ON FARBER'S INVARIANTS FOR SIMPLE $2q$-KNOTS

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ABSTRACT. Let K be a simple $2q$-knot with exterior X. We show directly how the Farber quintuple $(A, \Pi, \alpha, \ell, \psi)$ determines the homotopy type of X if the torsion subgroup of $A = \pi_q(X)$ has odd order. We comment briefly on the possible role of the EHP sequence in recovering the boundary inclusion from the duality pairings ℓ and ψ. Finally we reformulate the Farber quintuple as an hermitian self-duality of an object in an additive category with involution.

In a series of papers in the early 1980s Farber showed that stable high dimensional knots could be classified in terms of stable homotopy theory, and in particular that if $q \geq 4$ simple $2q$-knots may classified up to isotopy by quintuples $(A, \Pi, \alpha, \ell, \psi)$, where A, Π are $\mathbb{Z}[Z]$-modules, α is a monomorphism and ℓ and ψ are sesquilinear pairings [6]. Certain significant special cases were dealt with earlier by Kearton [14] and Kojima [17].

Farber's argument for showing that his invariant determined the knot involves Spanier-Whitehead duality for highly connected Seifert hypersurfaces and Wall's embedding theorem (to interpolate between different choices of such hypersurfaces). An alternative approach (avoiding Seifert arguments) is suggested by the work of Lashof and Shaneson, who used surgery over $\pi_1 = Z$ to show that the exterior X of a high dimensional knot with group Z is determined up to homeomorphism by the homotopy type of the pair $(X, \partial X)$ [18]. There remains a possible ambiguity of order 2, and a further argument is needed to show that the exterior determines the knot if $(X, \partial X)$ is highly connected [23].

In this note we shall provide some evidence for the feasibility of such a homotopy-theoretic approach. We show first that the homotopy type of X is determined by the modules $A = \pi_q(X)$ and $\pi_{q+1}(X)$ and a k-invariant $\kappa(X)$. We then show that these invariants may be derived algebraically from the Farber quintuple. (In the case of $\kappa(X)$ we need to assume that the $\mathbb{Z}$-torsion submodule $T(A)$ has odd order.) The work of Farber implies that the duality pairings ℓ and ψ determine the inclusion of the boundary. We have not yet been able to find a homotopy-theoretic proof of this implication. However the longitude (i.e., the lift of the boundary inclusion to infinite cyclic covers) has trivial suspension, and the EHP sequence of Chapter XII of [28] suggests a close connection between the longitude and these duality pairings. Further progress appears to depend on having a more direct construction of the pairing ψ (or at least of the version Ψ used below for the case with $T(A)$ odd).

Finally, Quebbemann, Scharlau and Schulte have codified the notion of hermitian pairing in an additive category $\mathcal{C}$ with an involution $*$ [22], and we shall show that Farber's invariant may be described in such terms. This approach has proven useful

in studying the factorization of some nontrivial classes of higher-dimensional knots into sums of indecomposable knots [8, 10]. It is also of interest in connection with criteria for a knot to be doubly null concordant [7, 8].

1. Knots

An *n-knot* is a locally flat embedding $K : S^n \to S^{n+2}$. We shall assume that S^n has the standard orientation as the boundary of the unit ball in R^{n+1}, for all $n > 0$. The *exterior* of K is $X = X(K) = S^{n+2} \setminus N(K)$, where $N(K) \cong S^n \times R^2$ is an open regular neighbourhood of the image of K, and $\partial X \cong S^n \times S^1$. The orientations of the spheres determine an orientation for X and an isotopy class of *meridians* $S^1 \subset \partial X$. The inclusion of a meridian induces an isomorphism on integral homology, and so there is a canonical generator t for $H = H_1(X(K); \mathbb{Z}) \cong Z$. Hence we may identify $\mathbb{Z}[H]$ with $\mathbb{Z}[Z] = \Lambda = \mathbb{Z}[t, t^{-1}]$.

Reattaching $S^n \times D^2$ to $X = X(K)$ along $\partial X \cong S^n \times S^1$ using the nontrivial twist map of $S^n \times S^1$ gives a homotopy $(n+2)$-sphere. The embedding of $S^n \times \{0\}$ in this sphere gives an n-knot K^*, called the *Gluck reconstruction* of K, and any knot with exterior homeomorphic to X is isotopic to K or to K^* (up to change of orientations).

Let $p : X' \to X$ be the infinite cyclic cover space corresponding to the commutator subgroup π'. The modules $H_*(X; \Lambda) = H_*(X'; \mathbb{Z})$ are finitely generated, since X is compact and Λ is noetherian. Moreover $t - 1$ acts invertibly on $H_*(X; \Lambda)$, since X is an homology circle, and so these are torsion modules. The *longitude* $\lambda_K \in \pi_n(X)$ is the class of the inclusion $S^n \times \{1\} \to S^n \times S^1 \cong \partial X \subset X$. If V is a Seifert hypersurface for K then λ_K factors though the inclusion of $\partial V \cong S^n$ into V. Let $k : V/\partial V \to S^{n+1}$ be the cofibre of the natural map from V to $V/\partial V$. Then the Pontrjagin-Thom collapse of S^{n+2} onto $S(V/\partial V)$ provides a section to Sk. It follows easily that the suspension of λ_K is trivial [25].

The knot K is *r-simple* if $\pi_1(X) \cong Z$ and $\pi_j(X) = 0$ for $1 < j \leq r$. The *sum* $K \# \widetilde{K}$ of r-simple n-knots K and $\widetilde{K}$ is r-simple. If K is 1-simple then X is determined up to homeomorphism by the homotopy type of the pair $(X, \partial X)$ [18]. The knot K is *simple* if it is $[(n-1)/2]$-simple and $n \geq 3$.

It is a consequence of Farber's work on stable knots that an r-simple n-knot K is determined by its exterior (i.e., $K \cong K^*$) if $3r > n$. In [23] Richter shows that if $3r \geq n$ and K is an r-simple n-knot the twist map of ∂X extends to a self homotopy equivalence of X and hence that K is determined by its complement, as a straightforward consequence of Williams' Poincaré Embedding Theorem. Richter's approach is close to the one taken here, in that he strives to work as far as possible in the homotopy category, before appealing to geometric topology through surgery. In particular, he subsequently gave a purely homotopy theoretic proof of William's theorem [24]. (The application to knot theory does however use the fact that K has a highly connected Seifert hypersurface, in an essential way.) Is there an *ad hoc* simplification for the case of simple knots?

2. Modules and extensions

If M is a finitely generated left Λ-module let $T(M)$ be the maximal finite submodule of M. In the cases of interest here $T(M)$ is the $\mathbb{Z}$-torsion submodule of M. The quotient $M/T(M)$ is $\mathbb{Z}$-torsion free, and has projective dimension at most 1

$(p.d._\Lambda M/T(M) \leq 1)$. If R is a *right* Λ-module let $\overline{R}$ denote the left module with the conjugate structure determined by $t.r = rt^{-1}$ for all $r \in R$.

The extension modules $Ext^i_\Lambda(M, \Lambda)$ are naturally right modules. Let $e^i M = \overline{Ext^i_\Lambda(M, \Lambda)}$. If M is a finite Λ-module then $e^0 M = e^1 M = 0$, and so the inclusion $T(M) \leq M$ induces isomorphisms $e^i M \cong e^i(M/T(M))$ for $i = 0, 1$ and $e^2 M \cong e^2 T(M)$. More generally, $Ext^2_\Lambda(M, N) \cong Ext^2_\Lambda(T(M), N)$, for any Λ-module N. If $p.d._\Lambda M = d$ then $Ext^d_\Lambda(M, N)$ and $Ext^d_\Lambda(M, \Lambda) \otimes_\Lambda N$ are naturally isomorphic, for if P_* is a finitely generated free resolution of M then $Hom_\Lambda(P_*, N) \cong Hom_\Lambda(P_*, \Lambda) \otimes_\Lambda N$, and $- \otimes_\Lambda N$ is right exact.

If P_* is a chain complex of finitely generated free left Λ-modules let $D(P)_j = \overline{Hom_\Lambda(P_{-j}, \Lambda)}$ be the dual complex, and let $P[n]_*$ be the n-fold suspension, with $P[n]_j = P_{j-n}$ for all j. Note that $D(D(P))_*$ is naturally isomorphic to P_*. If $H_*(P) = 0$ for $j \neq q$ or $q + 1$ then P is determined up to chain homotopy equivalence by the modules $A = H_q(P_*)$ and $B = H_{q+1}(P_*)$ and a k-invariant $\kappa(P_*)$ in $Ext^2_\Lambda(A, B)$ [4]. This may be identified with the class of the sequence

$$0 \to B \to P_{q+1}/\partial P_{q+2} \to \mathrm{Ker}(\partial_q) \to A \to 0$$

in $Ext^2_\Lambda(A, B) \cong Ext^2_\Lambda(T(A), B)$. Every such class is realizable by a free complex concentrated in degrees $[q, q+2]$.

A *simple Λ-complex concentrated in degrees* $[q, q+2]$ is a finitely generated free Λ-complex C_* whose homology modules are Λ-torsion modules and are 0 except in degrees q and $q + 1$. Let $A = H_q(C_*)$, $T = T(A)$ and $B = H_{q+1}(C_*)$ and let $k(A) \in Ext^1_\Lambda(A/T, T) \cong \overline{e^1 A} \otimes T$ and $\kappa(C_*) \in Ext^2_\Lambda(A, B) \cong \overline{e^2 T} \otimes B$ be the associated extension classes.

Theorem 1. *Let C_* be a simple Λ-chain complex concentrated in degrees $[q, q+2]$.*

   (1) C is determined up to chain homotopy equivalence by the quintuple
       $(A/T(A), T(A), B, k(A), \kappa(C_*))$;
   (2) any such system of invariants with A and B finitely generated Λ-torsion modules and $p.d._\Lambda B \leq 1$ is realized by a simple Λ-complex;
   (3) $D_* = D(C)_*$ is a simple complex concentrated in degrees $[-q-2, -q]$, with invariants $(e^1 B, e^2 A, e^1 A, k_D, \kappa(D_*))$ where k_D and $\kappa(D_*)$ are determined by $\kappa(C_*)$ and $k(A)$, respectively.

Proof. The module A is determined by $T = T(A)$, A/T and $k(A)$, and so the first assertion follows from the result of [4] cited above. Since A is a torsion module D_* is easily seen to be a simple Λ-complex. If $A_D = H_{-q-2}(D_*)$ and $B_D = H_{-q-1}(D_*)$ then $B_D \cong e^1 A$, $T_D = T(A_D) \cong e^2 A = e^2 T$ and $A_D/T_D \cong e^1 B$.

There are natural isomorphisms $T \cong e^2 e^2 A = \overline{e^2 T_D}$ and $B \cong e^1 A_D$. Hence the transpositions $\tau_1 : \overline{e^1 A} \otimes T \cong \overline{T} \otimes e^1 A$ and $\tau_2 : \overline{e^2 T} \otimes B \cong \overline{B} \otimes e^2 T$ given by interchange of factors induce natural isomorphisms $\overline{e^1 A} \otimes T \cong \overline{e^2 T_D} \otimes B_D$ and $\overline{e^2 T} \otimes B \cong \overline{e^1 A_D} \otimes T_D$ between the abelian groups in which the extension classes for A and D_* and for C_* and A_D lie.

To verify (2) and see that the extension classes correspond as asserted we shall construct an explicit representative for the chain homotopy type determined by the invariants $(A/T, T, B, k(A), \kappa(C_*))$. We shall assume for simplicity of notation that $q = 0$. Let $P_1 \to P_0$, $Q_1 \to Q_0$ and $E_2 \to E_1 \to E_0$ be finite free resolutions of A/T, B and T, respectively, and let $k : P_1 \to E_0$ and $\kappa : E_2 \to Q_0$ be homomorphisms representing $k(A)$ and $\kappa(C_*)$, respectively. Then we may using a mapping cone

construction (twice) as in Satz 7.6 of [4] to construct a complex

$$\text{(1)} \qquad\qquad E_2 \oplus Q_1 \xrightarrow{\Delta_2(\kappa)} P_1 \oplus E_1 \oplus Q_0 \xrightarrow{\Delta_1(k)} P_0 \oplus E_0$$

with these invariants. Here $\Delta_2(\kappa)$ and $\Delta_1(k)$ are matrices whose nonzero entries involve the differentials of the constituent complexes and the homomorphisms k and κ. It is clear that the dual complex is isomorphic to the complex obtained by taking $e^0\kappa$ and $e^0 k$ as representatives for k_D and $\kappa(D_*)$, respectively. $\qquad\square$

3. Duality for finite $\mathbb{F}\Lambda$-modules

Let $\mathbb{F}$ be a field and let $\mathbb{F}\Lambda = \mathbb{F}[t, t^{-1}]$. If M is a finite dimensional $\mathbb{F}\Lambda$-module let $E(M) = \overline{Ext^1_{\mathbb{F}\Lambda}(M, \mathbb{F}\Lambda)}$, $F(M) = \overline{Hom_{\mathbb{F}\Lambda}(M, \mathbb{F}(t)/\mathbb{F}\Lambda)}$ and $M^* = Hom_{\mathbb{F}}(M, \mathbb{F})$. (The left Λ-module structure on the latter group is given by $(t\phi)(m) = \phi(t^{-1}m)$ for all $m \in M$ and $\phi : M \to \mathbb{F}$.) These modules are each non-canonically isomorphic to M, since $\mathbb{F}\Lambda$ is a PID. The functors $E(-)$, $F(-)$ and $(-)^*$ each define an involution on the category of finite dimensional $\mathbb{F}\Lambda$-modules. They are in fact naturally equivalent. We shall give a simple proof of this for the case when $\mathbb{F}$ is finite.

If $f \in \mathbb{F}\Lambda$ then $\text{Res}(f\frac{dt}{t}, \infty) + \text{Res}(f\frac{dt}{t}, 0) = 0$, since the only poles of $f\frac{dt}{t}$ are at 0 and ∞. Thus we may define an $\mathbb{F}$-linear function $R : \mathbb{F}(t)/\mathbb{F}\Lambda \to \mathbb{F}$ by setting $R(q + \mathbb{F}\Lambda) = \text{Res}(q\frac{dt}{t}, \infty) + \text{Res}(q\frac{dt}{t}, 0)$ for all $q \in \mathbb{F}(t)$. In particular, if $q = \frac{f}{t^n - 1}$ where $f \in \mathbb{F}[t]$ has degree $< n$ then $R(q + \mathbb{F}\Lambda) = -f(0)$.

Theorem 2. *There are natural equivalences $E(M) \cong F(M) \cong M^*$ on the category of finite dimensional $\mathbb{F}\Lambda$-modules, if $\mathbb{F}$ is a finite field.*

Proof. Let $0 \to P_1 \to P_0 \to M \to 0$ be a free resolution of M. If $\phi \in F(M)$ let $\phi_0 : P_0 \to \mathbb{F}(t)$ be a lift of ϕ. Then $\phi_0 \partial_1$ has image in $\mathbb{F}\Lambda$, and so defines a homomorphism $\phi_1 : P_1 \to \mathbb{F}\Lambda$ such that $\phi_1 \partial_2 = 0$. Consideration of the short exact sequence of complexes

$$0 \to Hom_{\mathbb{F}\Lambda}(P_*, \mathbb{F}\Lambda) \to Hom_{\mathbb{F}\Lambda}(P_*, \mathbb{F}(t)) \to Hom_{\mathbb{F}\Lambda}(P_*, \mathbb{F}(t)/\mathbb{F}\Lambda) \to 0$$

shows that $\delta_M(\phi) = [\phi_1]$, where $\delta_M : F(M) \to E(M)$ is the Bockstein homomorphism associated to the coefficient sequence. (The extension corresponding to $\delta_M(\phi)$ is the pullback over ϕ of the sequence $0 \to \mathbb{F}\Lambda \to \mathbb{F}(t) \to \mathbb{F}(t)/\mathbb{F}\Lambda \to 0$.) Since M is a torsion $\mathbb{F}\Lambda$-module and $\mathbb{F}(t)$ is divisible δ_M is an isomorphism.

Define $\chi_M : F(M) \to M^*$ by $\chi_M(\phi)(m) = R(\phi(m))$ for all $\phi \in F(M)$ and $m \in M$. Then χ_M is $\mathbb{F}\Lambda$-linear. Suppose now that M has finite order n. If $g \in M^*$ let $L_M(g)(m) = \frac{\hat{g}(m)}{t^{n!}-1} + \mathbb{F}\Lambda$, where $\hat{g}(m) = \Sigma_{0 \le k < n!} g(t^k m) t^{-k}$ for all $m \in M$. Then $L_M(g)$ is $\mathbb{F}\Lambda$-linear, since $(t^{n!} - 1)M = 0$, and $\chi_M(L_M(g)) = g$. (This is the only point at which we need $\mathbb{F}$ and M to be finite). Since $F(M)$ and M^* have the same dimension and χ_M is $\mathbb{F}\Lambda$-linear χ_M and L_M are mutually inverse isomorphisms. $\quad\square$

Let $\tau_M = \chi_M \delta_M^{-1} : E(M) \to M^*$ be the composite isomorphism.

Theorem 2 holds also for infinite fields, but verifying that χ is an equivalence involves slightly more work. See [20, 26] for the case $\mathbb{F} = \mathbb{Q}$.

There is also a corresponding result over Λ, i.e., with integer coefficients. Levine showed that if M is a finite Λ-module such that $(t^k - 1)M = 0$ then $e^2 M \cong \overline{Ext^1_{\Lambda}(M, \Lambda/(t^k - 1))} \cong \overline{Hom_{\Lambda}(M, \mathbb{Q}/\mathbb{Z} \otimes \Lambda/(t^k - 1))}$ (via appropriate Bockstein homomorphisms), which is in turn isomorphic to $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ (via an explicit homomorphism similar to L_M) [19]. In particular, if M has exponent n as an

abelian group then $e^2M \cong Hom_{\mathbb{Z}}(M, Z/nZ)$. If A is a finitely generated Λ-module let $\sigma_A : e^2A \to Hom_{\mathbb{Z}}(T(A), \mathbb{Q}/\mathbb{Z})$ be the composite of these isomorphisms with the natural isomorphism $e^2A = e^2T(A)$.

4. Algebraic invariants for the exterior of a simple $2q$-knot

Let K be a simple $2q$-knot with $q > 2$. Then $\pi_1(X) = Z$ and $H_i(X; \Lambda) = 0$ for $0 < i < q$. Let $A = H_q(X; \Lambda)$ and $B = H_{q+1}(X; \Lambda)$. As observed in §1, these are torsion Λ-modules on which $t - 1$ acts invertibly. The Universal Coefficient spectral sequence and Poincaré duality give isomorphisms $e^1A = e^1(A/T(A)) \cong B$ and $e^2A \cong T(\overline{H^{q+2}(X; \Lambda)}) \cong T(A)$, while $H_i(X; \Lambda) = 0$ for $i > q+1$. In particular, there is a nonsingular pairing $\ell : T(A) \times T(A) \to \mathbb{Q}/\mathbb{Z}$ and $p.d._{\Lambda}B \leq 1$. Since $B = (t-1)B$ it follows that B is $\mathbb{Z}$-torsion free [19]. The exterior X fibres over S^1 if and only if A is finitely generated as an abelian group [3].

We may identify A with the Λ-module $\pi_q(X) = \pi_q(X')$, by the Hurewicz Theorem. Since $\pi_1(X) \cong Z$ and $\pi_j(X) = 0$ for $1 < j < q$ the Postnikov q-stage of X is the generalized Eilenberg-Mac Lane space $L_Z(A, q)$ determined by the Λ-module A. (See page 214 of [2].) Let $\tilde{\pi}_{q+1} = \pi_{q+1}(X') = \pi_{q+1}(X)$, considered as a Λ-module, and let $\kappa(X) \in H^{q+2}(L_Z(A, q); \tilde{\pi}_{q+1})$ be the k-invariant for the next stage. Let $\eta \in \pi_3(S^2)$ be the Hopf map. Then $\eta_q = \Sigma^{q-2}\eta$ generates $\pi_{q+1}(S^q) = \pi_1^{st} = Z/2Z$ for all $q > 2$.

Since X' is $(q-1)$-connected and $q > 2$ there is an exact sequence

$$(2) \quad \pi_{q+2}(X') \to H_{q+2}(X'; \mathbb{Z}) \xrightarrow{\ b\ } A/2A \xrightarrow{\ \eta_q^*\ } \tilde{\pi}_{q+1} \to H_{q+1}(X'; \mathbb{Z}) \to 0,$$

by Theorem XII.3.12 of [28]. Here η_q^* is induced by composition with η_q and the unlabeled homomorphisms are Hurewicz homomorphisms. This sequence can also be derived from the Atiyah-Hirzebruch spectral sequence for $\pi_*^{st}(X')$, which gives another exact sequence

$$(3) \qquad 0 \to H_q(X'; \pi_2^{st}) \to \Pi = \pi_{q+2}^{st}(X') \xrightarrow{\ \theta\ } H_{q+1}(X'; \pi_1^{st}) \to 0.$$

These are sequences of Λ-modules, by the naturality of the spectral sequence.

In our situation $H_{q+2}(X'; \mathbb{Z}) = 0$ and the right-hand portion of sequence (2) reduces to a short exact sequence of Λ-modules. Thus $\tilde{\pi}_{q+1}$ is determined by A and a class in $Ext_{\Lambda}^1(B, A/2A) = Ext_{\Lambda}^1(e^1(A/T(A)), A/2A)$. (See Theorem XII.4.16 of [28] for more on the corresponding class in $Ext_{\mathbb{Z}}^1(B, A/2A)$.) Moreover, $\tilde{\pi}_{q+1}$ is stable: $\pi_{q+1}^{st}(X') \cong \pi_{q+1}(X')$, since X' is highly connected. There is an analogous sequence when $q = 2$, involving the universal quadratic functor $\Gamma(A)$ instead of $A/2A$. In this case $\tilde{\pi}_{q+1} = \pi_3(X)$ is not a stable homotopy group, there are nontrivial Whitehead products in the image of $\Gamma(A)$ and η has infinite order. (See [11] for invariants for simple 4-knots.)

We may identify $H_q(X'; \pi_1^{st})$ and $H_q(X'; \pi_2^{st})$ with $A/2A$, since $H_{q-1}(X'; \mathbb{Z}) = 0$ and $\pi_1^{st} = \pi_2^{st} = Z/2Z$. Composition (on the right) with η_{q+1} induces a homomorphism $\eta_{q+1}^* : \tilde{\pi}_{q+1} \to \Pi$, which factors through $\tilde{\pi}_{q+1}/2\tilde{\pi}_{q+1}$, since η_{q+1} has order 2. Reduction mod (2) induces a monomorphism ρ_2 from $B/2B$ with cokernel $Tor(A, Z/2Z)$, by the Universal Coefficient Theorem for homology. Together, the

sequences (2) and (3) give a commutative diagram

$$
\begin{array}{ccccccccc}
 & 0 & \longrightarrow & A/2A & \xrightarrow{\eta_q^*} & \widetilde{\pi}_{q+1}/2\widetilde{\pi}_{q+1} & \longrightarrow & B/2B & \longrightarrow & 0 \\
(4) & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \eta_{q+1}^*} & & \downarrow{\scriptstyle \rho_2} & & \downarrow \\
 & 0 & \longrightarrow & A/2A & \xrightarrow{\alpha} & \Pi & \xrightarrow{\theta} & H_{q+1}(X';\mathbb{F}_2) & \longrightarrow & 0.
\end{array}
$$

The chain complex of the covering space X' is naturally a complex of Λ-modules, and is chain homotopy equivalent to a finitely generated free complex C_*. As $H_j(C_*) = 0$ if $j > q + 2$ and $p.d._\Lambda H_{q+1}(C_*) \leq 1$ we may assume that C_* is the direct sum of the standard resolution of the augmentation Λ-module $\mathbb{Z}$ and a simple Λ-complex C_*^{rel} concentrated in degrees $[q, q+2]$. (This is homotopy equivalent to the equivariant chain complex of the pair (X, S^1) where S^1 is the meridian.) Therefore X is homotopy equivalent to a CW complex of dimension $q + 2$ [27]. Poincaré duality for $(X, \partial X)$ reduces to a chain homotopy equivalence $D(C^{rel})[-2q - 2]_* \simeq C_*^{rel}$, and so it follows from Theorem 1 that C_*^{rel} and C_* are determined up to chain homotopy equivalence by A.

The universal covering space of $L_Z(A, q)$ is $K(A, q)$, and the spectral sequence for the classifying map from $L_Z(A, q)$ to $S^1 = K(Z, 1)$ gives an exact sequence

$$
(5) \qquad 0 \to Ext_\Lambda^2(A, \widetilde{\pi}_{q+1}) \to H^{q+2}(L_Z(A, q); \widetilde{\pi}_{q+1}) \to Hom_\Lambda(A/2A, \widetilde{\pi}_{q+1}) \to 0.
$$

The image of $\kappa(X)$ in $Hom_\Lambda(A/2A, \widetilde{\pi}_{q+1})$ is the homomorphism η_q^*. (It suffices to check the corresponding assertion for $\kappa(X')$, which follows from Lemma 4.15 of Chapter XII of [28], since $Hom_\Lambda(A/2A, \widetilde{\pi}_{q+1}) \leq Hom_\mathbb{Z}(A/2A, \widetilde{\pi}_{q+1})$.) The corresponding sequence with coefficients B reduces to an isomorphism $Ext_\Lambda^2(A, B) \cong H^{q+2}(L_Z(A, q); B)$, since B is $\mathbb{Z}$-torsion free, and the image of $\kappa(X)$ under the coefficient homomorphism induced by h is $\kappa(C_*)$.

Theorem 3. *Let A and $\widetilde{\pi}_{q+1}$ be finitely generated Λ-torsion modules and $\kappa \in H^{q+2}(L_Z(A, q); \widetilde{\pi}_{q+1})$. Suppose that the image of κ in $Hom_\Lambda(A/2A, \widetilde{\pi}_{q+1})$ is a monomorphism with cokernel B, and that $p.d._\Lambda B \leq 1$. Then there is a finite $(q+2)$-complex X_κ such that $\pi_1(X_\kappa) \cong Z$, $\pi_q(X_\kappa) \cong A$, $\pi_{q+1}(X_\kappa) \cong \widetilde{\pi}_{q+1}$, $\widetilde{H}_j(X_\kappa; \Lambda) = 0$ if $j \neq q$ or $q+1$, and $\kappa(X) = \kappa$. The homotopy type of X is determined by A, $\widetilde{\pi}_{q+1}$ and κ.*

Proof. These invariants determine a Postnikov $(q + 1)$-stage $P = P_{q+1}(\kappa)$. We may assume the k-skeleton $P^{[k]}$ is finite for all $k \geq 0$, since A and $\widetilde{\pi}_{q+1}$ are finitely generated Λ-modules and Λ is noetherian. The hypothesis on κ implies that $H_{q+1}(P^{[q+2]}; \Lambda) = H_{q+1}(P; \Lambda) \cong B$ and that composition with η_q determines a monomorphism from $A/2A$ to $\widetilde{\pi}_{q+1}$. Hence the Hurewicz homomorphism for $P^{[q+2]}$ in degree $q + 2$ is onto. Since $p.d._\Lambda B \leq 1$ and $P^{[q+2]}$ is a finite $(q + 2)$-complex $H_{q+2}(P^{[q+2]}; \Lambda)$ is a finitely generated free module. Attaching $(q+3)$-cells to $P^{[q+2]}$ along representatives for a basis of $H_{q+2}(P^{[q+2]}; \Lambda)$ gives a finite $(q + 3)$-complex with trivial homology in degrees $\geq q + 2$, and which is therefore homotopy equivalent to a finite $(q + 2)$-complex X_κ, by [27].

Let Y be a $(q+2)$-complex with Postnikov $(q+1)$-stage $f : Y \to P$ and such that $H_{q+2}(Y; \Lambda) = 0$. Then f factors through a map $g : Y \to X_\kappa$, since we may construct P by adjoining cells of dimension $\geq q + 3$ to X_κ, to kill the higher homotopy groups. The lift of g to the universal covers induces isomorphisms on homology and so is a

homotopy equivalence, by the Hurewicz and Whitehead Theorems. Therefore g is a homotopy equivalence. □

It is easy to construct a finite $(q + 2)$-complex with equivariant chain complex of the required chain homotopy type, using only the surjectivity of the Hurewicz homomorphism in degree $q + 1$ (and not appealing to [27]).

However not all such complexes admit Poincaré duality isomorphisms. Let $D_* = D(C^{rel})[-2q - 2]_*$. It is clear from Theorem 1 that there is a chain homotopy equivalence $D_* \simeq C_*^{rel}$ if and only if $k(A)$ and $\kappa(C_*)$ correspond.

An analogous but simpler argument shows that if K is a simple $(2q-1)$-knot the homotopy type of $X = X(K)$ is determined by the module $A = H_q(X; \Lambda)$ alone.

5. Farber quintuples and Kearton F-forms

In this section we shall define the Farber quintuple for simple $2q$-knots. This reduces to the torsion linking pairing in the odd finite case, and to an F-form in the torsion-free case. These cases were studied slightly earlier [14, 17]. However, in the latter case it is not clear that it is the *same F-form*. In other, earlier work Kearton and Kojima applied different systems of invariants to the odd torsion case and to torsion-free, fibred simple knots, respectively [13, 16]. Central to most of these approaches is the homotopy linking pairing

$$\mathcal{L}_V : \pi_{q+1}(V) \times \pi_{q+1}(V) \to \pi_1^{st} = Z/2Z,$$

for V a $(q-1)$-connected Seifert hypersurface for K. (The group $\pi_{q+1}(V)$ is stable, and homotopy classes $u, v \in \pi_{q+1}(V)$ can be represented by embedded spheres, since V is highly connected. These are unknotted in S^{2q+2}, since they have codimension $q + 1 > 2$, and so the pushoff of v defines an element of $\pi_{q+1}(S^{2q+2} \setminus S^{q+1}) = \pi_1^{st}$.) The invariants of [13, 16] involve complicated equivalence relations, and the connection with Farber's work is again not clear.

An *F-form* is a triple $(A, \mathcal{E}, [-, -])$, where A is a finitely generated Λ-module with $T(A)$ odd and $t - 1$ acting invertibly, $\mathcal{E}$ is an exact sequence of Λ-modules

$$\mathcal{E} : 0 \to A/2A \to \Pi \to B/2B \to 0,$$

and $[-, -] : \Pi \times \Pi \to \mathbb{F}_2$ is a nonsingular pairing on which t acts isometrically, and such that the image of $A/2A$ is self-annihilating ($[a, b] = 0$ for all $a, b \in A/2A$) and the induced pairing $\langle -, - \rangle : A/2A \times B/2B \to \mathbb{F}_2$ is nonsingular. In particular, Π has exponent 2 and $|A/2A| = |B/2B|$. (We have used the natural transformation χ_Π to modify Kearton's formulation; he used a nonsingular hermitean pairing into $\mathbb{F}_2(t)/\mathbb{F}_2\Lambda$ instead.)

The F-form associated to a simple $2q$-knot K has $A = H_q(X; \Lambda)$, $\Pi = \tilde{\pi}_{q+1}/2\tilde{\pi}_{q+1}$ and $B = H_{q+1}(X; \Lambda)$, and the sequence $\mathcal{E}$ is the *mod (2)* reduction of sequence (2) above. The pairing $[-, -]$ is defined for any simple $2q$-knot with $q \geq 3$, and is nonsingular if $T(A)$ is odd. The latter hypothesis is needed only for Lemmas 1.10 and 1.13 of §1 of [14].) The pairing $\langle -, - \rangle$ is the Milnor duality pairing for X' with coefficients $\mathbb{F}_2$, if $T(A)$ is odd. Kearton showed that the F-form is a complete invariant for torsion-free simple $2q$-knots with $q \geq 4$ [14]; this was extended to torsion-free simple 6-knots in [9].

A *Farber quintuple* $(A, \Pi, \alpha, \ell, \psi)$ (in dimension q) consists of a pair of finitely generated Λ-modules A and Π such that $t-1$ acts invertibly on A, a monomorphism

$\alpha : A/2A \to \Pi$ and nonsingular $(-1)^{q+1}$-symmetric pairings

$$\ell : T(A) \times T(A) \to \mathbb{Q}/\mathbb{Z} \quad \text{and} \quad \psi : \Pi \times \Pi \to Z/4Z$$

on which t acts isometrically. In other words the *adjoint* functions $Ad(\ell)$ and $Ad(\psi)$ are Λ-isomorphisms, where $Ad(\ell) : T(A) \to Hom_{\mathbb{Z}}(T(A), \mathbb{Q}/\mathbb{Z})$ is given by $Ad(\ell)(b)(a) = \ell(a, b)$ for all $a, b \in T(A)$, and $Ad(\psi)$ is defined similarly. Let $\beta = \alpha^* Ad(\psi) : \Pi \to Hom_{\mathbb{F}_2}(A/2A, \mathbb{F}_2)$, so that $\beta(p)(a) = \psi(p, \alpha(a))$ for all $a \in A/2A$ and $p \in \Pi$. Then β is an epimorphism and $\mathrm{Ker}(\beta) = \mathrm{Im}(\alpha)$. These pairings interact in the following way. If M is a Λ-module and $m \in M$ let $[m]_2$ be the image of m in $M/2M$. Let $\gamma : \Pi \to T(A)$ be the homomorphism determined by $\psi(p, \alpha([x]_2)) = \ell(\gamma(p), x)$ for all $x \in T(A)$ and $p \in \Pi$ and nonsingularity of ℓ. Then $\alpha([\gamma(p)]_2) = 2p$, for all $p \in \Pi$. In particular, Farber's Π has exponent 4, and $\gamma = 0$ if and only if $T(A)$ has odd order.

The Farber quintuple of K is $q_F(K) = (A, \Pi, \alpha, \ell, \psi)$, where $A = H_q(X; \Lambda)$, $\Pi = \pi_{q+2}^{st}(X')$, $\alpha : A/2A \to \Pi$ is the monomorphism determined by composition with $\eta_q \eta_{q+1}$, $\ell : T(A) \times T(A) \to \mathbb{Q}/\mathbb{Z}$ is the torsion linking pairing and $\psi : \Pi \times \Pi \to Z/4Z < \pi_3^{st}$ is a bilinear pairing derived from Spanier-Whitehead duality for a $(q-1)$-connected Seifert hypersurface V for K [6].

In outline, Farber's construction of ψ is as follows. Let $N(V)$ be a regular neighbourhood of V in S^{2q+2} and let $Y = S^{2q+2} \setminus N(V)$. We may assume that $V \cup Y \subset \mathbb{R}^{2q+2} = S^{2q+2} \setminus \{\infty\}$. Then $(v, y) \mapsto \frac{v-y}{||v-y||}$ defines a map from $V \times Y$ to S^{2q+1} which is nullhomotopic on $V \vee Y$ and hence induces a (Spanier-Whitehead) map $SW_V : V \wedge Y \to S^{2q+1}$. The orientations for K and S^{2q+2} determine an orientation for V and an oriented normal vectorfield, and hence a preferred pushoff map $i_+ : V \to Y$. Let $u : V \wedge V \to S^{n+1}$ and $z : V \to V$ be stable maps corresponding to the duality pairing and carving map of [6], respectively. Let

$$\psi_V(\alpha, \beta) = u \circ (\alpha \wedge \beta) \in \pi_{2q+4}^{st}(S^{2q+1}) = \pi_3^{st} \cong Z/24Z$$

for all $\alpha, \beta \in \pi_{q+2}^{st}(V)$. Let $P = \mathbb{Z}[z]$, $\bar{z} = 1 - z$ and $L = P[(z\bar{z})^{-1}]$. If we identify $t \in \Lambda$ with $1 - z^{-1} \in L$ then we also have $L = \Lambda[(1-t)^{-1}]$. Then $\pi_{q+2}^{st}(X') \cong L \otimes_P \pi_{q+2}^{st}(V)$ and ψ_V extends to a pairing ψ on $\pi_{q+2}^{st}(X')$ which is independent of the choice of Seifert hypersurface V. (See §10 of [6].)

Suppose now that $T(A)$ has odd order. Then Π has exponent 2, ℓ and ψ are independent and ψ is symmetric. In this case, a Farber quintuple is algebraically equivalent to an F-form together with a torsion linking pairing. If $A = 2A$ then $\Pi = 0$ and $q_F(K)$ reduces to the pair (A, ℓ). In particular, if A is finite this gives Kojima's classification of odd finite simple $2q$-knots [17]. If A is torsion-free $q_F(K)$ reduces to an F-form. We can show that the exact sequence $\mathcal{E}$ of this reduction agrees with that of Kearton, but it is not clear how the pairings ψ and $[-, -]$ correspond.

Farber's construction gives an analogous pairing Ψ on $\pi_{q+1}^{st}(X') \cong L \otimes_P \pi_{q+1}^{st}(V)$. Let

$$\Psi_V(\alpha, \beta) = u \circ (\alpha \wedge \beta) \in \pi_{2q+2}^{st}(S^{2q+1}) = \pi_1^{st} = Z/2Z$$

for all $\alpha, \beta \in \pi_{q+1}^{st}(V)$. Then Ψ_V extends to a pairing Ψ on $\pi_{q+1}^{st}(X')$, which is again independent of the choice of Seifert hypersurface V.

Lemma 4. *If $T(A)$ has odd order then $\widetilde{\pi}_{q+1}/2\widetilde{\pi}_{q+1} \cong \Pi = \pi_{q+2}^{st}(X')$, and ψ and Ψ are equivalent.*

Proof. Since $T(A)$ has odd order, $Tor(A, Z/2Z) = 0$, and so the homomorphism ρ_2 in diagram (4) is an isomorphism. Hence $\eta_{q+1}^* : \widetilde{\pi}_{q+1}/2\widetilde{\pi}_{q+1} \cong \Pi$ is also an isomorphism. Composition on the right with η^2 induces a monomorphism from π_1^{st} to π_3^{st}, and $\eta \wedge \eta = \eta^2 \in \pi_2^{st}$, by X.8.12 of [28]. Hence $\Psi(a, b) = \psi(a\eta_{q+1}, b\eta_{q+1})$ for all $a, b \in \pi_{q+1}^{st}(X')$, and so η_{q+1}^* is an isometry. $\square$

We expect that $\Psi = [-, -]$. If so, then the Farber quintuple and the Kearton F-form agree for torsion-free simple $2q$-knots.

The F-form and the torsion linking pairing have the advantage of being defined in terms of invariants of the exterior. Are there direct definitions of ψ or Ψ which does not involve the choice of a Seifert hypersurface? For instance, can they be defined directly in terms of a Z-equivariant SW duality? (See also [5] for the case of fibred simple $2q$-knots.)

These invariants are defined for $q \geq 1$, but simple 2-knots are topologically trivial, and the natural invariants for simple 4-knots are not stable, so it is unlikely that the Farber quintuple is a complete invariant when $q = 2$. (See instead [11].) It may however suffice when $q = 3$. There is an obvious notion of sum of Farber quintuples, and $q_F(K \# \widetilde{K}) = q_F(K) \oplus q_F(\widetilde{K})$. Can the Farber invariant be extended to all even-dimensional knots, as an additive invariant?

6. Recovering the homotopy type of X from $q_F(K)$

The first three ingredients of $q_F(K)$ are determined by the homotopy type of X alone, while ℓ and ψ are manifestations of Poincaré duality for the pair $(X, \partial X)$. We shall show that, conversely, $q_F(K)$ determines $\widetilde{\pi}_{q+1}$ and $\kappa(X)$ and hence the homotopy type of X, at least if $T(A)$ has odd order. We shall consider the duality pairings and how they interact with the inclusion of ∂X into X in the next two sections.

Let $pd_Z : B = H_{q+1}(X; \Lambda) \to e^1 A$ and $pd_2 : H_{q+1}(X; \mathbb{F}_2\Lambda) \to E(A/2A) = Ext^1_{\mathbb{F}_2\Lambda}(A/2A, \mathbb{F}_2\Lambda)$ be the isomorphisms determined by Poincaré duality and the Universal Coefficient spectral sequence, and let $\tau : E(A/2A) \cong Hom_{\mathbb{F}_2}(A/2A, \mathbb{F}_2)$ be the isomorphism defined in §2. Reduction *mod* (2) defines a homomorphism $\rho : B \to H_{q+1}(X; \mathbb{F}_2\Lambda)$. It also induces a natural isomorphism $\mathbb{F}_2 \otimes_{\mathbb{Z}} e^1(A/T(A)) \cong E(A/(T(A), 2A))$, and hence a natural transformation

$$e : e^1 A \cong e^1(A/T(A)) \to E(A/(T(A), 2A)) \to E(A/2A),$$

and the diagram

(6)
$$\begin{array}{ccccc} B & \longrightarrow & \overline{H^{q+1}(X; \Lambda)} & \longrightarrow & e^1 A \\ \downarrow{\scriptstyle \rho} & & \downarrow & & \downarrow{\scriptstyle e} \\ H_{q+1}(X; \mathbb{F}_2\Lambda) & \longrightarrow & \overline{H^{q+1}(X; \mathbb{F}_2\Lambda)} & \longrightarrow & E(A/2A) \end{array}$$

is commutative. (The horizontal maps are all isomorphisms.)

Lemma 5. *The module $\widetilde{\pi}_{q+1}$ is the pullback of τe and β over $Hom_{\mathbb{F}_2}(A/2A, \mathbb{F}_2)$, i.e., it is the fibre sum $\widetilde{\pi}_{q+1} \cong \{(b, p) \in e^1 A \times \Pi \mid \tau e(b) = \beta(p)\}$.*

Proof. This follows easily from diagrams (4) and (6), since $\beta = \tau pd_2 \theta$ (see Theorem 3.3(b) of [6]). $\square$

Theorem 6. *If $T(A)$ has odd order the homotopy type of X is determined by $q_F(K)$.*

Proof. If $T(A)$ has odd order $Ext^2_\Lambda(A, \widetilde{\pi}_{q+1}) \cong Ext^2_\Lambda(T(A), e^1(A/T(A)))$ is the odd-order summand of $H^{q+2}(L_Z(A, q); \widetilde{\pi}_{q+1})$. Since the image of $\kappa(X)$ in the 2-primary summand is η^*_q it follows that $[\kappa(X)]$ is determined by $[\kappa(C^{rel}_*)]$ and α, and hence by A and α. Thus the homotopy type of X is determined directly by $q_F(K)$, as in Theorems 3 and 4. $\square$

If $T(A)$ is odd α is the composite of the monomorphism η^*_q with reduction *mod* (2) and $q_F(K)$ is algebraically equivalent to the torsion linking pairing ℓ together with an F-form. If $A = 2A$ then $\Pi = 0$, $\widetilde{\pi}_{q+1} \cong e^1A$ and $\kappa(X) = \kappa(C^{rel}_*) \in Ext^2_\Lambda(A, e^1A)$. If instead A is torsion free $Ext^2_\Lambda(A, \widetilde{\pi}_{q+1}) = 0$ and so $\kappa(C^{rel}_*) = 0$. Hence $\kappa(X)$ is determined by α, which lifts to a monomorphism $A/2A \cong \{(0, p) \in B \times \Pi \mid \beta(p) = 0\} \leq \widetilde{\pi}_{q+1}$ in the above fibre sum.

7. Inclusion of the boundary

Let $Y = S^{2q} \times S^1$ have the product orientation and let $J_K : Y \cong \partial X \subset X$ be the inclusion of the boundary. The oriented homotopy type of X and the class $[J(K)]$ of J_K modulo homotopy and composition with orientation preserving self homotopy equivalences of X and Y together determine the oriented homotopy type of the pair $(X, \partial X)$. The difficulty is in giving a practical characterization of $[J(K)]$. In this section we shall mention two of the known constraints.

Let $J : Y \to X$ be a map which induces an isomorphism on fundamental groups. Let $M(J)$ be the mapping cylinder of J. There is a canonical homotopy equivalence $M(J) \simeq X$. The orientation for Y determines a generator μ_J for $H_{2q+2}(M(J), Y; \mathbb{Z}) \cong H_{2q+1}(Y; \mathbb{Z})$. Since the universal covers of $M(J) \simeq X$ and Y are highly connected cap product with μ_J determines homomorphisms $\overline{H^{q+1}(X; \Lambda)} \to H_{q+1}(X; \Lambda)$ and $\overline{H^{q+2}(X; \Lambda)} \to H_q(X; \Lambda)$. The pair $(M(J), Y)$ is a PD_{2q+2}-pair if and only if these homomorphisms are isomorphisms. (The corresponding homomorphisms in other degrees are isomorphisms, in all cases.)

The map $J' : Y' = S^{2q} \times R \to X'$ covering J determines an element $\lambda_J \in \pi_{2q}(X') = \pi_{2q}(X)$, such that $(t - 1)\lambda_J = 0$. In the fibred case the Poincaré duality constraints depend only on λ_J. The longitude λ_K corresponds to J_K. Lemma 3 of [11] asserted that *if X is an n-dimensional homology circle with $H_n(X; \Lambda) = 0$ and two maps $j_1, j_2 : S^n \times S^1 \to X$ induce isomorphisms on π_1 and agree on $S^n \vee S^1$ then there is a self-homotopy equivalent $f : X \to X$ such that $j_2 \sim fj_1$.* If this were correct it would follow not only that λ_K determines J_K for 1-simple knots but also that such knots are determined by their exteriors. The proof of Lemma 3 in [11] is wrong, and thus the later assumptions in that paper that the homotopy quadruple discussed there is a complete invariant for simple 4-knots are unjustified. (We thank W.Richter for pointing out our error). Nevertheless it remains possible that the result may be true, at least if X' is sufficiently highly connected.

The inclusion of the boundary of a Seifert hypersurface V for K has trivial suspension in $\pi_{2q+1}(SV)$ [25]. Therefore λ_K is in the kernel of the suspension $E : \pi_{2q}(X') \to \pi_{2q+1}(SX')$. The group $\pi_{2q}(X')$ is just outside the stable range, since X' is $(q - 1)$-connected, and sits near one extremity of the EHP sequence (Theorem XII.2.2 of [28]):

$$\pi_{3q-2}(X') \to \cdots \to \pi_{2q+2}(SX') \to \pi_{2q+2}(X' * X') \to \pi_{2q}(X') \to \pi_{2q+1}(SX') \to 0.$$

Moreover $X' \simeq M_A \vee M_B$ is a wedge of Moore spaces $M_A = M(A, q)$ and $M_B = M(B, q + 1)$, since X' is $(q - 1)$-connected, $H_j(X'; \mathbb{Z}) = 0$ for $j > q + 1$ and $q > 2$. Hence $\pi_{2q}(X') \cong \pi_{2q}(M_A) \oplus \pi_{2q}(M_B) \oplus [A, B]$, where $[A, B]$ is the subgroup generated by all Whitehead products $[x, y]$ with $x \in \pi_q(M_A) \cong A$ and $y \in \pi_{q+1}(M_B) \cong B$. The summand $\pi_{2q}(M_B)$ is stable, while all Whitehead products have trivial suspension.

Suppose now that K is fibred and $A \cong Z^\beta$ is torsion-free. Then $M_A \simeq \vee^\beta S^q$ and $M_B \simeq \vee^\beta S^{q+1}$. Let $\{i_q^m\}_{m \leq \beta}$ and $\{i_{q+1}^n\}_{n \leq \beta}$ be bases for A and the image of $B \cong \pi_{q+1}(\vee^\beta S^{q+1})$ in $\pi_{q+1}(X')$, respectively. In this case $\mathrm{Ker}(E)$ is generated by all Whitehead products $i_{mm'} = [i_q^m, i_q^{m'} \eta_{q+1}]$ and $j_{mn} = [i_q^m, i_{q+1}^n]$ for all $m, m', n \leq \beta$. Thus $\lambda_K = \Sigma c_{mm'} i_{mm'} + \Sigma d_{mn} j_{mn}$ for some coefficients $c_{mm'} \in \mathbb{F}_2$ and $d_{mn} \in \mathbb{Z}$. The integral coefficients d_{mn} may be detected by computing cup products in $(S^q \vee S^{q+1}) \cup_{p\lambda} e^{2q+1}$ for suitable projections $p : X' \to S^q \vee S^{q+1}$. (See [12], and note that $S^q \vee S^{q+1} \cup_{[i,j]} e^{2q+1} \simeq S^q \times S^{q+1}$.) Hence the matrix $D = [d_{mn}]$ is invertible, since (X', λ_K) is a PD_{2q+1}-complex, and we may choose the basis for B so that $D = I_\beta$.

Question. *How are the coefficients $c_{mm'}$ determined by the pairing Ψ?*

An analogous but simpler argument applies if K is a fibred simple $(4k-1)$-knot. For then $X' \simeq \vee^\beta S^{2k}$ and $\mathrm{Ker}(E) \leq \pi_{4k-1}(X')$ is generated by the Whitehead products $[i_{2k}^m, i_{2k}^n]$ for $m, n \leq \beta$. The coefficients e_{mn} of $\lambda_K = \Sigma e_{mn}[i_{2k}^m, i_{2k}^n]$ may be detected by cup products in $S^{2k} \cup_{p\lambda} e^{4k}$ or $(S^{2k} \vee S^{2k}) \cup_{p\lambda} e^{4k}$ for suitable projections $p : X' \to S^{2k}$ (and using the Hopf invariant, as on page 211 of [21]) or $p : X' \to S^{2k} \vee S^{2k}$ (and using [12]).

When $T(A) \neq 0$ there may be elements of $\mathrm{Ker}(E)$ which are not Whitehead products, as can be seen already when $A = Z/pZ$. This is the situation that arises for odd finite simple $2q$-knots, for which $A = T(A)$ is finite of odd order and the Farber quintuple reduces to the torsion linking pairing ℓ. (See [17].)

8. Hermitian self-dualities

Let $\mathcal{C}$ be an additive category with an involution $*$ and let $\varepsilon = \pm 1$. An ε-hermitian pairing on an object C of $\mathcal{C}$ is defined to be an isomorphism $\phi : C \to C^*$ such that $\phi^* = \varepsilon\phi$. This categorical point of view is particularly useful if the Krull-Schmidt Theorem holds for the objects of $\mathcal{C}$ and the endomorphism rings of objects are radically complete. For then the analysis of the decompositions of hermitian pairings into orthogonal direct sums may be reduced to corresponding questions for pairings over (skew) fields [22].

Consider triples $Q = (A, \theta, \mathcal{E})$ where A is a finitely generated Λ-module on which $t - 1$ acts invertibly, $\theta : T(A) \to e^2 A$ is an isomorphism and $\mathcal{E}$ is an exact sequence of Λ-modules

$$(7) \qquad 0 \to A/2A \xrightarrow{\alpha} \Pi \xrightarrow{\omega} e^2(A/2A) \to 0.$$

A morphism between two such quadruples is a triple $\phi = (f, g, h)$ where $f, g : A \to A$ and $h : \Pi \to \Pi'$, such that $\theta' f|_{T(A)} = e^2 g \theta$, $h\alpha = \alpha'[f]_2$ and $\omega' h = e^2[g]_2 \omega$. The category $\mathcal{Q}$ of such quadruples is additive.

Define a duality functor by $Q^* = (A, \theta, \mathcal{E})^* = (A, e^2\theta, e^2\mathcal{E})$ and $(f, g, h)^* = (g, f, e^2h)$. An isomorphism $\phi = (f, g, h)$ from Q to Q^* is an *ε-hermitian self-duality* of Q if $\phi^* = \varepsilon\phi$, in which case f and g are automorphisms of A, $f = \varepsilon g$ and $h = \varepsilon e^2 h$. A pair (Q, ϕ) with ϕ an ε-hermitian self-duality of q is *hyperbolic* if there

is an object N in $\mathcal{Q}$ such that $Q \cong N \oplus N^*$ and ϕ has matrix $\begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$ with respect to this decomposition.

Theorem 7. *Farber quintuples for simple $2q$-knots correspond bijectively to pairs (Q, ϕ) where $Q = (A, \theta, \mathcal{E})$ is an object in $\mathcal{Q}$ and $\phi = (f, g, h)$ is a $(-1)^{q+1}$-hermitian self-duality of Q such that $\alpha[\gamma(p)]_2 = 2p$ for all $p \in \Pi$, where $\gamma(p)$ is determined by $e^2 g \theta(\gamma(p))(x) = \sigma_\Pi h(\alpha([x]_2)(p))$ for all $x \in T(A)$ and $p \in \Pi$.*

Proof. Let $(A, \Pi, \alpha, \ell, \psi)$ be a Farber quintuple with $(-1)^{q+1}$-symmetric pairings ℓ and ψ and let β be the associated epimorphism. Let $\theta = \sigma_A^{-1} Ad(\ell)$, $\omega = \sigma_A^{-1} \beta$ and $\mathcal{E}$ be the exact sequence determined by α and ω. Then $Q = (A, \theta, \mathcal{E})$ is in $\mathcal{Q}$ and $(1_A, 1_A, \sigma_\Pi^{-1} Ad(\psi))$ is a $(-1)^{q+1}$-hermitian self-duality of Q.

Conversely, if $\phi = (f, g, h)$ is a $(-1)^{q+1}$-hermitian self-duality of $(A, \theta, \mathcal{E})$ we may define pairings ℓ on $T(A)$ and ψ on Π by setting $\ell(a, a') = e^2 g \theta(a')(a)$ for all $a, a' \in T(A)$ and $\psi(p, q) = \sigma_\Pi(h(q))(p)$ for all $p, q \in \Pi$. (We then have $\beta = \sigma_A e^2 g \omega$.) Then $(A, \Pi, \alpha, \ell, \psi)$ is almost a Farber quintuple; we need only the additional condition that $\alpha[\gamma(p)]_2 = 2p$ for all $p \in \Pi$, where γ is defined as in §5. The equation in the statement of the theorem is a reformulation of this condition in terms of the constituents of ϕ. $\qquad\square$

The Krull-Schmidt Theorem holds in any additive category in which

(1) all idempotents split; and
(2) every object is a finite sum of objects whose endomorphism rings are local rings. (Such summands are necessarily indecomposable).

It is easily verified that all idempotents in $\mathcal{Q}$ split. We shall use the next lemma to determine the objects of $\mathcal{Q}$ which satisfy the second condition.

Lemma 8. *Let R be a local Λ-algebra which is finitely generated as a Λ-module. Then R is finite.*

Proof. Let $p : \Lambda \to R$ be the natural homomorphism. Let $J = \mathrm{rad}(R) = R \setminus R^\times$ and $m = p^{-1}(J)$. The skewfield R/J is finitely generated as a Λ/m-module. It follows easily that Λ/m is a field. In particular m is a maximal ideal (and R/J is a finite field). Since $p(\Lambda)_m \leq R$ it is an integral extension of $p(\Lambda)$, and every prime ideal of $p(\Lambda)$ is the restriction of a prime ideal of $p(\Lambda)_m$, by Theorem 5.10 of [1]. Thus $p(\Lambda)$ is also a local ring, and so m is the unique maximal ideal of Λ which contains $K = \mathrm{Ker}(p)$. Consideration of the primary decomposition of K shows that K must be m-primary and $p(\Lambda) = \Lambda/K$ finite. Hence R is also finite. $\qquad\square$

An object N in an abelian category $\mathcal{C}$ is a *torsion* object if the ring $End_\mathcal{C}(N) = \mathcal{C}(N, N)$ is finite.

Theorem 9. *Let $Q = (A, \theta, \mathcal{E})$ in $\mathcal{Q}$. Then the following are equivalent*

(1) *A is a finite Λ-module;*
(2) *Q is a torsion object;*
(3) *Q is a finite sum of objects whose endomorphism rings are local rings.*

The Krull-Schmidt Theorem holds in the full subcategory of $\mathcal{Q}$ determined by the torsion objects, and the endomorphism rings of such objects are radically complete.

Proof. It is immediate that $End_\mathcal{Q}(Q)$ is commensurable with $End_\Lambda(A) \times End_\Lambda(A)$. Thus $End_\mathcal{Q}(Q)$ is finite if and only if $End_\Lambda(A)$ is finite. It follows easily that Q is a torsion object in $\mathcal{Q}$ if and only if A is finite. Thus $(1) \Leftrightarrow (2)$.

In a finite ring some power of each element is idempotent. Thus a finite ring with no nontrivial idempotents is local. Since idempotents in $\mathcal{Q}$ split it follows that $(2) \Rightarrow (3)$.

If $End_{\mathcal{Q}}(Q)$ is local then it is finite, by the lemma, and so Q is a torsion object. Thus $(3) \Rightarrow (2)$.

The final assertion is now clear, since the radical of an Artinian ring is nilpotent.
$\square$

If $(A, \theta, \mathcal{E})$ is a torsion object in $\mathcal{Q}$ then $\theta : A \cong e^2 A$ and $e^2(A/2A) \cong {}_2A$, and so the full subcategory of $\mathcal{Q}$ determined by the torsion objects is essentially the category considered in [8].

In [7, 8] and [10] certain classes of knots were classified by similar reductions to pairings over finite fields, and in [7] and [8] these invariants were applied to questions related to double null concordance for such knots. Kearton had earlier shown that a torsion-free simple $2q$-knot is doubly null concordant if and only if its F-form was hyperbolic, in an appropriate sense [15]. Thus we expect that *a simple $2q$-knot K with $q \geq 4$ is doubly null concordant if and only if $q_F(K)$ is hyperbolic.* Moreover, *a simple $2q$-knot K with $q \geq 4$ is stably doubly null concordant if and only if the Witt class of $q_F(K)$ is trivial.* Establishing the latter result may require extending $q_F(-)$ to an invariant of all even-dimensional knots.

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