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Mapping Surgery to Analysis II: Geometric Signatures*

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Abstract. We give geometric constructions leading to analytically controlled Poincaré complexes in the sense of the previous paper. In the case of a complete Riemannian manifold we identify the signature of the associated complex with the coarse index of the signature operator.

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1. Introduction

This is the second of a series of three papers whose objective is to describe a C^*-algebraic counterpart to the 'surgery exact sequence' of Browder, Novikov, Sullivan and Wall. In the first paper [8], we defined an *analytic signature* invariant in C^*-algebra K-theory. Such an invariant is associated to any *analytically controlled Hilbert–Poincaré complex*, and it has 'homotopy invariance' and 'bordism invariance' properties in this analytic context.

In this second paper we will show that analytically controlled Hilbert–Poincaré complexes arise naturally from various *geometric constructions*. Among these are:

(a) the de Rham complex of a complete, bounded geometry Riemannian manifold (in particular, of a closed manifold);
(b) the simplicial chain complex of a uniform triangulation of a bounded geometry PL manifold;
(c) the simplicial chain complex of a uniform triangulation of an appropriately bounded Poincaré complex;
(d) various modifications of the above constructions that take into account covering spaces and the fundamental group.

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Furthermore, we will show that natural compatibility relations hold among the above constructions, so that (for instance) the analytic signature associated to the de Rham complex of a smooth manifold is the same as the signature associated to a triangulation of the underlying PL structure, and also agrees with the appropriate 'higher index' of the *signature operator* in the sense of Atiyah and Singer [2].

In the case of a Poincaré complex which is *not* a manifold, there may be *no* pseudolocal 'signature operator' of which the analytic signature is the index. This phenomenon – that the signature of a non-manifold Poincaré space need not belong to the image of the assembly map – will be discussed at length in paper III, where it underlies the construction of the analytic surgery exact sequence.

We will freely make use of the notation and terminology of paper I [8].

2. Notations and Definitions from Analysis

In this section we will describe the 'control categories' to which we will apply the theory of paper I in our geometric examples. Most of the following definitions and theorems are adapted from [9], and we refer the reader to that paper and to the book [10] for further details and examples.

A metric space is *proper* if its closed and bounded sets are compact. If X is a proper metric space then an *X-module* is a separable Hilbert space H equipped with a non-degenerate representation of the C^*-algebra $C_0(X)$ of continuous, complex-valued functions on X which vanish at infinity (*non-degenerate* means that $C_0(X)H$ is dense in H).

If H_0 and H_1 are X-modules then the *support* of a bounded linear operator $T\colon H_0 \to H_1$ is the complement of the set of points $(x_0, x_1) \in X \times X$ for which there exist functions $f_0 \in C_0(X)$ and $f_1 \in C_0(X)$ such that

$$f_0 T f_1 = 0, \quad f(x_0) \neq 0, \quad \text{and} \quad f_1(x_1) \neq 0.$$

The operator T has *finite propagation* if

$$\sup\{\, d(x_0, x_1) : (x_0, x_1) \in \mathrm{Supp}(T)\,\} < \infty.$$

The collection of all finite propagation operators is closed under addition, composition and adjunction.

DEFINITION 2.1. Let X be a proper metric space. Define $\mathfrak{A}^*(X)$ to be the C^*-category whose objects are the Hilbert X-modules and whose morphisms are the norm limits of finite propagation operators.

DEFINITION 2.2. A bounded operator $T\colon H_0 \to H_1$ between X-modules is *locally compact* if the operators fT and Tf are compact, for every

$f \in C_0(X)$. Define $\mathfrak{C}^*(X)$ to be the C^*-category whose objects are the Hilbert X-modules and whose morphisms are the norm limits of finite propagation, locally compact operators. It is an ideal in $\mathfrak{A}^*(X)$.

Remark 2.3. We say that an X-module H is *locally finite-dimensional* if, for every compactly supported function f on X, the operator on H given by multiplication by f is of finite rank. This happens, for example, if X is discrete and $H = \ell^2(X)$. In this case every bounded operator on H is locally compact.

Remark 2.4. $\mathfrak{C}^*(X)$ does not have identity morphisms for all objects, so it is not a category in the usual sense; it is however a 'non-unital C^*-category' in the sense of Mitchener. The C^*-categories $\mathfrak{A}^*(X)$ and $\mathfrak{C}^*(X)$, defined above, are not small. Their K-theory groups however may still be defined by using small cofinal subcategories. See [13, 14] for more on these matters.

Remark 2.5. For brevity we will use the phrase 'analytically controlled over X' to refer to the objects and morphisms of the categories $\mathfrak{A}^*(X)$ and $\mathfrak{C}^*(X)$ that we have defined. Thus, for example, an 'analytically controlled Hilbert space over X' is an object of the category $\mathfrak{A}^*(X)$; an 'analytically controlled linear map' is a morphism of $\mathfrak{A}^*(X)$; an 'analytically controlled Hilbert–Poincaré complex over X' is analytically controlled over $(\mathfrak{A}^*(X), \mathfrak{C}^*(X))$ in the sense of Definition 5.3 of the previous paper; and so on.

2.1. EQUIVARIANT THEORY

We will need to generalize our C^*-categories in order to take into account the possibility of a group action.

DEFINITION 2.6. Let π be a finitely generated discrete group. By a π-presented space X we will understand a proper geodesic metric space which is presented as the quotient $\tilde{X}/\pi$ of a proper geodesic metric space $\tilde{X}$ by an isometric, free and proper action of π. For a fixed π the π-presented spaces form a category, whose morphisms are equivariant maps on the presentation covers.

EXAMPLE 2.7. Let X be a compact path metric space which is locally simply connected, and let $\pi = \pi_1 X$. Then the universal cover $\tilde{X}$ of X is a locally compact path metric space on which π acts by isometries, with quotient X; so X is a π-presented space. This is the most significant example, but it is convenient to have the general language.

DEFINITION 2.8. Let X be a π-presented space. An *equivariant* X-module is an $\tilde{X}$-module H equipped with a compatible unitary representation of π. Specifically, we assume that for each $\gamma \in \pi$ there is given a unitary transformation $H \to H$, which we write as $v \mapsto v^\gamma$, such that:

(a) The unitaries $v \mapsto v^\gamma$ give a right action of γ, that is,

$$v^{\gamma\gamma'} = (v^\gamma)^{\gamma'},$$

(b) If $f \in C_0(X)$ then

$$(f \cdot v)^\gamma = f^\gamma \cdot v^\gamma,$$

where the dot denotes module multiplication and $f^\gamma(x) = f(\gamma x)$.

Remark 2.9. There is a natural 'induction' procedure from unequivariant to equivariant X-modules. To describe it, note that by standard spectral theory, every unequivariant X-module is a finite or infinite direct sum of modules of the form $L^2(X, \mu)$ where μ is a regular Borel measure on X. Each such measure lifts to a π-invariant measure $\tilde{\mu}$ on $\widetilde{X}$ by the formula

$$\tilde{\mu}(B) = \int_X \#(\pi^{-1}\{x\} \cap B)\, \mathrm{d}\mu(x),$$

where $\pi\colon \widetilde{X} \to X$ is the covering projection. Then $L^2(\widetilde{X}, \tilde{\mu})$ is an equivariant X-module, which we say is induced from $L^2(X, \mu)$. It is also easy to reverse this argument and conclude that every equivariant X-module is induced from an unequivariant one.

DEFINITION 2.10. Let X be a π-presented space. Denote by $\mathfrak{A}^*(X)$ the C^*-category whose objects are equivariant X-modules and whose morphisms are the norm limits of finite propagation, π-equivariant operators. Similarly denote by $\mathfrak{C}^*(X)$ the ideal with the same objects, and morphisms being norm limits of finite propagation, locally compact, π-equivariant operators.

Remark 2.11. It is a technical question in functional analysis whether the processes of taking the π-equivariant part and taking the norm limit can be interchanged in the definition above (see [20]). However, the formulation that we have given allows one to avoid that question.

LEMMA 2.12. *Let X be a compact π-presented space. Then the C^*-algebra of endomorphisms of any non-trivial object of $\mathfrak{C}^*(X)$ is Morita equivalent to $C^*_r(\pi)$. Consequently, the K-theory of the category $\mathfrak{C}^*(X)$ is isomorphic to the K-theory of $C^*_r(\pi)$.*

Proof. Let H be an equivariant X-module. We say that $v \in H$ is *compactly supported* if there is a compactly supported function f on $\tilde{X}$ such that $f \cdot v = v$. A $\mathbb{C}[\pi]$-valued inner product is defined on the vector space of compactly supported elements of H by

$$\langle\langle v, w\rangle\rangle = \sum_{\gamma} \langle v^{\gamma}, w\rangle[\gamma],$$

and it is not hard to see that (if H is non-trivial) every element of $\mathbb{C}[\pi]$ can arise in this way. Complete this vector space to a Hilbert module E over $C_r^*(\pi)$; then one can show that the algebra of 'compact' operators on E (in the sense of Hilbert modules) is isomorphic to the algebra of endomorphisms of H in the category $\mathfrak{C}^*(X)$. For more details of this argument see [11] or [19].

The second statement of the lemma follows from the definition of K-theory for C^*-categories (see [13]). □

We shall use the phrase 'equivariant analytic control over X' to describe the objects and morphisms of the categories $\mathfrak{A}^*$ and $\mathfrak{C}^*$, in the same way as in Remark 2.5 above.

3. Geometric Hilbert–Poincaré Complexes

Let X be a simplicial complex. In what follows we shall use the same notation both for the abstract simplicial complex X and for its geometric realization. This should not cause any confusion.

DEFINITION 3.1. A simplicial complex X is of bounded geometry if there is a number N such that each of the vertices of X lies in at most N different simplices of X.

DEFINITION 3.2. Let X be a connected simplicial complex. The *path metric* on the geometric realization of X is as follows:

(i) each simplex in X is given the metric of the standard Euclidean simplex;
(ii) if two points belong to different simplices then the distance between them is the length of the shortest piecewise linear path connecting them (the length of each linear segment within a single simplex is measured using the Euclidean metric there).

We shall work exclusively with connected, bounded geometry simplicial complexes; equipped with the path metric they are proper metric spaces.

Denote by $C_*(X)$ the space of (finitely supported) simplicial chains on X, with complex coefficients. Each vector space $C_p(X)$ has a natural basis, comprised of the p-simplices in X, and by requiring this basis to be orthonormal we may complete $C_p(X)$ to obtain the Hilbert space $C_p^{\ell^2}(X)$ of square integrable simplicial p-chains on X. It is an X-module in a natural way: if $f \in C_0(X)$ and if $c = \sum c_\sigma[\sigma]$ is an ℓ^2-p-chain then we define

$$f \cdot c = \sum f(b_\sigma) c_\sigma[\sigma],$$

where b_σ is the barycenter of σ. The simplicial differential[1] $b\colon C_p(X) \to C_{p-1}(X)$ extends to a bounded operator on ℓ^2-chains (thanks to our requirement that X be of bounded geometry) and we obtain a complex of Hilbert spaces

$$C_0^{\ell^2}(X) \xleftarrow{b} \cdots \xleftarrow{b} C_n^{\ell^2}(X)\,. \tag{1}$$

We shall call this complex the *ℓ^2-chain complex*, and we shall call its adjoint

$$C_0^{\ell^2}(X) \xrightarrow{b^*} \cdots \xrightarrow{b^*} C_n^{\ell^2}(X)\,. \tag{2}$$

the *ℓ^2-cochain complex*. It is easily checked that the differentials b and b^* are analytically controlled over X. Furthermore they are locally compact, as indeed are all bounded operators on the Hilbert spaces $C_*^{\ell^2}(X)$. So the ℓ^2-complexes (1) and (2) are analytically controlled over X.

Remark 3.3. One can alternatively define the ℓ^2-cochain complex by first considering the complex of *finitely supported* simplicial cochains on X, and then completing to a Hilbert space by the requirement that the natural basis be orthonormal. It is clear that this process yields an isomorphic complex to that described above.

We want to consider under what geometric conditions on X the complex (1) admits a Hilbert–Poincaré structure. This is not an altogether simple matter, combining as it does issues of analysis (entering through the ℓ^2-completion process) with issues in topology and geometry. We shall consider only some special cases which are adequate for our purposes. They rely upon the following definitions (compare [3,4]):

DEFINITION 3.4. Let X be a proper metric space. A complex vector space V is *geometrically controlled* over X if it is provided with a basis $B \subset V$ and a function $c\colon B \to X$ with the following property: for every $R > 0$ there is an $N < \infty$ such that if $S \subset X$ has diameter less than R then $c^{-1}[S]$ has cardinality less than N. We shall call the function c the *control map* for V.

[1]To take care of the signs, we fix an ordering of the vertices of X.

DEFINITION 3.5. A linear transformation $T\colon V \to W$ is *geometrically controlled* over X if

(i) V and W are geometrically controlled, and
(ii) the matrix coefficients of T with respect to the given bases of V and W are uniformly bounded, and
(iii) there is some $C > 0$ such that the (v, w)-matrix coefficient of T is zero whenever $d(c(v), c(w)) > C$.

Remark 3.6. Conditions (i) and (iii) above are a standard part of the boundedly controlled algebra used by topologists (see [15] for instance); condition (ii) is an additional bounded geometry condition which allows us to relate the algebra to analysis.

EXAMPLE 3.7. Obviously the main example of a vector space geometrically controlled over X is the space $C_p(X)$ of simplicial chains of a bounded geometry simplicial complex X. The differential $b\colon C_p(X) \to C_{p-1}(X)$ is a geometrically controlled linear transformation. Similar remarks apply to the complex of finitely supported simplicial cochains.

If V is geometrically controlled over X then there is a natural completion of V to a Hilbert space $\overline{V}$ in which the given basis of V becomes an orthonormal basis for $\overline{V}$. This is of course precisely how we obtain the Hilbert spaces $C_p^{\ell^2}(X)$ from the simplicial p-chains on X.

LEMMA 3.8. *A geometrically controlled linear transformation $T\colon V \to W$ extends to a bounded and analytically controlled linear operator $\overline{T}\colon \overline{V} \to \overline{W}$.*

Proof. A routine calculation using the Cauchy–Schwartz inequality. □

PROPOSITION 3.9. *A chain equivalence $T\colon V \to W$ in the category of complexes of geometrically controlled vector spaces and geometrically controlled linear transformations extends by completion to a chain equivalence $\overline{T}\colon \overline{V} \to \overline{W}$ in the category of X-modules and analytically controlled bounded linear operators.*

Proof. This is immediate from the lemma since all the linear transformations involved in the chain homotopy (namely T, its homotopy inverse, and the chain homotopy itself) extend by continuity to the Hilbert space completions. □

Remark 3.10. Contrary to the case of Hilbert space, it is not always true that a geometrically controlled homology isomorphism is a chain homotopy equivalence.

We need to make some standard remarks on cup and cap products. Let φ and ψ be finitely supported simplicial cochains, of degrees k and l respectively, for the bounded geometry simplicial complex X. Their *cup product* $\varphi \smile \psi$ is defined by the explicit (Alexander–Whitney) formula

$$\varphi \smile \psi([v_0 \dots v_{k+l}]) = \varphi([v_0 \dots v_k])\psi([v_k \dots v_{k+l}]),$$

where the notation $[v_0 \dots v_k]$ denotes the simplex with ordered vertices $v_0, \dots, v_k$. The *cap product* is the map from cochains to chains defined by dualizing the cup product. Explicitly, if ψ is a finitely supported l-cochain, and $\sigma = [v_0 \dots v_{k+l}]$ is a $(k+l)$-simplex, then the cap product $\psi \frown \sigma$ is the k-chain defined by

$$\psi \frown \sigma = \psi([v_k \dots v_{k+l}])[v_0 \dots v_k];$$

we extend by linearity to a product between cochains and chains. We then have the adjunction relationship

$$(\varphi \smile \psi)(x) = \varphi(\psi \frown x).$$

Remark 3.11. The same formulae define the cap-product of a finitely supported cochain with a *locally finite* chain x; that is, an *infinite* formal linear combination of simplices. The resulting cap-product will be an ordinary (finitely supported) chain. Moreover, if the coefficients of the chain x are uniformly bounded, the linear operator (from cochains to chains) given by cap-product with x will be geometrically controlled.

The cup and cap products are related to the boundary and coboundary maps by the standard formulae

$$b^*(\varphi \smile \psi) = b^*\varphi \smile \psi + (-1)^k \varphi \smile b^*\psi$$

and

$$b(\psi \frown x) = \psi \frown bx - (-1)^k b^*\psi \frown x.$$

Thanks to these remarks we obtain:

LEMMA 3.12. *Let X be a bounded geometry simplicial complex and suppose that $[X]$ is a locally finite and uniformly bounded simplicial n-cycle for X. Then the map $\mathbb{P}$ from p-cochains to $(n-p)$-chains defined by*

$$\mathbb{P}\psi = \psi \frown [X]$$

is geometrically controlled and satisfies

$$b\mathbb{P}\psi = (-1)^{n-p}\mathbb{P}b^*\psi.$$

Moreover, if we identify chains and cochains via the canonical inner product, the geometrically controlled chain maps $\mathbb{P}$ *and* $(-1)^{p(n-p)}\mathbb{P}^*$ *are chain homotopic* (*in the geometrically controlled category*).

We call $\mathbb{P}$ the *duality chain map* associated to $[X]$.

Proof. These are standard calculations which may be found for example in Hatcher's textbook [7, p. 215–217]. It is necessary to check that the explicit chain homotopy which verifies the graded commutativity of the cup-product is geometrically controlled. □

DEFINITION 3.13. Let X be a bounded geometry simplicial complex. We shall say that X is a *geometrically controlled Poincaré complex* of dimension n if it is provided with an n-dimensional uniformly finite simplicial cycle $[X]$ (called the *fundamental cycle*) for which the associated duality chain map $\mathbb{P}$ is a chain equivalence in the geometrically controlled category.

THEOREM 3.14. *Let* X *be a geometrically controlled Poincaré complex. Complete the simplicial chain and cochain complexes to complexes of Hilbert spaces as in displays* 1 *and* 2*. Then the duality chain map* $\mathbb{P}$ *extends by continuity to a bounded operator* P *on Hilbert space, and the operator*

$$T = \tfrac{1}{2}\left(P^* + (-1)^{p(n-p)}P\right)$$

provides $(C_*^{\ell^2}(X), b)$ *with the structure of an analytically controlled Hilbert–Poincaré complex over* X.

Proof. It follows from Proposition 3.9 that $\mathbb{P}$ extends to an analytically controlled chain equivalence P. By Lemma 3.12, P and $(-1)^{p(n-p)}P^*$ are chain homotopic (in the analytically controlled category). The average T is therefore an analytically controlled chain equivalence also, and $T^* = (-1)^{p(n-p)}T$ by construction. □

3.1. THE EQUIVARIANT CASE

In the above discussion we have not taken into account the possibility of a group action. It is however easy to do so. Suppose that X is a bounded geometry simplicial complex in the category of π-presented spaces; this means that there is given a free simplicial action of π on a complex $\tilde{X}$, whose quotient $\tilde{X}/\pi$ is identified with X. (Notice that it is then automatic that $\tilde{X}$ is of bounded geometry.) The simplicial chain and cochain complexes of $\tilde{X}$ are then complexes of geometrically controlled vector spaces on which π acts (compatibly with the control map); the action of π is by permuting the basis elements, and is therefore unitary. It now follows by the same arguments as before that the Hilbert space completions of these complexes are equivariantly analytically controlled.

Let $[X]$ be a locally finite and uniformly bounded π-invariant n-cycle for $\tilde{X}$. Cap-product with $[X]$ then defines an equivariant chain map from the cochain complex of $\tilde{X}$ to the corresponding chain complex, and the analog of Lemma 3.12 holds.

DEFINITION 3.15. We will say that X is a *geometrically controlled Poincaré complex* in the π-category if it is provided with an n-dimensional π-invariant locally finite and uniformly bounded simplicial cycle $[X]$ (called the *fundamental cycle*) for which the associated duality chain map $\mathbb{P}$ is a chain equivalence in the category of geometrically controlled π-equivariant maps.

The analog of Theorem 3.14 states that a geometrically controlled Poincaré complex in the π-category can be completed to yield an equivariantly analytically controlled Hilbert–Poincaré complex. The proof is the same as that of the unequivariant version.

3.2. POINCARÉ PAIRS

Now let (X, Y) be a pair of bounded geometry simplicial complexes. Let $[X]$ be a locally finite and uniformly bounded n-chain for X whose boundary lies in the subcomplex $C_{n-1}(Y) \subseteq C_{n-1}(X)$. Analogously to Lemma 3.12 above, we can then prove

LEMMA 3.16. *With hypotheses as above, the map* $\mathbb{P}$ *from* q*-cochains to* $(n-q)$*-chains defined by*

$$\mathbb{P}\psi = \psi \frown [X]$$

is geometrically controlled and satisfies

$$\mathrm{Image}(b\mathbb{P}\psi - (-1)^{n-q}\mathbb{P}b^*\psi) \subseteq C_*(Y);$$

in fact, the parenthesized map is simply the operation of cap-product with $b[X]$*. In particular,* $\mathbb{P}$ *gives rise to a chain map from* $C^*(X)$ *to the relative chain complex* $C_{n-*}(X, Y)$*. Moreover, if we identify chains and cochains via the canonical inner product, the geometrically controlled chain maps* $\mathbb{P}$ *and* $(-1)^{p(n-p)}\mathbb{P}^*$ *are chain homotopic (in the geometrically controlled category) as maps from* $C^*(X)$ *to* $C_{n-*}(X, Y)$.

DEFINITION 3.17. Let (X, Y) be a bounded geometry simplicial pair. We shall say that X is a *geometrically controlled Poincaré pair* of dimension

n if it is provided with an n-dimensional uniformly finite simplicial chain $[X]$, with $b[X] \in C_{n-1}(Y)$, for which the associated duality chain map $\mathbb{P}$ is a chain equivalence from $C^*(X, Y)$ to $C_{n-*}(X)$ in the geometrically controlled category.

The analog of Theorem 3.14 is then

THEOREM 3.18. *Let (X, Y) be a geometrically controlled Poincaré pair. Complete the simplicial chain and cochain complexes to complexes of Hilbert spaces. Then the duality chain map $\mathbb{P}$ extends by continuity to a bounded operator P on Hilbert space, and the operator*

$$T = \tfrac{1}{2}\left(P^* + (-1)^{p(n-p)} P\right),$$

together with the orthogonal projection operator onto the closed subcomplex of ℓ^2-chains on Y, provides $(C_*^{\ell^2}(X), b)$ with the structure of an analytically controlled Hilbert–Poincaré pair over X.

For the definition of analytically controlled Hilbert–Poincaré pair, see Section 7 of the first paper of this series.

Remark 3.19. There is of course an equivariant version of the theory of geometrically controlled Poincaré pairs, but we will not formulate that in detail here. Notice however that X and Y must belong to the π-category for the *same* group π. In the usual case where π is the fundamental group, this corresponds to the π–π *condition* that $\pi_1(Y) \to \pi_1(X)$ should be an isomorphism, which is familiar from surgery theory [22].

4. Examples of Geometrically Controlled Poincaré Complexes

4.1. COMBINATORIAL MANIFOLDS

The main examples of geometrically controlled Poincaré complexes are the bounded geometry combinatorial manifolds. We recall that a *combinatorial manifold* is a simplicial complex in which the star of each simplex is a combinatorial ball. A *bounded geometry combinatorial manifold* is just a bounded geometry simplicial complex which is also a combinatorial manifold.

Suppose that X is an oriented, n-dimensional bounded geometry combinatorial manifold. The orientation of X provides a canonical n-cycle $[X]$, which assigns the scalar $+1$ to each properly oriented n-simplex in X. Clearly, this cycle is uniformly bounded.

PROPOSITION 4.1. *Let X be an oriented, n-dimensional bounded geometry combinatorial manifold. Then the fundamental class $[X]$ provides X with the structure of a geometrically controlled Poincaré complex.*

To prove this we must show that the cap-product with $[X]$ provides a geometrically controlled chain equivalence between the cochain and chain complexes. Without the statement of geometric control, this is just classical Poincaré duality; to verify the proposition one must repeat one of the classical proofs of Poincaré duality, paying attention to the issue of geometric control. For the sake of completeness we sketch below one means of carrying out this program.

LEMMA 4.2. *Let*

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D' \arrow[r] & D \arrow[r] & D'' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of semisplit short exact sequences of finite, geometrically controlled chain complexes. If the first and third vertical maps are geometrically controlled chain equivalences, then so is the second vertical map.

We say that a short exact sequence of geometrically controlled complexes is *semisplit* if it splits as a short exact sequence of geometrically controlled graded vector spaces.

Proof. By working with mapping cone complexes, we can reduce the desired statement to the following one: if

$$0 \to C' \to C \to C'' \to 0$$

is a semisplit short exact sequence of finite, geometrically controlled chain complexes, and C', C'' are chain contractible (in the geometrically controlled category), then C is chain contractible too.

Choose a splitting of the exact sequence (as geometrically controlled vector spaces). Relative to this splitting one can write

$$d = \begin{pmatrix} d' & \alpha \\ 0 & d'' \end{pmatrix},$$

where $\alpha\colon C''_* \to C'_{*-1}$ is geometrically controlled and is a chain map up to sign (that is $\alpha d'' + d'\alpha = 0$; this follows because $d^2 = 0$).

Let s' and s'' be chain contractions of C' and C'' respectively, so that $s'd' + d's' = 1$, and so on. We seek a chain contraction of C of the form

$$s = \begin{pmatrix} s' & \beta \\ 0 & s'' \end{pmatrix},$$

where $\beta\colon C''_* \to C'_{*+1}$. For this to be a chain contraction we must have

$$d'_{n+1}\beta_n = -\beta_{n-1}d''_n + \alpha_{n+1}s''_n + s'_{n-1}\alpha_n. \tag{3}$$

Our convention is that the subscript on a map denotes the degree in the *domain*. We shall define β_n by induction on n. Denote by Θ_n the right-hand side of Equation (3). Then an obvious necessary condition for the existence of the map β_n is that $d'_n\Theta_n = 0$. Moreover, this necessary condition is actually *sufficient* because if it is satisfied we may simply define

$$\beta_n = s'_n\Theta_n;$$

then $d'_{n+1}\beta_n = d'_{n+1}s'_n\Theta_n + s'_{n-1}d'_n\Theta_n = \Theta_n$.

Suppose our complexes begin in degree zero. Then $d'_0\Theta_0 = 0$. Thus we can start the induction.

Suppose inductively that $\beta_0, \ldots, \beta_{n-1}$ have been defined to satisfy Equation (3). Then we must consider $d'_n\Theta_n$. By induction,

$$-d'_n\beta_{n-1} = \beta_{n-2}d''_{n-1} - \alpha_n s''_{n-1} - s'_{n-2}\alpha_{n-1}.$$

Substitute this into the definition of Θ to obtain

$$d'_n\Theta_n = -\alpha_n s''_{n-1}d''_n - s'_{n-2}\alpha_{n-1}d''_n - d'_n\alpha_{n+1}s''_n + d'_n s'_{n-1}\alpha_n.$$

Using the chain map property of α this becomes

$$d'_n\Theta_n = -\alpha_n\left(d'_{n+1}s''_n + s''_{n-1}\right) + \left(s'_{n-2}d''_{n-1} + d'_n s'_{n-1}\right)\alpha_n$$

and this vanishes because s', s'' are chain contractions.

Since our complexes are finite the induction terminates after finitely many steps and produces a geometrically controlled chain contraction. □

This lemma sets the stage for a 'Mayer–Vietoris' proof of Poincaré duality. Let Y be a subcomplex of X. Cap-product with $[X]$ defines a relative duality chain map

$$\mathbb{P}_Y\colon (C^*(X,Y), b^*) \to (C_{n-*}(\overline{X\setminus Y}), b). \tag{4}$$

Using the lemma above, one shows that if $\mathbb{P}_{Y_1}$, $\mathbb{P}_{Y_2}$, and $\mathbb{P}_{Y_1\cup Y_2}$ are all geometrically controlled chain equivalences, then $\mathbb{P}_{Y_1\cap Y_2}$ is such an equivalence also.

Now by bounded geometry, there is a finite cover of X by subcomplexes $L_1, \ldots, L_N$, each of which is a (possibly infinite) disjoint union of stars of vertices. The multiple intersections of the L_i are then either empty, or else disjoint unions of stars of simplices. Because X has bounded geometry, there are only finitely many possible combinatorial types of stars of simplices. From this it is easy to see that $\mathbb{P}_Y$ is a geometrically controlled chain equivalence whenever Y is the complement of the interior of one of the L_i or a union of some such complements. From the Mayer-Vietoris argument we deduce that $\mathbb{P}_Y$ is a geometrically controlled chain equivalence when $Y = \emptyset$; this completes our sketch proof of Proposition 4.1.

Remark 4.3. The argument sketched above also goes through equivariantly to show that if X is a bounded geometry combinatorial manifold in the π-category, then the fundamental class for $\tilde{X}$ provides X with the structure of a π-equivariant geometrically controlled Poincaré complex.

4.2. MANIFOLDS WITH BOUNDARY

Suppose now that X is a bounded geometry combinatorial manifold with boundary Y. We want to show that (X, Y) is a geometrically controlled Poincaré pair, in the sense of Definition 3.17. Moreover, we need to see that the boundary of this pair (in the algebraic sense of paper I) is in fact the geometric Poincaré complex associated to the boundary manifold Y.

To make use of the work that we have already done let us embed X in the bounded geometry combinatorial manifold (without boundary)

$$\hat{X} = X \cup_Y Y \times [0, \infty),$$

and let $\hat{Y} \subseteq \hat{X}$ be $Y \times [0, \infty)$. We may identify the chain complex $C_*(X)$ with a subcomplex of $C_*(\hat{X})$, and we may identify the relative cochain complex $C^*(X, Y) = C^*(\hat{X}, \hat{Y})$ with a subcomplex of $C^*(\hat{X})$. An orientation for X gives rise to an orientation for $\hat{X}$, and cap-product with the associated fundamental class gives a duality map

$$C^*(X, Y) \to C_{n-*}(X);$$

this is just the relative duality map $\mathbb{P}_{\hat{Y}}$ appearing in Equation (4) above. Our discussion of relative duality yields a proof that $\mathbb{P}_{\hat{Y}}$ is a geometrically controlled chain equivalence.

Define the fundamental chain $[X]$ by restricting the fundamental cycle $[\hat{X}]$ to $C_n(X)$. This restriction process is *not* a chain map, and thus $[X]$ has a non-trivial boundary, which is in fact the fundamental cycle $[Y]$ for Y. Moreover, cap-product with $[X]$ when applied to $C^*(X, Y)$ is just the relative duality map $\mathbb{P}_{\hat{Y}}$, and is therefore a geometrically controlled chain equivalence. This shows that (X, Y) is indeed a geometrically controlled Poincaré pair with the correct boundary.

From Theorem 7.7 of paper I we therefore obtain the following bordism invariance result:

PROPOSITION 4.4. *Let X be a bounded geometry combinatorial manifold with boundary Y. Then the signature* $\mathrm{Sign}(Y) \in K_*(\mathfrak{C}^*(Y))$ *vanishes under the map* $K_*(\mathfrak{C}^*(Y)) \to K_*(\mathfrak{C}^*(X))$ *induced by the inclusion* $Y \to X$.

4.3. FINITE POINCARÉ COMPLEXES

In classical surgery theory a *finite Poincaré complex* is a finite simplicial complex K equipped with a fundamental class $[K] \in H_n(K)$, which has the following property: the cap-product with $[K]$ induces an isomorphism

$$H^*(K; \mathbb{Z}[\pi]) \to H_{n-*}(K; \mathbb{Z}[\pi]),$$

where $\pi = \pi_1(K)$. See [21,22]; we are assuming for simplicity that the 'orientation character' w is trivial.

To see that $[K]$ provides $\tilde{K}$ with the structure of a geometrically controlled Poincaré complex we simply note the following fact:

LEMMA 4.5. *Let X be a proper metric space on which π acts properly, freely and with compact quotient. Let V and W be geometrically controlled vector spaces over X which are equipped with compatible actions of π (this means that π should act by permutations on the given bases and that the control maps should be equivariant). Then any equivariant linear transformation $T: V \to W$ is geometrically controlled.*

Proof. If Z is a compact subset of X whose π-saturation is all of X then there are only finitely many basis elements of V which belong to Z. The conditions in Definition 3.5 obviously hold when v is restricted to this finite subset of the basis of V. By equivariance, the conditions then hold (with the same constants) for all basis elements of V. □

It is clear that in the above discussion we may take π to be any homomorphic image of $\pi_1 K$, rather than $\pi_1 K$ itself.

Remark 4.6. In the next paper of this series we shall in fact need to consider the signatures of certain *non-compact* Poincaré spaces, obtained by gluing together noncompact bounded-geometry manifolds with compact boundary by homotopy equivalences of the boundary components. It should be clear to the reader that our discussion will extend to this case, but we postpone a detailed consideration until paper III.

5. Riemannian Manifolds

5.1. THE DE RHAM COMPLEX

Let X be an oriented, complete Riemannian n-manifold and form the de Rham complex

$$\Omega^0(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X) \tag{5}$$

of smooth, compactly supported differential forms on X. Passing to L^2-completions, and taking the adjoint b (in the sense of operator theory) of the de Rham differential, we obtain a complex of Hilbert spaces

$$\Omega^0_{L^2}(X) \xleftarrow{b} \cdots \xleftarrow{b} \Omega^n_{L^2}(X). \tag{6}$$

The domain of b consists of those L^2-forms ω for which the formal adjoint of the de Rham differential, applied to ω in the sense of distribution theory, produces an L^2-form.

Since it is convenient to work with *closed* unbounded operators, from here on we shall denote by d the operator-closure of the de Rham differential on $\Omega^*_{L^2}(X)$.

Let us recall that if D is a partial differential operator (on differential forms, say) then its *maximal domain* is the set of all L^2-forms such that the distribution $D\omega$ is L^2, and its *minimal domain* is the collection of all L^2 forms ω for which there exist smooth compactly supported forms ω_n such that $\omega_n \to \omega$ and $D\omega_n \to D\omega$ in L^2. The following calculation is well known:

THEOREM 5.1. *If D is the de Rham differential d on the complete manifold X, its formal adjoint, or the sum of the two, then its minimal domain is equal to its maximal domain.*

Recall that the *Hodge operator*

$$T\colon \Omega^p_{L^2}(X) \to \Omega^{n-p}_{L^2}(X)$$

is defined by

$$\langle T\alpha, \beta\rangle = \int_M \alpha \wedge \overline{\beta}.$$

It is an isometric Hilbert space operator and if α is any $L^2 p$-form then

$$T^*\alpha = (-1)^{(n-p)p} T\alpha.$$

If α is a smooth and compactly supported form then one calculates that

$$Td\alpha + (-1)^p d^* T\alpha = 0,$$

or in other words

$$Tb^*\alpha + (-1)^p bT\alpha = 0. \tag{7}$$

Theorem 5.1 and an approximation argument show that T maps the domain of b^* into the domain of b, and that (7) holds for all α in the domain of b^*.

A further calculation shows that

$$T^2\alpha = (-1)^{np+p}\alpha,$$

for any L^2 p-form α, and thus T not only maps the complex $(\Omega^*_{L^2}, b^*)$ to the complex $(\Omega^*_{L^2}, b)$ but it also defines a map the other way, which is, up to signs, an inverse. So T is an isomorphism on homology, and hence provides the L^2 de Rham complex of X with the structure of a Hilbert–Poincaré complex.

DEFINITION 5.2. We shall call the Hilbert–Poincaré complex so defined the *Hodge–de Rham complex* of X.

PROPOSITION 5.3. *Let X be an oriented, complete Riemannian manifold. The Hodge–de Rham complex of X is analytically controlled over X.*

Proof. The Hilbert spaces $\Omega^*_{L^2}(X)$ are X-modules in the obvious way, and the operator T is analytically controlled over X since its support is the diagonal in $X \times X$. It remains to note that the resolvents of the operator $B = b^* + b$ are analytically controlled and locally compact. Analytic control is a simple consequence of the finite propagation speed property of the Dirac-type operator B; local compactness is a consequence of the ellipticity of B combined with the Rellich lemma of Sobolev space theory. See [10, Chapter X] or [16] for details. □

5.2. THE SIGNATURE OPERATOR

Let X be an oriented, complete Riemannian manifold. The fact that $T^2 = \pm I$ in the Hodge–de Rham complex allows us to associate to X a *signature operator,* as follows.

DEFINITION 5.4. Form $S = i^{p(p-1)+l}T$ as in Section 3, and observe that it is a self-adjoint unitary which *anticommutes* with the operator B. If the dimension of X is even then the signature operator on X is simply $B = d + d^*$ itself, viewed as an operator which is graded by the symmetry S. If the dimension of X is odd then the signature operator is the self-adjoint operator iBS, viewed as acting on acting on even degree differential forms.

The definition in the even-dimensional case is standard. In the odd-dimensional case one checks easily that

$$D\omega_{2p} = i^{2p+l+1}(d* - *d)\omega_{2p},$$

where $*$ is the Hodge operator. So our definition is consistent with others appearing in the literature [1].

In either case, the signature operator has an *index* lying in the C^*-algebra K-theory group $K_n(\mathfrak{C}^*(X))$. We refer the reader to [17, 18] for details of this construction (although it will be reviewed in the course of the proof of the next theorem). Our aim in this subsection is to prove the following result:

THEOREM 5.5. *Let X be an oriented, complete Riemannian manifold. The index of its signature operator is equal to the signature of the algebraic Hilbert–Poincaré complex associated to the de Rham complex of X.*

5.2.1. *Even-dimensional case*

We begin by recalling the definition of the K-theoretic abstract index. The formulation used here is borrowed from [17], but the construction goes back to Milnor's description of the boundary map in algebraic K-theory [12].

Let A be a unital C^*-algebra, J an ideal in A, and suppose that S_1 and S_2 are symmetries in A whose anticommutator $S_1S_2 + S_2S_1$ belongs to J. Then $-S_2S_1S_2$ is a symmetry differing from S_1 by an element of J, and therefore the formal difference

$$[-S_2S_1S_2] \ominus [S_1]$$

defines a class in $K_0(J)$, which we denote by $i(S_1; S_2)$.

This quantity is related to the analytic index in the following manner. Suppose that A and J act on a Hilbert space H, and that D is an unbounded self-adjoint operator which is analytically controlled in the sense that $f(D) \in J$ for all $f \in C_0(\mathbb{R})$ and $g(D) \in A$ for all $g \in C[-\infty, \infty]$. Suppose further that A contains a symmetry S, the grading operator, which anticommutes with D. If we choose a *normalizing function* g which is odd and tends to ± 1 at $\pm\infty$, then the operator $g(D) \in A$ is a 'symmetry modulo J' and it therefore defines a symmetry in the quotient algebra A/J. Because this symmetry is odd relative to the grading S, it has the matrix form

$$\begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

relative to the grading; the unitary U defines a class in $K_1(A/J)$ and the image of this class under the boundary map $\partial\colon K_1(A/J) \to K_0(J)$ is the index. Examination of the explicit formula for the boundary map in K-theory shows that this index can be expressed as $i(S_1; S_2)$, where $S_1 = S$ is the grading operator and S_2 is any symmetry in A which differs from $g(D)$ by an element of J. An explicit formula for such a symmetry is

$$S_2 = g(D) + Sf(D), \tag{8}$$

where $f(\lambda) = \sqrt{1 - g(\lambda)^2}$.

LEMMA 5.6. *With notation as above, suppose that $S_1 + S_2$ is invertible. Then $i(S_1; S_2) = i(S_2; S_1)$.*

Proof. Let U be the unitary $S_1 S_2$. The identity $1 + U = S_1(S_1 + S_2)$ shows that $1 + U$ is invertible, and therefore that -1 does not belong to the spectrum of U. Therefore, we may form a square root $V = U^{1/2}$, applying the functional calculus to a branch of the function $z \mapsto z^{1/2}$ defined in the plane cut along the negative real axis. Note that

$$S_1 U = U^* S_1, \quad S_2 U = U^* S_2$$

and so

$$S_1 V = V^* S_1, \quad S_2 V = V^* S_2.$$

Finally let W be the unitary $V S_2$. It is now easy to check that

$$W^* S_1 W = S_2 V^* S_1 V S_2 = S_2 S_1 V^2 S_2 = S_2 S_1 (S_1 S_2) S_2 = S_2$$

and

$$W^* S_2 W = S_2 V^* S_2 V S_2 = S_2 S_2 V^2 S_2 = S_2 S_2 (S_1 S_2) S_2 = S_1.$$

The result now follows from the obvious invariance of the quantity $i(S_1; S_2)$ under conjugation. □

Proof of Theorem 5.5. (*Even-dimensional case*). Let D be the signature operator and S the grading operator. Take the C^*-algebras A and J to be the algebras of endomorphisms, in the controlled categories $\mathfrak{A}^*(X)$ and $\mathfrak{C}^*(X)$ respectively, of the Hilbert space of L^2-forms on X (or on $\tilde{X}$ in the equivariant case). Note that the hypothesis of Lemma 5.6 is satisfied when $S_1 = S$ and S_2 is defined by Equation (8). Indeed,

$$(S_1 + S_2)^2 = (g(D) + S(1 + f(D)))^2 = 2(1 + f(D)) \geqslant 2$$

since f is a positive function and $f(D)$ is therefore a positive operator. Thus $i(S_1; S_2)$, which is by definition the index of the signature operator, is equal to $i(S_2; S_1)$. Let us make the particular choice $g(\lambda) = \lambda(1+\lambda^2)^{-1/2}$, so that $f(\lambda) = (1+\lambda^2)^{-1/2}$. Thus

$$S_2 = (D+S)(1+D^2)^{-1/2} = (D+S)|D+S|^{-1/2}$$

is the 'phase' of $D+S$, and since S anticommutes with D,

$$-S_1 S_2 S_1 = -S(D+S)S(1+D^2)^{-1/2} = (D-S)|D-S|^{-1/2}$$

is the phase of $D-S$. The difference between these two phases is our definition of the signature [8, Definition 5.11], expressed in terms of symmetries rather than projections; so we have completed the proof that our signature is equal to the index of the signature operator, in the even-dimensional case. □

5.2.2. *Odd-dimensional case*

Again, let us describe the construction of the analytic index. Let the notations A, J, D, and g have the same meaning as in the previous section; this time, however, we do not assume the presence of a symmetry which acts as a grading operator. Then $g(D) \in A$ is a symmetry modulo J; its image in A/J defines a class in $K_0(A/J)$; and the image of that symmetry under the boundary map $\partial\colon K_0(A/J) \to K_1(J)$ is the odd-dimensional index that we require. Using the explicit description of this boundary map in terms of the exponential function, we see that the index is represented by the unitary $-\exp(i\pi g(D))$ in the unitalization of J. It is convenient to choose g so that this representative becomes the *Cayley transform* $(D+i)(D-i)^{-1}$ of D.

Proof of Theorem 5.5. (*Odd-dimensional case*). The index of D is the class in $K_1(J)$ of the Cayley transform $(D+i)(D-i)^{-1}$ But note that

$$(D \pm i) = (iBS \pm i) = (B \pm S)(iS),$$

so that

$$(D+i)(D-i)^{-1} = (B+S)(iS)(iS)^{-1}(B-S)^{-1} = (B+S)(B-S)^{-1},$$

which shows that the index and the signature are equal at the level of cycles for the group $K_1(J)$. □

5.3. COMPARISON WITH SIMPLICIAL COHOMOLOGY

Following roughly the approach of Whitney [24], we are going to prove an analogue of the de Rham theorem for the signature of a complete Riemannian manifold. A very similar result is proved by Dodziuk in [6], and since his argument needs only minor adaptation to suit our purposes we shall be brief.

Perhaps it is worth noting that the case of the universal cover of a closed manifold is rather simpler than the general case of our theorem. One should compare the calculations of [5] to those of [6].

We require the following hypothesis.

DEFINITION 5.7. A complete Riemannian manifold has *bounded geometry* if it has positive injectivity radius and the curvature tensor is uniformly bounded, as is each of its covariant derivatives.

It is probably harmless to drop the uniform boundedness of the covariant derivatives, but in any case, the additional assumption will not stand in the way of the applications we have in mind.

DEFINITION 5.8. Let X be a bounded geometry, complete Riemannian manifold. A smooth triangulation of X is of *bounded geometry* if it is a bounded geometry simplicial complex, and if the identity map from the complex X into the manifold X is a bi-Lipschitz homeomorphism.

Remark 5.9. An unpublished theorem of Calabi is said to assert that every complete Riemannian manifold of bounded geometry has a smooth triangulation of bounded geometry. The proof is outlined in [3]. In our applications in paper 3, the Riemannian manifolds in question will have a simple structure at infinity (they will be cones over compact manifolds) and for these examples it is easy to construct bounded geometry triangulations directly, without appealing to the general result.

Of course, some examples arise immediately:

EXAMPLE 5.10. If X is the universal cover of a closed Riemannian manifold V then X has bounded geometry, and any triangulation of X which is lifted from a triangulation of V is of bounded geometry.

Let us suppose now that X is a complete Riemannian manifold of bounded geometry, equipped with a smooth triangulation of bounded geometry.

If $\Delta = (d + d^*)^2$ then the heat kernel $e^{-\Delta}$ is a chain equivalence from $\Omega^*_{L^2}$ to itself, for either the de Rham differential or its adjoint – in other words for either homology or cohomology. By elliptic regularity theory, its image consists entirely of smooth forms on X. By the finite propagation property of the first order operator $d + d^*$, the map $e^{-\Delta}$ is analytically controlled. It follows from [6] that the supremum norm of $e^{-\Delta}\omega$ is bounded by a multiple of the L^2 norm of ω (here is where we use the full strength of the definition of bounded geometry). These remarks, together with Stokes' theorem, show that by mapping an L^2 p-form ω to the function $\sigma \mapsto \int_\sigma e^{-\Delta}\omega$ on the p-simplices in the triangulation of X we obtain a chain map Ψ from $(\Omega^*_{L^2}(X), d)$ to the complex $(C^{\ell^2}_*(X), b^*)$ computing ℓ^2-simplicial cohomology.

THEOREM 5.11. *The chain map* $\Psi\colon \left(\Omega^*_{L^2}(X), d\right) \to \left(C^{\ell^2}_*(X), b^*\right)$ *is a homology isomorphism.*

Proof. We shall essentially follow the argument of Dodziuk [6], who in turn follows an argument of Whitney [24]. In the context of smooth closed manifolds, Whitney constructs a right inverse Φ to the chain map which integrates smooth forms over simplices. The main ingredient for this is a smooth partition of unity subordinate to the star-neighborhoods of the vertices in the triangulation of X. If φ_v is the smooth function associated to the vertex v, and if σ is the simplex in X with vertices $v_0, \dots, v_p$ then Whitney defines

$$\Phi\sigma = p! \sum_{i=0}^{p} (-1)^i \varphi_{v_i} d\varphi_{v_1} \dots d\hat{\varphi}_{v_i} \dots d\varphi_{v_p},$$

where the 'hat' denotes omission of the specified term. See [24, Chapter IV, Section 27]. This is right inverse to integration over simplices at the chain level, and is left inverse up to chain homotopy: the homotopy is constructed using the Poincaré lemma on the simplices of the triangulation. Dodziuk shows that in the context of bounded geometry manifolds and L^2-cohomology, Whitney's formula continues to provide a map from ℓ^2-cochains to L^2-forms, provided the functions φ_v are chosen with uniformly (over the set of vertices) bounded covariant derivatives. See [6, Section 2]. Since our chain map I contains the smoothing operator $e^{-\Delta}$, the composition of Φ, followed by Ψ, is not the identity. But since $e^{-\Delta}$ is chain homotopic to the identity, so is the composition $\Psi \circ \Phi$. Whitney's argument (which Dodziuk chooses not to follow) now shows that the reverse composition is chain homotopic to $e^{-\Delta}$, which is in turn chain homotopic to the identity (we use here the bounded geometry of X and its triangulation once more, which ensures that the chain homotopy implicit in the Poincaré lemma may be made uniformly bounded in norm, over the simplices of X). $\square$

Let us now equip the complex of ℓ^2-simplicial chains on X with the Hilbert–Poincaré structure derived from the structure of X as a combinatorial manifold.

Form the *adjoint*

$$A = \Psi^*\colon \left(C_*^{\ell^2}(X), b\right) \to \left(\Omega^*_{L^2}(X), b\right)$$

of the cohomology isomorphism considered above. It induces an isomorphism from ℓ^2-simplicial homology to L^2-de Rham homology.

THEOREM 5.12. *The chain map A is a homotopy equivalence from the Hilbert–Poincaré complex $(C_*^{\ell^2}(X), b)$ to the Hodge–de Rham complex of X.*

Proof. All that needs to be checked is that the duality operators on the two complexes are compatible with one another, via A, in the sense of [8, Definition 4.1]. Since the duality operator on $(C_*^{\ell^2}(X), b)$ is chain homotopic to the operator T, as in Proposition 4.1 and its proof, we may as well work with T. The theorem now follows from the assertion that the cup product of simplicial cochains is compatible with the wedge product of forms, at the level of ℓ^2-cohomology and via the map Φ. For closed manifolds this is well known, and is proved for instance in [24, Chapter IV, Section 29] (see also [23]). The argument there adapts to the present context, and is omitted. □

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