \mathcal{T}_2$ is an equivalence (4.1) because both have $K_0 = 0$ (as $K_0(C\mathcal{A}-\mathrm{proj}) = 0$ and $K_0(C(f)-\mathrm{proj}) = 0$). Replacing in the exact sequences up to direct factors the third term with $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, we obtain exact sequences, and thus associated long exact sequences of triangular Witt groups (4.19). Since $W_B^*(P(f))$ is an isomorphism, we have $W_B^*(C(f)-\mathrm{proj}) = 0$. Hence ${}_\epsilon W_i^K(C(f)-\mathrm{proj}) = 0$ (A.3), $i \leq 0$. With $K_*(C(f)) = 0$ we have ${}_\epsilon K^h{}_i(C(f)) = 0$ for $i \leq 0$. So $K_i^h(\mathcal{A}) \to K_i^h(\mathcal{B})$ is an isomorphism for $i < 0$ (3.17). Using part a) we are done. □

The proof of this Lemma is easier for regular rings as then one may use Lemma A.5.

6. Proof of Localization

To enhance readability, we will frequently drop the index ϵ from our notation in this section if no confusion may arise, *e.g.*, a statement about K^h is actually a statement about ${}_\epsilon K^h$ both for $\epsilon = 1$ and $\epsilon = -1$.

6.1. *Outline of the proof.* We introduce a simplicial additive category with duality G_* and a sequence of simplicial additive categories with duality (6.2)

$$P(A) \overset{\iota_*}{\to} G_* \overset{\rho_*}{\to} \mathcal{R}_*\mathcal{T}_\Sigma.$$

In each degree, the sequence will induce a homotopy fibration of hermitian K-theory spaces (6.7). This is deduced from the corresponding localization theorems in ordinary K-theory, in Balmer's theory of Witt groups and Karoubi induction (which in turn relies on cofinality.) The Bousfield-Friedlander theorem [BF78] will imply that after topological realization

$$K^h(A) \overset{\iota_*}{\to} |K^h(G_*)| \overset{\rho_*}{\to} |K^h(\mathcal{R}_*\mathcal{T}_\Sigma)|$$

we still have a homotopy fibration (6.10). The last step consists of identifying (up to π_0) $|K^h(G_*)|$ and $|K^h(\mathcal{R}_*\mathcal{T}_\Sigma)|$ with $K^h(\Sigma^{-1}A)$ and $\mathcal{W}(\mathcal{T}_\Sigma)$ (6.11, 6.10).

6.2 Definition. We define a simplicial additive category with duality $(G_*, \sharp, id)$. Recall the cosimplicial category with duality $[n] \mapsto \mathbf{n}$ (1.8). The simplicial additive category G_* is the full subcategory of the category of additive functors $P : \mathbf{n} \to P(A)$ of those objects P such that $P(i \leq j) : P_i \to P_j$ is an inclusion with Σ-torsion cokernel. The duality $\sharp$ on A induces a duality $\sharp$ on G_n by $P^\sharp(i \leq j) = (P(j' \leq i'))^\sharp$ such that $(G_n, \sharp, id)$ becomes an additive category with duality. The cosimplicial structure $[n] \mapsto \mathbf{n}$ gives $(G_*, \sharp, id)$ the structure of a simplicial additive category with duality.

There is an additive duality preserving functor $\iota_n : P(A) \to G_n$ sending Q to the constant diagram $P(i \leq j) = id_Q$. Furthermore, there is an additive duality preserving functor $\rho_n : G_n \to \mathcal{R}_n\mathcal{T}_\Sigma$ sending P to $\rho_n(P)$ with $\rho_n(P)_{i,j} = P(i \leq j)$ and maps $\rho_n(P)_{i,j} \to \rho_n(P)_{k,l}$ given by the inclusions $P(i) \subset P(j), P(k) \subset P(l)$. Considering $P(A)$ as a constant simplicial additive category with duality these two functors assemble to give functors of simplicial additive categories with duality

$$P(A) \stackrel{\iota_*}{\to} G_* \stackrel{\rho_*}{\to} \mathcal{R}_*\mathcal{T}_\Sigma. \tag{6.3}$$

Note that the composition of the two functors is trivial.

6.4 Proposition. *For any integer $n \geq 0$, the maps 6.3 induce a short exact sequence of triangulated categories with duality*

$$D_b(P(A)) \stackrel{D_b(\iota_n)}{\to} D_b(G_n) \stackrel{D_b(\rho_n)}{\to} D_b(\mathcal{R}_n\mathcal{T}_\Sigma).$$

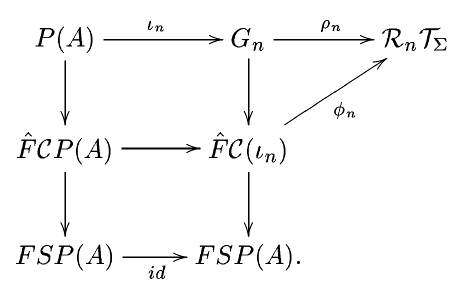
Proof. The inclusion $\iota_n : P(A) \subset G_n$ admits a right adjoint, namely $G_n \to P(A) : P \mapsto P(n')$. The counit is monic since all maps $P(n' \leq m)$ are monic. It is simple homological algebra to check that the map $G_n/P(A) \to \mathcal{R}_n\mathcal{T}_\Sigma$ is an equivalence of categories. Now the claim follows from 4.17 since $D_b(P(A))$ is the kernel category of $D_b(G_n) \to D_b(G_n)/D_b(P(A))$. This is because $D_b(P(A))$ is idempotent complete as $P(A)$ is [BS01]. □

6.5. Recall the definition of the cone of a duality preserving map of preadditive categories with duality (3.17). Since the composition $P(A) \stackrel{\iota_k}{\to} G_k \to \mathcal{R}_k\mathcal{T}_\Sigma$ is trivial, the universal property of push-outs yields a functor $\phi_k : \mathcal{C}(\iota_k) \to \mathcal{R}_k\mathcal{T}_\Sigma$ of preadditive categories with duality sending $\mathcal{C}P(A)$ to the base point 0.

6.6 Proposition. *If 2 is invertible, then the functor $\phi_n : C(\iota_n) \to \mathcal{R}_n\mathcal{T}_\Sigma$ induces homotopy equivalences of non-connective hermitian K-theory spectra*

$${}_\epsilon\mathbf{K}^h C(\iota_n) \stackrel{\sim}{\to} {}_\epsilon\mathbf{K}^h(\mathcal{R}_n\mathcal{T}_\Sigma).$$

Proof. Since $\mathcal{R}_n\mathcal{T}_\Sigma$ is idempotent complete, the functor $\phi_n : \mathcal{C}(\iota_n) \to \mathcal{R}_n\mathcal{T}_\Sigma$ extends to $\phi_n : P\mathcal{C}(\iota_n) \to \mathcal{R}_n\mathcal{T}_\Sigma$ and all its subcategories. Consider the commutative diagram of additive categories with duality



The first horizontal and the two vertical lines induce exact sequences of bounded derived categories with duality (4.11, 6.4). We therefore have associated homotopy fibrations of non-connective K-theory spectra (4.18) and long exact sequences for triangular Witt groups (4.19).

As the vertical homotopy fibrations have the same base, the top square induces a homotopy cartesian square of K-theory spectra. It maps to the homotopy cartesian square belonging to the homotopy fibration of the top row. As $\mathbf{K}(\mathcal{C}P(A)) \simeq *$ by the Eilenberg swindle, the map ϕ_n induces a homotopy equivalence of non-connective spectra, by the five lemma. Moreover, the existence of a retraction to ι_n as additive categories, namely $G_n \to P(A) : P \mapsto P(0)$, implies that $\mathbf{K}(G_n) \to \mathbf{K}(\mathcal{C}(\iota_n))$ is surjective on all π_i. Thus the composition of usual K-groups (no idempotent completion before taking K_0!) $K_0(G_n) \to K_0(\hat{F}(C(\iota_n))) \to K_0(P(C(\iota_n)))$ is surjective. In particular, the last map is surjective, hence bijective as it is always injective by cofinality. So the functor $D_b(\hat{F}\mathcal{C}(\iota_n)) \to D_b(P\mathcal{C}(\iota_n))$ is an equivalence (4.1). In particular, replacing $\hat{F}\mathcal{C}(\iota_n)$ with $P\mathcal{C}(\iota_n)$ doesn't change neither (usual) K-groups nor triangular Witt-groups.

There is a map from the Mayer-Vietoris long exact sequence for triangular Witt-groups associated with the top square to the long exact sequence of triangular Witt-groups associated with the top row. Commutativity for the boundary maps (up to sign) was checked in 4.22. Again by the Eilenberg swindle and the five lemma we conclude that $W^*(\phi_n)$ is an isomorphism. Since $\hat{F}\mathcal{C}(\iota_n) \to P\mathcal{C}(\iota_n)$ and $\mathcal{R}_n\mathcal{T}_\Sigma \to P\mathcal{R}_n\mathcal{T}_\Sigma$ induce isomorphisms for all triangular Witt groups, we may apply Karoubi induction (5.20 b)) to obtain the claim. □

6.7 Corollary. *For any integer $n \geq 0$, the maps defined in 6.2 induce a homotopy fibration*

$$_\epsilon(iP(A))^+_h \stackrel{\iota_n}{\to} {}_\epsilon(iG_n)^+_h \stackrel{\rho_n}{\to} {}_\epsilon(i\mathcal{R}^h_n\mathcal{T}_S)^+.$$

Proof. It follows from 3.16 that applying non-connective hermitian K-theory spectra to the top square in the diagram of the proof of 6.6 yields a homotopy cartesian square. It maps to the square that belongs to the top row of the diagram with 0 in the left lower corner. Since all of these maps are hermitian K-theory equivalences, it follows that the top row induces a homotopy fibration of non-connective K-theory spectra. As $P(A)$, G_n and $\mathcal{R}_n\mathcal{T}_\Sigma$ are idempotent complete, the claim is just -1 connected cover of this homotopy fibration. □

6.8. We would like to apply the Bousfield-Friedlander Theorem [BF78, Theorem B.4] to conclude that the topological realization

$$(iP(A))^+_h \stackrel{\iota_*}{\to} |(iG_*)^+_h| \stackrel{\rho_*}{\to} |(i\mathcal{R}^h_*\mathcal{T}_\Sigma)^+|$$

is still a homotopy fibration. For this we have to check the two conditions of [BF78, Theorem B4]. The "π_*-Kan-condition" holds because we are dealing with simplicial H-groups. But the morphism $\pi_0((iG_n)^+_h) \stackrel{\rho_n}{\to} \pi_0(i\mathcal{R}^h_n\mathcal{T}_\Sigma)^+$ might not be surjective, in general. If we write $\hat{\mathcal{T}}_n$ for the full subcategory of $i(\mathcal{R}^h_n\mathcal{T}_\Sigma)^+$ consisting of those components lying in the image of $i(G_n)^+_h \to i(\mathcal{R}^h_n\mathcal{T}_\Sigma)^+$, then we can prove the following:

6.9 Lemma. *If 2 is invertible, then there is a homotopy fibration*

$$|\hat{\mathcal{T}}_*| \to |(i\mathcal{R}^h_*\mathcal{T}_\Sigma)^+| \to |L_*|$$

where L_* is a constant simplicial abelian group.

Proof. We define the simplicial abelian group L_* by $L_n := \operatorname{coker}[\pi_0(\hat{\mathcal{T}}_n) \to \pi_0((i\mathcal{R}^h_n\mathcal{T}_\Sigma)^+)] \cong \operatorname{coker}[K^h_0(G_n) \to K^h_0(\mathcal{R}_n\mathcal{T}_\Sigma)]$. By the Bousfield-Friedlander Theorem, we have a homotopy fibration $|\hat{\mathcal{T}}_*| \to |(i\mathcal{R}^h_*\mathcal{T}_\Sigma)^+| \to |L_*|$. Denote by L^W_n the cokernel of $W^0_B(G_n) \to W^0_B(\mathcal{R}_n\mathcal{T}_\Sigma)$. The induced map on cokernels $L_n \to L^W_n$ is an isomorphism since $K_0(G_n) \to K_0(\mathcal{R}_n\mathcal{T}_\Sigma)$ is an epimorphism (6.4). Consider L_0 as a constant simplicial abelian group. We claim that the natural map of simplicial groups $\bar{\sigma}_* : L_0 \to L_*$ induced by $\sigma_n : [n] \to [0]$ is an isomorphism. Considering $W^1_B(P(A))$ as a constant simplicial abelian group, the residue homomorphisms of Balmer's localization sequence 4.20 (associated to the short exact sequences of Proposition 6.4) $\delta_* : W^0_B(\mathcal{R}_*\mathcal{T}_\Sigma) \to W^1_B(P(A))$ yield a simplicial factorization $W^0_B(\mathcal{R}_*\mathcal{T}_\Sigma) \twoheadrightarrow L^W_* \rightarrowtail W^1_B(P(A))$. It follows that for any $\theta : [n] \to [m]$ in Δ the induced map $\bar{\theta} : L_m \to L_n$ is a monomorphism. In particular, $\bar{\sigma}_n$ and $\bar{\eta}$ are monomorphisms for any $\eta : [0] \to [n]$. Using the identity $\bar{\eta} \circ \bar{\sigma}_n = id_{L_0}$, we conclude that the map $\bar{\sigma}_n$ is an isomorphism for all n. □

6.10. Since G_* and $\mathcal{R}_*\mathcal{T}_\Sigma$ are simplicial strict symmetric monoidal categories (as $P(A)$ is), the natural maps $|(iG_*)_h| \to |(iG_*)^+_h|$ and $|i\mathcal{R}^h_*\mathcal{T}_\Sigma| \stackrel{\simeq}{\to} |(i\mathcal{R}^h_*\mathcal{T}_\Sigma)^+|$ are group completions (e.g. [Schb]). The latter map is already a homotopy equivalence because $\pi_0(|i\mathcal{R}^h_*\mathcal{T}_\Sigma|)$ is the usual Witt group and thus a group (1.14). It follows that we have homotopy fibrations

$$iP(A)^+_h \to i(G_*)^+_h \to \mathcal{W}(\mathcal{T}_\Sigma) \;\text{ and thus }\; U(\mathcal{T}_\Sigma) \to iP(A)^+_h \to i(G_*)^+_h.$$

Denote by $P'(\Sigma^{-1}A)$ the full subcategory of $P(\Sigma^{-1}A)$ of objects which are localizations of finitely generated projective A-modules. By cofinality, the inclusion $P'(\Sigma^{-1}A) \to P(\Sigma^{-1}A)$ induces a map of hermitian K-theory spaces $iP'(\Sigma^{-1}A)^+_h \to iP(\Sigma^{-1}A)^+_h$ which is a monomorphism on π_0 and an isomorphism on π_i, $i \geq 1$. Theorem 1.15 now follows from the following proposition.

6.11 Proposition. *There is a homotopy equivalence*

$$|i(G_*)_h|^+ \xrightarrow{\sim} iP'(\Sigma^{-1}A)^+_h,$$

and if $K_0(A) \to K_0(\Sigma^{-1}A)$ is surjective, there is a homotopy equivalence

$$|i(G_*)_h|^+ \xrightarrow{\sim} iP(\Sigma^{-1}A)^+_h,$$

such that composition with $iP(A)^+_h \to |i(G_*)_h|^+$ yields the localization map induced by $A \to \Sigma^{-1}A$.

Proof. For any category $\mathcal{C}$, we denote by $N_*\mathcal{C}$ its nerve (a simplicial set), and by $\mathcal{N}_*\mathcal{C}$ the simplicial category with $\mathcal{N}_n\mathcal{C} = Fun([n],\mathcal{C})$, the category of functors from $[n]$ to $\mathcal{C}$ and natural transformations as maps. Note that $N_*\mathcal{C} = Ob\mathcal{N}_*\mathcal{C}$. The bisimplicial set $N_*i\mathcal{N}_*\mathcal{C}$ is isomorphic to the nerve of the full simplicial subcategory of $\mathcal{N}_*\mathcal{C}$ with objects those functors $[n] \to \mathcal{C}$ that send maps in $[n]$ to isomorphisms in $\mathcal{C}$. Considering $\mathcal{C}$ as a constant simplicial category, we have a map from $\mathcal{C}$ to this full simplicial subcategory which sends an object $C \in \mathcal{C}$ to the string of isomorphisms consisting only of identity maps id_C. This map is degree-wise an equivalence of categories. Hence we have a homotopy equivalence $\mathcal{C} \xrightarrow{\sim} i\mathcal{N}_*\mathcal{C}$.

For $(\mathcal{C},\sharp,id)$ a category with duality, let $\mathcal{C}_{hd}$ denote the category of all hermitian objects including the degenerate ones. More precisely, objects are pairs (M,ϕ) with $\phi = \phi^\sharp : M \to M^\sharp$ not necessarily an isomorphism, and maps are as in Definition 1.5. Furthermore, we write $(\mathcal{N}^e_*\mathcal{C},\sharp,id)$ for the simplicial category with duality with $\mathcal{N}^e_n\mathcal{C}$ the category of functors $[n] \mapsto Fun(\mathbf{n},\mathcal{C})$. For $F : \mathbf{n} \to \mathcal{C}$, its dual is defined by $F^\sharp(i \leq j) = (F(j' \leq i'))^\sharp$. Note that there is an isomorphism of simplicial categories $i\mathcal{N}^e_*(\mathcal{C})_h \cong i\mathcal{N}_*[(\mathcal{C})_{hd}]$.

We write $m\mathcal{P} \subset P(A)$ for the subcategory with the same objects and those morphisms which are monomorphisms with Σ-torsion cokernel. The inclusion $m\mathcal{P} \subset P(A)$ is closed under the duality, and thus $(m\mathcal{P},\sharp,id)$ is a category with duality. Composing the inclusion with the localization map $A \to \Sigma^{-1}A$ yields a map $L : m\mathcal{P} \to P'(\Sigma^{-1}A)$ of categories with duality which sends all maps to isomorphisms. Note that $i(G_*)_h = i(\mathcal{N}^e_*m\mathcal{P})_h$.

The first homotopy equivalence of the proposition follows once we show that the last map in the diagram

$$i(G_*)_h = i(\mathcal{N}^e_*m\mathcal{P})_h \cong i\mathcal{N}_*[(m\mathcal{P})_{hd}] \xleftarrow{\sim} (m\mathcal{P})_{hd} \xrightarrow{L} iP'(\Sigma^{-1}A)_h$$

is a homotopy equivalence.

Let $M \in P'(\Sigma^{-1}A)$ be a localization of a finitely generated projective A-module. A lattice in M is a finitely generated projective sub-A-module $P \subset M$ such that the induced map $\Sigma^{-1}P \to M$ is an isomorphism. Lattices form a non-empty partially ordered set under inclusion. Given two lattices $P,Q \subset M$, the A-module $P/(P \cap Q)$ is a finitely generated and Σ-torsion. We can find an $s \in \Sigma$ with $[P/(P \cap Q)]s = 0$. Thus $P \cong Ps \subset P \cap Q$ is a lattice contained in P and Q.

Given a non-degenerate ϵ-hermitian form $\phi : M \to M^\sharp$ on M, the dual lattice $P^\vee$ of P is the set of all $x \in M$ such that $< x, y > \in A$ for all $y \in P$ where $< , >$ denotes the ϵ-hermitian bilinear form associated with ϕ. It is identified with $P^\sharp \subset \Sigma^{-1}(P^\sharp) \cong M^\sharp$ under the map ϕ and thus is indeed a lattice. Note that $P^{\vee\vee} = P$. Let $\mathcal{L}(M,\phi)$ be the partially ordered set of lattices in M which are contained in their dual $P \subset P^\vee$. It is not empty as any lattice in $P \cap P^\vee$ is contained in its dual, for P a lattice. It is filtering as any two lattices contained in their dual have a common sub-lattice which is necessarily contained in its dual. Thus $\mathcal{L}(M,\phi)$ is contractible.

The functor $\mathcal{L}(M,\phi) \to (L \downarrow (M,\phi))$ sending $i : P \subset M$ to $\phi \circ i : P \to P^\sharp$ together with the isomorphism $\Sigma^{-1}P \to M$ induced by $P \subset M$ is an equivalence of categories. Thus all categories $(L \downarrow (M,\phi))$ are contractible and $L : (m\mathcal{P})_{hd} \to iP'(\Sigma^{-1}A)_h$ is a homotopy equivalence by Quillen's theorem A.

We show the second homotopy fibration. By the first homotopy fibration and cofinality, we only need to show that $K_0^h(P'(\Sigma^{-1}A)) \to K_0^h(P(\Sigma^{-1}A))$ is surjective Given an object (P,φ) of $P(\Sigma^{-1}A)_h$, our assumption implies that $K_0(P'(\Sigma^{-1}A)) \to K_0(P(\Sigma^{-1}A))$ is an isomorphism. Therefore, there is an object P' of $P'(\Sigma^{-1}A)$ such that $P \oplus P'$ is in $P(\Sigma^{-1}A)$. Then $H(P') \oplus (P,\varphi)$ is in $P'(\Sigma^{-1}A)_h$ and the element $[H(P') \oplus (P,\varphi)] - [H(P')]$ of $K_0^h(P'(\Sigma^{-1}A))$ maps to $[M,\varphi]$. □

For the rest of the section, keep the hypothesis of 1.15 and assume that $K_0(A) \to K_0(\Sigma^{-1}A)$ is surjective. The above discussion immediatly yields the following variant of Theorem 1.15.

6.12 Theorem. *Under the hypothesis of 1.15 assume furthermore that $K_0(A) \to K_0(\Sigma^{-1}A)$ is surjective, e.g., A regular. Then there is a homotopy fibration*

$$ {}_\epsilon K^h(A) \to {}_\epsilon K^h(\Sigma^{-1}A) \to {}_\epsilon \mathcal{W}(\mathcal{T}_\Sigma). $$

□

6.13. *The map $K_0^h(\Sigma^{-1}A) \to W_0(\mathcal{T}_\Sigma)$.* We give a description of the map $K_0^h(\Sigma^{-1}A) \to W_0(\mathcal{T}_\Sigma)$ under the hypothesis of Theorem 6.12. Inclusion of zero simplices into the simplicial fibration of Corollary 6.7 yields a commutative square of spaces

$$ \begin{array}{ccc} K^h(G_0) & \longrightarrow & K^h(\mathcal{T}_\Sigma^\oplus) \\ \downarrow & & \downarrow \\ K^h(\Sigma^{-1}A) & \longrightarrow & \mathcal{W}(\mathcal{T}_\Sigma). \end{array} $$

When applying π_0, the vertical maps are surjective, the right one obviously, the left one because of the proof of 6.11. This defines the map $K_0^h(\Sigma^{-1}A) \to W_0(\mathcal{T}_\Sigma)$.

6.14. *The isomorphism $\iota : W_B^i(\mathcal{T}_\Sigma, Ext^1) \xrightarrow{\sim} W_B^{i+1}(D_b^\Sigma(A))$.* We now construct a duality preserving equivalence of triangulated categories $\iota : (D_b(\mathcal{T}_\Sigma), Ext^1) \to (D_b^\Sigma P(A), T\sharp)$ in the sense of [Gil02, Definition 2.6]. This yields an isomorphism $W_B^i(\mathcal{T}_\Sigma, Ext^1) \xrightarrow{\sim} W_B^{i+1}(D_b^\Sigma(A))$. Note that the first category has a 1-duality and the second a -1-duality!

The modified cone of a map of complexes $f_* : (A_*, d_*^A) \to (B_*, d_*^B)$ is the complex $(\bar{C}(f)_*, d_*)$ with

$$ \bar{C}(f)_i = B_i \oplus A_{i-1}, \quad d_i = \begin{pmatrix} d_i^B & (-1)^{i-1} f_{i-1} \\ 0 & d_{i-1}^A \end{pmatrix}. $$

Consider a chain complex in G_0 as a map of chain complexes in $P(A)$, then taking the modified cone yields a map $F : Ch_b G_0 \to Ch_b P(A)$. The map F preserves degree-wise split exact sequences and contractible chain complexes, hence homotopies. It therefore induces an exact functor of homotopy categories $F : D_b G_0 \to D_b P(A)$ which obviously has image in $D_b^\Sigma P(A)$. Note that $T^{-1} \circ F = F \circ T^{-1}$. The functor F is duality preserving if we choose as natural isomorphism $\eta : F \circ \sharp \to (T\sharp) \circ F$ multiplication with $(-1)^{\deg}$.

Next, the composition $D_b PA \to D_b G_0 \to D_b^\Sigma P(A)$ is zero, thus induces a duality preserving map $(D_b \mathcal{T}_\Sigma^\oplus, Ext^1) \to (D_b^\Sigma P(A), \sharp)$ by 6.4. As quasi-isomorphisms in $Ch_b \mathcal{T}_\Sigma$ are isomorphisms in $D_b P(A)$, the map induces a duality preserving map $\iota : (D_b \mathcal{T}_\Sigma, Ext^1) \to (D_b^\Sigma P(A), \sharp)$ which is an equivalence of triangulated categories (4.10).

6.15. The following diagram commutes

$$\begin{array}{ccc} W^i_B(G_0) & \longrightarrow & W^i_B(\mathcal{T}_\Sigma, Ext^1) \\ \downarrow & & \downarrow \iota \\ W^i_B(\Sigma^{-1}A) & \xrightarrow{\delta} & W^{i+1}_B(D^\Sigma_b P(A)) \end{array}$$

where δ is the boundary map of Balmer's localization sequence for the exact sequence 4.4. Going first horizontally and then vertically is the map F of 6.14. Inspecting the definition of δ [Bal00], the commutativity is obvious.

6.16. There is a commutative diagram

$$\begin{array}{ccccccccccc} K^h_0(A) & \longrightarrow & K^h_0(\Sigma^{-1}A) & \longrightarrow & W_0(\mathcal{T}_\Sigma, Ext^1) & \xrightarrow{j\circ\iota} & W^1_B(A) & \longrightarrow & W^1_B(\Sigma^{-1}A) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow \iota & & \downarrow id & & \downarrow id & & \\ W^0_B(A) & \longrightarrow & W^0_B(\Sigma^{-1}A) & \longrightarrow & W^1_B(D^\Sigma_b(P(A))) & \xrightarrow{j} & W^1_B(A) & \longrightarrow & W^1_B(\Sigma^{-1}A) & \longrightarrow & \cdots \end{array}$$

where the first two horizontal maps are the maps given by applying π_0 to the fibration of Theorem 6.12. The lower sequence is Balmer's localization sequence applied to the exact sequence 4.4 which is exact because of the K_0 assumption. Commutativity of the second square follows from 6.13 and 6.15. Since the lower sequence is exact, and since ι is an isomorphism (6.14), the upper sequence is also exact.

6.17. One checks that the boundary map $W^*(\mathcal{T}^\oplus_\Sigma, Ext^1) \to W^{*+1}(A)$ associated to the exact sequence of 6.4 equals the composition $W^*(\mathcal{T}^\oplus_\Sigma, Ext^1) \to W^*(\mathcal{T}_\Sigma, Ext^1) \to W^{*+1}(D^\Sigma_b(P(A))) \to W^{*+1}(A)$.

7. Proof of Dévissage

For this section, we fix a ring A (not necessarily commutative) with involution and a multiplicative subset $\Sigma \subset A$ of central non-zero divisors closed under the involution. However, the *dévissage* theorem we will prove in this section is only valid for commutative rings as the corresponding result for Witt groups is only known in this case. As for localization, the proof of *dévissage* uses Karoubi induction starting in negative degrees. What is new is that we will need functorial deloopings for the hermitian K-theory of a non-split exact category, namely the category of torsion modules $\mathcal{T}_\Sigma$. This is solved using appendix B and showing that the map $\pi_0 f$ below is an isomorphism.

7.1. Let B be a flat $\mathbb{Z}$-algebra with involution. Then the set $1_B \otimes \Sigma = \{1_B \otimes f \mid f \in \Sigma\}$ is a central multiplicative subset of $B \otimes A$ which consists of non-zero divisors because of the flatness assumption The set is closed under the involution. We have $(\Sigma \otimes 1_B)^{-1}(A \otimes B) = \Sigma^{-1}A \otimes B$. Let $B\mathcal{T}_\Sigma$ denote the category $\mathcal{T}_{\Sigma\otimes 1_B}$. If $B \to B'$ is a flat map of flat $\mathbb{Z}$-algebras with involution, then $B' \otimes_B \ : B\mathcal{T}_\Sigma \to B'\mathcal{T}_\Sigma$ is a well defined exact functor preserving dualities. In this way, suspensions $S\mathcal{T}_\Sigma$, cones $C\mathcal{T}_\Sigma$ of $\mathcal{T}_\Sigma$, the categories $U\mathcal{T}_\Sigma$, $V\mathcal{T}_\Sigma$ and various functors between them are defined (*cf.* 2.10, 2.12, 5.1, 5.6).

7.2. *The* $\mathbf{W}$*-spectrum.* For notations, the reader is referred to appendix B. The less "functorially minded" reader may just forget about appendix B, drop the index n of $K^h(\ )_n$ and take the definition of the hermitian K-theory spectrum given in 3.14.

For $\mathcal{E}$ an exact category with duality, we define the space ${}_\epsilon\mathcal{W}(\mathcal{E})_n$ to be the topological realization of the simplicial space ${}_\epsilon K^h(\mathcal{R}_*\mathcal{E})_n$. Note that ${}_\epsilon\mathcal{W}(\mathcal{T}_\Sigma)_0 = {}_\epsilon\mathcal{W}(\mathcal{T}_\Sigma)$ (if you drop indices then the inclusion ${}_\epsilon\mathcal{W}(\mathcal{T}_\Sigma) \to {}_\epsilon\mathcal{W}(\mathcal{T}_\Sigma)_0$ is a homotopy equivalence).

The exact functor $\otimes S : \mathcal{T}_\Sigma \to S\mathcal{T}_\Sigma$ induces a simplicial duality preserving functor $S\mathcal{R}_*\mathcal{T}_\Sigma \to \mathcal{R}_*S\mathcal{T}_\Sigma$. The structure maps (B.2) of the ${}_\epsilon\mathbf{K}^h$-theory spectrum ${}_\epsilon K^h(\mathcal{R}_i\mathcal{T}_\Sigma)_n \wedge S^1 \to {}_\epsilon K^h(S\mathcal{R}_i\mathcal{T}_\Sigma)_{n+1}$ yield maps

$${}_\epsilon\mathcal{W}(\mathcal{T}_\Sigma)_n \wedge S^1 = |\,{}_\epsilon K^h(\mathcal{R}_*\mathcal{T}_\Sigma)_n \wedge S^1| \to |\,{}_\epsilon K^h(S\mathcal{R}_*\mathcal{T}_\Sigma)_{n+1}| \to |\,{}_\epsilon K^h(\mathcal{R}_*S\mathcal{T}_\Sigma)_{n+1}| = {}_\epsilon\mathcal{W}(S\mathcal{T}_\Sigma)_{n+1}$$

which are the structure maps of a spectrum

$${}_\epsilon\mathbf{W}(\mathcal{T}_\Sigma) = \{\, {}_\epsilon\mathcal{W}(\mathcal{T}_\Sigma),\, {}_\epsilon\mathcal{W}(S\mathcal{T}_\Sigma)_1,\, {}_\epsilon\mathcal{W}(S^2\mathcal{T}_\Sigma)_2, ...\}.$$

It follows from our Localization Theorem 1.15 that $\Omega\,{}_\epsilon\mathbf{W}$ is an Ω-spectrum which fits into a homotopy fibration of Ω-spectra (actually, ${}_\epsilon\mathbf{K}^h$ is only an Ω-spectrum beyond the 0-th space, in any case, we have a homotopy fibration of spectra)

$$\Omega\,{}_\epsilon\mathbf{W}(\mathcal{T}_\Sigma) \to {}_\epsilon\mathbf{K}^h(A) \to {}_\epsilon\mathbf{K}^h(\Sigma^{-1}A). \tag{7.3}$$

The following proposition shows that under suitable hypothesis, the spectrum ${}_\epsilon\mathbf{W}$ is itself an Ω-spectrum.

7.4 Proposition. *Suppose A satisfies $K_i(A) = 0$ and $K_i(\Sigma^{-1}A) = 0$ for all $i < 0$ and $K_0(A) \to K_0(\Sigma^{-1}A)$ is surjective, e.g., $A = S^nR$, R a regular ring. Then the adjoints ${}_\epsilon\mathcal{W}(S^i\mathcal{T}_\Sigma)_n \to \Omega\,{}_\epsilon\mathcal{W}(S^{i+1}\mathcal{T}_\Sigma)_{n+1}$ of the structure maps for ${}_\epsilon\mathbf{W}$ defined in 7.2 are homotopy equivalences.*

Proof. It suffices to show the case $i = 0$. The general case is obtained from this case by replacing A with S^iA. The functorial construction of the spectra ${}_\epsilon\mathbf{K}^h$ and ${}_\epsilon\mathbf{W}$ applied to 6.3 and 6.11 yield a commutative diagram of spaces

$$\begin{array}{ccccc}
{}_\epsilon K^h(A)_n & \longrightarrow & {}_\epsilon K^h(\Sigma^{-1}A)_n & \longrightarrow & {}_\epsilon\mathcal{W}(\mathcal{T}_\Sigma)_n \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle f} \\
\Omega_\epsilon K^h(SA)_{n+1} & \longrightarrow & \Omega_\epsilon K^h(S\Sigma^{-1}A)_{n+1} & \longrightarrow & \Omega_\epsilon\mathcal{W}(S\mathcal{T}_\Sigma)_{n+1}
\end{array}$$

in which the columns are the adjoints of the structure maps and the rows are homotopy fibrations under our hypothesis (6.12). So Ωf is a homotopy equivalence. As all spaces in the diagram are H-groups, we are left with checking that $\pi_0 f$ is an isomorphism.

Consider the diagram

$$\begin{array}{ccccccccc}
{}_\epsilon K_0^h(A) & \longrightarrow & {}_\epsilon K_0^h(\Sigma^{-1}A) & \longrightarrow & {}_\epsilon W_0(\mathcal{T}_\Sigma) & \longrightarrow & {}_\epsilon W_B^1(A) & \longrightarrow & {}_\epsilon W_B^1(\Sigma^{-1}A) \\
{\scriptstyle \pi_0\alpha_1}\downarrow & & {\scriptstyle \pi_0\alpha_2}\downarrow & & {\scriptstyle \pi_0 f}\downarrow & (+) & {\scriptstyle \beta_1}\uparrow & & {\scriptstyle \beta_2}\uparrow \\
{}_\epsilon K_1^h(SA) & \longrightarrow & {}_\epsilon K_1^h(\Sigma^{-1}SA) & \longrightarrow & \pi_0\Omega_\epsilon\mathcal{W}(S\mathcal{T}_\Sigma) & \longrightarrow & {}_\epsilon K_0^h(SA) & \longrightarrow & {}_\epsilon K_0^h(\Sigma^{-1}SA)
\end{array}$$

where the upper row is the exact sequence of 6.16 and the lower row is the exact sequence of homotopy groups associated to the lower ("unlooped") homotopy fibration of the previous diagram. The maps β_1, β_2 are the isomorphisms of A.5. The maps $\pi_0\alpha_1$, $\pi_0\alpha_2$ are isomorphisms because the first two vertical arrows in the first diagram are homotopy equivalences. We now show commutativity of $(+)$ up to sign which yields the desired isomorphism by the five lemma.

Composing $(+)$ with the inclusion of zero simplices $\mathcal{R}_0 \to \mathcal{R}_*$ and with the hermitian K-theory equivalence $C(\iota_0) \to \mathcal{T}_\Sigma^\oplus$ (6.6) yields a diagram

$$\begin{array}{ccccccc}
{}_\epsilon K_0^h(C(\iota_0)) & \xrightarrow{\sim} & {}_\epsilon K_0^h(\mathcal{T}_\Sigma^\oplus) & \twoheadrightarrow & {}_\epsilon W_0(\mathcal{T}_\Sigma) & \longrightarrow & {}_\epsilon W_B^1(A) \\
\downarrow\cong & & \downarrow\cong & & {\scriptstyle \pi_0 f}\downarrow & (+) & {\scriptstyle \beta_1}\uparrow \\
{}_\epsilon K_1^h(SC(\iota_0)) & \xrightarrow{\sim} & {}_\epsilon K_1^h(S\mathcal{T}_\Sigma^\oplus) & \longrightarrow & \pi_0\Omega_\epsilon\mathcal{W}(S\mathcal{T}_\Sigma) & \longrightarrow & {}_\epsilon K_0^h(SA)
\end{array}$$

whose first two squares are commutative. Since ${}_\epsilon K_0^h(\mathcal{T}_\Sigma^\oplus) \to {}_\epsilon W_0(\mathcal{T}_\Sigma)$ is surjective, it suffices to show that the outer square commutes. By B.3, going first vertically then horizontally yields the map which is induced by the cofiltering map $C(\iota_0) \to SA$. By 6.17, the map ${}_\epsilon K_0^h(\mathcal{T}_\Sigma^\oplus) \to {}_\epsilon W^1(A)$ is the composition of the canonical surjection ${}_\epsilon K_0^h \to {}_\epsilon W_0$ composed with the boundary map associated to the exact sequence of triangulated categories 6.4. Recall (proof of A.5) that β_1 is the composition of the surjection ${}_\epsilon K_0^h \to {}_\epsilon W_0$ and the boundary map associated to $A \to \mathcal{C}A \to SA$. So we are left with checking that the two compositions ${}_\epsilon W_0(C(\iota_0)) \to {}_\epsilon W_0(\mathcal{T}_\Sigma^{\oplus}) \xrightarrow{\delta} {}_\epsilon W^1(A)$ and ${}_\epsilon W_0(C(\iota_0)) \to {}_\epsilon W_0(SA) \xrightarrow{\delta} {}_\epsilon W^1(A)$ coincide up to a sign. But this is 4.22 for the inclusion of triangulated categories with duality $D_b(P(A)) \subset D_b(G_0), D_b(\mathcal{C}P(A)) \subset D_b(C(\iota_0))$. □

7.5 Corollary. *Let R be a ring with trivial negative K-groups and let A, Σ be as in 7.4. Then for $n \leq 0$, there are isomorphisms $\pi_n({}_\epsilon\mathbf{W}(R)) \cong {}_\epsilon W_B^{-n}(P(R))$ and $\pi_n({}_\epsilon\mathbf{W}(\mathcal{T}_\Sigma)) \cong {}_\epsilon W_B^{-n}(\mathcal{T}_\Sigma)$ In particular, the map $\pi_n({}_\epsilon\mathbf{W}(A/f)) \to \pi_n({}_\epsilon\mathbf{W}(\mathcal{T}_\Sigma))$ is isomorphic to the map ${}_\epsilon W_B^{-n}(A/f) \to {}_\epsilon W_B^{-n}(\mathcal{T}_\Sigma)$.*

Proof. Let $\mathcal{E}$ be $P(R)$ or $\mathcal{T}_\Sigma$. Then the sequence $S^n\mathcal{E} \to CS^n\mathcal{E} \to S^{n+1}\mathcal{E}$ induces homotopy fibrations of $\mathbf{W}$-spectra with contractible total space. For rings, this is 3.16 and 1.12. For $\mathcal{E} = \mathcal{T}_\Sigma$, this follows from the ring case and 7.3. Thus $\pi_n\,{}_\epsilon\mathbf{W}(A/f) \to \pi_n\,{}_\epsilon\mathbf{W}(\mathcal{T}_\Sigma)$ is isomorphic to $\pi_0\,{}_\epsilon\mathbf{W}(S^{-n}A/f) \to \pi_0\,{}_\epsilon\mathbf{W}(S^{-n}\mathcal{T}_\Sigma)$ which is ${}_\epsilon W_0(S^{-n}A/f) \to {}_\epsilon W_0(S^{-n}\mathcal{T}_\Sigma)$ by 1.14 and 7.4.

The sequence also induces exact sequences $D_b(S^n\mathcal{E}\text{–proj}) \to D_b(CS^n\mathcal{E}\text{–proj}) \to D_b(S^{n+1}\mathcal{E}\text{–proj})$ of triangulated categories. For rings, this is 4.11 together with the fact that $D_b(CS^n\mathcal{E}\text{–proj}) \to D_b(S^{n+1}\mathcal{E}\text{–proj})$ is a localization as both categories have $K_0 = 0$ since R has trivial negative K-groups. For $\mathcal{E} = \mathcal{T}_\Sigma$, exactness up to direct factors follows from 4.16, 4.10. Exactness follows from the fact that negative K-groups of $\mathcal{T}_\Sigma$ is trivial since A is regular. By 4.20, the sequence induces long exact sequences of triangular Witt groups where all terms ${}_\epsilon W_B^* C\mathcal{E} = 0$. Thus ${}_\epsilon W_B^{-n}(A/f) \to {}_\epsilon W_B^{-n}(\mathcal{T}_\Sigma)$ is isomorphic to ${}_\epsilon W_B^0(S^{-n}A/f) \to {}_\epsilon W_B^0(S^{-n}\mathcal{T}_\Sigma)$ which is ${}_\epsilon W_0(S^{-n}A/f) \to {}_\epsilon W_0(S^{-n}\mathcal{T}_\Sigma)$ (4.19). □

Proof of Dévissage. Let $\mathcal{E}$ be $\mathcal{T}_\Sigma$ or $P(R)$ with R a regular ring. Recall that the rings C (2.10), S (2.12), U (5.10) and V (5.5) are flat $\mathbb{Z}$-algebras, and that all maps in the digrams (5.2) and (5.7) are flat. So the categories with dualities $C\mathcal{E}$, $S\mathcal{E}$, $U\mathcal{E}$, $V\mathcal{E}$ and various maps between them are defined (7.1). The diagrams (5.2) and (5.7) induce homotopy fibrations of spectra

$$ {}_\epsilon\mathbf{W}(\mathcal{E}) \to {}_\epsilon\mathbf{W}(\mathcal{E} \times \mathcal{E}^{op}) \to {}_\epsilon\mathbf{W}(V\mathcal{E}), $$

$$ {}_\epsilon\mathbf{W}(\mathcal{E} \times \mathcal{E}^{op}) \to {}_\epsilon\mathbf{W}(\mathcal{E}) \to {}_\epsilon\mathbf{W}(U\mathcal{E}). $$

For rings, this follows from 1.12, and the corresponding statements about hermitian K-theory. For $\mathcal{E} = \mathcal{T}_\Sigma$, this follows from the ring case and the localization homotopy fibration of spectra (7.3). Recall also that ${}_\epsilon\mathbf{W}(\mathcal{E} \times \mathcal{E}^{op})$ is a delooping of the K-theory spectrum of $\mathcal{E}$ as $i\,{}_\epsilon(\mathcal{E} \times \mathcal{E})_h$ is equivalent to $i\mathcal{E}$.

As in 5.13, tensoring with the hermitian modules (P_i, λ_i) of 5.13 induces maps of categories with dualities and thus maps f_i of $\mathbf{W}$-spectra. We have a commutative diagram of spectra

$$ \begin{array}{ccc} {}_\epsilon\mathbf{W}(U(A/f)) & \longrightarrow & {}_\epsilon\mathbf{W}(U\mathcal{T}_\Sigma) \\ \Big\downarrow{\scriptstyle f_1 - f_2} & & \Big\downarrow{\scriptstyle f_1 - f_2} \\ {}_\epsilon\mathbf{W}(V^2U(A/f)) & \longrightarrow & {}_\epsilon\mathbf{W}(V^2U\mathcal{T}_\Sigma) \end{array} $$

with vertical maps homotopy equivalences. For A/f this is Karoubi's fundamental theorem 5.14, and for $\mathcal{T}_\Sigma$, this follows from Karoubi's fundamental theorem for the rings UA and $U\Sigma^{-1}A$ and the homotopy fibration of spectra (7.3) applied to the localization $UA \to U\Sigma^{-1}A$. By the arguments of 5.16 which also apply to $\mathcal{T}_\Sigma$, there is a natural homotopy equivalence between the functors $\otimes S$ and $\otimes U \otimes V$. Moreover, $\Omega\,{}_\epsilon\mathbf{W}(S\mathcal{E}) \simeq {}_\epsilon\mathbf{W}(\mathcal{E})$.

By Corollary 7.5, Lemma 7.6 below and ${}_\epsilon W_B^n = {}_{-\epsilon}W_B^{n+2}$, the map ${}_\epsilon\mathbf{W}(A/f) \to {}_\epsilon\mathbf{W}(\mathcal{T}_\Sigma)$ is an isomorphism on π_n, $n \leq 0$. Moreover, $A/f \to \mathcal{T}_\Sigma$ is a K-theory equivalence by Quillen's *dévissage* theorem [Qui73]. Karoubi's induction principle 5.20 a) applies now to the functor $A/f \to \mathcal{T}_\Sigma$ with ${}_\epsilon\mathbf{K}^h$ replaced by ${}_\epsilon\mathbf{W}$. This yields the homotopy equivalence ${}_\epsilon\mathbf{W}(A/f) \to {}_\epsilon\mathbf{W}(\mathcal{T}_\Sigma)$. □

7.6 Lemma. *The map $j : A/f \to \mathcal{T}_\Sigma$ of 1.18 induces isomorphisms of triangular Witt groups for $n \in \mathbb{Z}$*

$$ W_B^n(A/f) \to W_B^n(\mathcal{T}_\Sigma). $$

Proof. We just treat the case $n = 0$, the general case is similar composing everything with T^n and changing signs following [Bal00]. The commutative square of triangulated categories

$$\begin{array}{ccc} D_b(P(A/f)) & \xrightarrow{j} & D_b(\mathcal{T}_\Sigma) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \iota} \\ D_b(A/f - mod) & \xrightarrow{\alpha} & D_b^\Sigma(A - mod) \end{array}$$

with i and ι the obvious inclusions extends to a commutative diagram of triangulated categories with dualities

$$\begin{array}{ccc} (D_b(P(A/f)), Ext^1, 1, \eta) & \xrightarrow{(j,id)} & (D_b(\mathcal{T}_\Sigma), Ext^1, 1, \eta) \\ \downarrow{\scriptstyle (id,c)} & & \\ (D_b(P(A/f)), \sharp/f, 1, ev) & & \downarrow{\scriptstyle (\iota,\rho)} \\ \downarrow{\scriptstyle (i,id)} & & \\ (D_b(A/f - mod), \sharp/f, 1, ev) & \xrightarrow{(\alpha,\tau)} & (D_b^\Sigma(A - mod), T\sharp, -1, -ev) \end{array}$$

in the sense of [Gil02, Definition 2.6]. Here we define ρ and τ using the same sign conventions as for η in [Gil02, p.129], compare also 6.14. See *e.g.* [Kar74] (and 1.7) for the isomorphism c between Ext^1 and $\sharp/f = hom_{A/f}(\ \ , A/f)$ which already exists on the category $P(A/f)$, so no signs appear. The functors i and ι induce isomorphisms on W by resolution and [Gil02, Theorem 2.7], and α induces an isomorphism on Witt groups by *dévissage* [Gil02, Theorem 4.1]. Hence j also induces an isomorphism on W_B^n as claimed. □

Appendix A. Cofinality for Triangular Witt groups

A.1. For what follows the reader is advised to have a copy of [Bal00] at hand as we will make frequent use of its terminologies and results. Let $(\mathcal{B}, \sharp, \omega, \delta)$ be a triangulated category with δ-duality. Let $\mathcal{A} \subset \mathcal{B}$ be a full triangulated subcategory invariant under the duality functor. Suppose $\mathcal{A}$ is *cofinal* in $\mathcal{B}$, *i.e.,* every object of $\mathcal{B}$ is a direct factor of an object of $\mathcal{A}$. Let $K(\mathcal{B}, \mathcal{A})$ be the monoid of isomorphism classes of objects of $\mathcal{B}$ under direct sum operation modulo the monoid of isomorphism classes of objects of $\mathcal{A}$. The identity map on generators yields an isomorphism $K(\mathcal{B}, \mathcal{A}) \to K_0(\mathcal{B})/K_0(\mathcal{A})$. Moreover, an object B of $\mathcal{B}$ yields the trivial class $0 = [B] \in K(\mathcal{B}, \mathcal{A})$ iff B is isomorphic to an object of $\mathcal{A}$.

The duality functor $\sharp$ induces a $\mathbb{Z}/2\mathbb{Z}$-action on $K(\mathcal{B}, \mathcal{A})$. Write σ for the generator of $\mathbb{Z}/2\mathbb{Z}$. Let $\hat{H}^i(\mathbb{Z}/2\mathbb{Z}, K(\mathcal{B}, \mathcal{A}))$ be the i-th Tate cohomology group of $\mathbb{Z}/2\mathbb{Z}$ with coefficients in $K(\mathcal{B}, \mathcal{A})$. It is the i-th cohomology group of the complex

$$\cdots \xrightarrow{1-\sigma} K(\mathcal{B}, \mathcal{A}) \xrightarrow{1+\sigma} K(\mathcal{B}, \mathcal{A}) \xrightarrow{1-\sigma} K(\mathcal{B}, \mathcal{A}) \xrightarrow{1+\sigma} \cdots$$

where the middle term is placed in cohomological degree zero.

A.2 Theorem. *If $\mathcal{B}$ is a triangulated category with δ-duality such that $\frac{1}{2} \in \mathcal{B}$ and if $\mathcal{A}$ a cofinal full triangulated subcategory of $\mathcal{B}$ invariant under the duality, then there is a natural (12-term periodic) long exact sequence*

$$\cdots \longrightarrow W_B^i(\mathcal{A}) \longrightarrow W_B^i(\mathcal{B}) \longrightarrow \hat{H}^i(\mathbb{Z}/2\mathbb{Z}, K(\mathcal{B}, \mathcal{A})) \longrightarrow W_B^{i+1}(\mathcal{A}) \longrightarrow W_B^{i+1}(\mathcal{B}) \longrightarrow \cdots$$

Proof. Write $*$ for the $(-\delta)$-duality $T \circ \sharp$ and $\eta = -\delta\omega$, then $(*, \eta) = T(\sharp, \omega)$ [Bal00, 2.8. Definition]. By definition of triangular Witt groups [Bal00, 2.13. Definition] it suffices to construct a natural exact sequence

$$(1) \qquad W(\mathcal{A},\sharp,\omega) \xrightarrow{i} W(\mathcal{B},\sharp,\omega) \xrightarrow{\alpha} \tfrac{\ker(1-\sigma)}{\mathrm{Im}(1+\sigma)} \xrightarrow{\beta} W(\mathcal{A},*,\eta) \xrightarrow{j} W(\mathcal{B},*,\eta).$$

The maps i, j in the sequence are induced by the inclusion $\mathcal{A} \subset \mathcal{B}$.

The map α is defined as follows. Let (B,φ) be a symmetric space in $(\mathcal{B},\sharp,\omega)$, then $[B] \in \ker(1-\sigma)$ as B is isomorphic to $B^\sharp$ via φ. If (B,φ) is neutral, *i.e.,* the cone of some symmetric map $u : C \to T^{-1}(C^\sharp)$, then $[B] = [TC]+[(TC)^\sharp] \in K_0(\mathcal{B})$ as $[TC] = -[C]$ and $(TC)^\sharp \simeq T^{-1}(C^\sharp)$. It follows that $[B] \in \mathrm{Im}(1+\sigma)$ and the map α with $\alpha([B,\varphi]) = [B]$ is well-defined.

The map β is defined as follows. Let $[B] \in \ker(1-\sigma)$, then $B \oplus T(B^\sharp) = B \oplus B^*$ is an object of $\mathcal{A}$. Let $H(B)$ be the hyperbolic form associated with B and the $(-\delta)$-duality $(*,\eta)$, *i.e.,*

$$H(B) = \left(B \oplus B^*, \begin{pmatrix} 0 & 1 \\ \eta_B & 0 \end{pmatrix}\right).$$

Then $H(B)$ is a symmetric space in $(\mathcal{A},*,\eta)$. Suppose $[B] \in \mathrm{Im}(1+\sigma)$, *i.e.,* $[B] = [C \oplus C^\sharp] = -[TC] - [T^{-1}C^\sharp]$ in $K(\mathcal{B},\mathcal{A})$ for some C in $\mathcal{B}$. So $[B \oplus TC \oplus T^{-1}C^\sharp] = 0$, hence $B \oplus TC \oplus T^{-1}C^\sharp$ is an object of $\mathcal{A}$. Now $H(B)$ is the cone of the $(\sharp,\omega)$-symmetric morphism [Bal00, 2.10 Definition] in A

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & T\omega_C & 0 \end{pmatrix} : B^\sharp \oplus TC \oplus T^{-1}C^\sharp \to B^{\sharp\sharp} \oplus T^{-1}C^\sharp \oplus TC^{\sharp\sharp}.$$

Therefore, $H(C)$ is neutral in the triangulated category $\mathcal{A}$ with $-\delta$ duality $(*,\eta)$ and the map β given by $\beta([B]) := [H(B)]$ is well-defined.

We need to show exactness of the sequence (1). First, the composition of consecutive maps is zero. This is obvious for $\alpha \circ i$ and $j \circ \beta$. For $\beta \circ \alpha$, let (B,ϕ) be a symmetric space for $(\mathcal{B},\sharp,\omega)$, then $\beta \circ \alpha(B,\phi) = [H(B)]$. But $H(B)$ is the cone on the symmetric morphism in $\mathcal{A}$

$$\begin{pmatrix} (\phi^\sharp)^{-1} & 0 \\ 0 & 0 \end{pmatrix} : B^\sharp \oplus T^{-1}B \to B^{\sharp\sharp} \oplus TB^\sharp.$$

Consequently, $H(B)$ is neutral in $(\mathcal{A},*,\eta)$ and so $\beta \circ \alpha = 0$.

Exactness at $W(\mathcal{B},\sharp,\omega)$. Let (B,ϕ) be a symmetric space for $(\mathcal{B},\sharp,\omega)$ and suppose $\alpha([B,\phi]) = [B] = 0$. Then $[B] = [C \oplus C^\sharp]$ in $K(\mathcal{B},\mathcal{A})$, hence $B \oplus TC \oplus (TC)^\sharp$ is in $\mathcal{A}$ as its class is zero in $K(\mathcal{B},\mathcal{A})$. Write $L(C)$ for the neutral symmetric space in $(\mathcal{B},\sharp,\omega)$

$$L(C) = \left(C \oplus C^\sharp, \begin{pmatrix} 0 & 1 \\ \omega_C & 0 \end{pmatrix}\right).$$

Then $(B,\phi) \oplus L(TC)$ is a symmetric space in $(\mathcal{A},\sharp,\omega)$ which is equivalent to (B,ϕ) in $W(\mathcal{B},\sharp,\omega)$.

Exactness at the middle term. Let B be an object of $\mathcal{B}$ with $[B] = [B^\sharp]$ in $K(\mathcal{B},\mathcal{A})$ such that $\beta(B) = H(B)$ is Witt-trivial, hence neutral in $(\mathcal{A},\sharp,\eta)$ [Bal00, 3.5. Theorem]. This means that there is a symmetric morphism $u : A \to A^\sharp$ in $(\mathcal{A},\sharp)$ $(u = u^\sharp \circ \omega_A)$ such that $H(B)$ is the symmetric cone on u [Bal00, 2.12. Definition, 1.6. Theorem]. Eliminating redundant variables, this means that there is an exact triangle in $\mathcal{A}$

$$A \xrightarrow{u} A^\sharp \xrightarrow{\begin{pmatrix} \delta\omega_B^{-1}b^\sharp \\ -Ta^\sharp \end{pmatrix}} B \oplus TB^\sharp \xrightarrow{(a \;\; Tb)} TA.$$

Let C be the cone of $\delta T^{-1}(\omega_B^{-1}b^\sharp)$. Apply the octahedron axiom TR4 ([Bal00, section 1]) to the composition

$$(1 \;\; 0) \circ \begin{pmatrix} \delta T^{-1}(\omega_B^{-1}b^\sharp) \\ -a^\sharp \end{pmatrix} = \delta T^{-1}(\omega_B^{-1}b^\sharp)$$

to find maps $g_1 : A \to C$ and $g_2 : C \to A^\sharp$ such that $g_2 g_1 = u$ and a map of exact triangles $(T^{-1}a, id_C, Ta^\sharp, a) : (g_1 T^{-1}a, -g_2, \delta\omega_B^{-1}b^\sharp) \to (g_1, -(Ta^\sharp)g_2, Tb)$. As $\sharp$ is δ-exact ([Bal00, 2.2. Definition]), the triangle $(-g_2^\sharp\omega_A(T^{-1}a), g_1^\sharp, \delta\omega_B^{-1}b^\sharp)$ is exact as well (after identification of $B^{\sharp\sharp}$ with B via ω_B).

By TR3, the partial map of exact triangles $(1, ?, 1, 1) : (g_1(T^{-1}a), -g_2, \delta\omega_B^{-1}b^\sharp) \to (-g_2^\sharp\omega_A(T^{-1}a), g_1^\sharp, \delta\omega_B^{-1}b^\sharp)$ can be completed to a map of triangles $(1, \phi, 1, 1)$ with ϕ necessarily an isomorphism (5 lemma). Putting these two maps of exact triangles together we obtain a commutative diagram in which the rows are exact triangles

$$\begin{array}{ccccccc}
T^{-1}B & \xrightarrow{-g_2^\sharp\omega_A(T^{-1}a)} & C^\sharp & \xrightarrow{g_1^\sharp} & A^\sharp & \xrightarrow{\delta\omega_B^{-1}b^\sharp} & B \\
{\scriptstyle T^{-1}a}\downarrow & & \uparrow{\scriptstyle \phi} & & \downarrow{\scriptstyle Ta^\sharp} & & \downarrow{\scriptstyle a} \\
A & \xrightarrow[g_1]{} & C & \xrightarrow[-(Ta^\sharp)g_2]{} & TB^\sharp & \xrightarrow[Tb]{} & TA.
\end{array}$$

One checks that $\phi^\sharp\omega_C$ can replace ϕ in the above commutative diagram, then $\varphi := \frac{1}{2}(\phi + \phi^\sharp\omega_C)$ can replace ϕ as well. We have $g_1^\sharp\phi g_1 = -g_2 g_1 = -u$, hence $g_1^\sharp(\phi^\sharp\omega_C)g_1 = -u$ (use $u = u^\sharp\omega_A$) and so $g_1^\sharp\phi g_1 = g_1^\sharp\varphi g_1 = -u$. Since ϕ is an isomorphism, so is φ ([Bal00, 4.6. Lemma]). By construction, $\varphi = \varphi^\sharp\omega_C$. It follows that (C, φ) is a symmetric space in $(\mathcal{B}, \sharp, \omega)$ and thus defines an element in $W(\mathcal{B}, \sharp, \omega)$. We have $\alpha(-[C, \varphi]) = -[C] = -[T(B^\sharp)] = [B^\sharp] = [B]$ in $K(\mathcal{B}, \mathcal{A})$. This shows exactness at the middle term.

Exactness at $W(\mathcal{A}, *, \eta)$. Let (A, ϕ) be a symmetric space for $(\mathcal{A}, *, \eta)$ which is Witt-trivial, hence neutral in $(\mathcal{B}, *, \eta)$, *i.e.*, , (A, ϕ) is the cone on some $(\sharp, \omega)$-symmetric morphism $u : B \to B^\sharp$. It follows that $[B] = [B^\sharp]$ in $K(\mathcal{B}, \mathcal{A})$, in particular, $H(B)$ is in $\mathcal{A}$. Since $H(B)$ is the cone on the symmetric morphism $0 : T^{-1}B \to TB^\sharp$, we see that $H(B) \oplus (A, \phi)$ is the cone on the symmetric morphism $0 \oplus u : T^{-1}B \oplus B \to TB^\sharp \oplus B^\sharp$ which is a morphism in $\mathcal{A}$. Hence, $H(B) \oplus (A, \phi)$ is neutral in $(\mathcal{A}, *, \eta)$, and so $[A, \phi] = -[H(B)] = -\beta([B]) = \beta(-[B])$ is in the image of β. □

A.3 Corollary. *Let $\mathcal{A} \to \mathcal{B}$ be a map of idempotent complete additive categories with duality. Suppose that $\frac{1}{2} \in \mathcal{A}$, $\mathcal{B}$ and that the map induces isomorphisms $W_B^*(\mathcal{A}) \to W_B^*(\mathcal{B})$, $* \in \mathbb{Z}$ and $K_i(\mathcal{A}) \to K_i(\mathcal{B})$, $i \leq 0$. Then it induces isomorphisms $W_i^K(\mathcal{A}) \to W_i^K(\mathcal{A})$, $i \leq 0$.*

Proof. By induction on n we show that $W_B^*(S^n\mathcal{A}-\text{proj}) \to W_B^*(S^n\mathcal{B}-\text{proj})$ are isomorphisms. The case $n = 0$ is the hypothesis on Balmer Witt groups. The induction step follows from the natural exact Cofinality sequence A.2 applied to the idempotent completion $S\mathcal{C}-\text{free} \to S\mathcal{C}-\text{proj}$, the isomorphism $W_B^*(S^n\mathcal{C}-\text{free}) \cong W_B^{*+1}(S^{n+1}\mathcal{C})$ (4.13, 4.20) for $\mathcal{C} = \mathcal{A}, \mathcal{B}$ and the five lemma. □

A.4 Lemma. *For any ring A in which 2 is invertible, there is a natural isomorphism*

$$W_n^K(A) \otimes \mathbb{Z}[1/2] \cong W_B^{-n}(A) \otimes \mathbb{Z}[1/2].$$

Proof. As Tate cohomology of $\mathbb{Z}/2\mathbb{Z}$ is 2-torsion, a consequence of the Cofinality Theorem A.2 is the isomorphism $W_B^i(\mathcal{A}) \otimes \mathbb{Z}[1/2] \overset{\sim}{\to} W_B^i(\widetilde{\mathcal{A}}) \otimes \mathbb{Z}[1/2]$ for any triangulated category with duality. By Balmer localization 4.20 applied to 4.13 we thus have isomorphisms $W_B^{*+1}(S^{n-1}A-\text{proj}) \otimes \mathbb{Z}[1/2] \cong W_B^*(S^nA-\text{free}) \otimes \mathbb{Z}[1/2] \cong W_B^*(S^nA-\text{proj}) \otimes \mathbb{Z}[1/2]$, $* \in \mathbb{N}$. Since by definition $W_{-i}^K(A) = W_B^0(S^iA-\text{proj})$, $i \geq 0$, induction on n starting with $n = 0$, proves the result for $n \leq 0$. For $n > 0$ we use the fact that W_B^* and $W_*^K \otimes \mathbb{Z}[1/2]$ are periodic of period 4 [Bal00], [Kar80]. □

The following result identifies Balmer's Witt groups with Karoubi's negative Witt groups for regular rings (not only up to 2-torsion):

A.5 Lemma. *Let $N \leq 0$ be an integer. Let $\mathcal{A}$ be an idempotent complete additive category with duality in which 2 is invertible such that $K_n(\mathcal{A}) = 0$ for all $N \leq n < 0$. Then $W_n^K(\mathcal{A}) \cong W_B^{-n}(\mathcal{A})$ for all $N \leq n \leq 0$. In particular, there is an isomorphism $K_n^h(\mathcal{A}) \overset{\sim}{\to} W_B^{-n}(\mathcal{A})$, $N \leq n < 0$ which is natural for additive categories satisfying the above condition.*

Proof. The proof is the same as the proof of A.4 without tensoring with $\mathbb{Z}[\frac{1}{2}]$ since $W^*_B(S^nA\text{−free}) \cong W^*_B(S^nA\text{−proj})$ for $-N \geq n \geq 0$. This is because of the assumption on negative K-groups. □

A.6 *Remark.* If one is interested only in regular rings, then it is possible to prove Corollary 6.6 applying this comparison lemma and a result of the second author on the vanishing of K_{-1} of abelian categories instead of cofinality for triangular Witt groups. This approach was discussed in preliminary versions of this article.

Appendix B. The non-connective hermitian K-theory spectrum

This appendix is devoted to the construction of functorial versions of non-connective hermitian K-theory spectra announced in definition 3.14. This construction uses Γ- spaces to produce strict deloopings. A similar construction of deloopings for ordinary algebraic K-theory can be found in [Jar97, section 5.1].

B.1. *The -1-connected hermitian K-theory spectrum and pairings.* Let Γ^{op} be the skeletal category of finite pointed sets. The object with n non-base points is denoted by n_+. The elements of n_+ are labeled 0 through n with 0 as base point. The category Γ^{op} is symmetric strict monoidal under the smash product of pointed sets $\wedge : \Gamma^{op} \times \Gamma^{op} \to \Gamma^{op}$ sending (m_+, n_+) to $m_+ \wedge n_+$ where $m_+ \wedge n_+$ is identified with $(mn)_+$ by ordering its elements $i \wedge j$ lexicographically. Recall that a Γ-object in a category is a covariant functor from Γ^{op} to that category.

Let $\mathcal{A}$ be in *Pad*. We construct Γ-objects $\mathcal{A}_k$, $k \in \mathbb{N}$, in *Pad* associated with $\mathcal{A}$ as follows. For $n \in \mathbb{N}$, the category $\mathcal{A}_k(n_+)$ has objects pairs (A, σ) with $A : \mathbb{N}^k \to Ob\mathcal{A}$ and $\sigma : \mathbb{N}^k \to n_+$ maps of sets such that for all but finitely many $i \in \mathbb{N}^k$ we have $\sigma(i) = 0$ and such that $A_i = 0$ whenever $\sigma(i) = 0$. A map $\alpha : (A, \sigma) \to (B, \rho)$ is a collection of maps $\alpha_{j,i} : A_i \to B_j$ in $\mathcal{A}$, $i, j \in \mathbb{N}^k$, such that $\alpha_{j,i} = 0$ whenever $\sigma(i) \neq \rho(j)$. We identify the objects $(0, \sigma)$ with the base point of $\mathcal{A}_k(n_+)$. For a map $\theta : n_+ \to m_+$ in Γ^{op}, the map $\mathcal{A}_k(\theta)$ sends (A, σ) to $(\theta A, \theta \circ \sigma)$ with $(\theta A)_i = 0$ for $\theta \circ \sigma(i) = 0$ and $(\theta A)_i = A_i$ for $\theta \circ \sigma(i) \neq 0$. It sends a map α to $\theta(\alpha)$ with $\theta(\alpha)_{j,i} = 0$ whenever $\theta \circ \sigma(i) = 0$ or $\theta \circ \sigma(j) = 0$ and $\theta(\alpha)_{j,i} = \alpha_{j,i}$ otherwise. Notice that a duality $\sharp$ on $\mathcal{A}$ induces a duality $\sharp$ on $\mathcal{A}_k$ by declaring $(A, \sigma)^\sharp = (A^\sharp, \sigma)$ and $(\alpha^\sharp)_{i,j} = (\alpha_{j,i})^\sharp$. This makes $\mathcal{A}_k$ into a Γ-object in *Pad* functorial in $\mathcal{A}$. Taking idempotent completions, hermitian objects, associated isomorphism categories and nerves yields a Γ-object in the category of pointed simplicial sets

$$B^k_{\mathcal{A}} = N_* i(P(\mathcal{A}_k))_h.$$

Given a Γ-object F in the category of pointed simplicial sets and given a *finite* pointed simplicial set K (*i.e.,* a functor $\Delta^{op} \to \Gamma^{op}$), the pointed simplicial set $F(K)$ is the diagonal of the bisimplicial set $m, n \mapsto F(K_m)_n$. For two finite pointed simplicial sets K, L, there is a map $F(K) \wedge L \to F(K \wedge L)$ which sends an n-simplex $x \wedge l$ to $F(\ \wedge l)(x)$ where $\wedge l$ is the map $X_n \to X_n \wedge L_n : x \mapsto x \wedge l$. When K, L run through the spheres $S^n = (S^1)^{\wedge n}$, these maps become the structure maps of a spectrum $\{F(S^0), F(S^1), F(S^2), ...\}$. If F is a special Γ-space, *i.e.,* the maps $F(n_+) \to F(1_+)^n$, induced by the various projections $n_+ \to 1_+$ sending all but one element to the base point, are homotopy equivalences, then the spectrum is an Ω-spectrum except possibly at $F(S^0)$, *i.e.,* the adjoints $|F(S^n)| \to \Omega|F(S^{n+1})|$ of the topological realizations of the structure maps are homotopy equivalences for $n > 0$ [BF78, Theorem 4.4]. The spectra obtained from special Γ spaces are -1-connected.

The Γ-space $B^k_{\mathcal{A}}$ is special for $k > 0$ as the map $\mathcal{A}_k(n_+) \to \mathcal{A}_k(1_+)^n$ is an equivalence of categories with duality for each n. The associated spectrum is therefore an Ω-spectrum beyond the 0-th space. In our case, the first structure map $|F(S^0)| \to \Omega|F(S^1)|$ of the associated spectrum is a group completion [May75, section 15] as $F(S^1)$ can be identified with the Bar construction on $F(S^0)$. Since $\mathcal{A}_k(1_+)$ is equivalent to $\mathcal{A}$−free as category with duality, the spectrum $\mathbf{k}^h(\mathcal{A})_k$ given by the Γ-space $B^k_{\mathcal{A}}$ represents -1-connected hermitian K-theory, $k > 0$.

For $k = 0$ we obtain the suspension spectrum of $|iP(\mathcal{A})_h|$ as $\mathcal{A}_0(n_+) = \mathcal{A} \vee \mathcal{A} \vee ...\mathcal{A}$ (n-times) and thus $B^0_{\mathcal{A}}(n_+) = N_* iP(\mathcal{A})_h \wedge n_+$.

Given $\mathcal{A}, \mathcal{B} \in Pad$, there are maps of categories

$$\mathcal{A}_k(m_+) \times \mathcal{B}_l(n_+) \longrightarrow (\mathcal{A} \otimes \mathcal{B})_{k+l}(m_+ \wedge n_+)$$

$$(A, \sigma), (B, \rho) \longmapsto (A \wedge B, \sigma \wedge \rho)$$

where $(A \wedge B)_{i,j} = A_i \wedge B_j$. On morphisms the maps are given by the ordinary tensor product of matrices. These maps induce maps,

$$B^i_{\mathcal{A}}(m_+) \wedge B^j_{\mathcal{B}}(n_+) \to B^{i+j}_{\mathcal{A} \otimes B}(m_+ \wedge n_+) \quad \text{functorially in } m_+,\ n_+, \text{ and thus}$$

$$B^i_{\mathcal{A}}(K) \wedge B^j_{\mathcal{B}}(L) \to B^{i+j}_{\mathcal{A} \otimes B}(K \wedge L) \quad \text{functorially in } K \text{ and } L.$$

Writing $K^h(\mathcal{A})_i$ for $\Omega^i|B^i_{\mathcal{A}}(S^i)|$, the maps induce pairing maps

$$K^h(\mathcal{A})_i \wedge K^h(\mathcal{B})_j \to K^h(\mathcal{A} \otimes B)_{i+j}$$

functorially in $\mathcal{A}$ and $\mathcal{B}$. For $i > 0$, the space $K^h(\mathcal{A})_i$ has the homotopy type of the hermitian K-theory space of $\mathcal{A}$.

More precisely, the object $\mathbb{Z}$ of $P(\mathbb{Z})$ equipped with the trivial form $id : \mathbb{Z} \to \mathbb{Z}^\sharp = \mathbb{Z}$ defines a point in $B^1_{\mathbb{Z}}(S^0) = N_* iP(\mathbb{Z})_h$, thus a map $S^0 \to B^1_{\mathbb{Z}}(S^0) \to \Omega B^1_{\mathbb{Z}}(S^1) = K^h(\mathbb{Z})_1$. Smashing with $K(\mathcal{A})_n$ and composing with the pairing map yields a map $c_n : K^h(\mathcal{A})_n = K^h(\mathcal{A})_n \wedge S^0 \to K^h(\mathcal{A})_n \wedge K^h(\mathbb{Z})_1 \to K^h(\mathcal{A} \otimes \mathbb{Z})_{n+1} = K^h(\mathcal{A})_{n+1}$ which is a homotopy equivalence for $n > 0$ and a group completion for $n = 0$ (exercise).

B.2. *The non-connective hermitian K-theory spectrum* [Kar80]. Recall that $\mathcal{C}$ denotes the cone category (2.10). The object (c, σ) of $\mathcal{C}_1$ with $\sigma(i) = 0$ for $i \neq 1$ and $\sigma(1) = 1$, $c_1 = \mathrm{Im}(p)$ and hermitian form $id : (C, p) \to (C, p)^\sharp = (C, p^\sharp = p)$ defines a zero simplex of $B^1_{\mathcal{C}}(S^0)$. It induces a map $\eta : S^0 \to |B^1_{\mathcal{C}}(S^0)| \to \Omega|B^1_{\mathcal{C}}(S^1)| = K^h(\mathcal{C})_1$. Choose a contraction $h : K^h(\mathcal{C})_1 \wedge I \to K^h(\mathcal{C})_1$, $h_1 = id$, $h_0 = *$. This is possible since the hermitian K-theory of $\mathcal{C}$ is trivial by the Eilenberg swindle. Define $\rho : S^1 \to K^h(S)_1$ by the commutativity of the diagram

$$\begin{array}{ccccc}
I & \longrightarrow & I/S^0 = S^1 & \xrightarrow{\ \rho\ } & K^h(S)_1 \\
{\scriptstyle \eta \wedge id}\downarrow & & & & \uparrow \\
K^h(\mathcal{C})_1 \wedge I & & \xrightarrow{\qquad h \qquad} & & K^h(\mathcal{C})_1
\end{array}$$

Smashing with $K^h(\mathcal{A})_n$ yields a commutative diagram

$$\begin{array}{ccccccc}
K^h(\mathcal{A})_n \wedge I & \longrightarrow & K^h(\mathcal{A})_n \wedge S^1 & \xrightarrow{1 \wedge \rho} & K^h(\mathcal{A})_n \wedge K^h(S)_1 & \longrightarrow & K^h(S\mathcal{A})_{n+1} \\
\downarrow & & & & \uparrow & & \\
K^h(\mathcal{A})_n \wedge K^h(\mathcal{C})_1 \wedge I & & \xrightarrow{\qquad 1 \wedge 1 \qquad} & & K^h(\mathcal{A})_n \wedge K^h(\mathcal{C})_1. & &
\end{array}$$

Write R for the map $K^h(\mathcal{A})_n \wedge S^1 \to K^h(S\mathcal{A})_{n+1}$ and H for the map $K^h(\mathcal{A})_n \wedge K^h(\mathcal{C})_1 \wedge I \to K^h(S\mathcal{A})_{n+1}$.

In the commutative diagram

$$\begin{array}{ccccc}
K^h(\mathcal{A})_n \wedge S^0 & \xrightarrow{1 \wedge \eta} & K^h(\mathcal{A})_n \wedge K^h(\mathcal{C})_1 & \longrightarrow & K^h(S\mathcal{A})_{n+1} \\
{\scriptstyle c_n}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\
K^h(\mathcal{A})_{n+1} & \longrightarrow & K^h(\mathcal{C}\mathcal{A})_{n+1} & \longrightarrow & K^h(S\mathcal{A})_{n+1}
\end{array}$$

the lower row is a homotopy fibration (3.6). Since the vertical maps are all homotopy equivalences (the middle one because both spaces are contractible) the top row is a also a homotopy fibration.

The map of homotopy fibrations

$$\begin{array}{ccccc} K^h(\mathcal{A})_n \wedge S^0 & \xrightarrow{1\wedge\eta} & K^h(\mathcal{A})_n \wedge K^h(\mathcal{C})_1 & \longrightarrow & K^h(S\mathcal{A})_{n+1} \\ \downarrow{\scriptstyle adj(R)} & & \downarrow{\scriptstyle adj(H)} & & \downarrow{\scriptstyle =} \\ \Omega K^h(S\mathcal{A})_{n+1} & \longrightarrow & PK^h(S\mathcal{A})_{n+1} & \longrightarrow & K^h(S\mathcal{A})_{n+1} \end{array}$$

is a homotopy equivalence on base and total spaces, hence on fibers as well. Therefore, $adj(R) : K^h(\mathcal{A})_n \to \Omega K^h(S\mathcal{A})_{n+1}$ is a homotopy equivalence. The sequence $\{K^h(\mathcal{A})_0, K^h(S\mathcal{A})_1, K^h(S^2\mathcal{A})_2, ...\}$ with the structure maps given by the adjoint of R is called the *non-connective hermitian K-theory spectrum of $\mathcal{A}$* and is denoted by $\mathbf{K}^h(\mathcal{A})$. By the preceding arguments, it is an Ω-spectrum beyond its 0-th space. The construction is functorial in $\mathcal{A}$.

B.3. Let $f : \mathcal{A} \to \mathcal{B}$ be a map of preadditive categories with duality. Recall (3.17) that

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \to C(f)$$

induces a homotopy fibration in hermitian K-theory and that we have the cofiltering map $C(f) \to S\mathcal{A}$. The following diagram commutes (exercise)

$$\begin{array}{ccc} K^h_n(C(f)) & & \\ \downarrow{\scriptstyle \cong} & \searrow & \\ K^h_{n+1}(SC(f)) & \xrightarrow[\delta]{} & K^h_n(S\mathcal{A}) \end{array}$$

where the vertical arrow is induced by the adjoint of the structure map, the horizontal map is the boundary map for the homotopy fibration and the diagonal map is induced by the cofiltering map $C(f) \to S\mathcal{A}$.

References

[Bal00] Paul Balmer. Triangular Witt groups. I. The 12-term localization exact sequence. *K-Theory*, 19(4):311–363, 2000.

[Bal01] Paul Balmer. Triangular Witt groups. II. From usual to derived. *Math. Z.*, 236(2):351–382, 2001.

[Bas68] Hyman Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[BF78] A. K. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, pages 80–130. Springer, Berlin, 1978.

[BS01] Paul Balmer and Marco Schlichting. Idempotent completion of triangulated categories. *Journal of Algebra*, 236:819–834, 2001.

[BW02] Paul Balmer and Charles Walter. A Gersten-Witt spectral sequence for regular schemes. *Ann. Sci. École Norm. Sup. (4)*, 35(1):127–152, 2002.

[CL86] Ruth Charney and Ronnie Lee. On a theorem of Giffen. *Michigan Math. J.*, 33(2):169–186, 1986.

[CP97] M. Cárdenas and E. K. Pedersen. On the Karoubi filtration of a category. *K-Theory*, 12(2):165–191, 1997.

[Gil02] Stefan Gille. On Witt groups with support. *Math. Ann.*, 322(1):103–137, 2002.

[Gra76] Daniel Grayson. Higher algebraic K-theory. II (after Daniel Quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 217–240. Lecture Notes in Math., Vol. 551. Springer, Berlin, 1976.

[GZ67] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Springer-Verlag New York, Inc., New York, 1967. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35.

[Hor] Jens Hornbostel. A^1-representability of hermitian K-theory and Witt groups. preprint 578 at `http://www.math.uiuc.edu/K-theory/`.

[Hor02] Jens Hornbostel. Constructions and dévissage in Hermitian K-theory. *K-Theory*, 26(2):139–170, 2002.

[Jar97] J. F. Jardine. *Generalized étale cohomology theories*, volume 146 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.

[Kar70] Max Karoubi. Foncteurs dérivés et K-théorie. In *Séminaire Heidelberg-Saarbrücken-Strasbourg sur la K-théorie (1967/68)*, pages 107–186. Lecture Notes in Math., Vol. 136. Springer, Berlin, 1970.

[Kar73] Max Karoubi. Périodicité de la K-théorie hermitienne. In *Algebraic K-theory, III: Hermitian K-theory and geometric applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 301–411. Lecture Notes in Math., Vol. 343. Springer, Berlin, 1973.
[Kar74] Max Karoubi. Localisation de formes quadratiques. I, II. *Ann. Sci. École Norm. Sup. (4)*, 7:359–403 (1975); ibid. (4) **8** (1975), 99–155, 1974.
[Kar80] Max Karoubi. Théorie de Quillen et homologie du groupe orthogonal and Le théorème fondamental de la K-théorie hermitienne. *Ann. of Math. (2)*, 112(2):207–257 and 259–282, 1980.
[Kel96] Bernhard Keller. Derived categories and their uses. In *Handbook of algebra, Vol. 1*, pages 671–701. North-Holland, Amsterdam, 1996.
[KV73] Max Karoubi and Orlando Villamayor. K-théorie algébrique et K-théorie topologique. II. *Math. Scand.*, 32:57–86, 1973.
[Mac71] Saunders MacLane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1971.
[May75] J. Peter May. Classifying spaces and fibrations. *Mem. Amer. Math. Soc.*, 1(1, 155):xiii+98, 1975.
[MH73] John Milnor and Dale Husemoller. *Symmetric bilinear forms*. Springer-Verlag, New York, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.
[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143, 1999.
[Nee01] Amnon Neeman. *Triangulated categories*. Princeton University Press, Princeton, NJ, 2001.
[Pop73] N. Popescu. *Abelian categories with applications to rings and modules*. Academic Press, London, 1973. London Mathematical Society Monographs, No. 3.
[PW89] Erik K. Pedersen and Charles A. Weibel. K-theory homology of spaces. In *Algebraic topology (Arcata, CA, 1986)*, pages 346–361. Springer, Berlin, 1989.
[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.
[Ran81] Andrew Ranicki. *Exact sequences in the algebraic theory of surgery*, volume 26 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1981.
[Scha] Marco Schlichting. Delooping the K-theroy of exact categories. submitted, available at `http://www.uni-essen.de/~bm0031/`.
[Schb] Marco Schlichting. Hermitian K-theory: On a theorem of Giffen. submitted, available at `http://www.uni-essen.de/~bm0031/`.
[Schc] Marco Schlichting. Negative K-theory of derived categories. preprint 636 at `http://www.math.uiuc.edu/K-theory/`.
[SY96] Jack M. Shapiro and Dongyuan Yao. Hermitian U-theory of exact categories with duality functors. *J. Pure Appl. Algebra*, 109(3):323–330, 1996.
[Tho82] Robert W. Thomason. First quadrant spectral sequences in algebraic K-theory via homotopy colimits. *Comm. Algebra*, 10(15):1589–1668, 1982.
[Tho97] R. W. Thomason. The classification of triangulated subcategories. *Compositio Math.*, 105(1):1–27, 1997.
[TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.
[Uri90] M. Uridia. U-theory of exact categories. In *K-theory and homological algebra (Tbilisi, 1987–88)*, pages 303–313. Springer, Berlin, 1990.
[Wal85] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, pages 318–419. Springer, Berlin, 1985.

Jens Hornbostel, Universität Regensburg, NWF I - Mathematik, 93040 Regensburg, Germany
E-mail address: `jens.hornbostel@mathematik.uni-regensburg.de`

Marco Schlichting, Universität Duisburg-Essen, Fb 6 Mathematik, 45117 Essen, Germany
E-mail address: `marco.schlichting@uni-essen.de`