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On Piecewise Regular n-Knots

By MORRIS W. HIRSCH and LEE P. NEUWIRTH

0. Introduction

This paper is divided into two parts. In the first part, we define piecewise regular *n*-knots in S^{n+2} in such a way as to include both smooth and piecewise linear knots. Various equivalence relations are defined. The main point is that two smooth knots are diffeomorphic if they are equivalent; see § 3. Thus smooth homotopy *n*-spheres which are embeddable in S^{n+2} are distinguishable by their knot types. In particular, if $A, B \subset S^{n+2}$ are smooth homotopy *n*-spheres which are not diffeomorphic, then $S^{n+2} - A$ and $S^{n+2} - B$ are not piecewise linearly homeomorphic. In § 4 various groups involving knots are defined.

In Part II, which is independent from Part I, certain algebraic invariants of n-knots are studied. One of these generalizes the Alexander polynomials. A particular smooth 7-knot is examined illustrating the technique.

PART I

1. Definition and background

By a smooth manifold we mean a differential manifold of class C^{∞} . We shall assume the reader is familiar with the ideas of diffeomorphism, smooth triangulation, etc. A good reference is the book [3] of Munkres. In addition, the basic notions of piecewise linear theory are needed; see for example Zeeman [1, 2]. In order to combine smooth and piecewise linear (PL) theory, we use piecewise regular (PR) techniques.

We indicate homotopy by \cong , diffeomorphism by \approx , and PL homeomorphism by \equiv . The identity map of X is 1_x .

By an embedding, we mean a map which is a homeomorphism onto its image. We define a PR embedding of a PL manifold K into a smooth manifold M to be an embedding $f: K \rightarrow M$ such that there exists a commuting diagram

$$(1) \qquad \qquad \begin{matrix} g \\ f \\ K \end{matrix} \begin{matrix} f \\ f \\ M \end{matrix}$$

where L is a PL manifold, $t: L \rightarrow M$ is a smooth triangulation, and $g: K \rightarrow L$ is a PL embedding onto a subcomplex of L. Thus f(K) is closed. We also define a PR embedding of a smooth manifold J into a smooth manifold M to be an embedding $h: J \rightarrow M$ such that there is a commuting diagram



where K is a PL manifold, s: $K \rightarrow J$ is a smooth triangulation, and $f: K \rightarrow M$ is a PR embedding as previously defined. A PR homeomorphism is therefore essentially the same as a smooth triangulation.

By a PR submanifold A of a smooth manifold M we mean the image of a PR embedding into M. Observe that A is necessarily closed.

Let K be a PL submanifold of codimension d of a PL manifold L. We say that K is *locally flat* (in L) if each point of K has a neighborhoood U in L such that putting $U \cap K = V$, we have $(V \times R^{a}, V \times 0) \equiv (U, V)$. Here R^{a} is euclidean d-space.

Referring to diagram (1) above, we say that the PR submanifold f(K) is *locally flat* if g(K) is a locally flat PL submanifold of L. We also say, in this case, that the embeddings f and g are locally flat.

Observe that, as a consequence of the invariance of domain, if K is locally flat in M then $K \cap \partial M = \partial K$ (where " ∂ " denotes boundary).

If K and L are topological manifolds, the definition of K being locally flat in L is just as in the PL case, except that the term "PL" is never mentioned.

Let *M* be a smooth manifold. A subcomplex of *M* is the image t(K) where $t: L \rightarrow M$ is a smooth triangulation, and $K \subset L$ is a subcomplex.

The following lemma can be proved by the theory of triangulating vector bundles; cf. Lashof—Rothenberg [8]. An equivalent theorem was stated by Cairns [6]. (For a dubious proof, see Hirsch [7].)

LEMMA 1.1. A smooth compact unbounded submanifold in the interior of a smooth manifold M is a locally flat PR submanifold of M.

Let M be a smooth manifold, K a PL manifold, and $f: K \rightarrow M$ a smooth triangulation. A PR *isotopy* of f is a PR homeomorphism $F: K \times I \rightarrow M \times I$ such that,

(a) $F(K \times t) = M \times t$

(b) F(x, 0) = (f(x), 0).

By (a), there are unique maps $F_t: K \to M$ such that $F(x, t) = (F_t(x), t)$; it is easy to see that each F_t is a PR homeomorphism. Two subcomplexes A, B of M are called *isotopic* if there is an F as above such that $F_1F_0^{-1}(A) = B$. A similar definition holds for two pairs of subcomplexes. The homotopy $F_tF_0^{-1}$: $M \to M$ is called a PR *isotopy of* M.

EXAMPLE. In the plane, the boundary of a triangle and a circle are isotopic

subcomplexes.

The following version of J. H. C. Whitehead's uniqueness theorem for smooth triangulations can be easily derived from his original paper [5], or Munkres' book [3].

THEOREM 1.2 (Whitehead). Let M be a smooth manifold, and $s: K \to M$, t: $L \to M$ smooth triangulations. There is a PR isotopy $F: K \times I \to M \times I$ of s such that $t^{-1}F_1: K \to L$ is a PL homeomorphism.

2. Smooth regular neighborhoods

In this section we develop the material needed for the diffeomorphism criteria for smooth knots of § 3.

We refer to Hirsch [7] for details on smooth regular neighborhoods (= SRN's). See also Mazur [9].

We need only the following facts from [7]. If M is a smooth manifold and $K \subset M - \partial M$ is a subcomplex of M, in any neighborhood of K there is a smooth submanifold N of M such that:

(a) N is a neighborhood of K;

(b) let P be the second barycentric regular neighborhood of K (in a smooth triangulation of M making K a subcomplex). There is a PR isotopy of M taking (P, K) onto (N, K). The isotopy leaves a neighborhood of K fixed.

(c) If K is a smooth submanifold of M, then N is a tubular neighborhood of K.

We call N an SRN of K.

Now suppose K meets the boundary of M, a case not covered in [7]. By an SRN of K we shall mean a PR submanifold N of M such that:

(a) N is a neighborhood of K;

(b) $N \cap \partial M$ is an SRN of $K \cap \partial M$, as previously defined;

(c) the closure $cl(\partial N - (N \cap \partial M))$ is a smooth manifold;

(d) if P is the second barycentric regular neighborhood of K in M, then (P, K) and (N, K) are PR isotopic in M. Thus N is smooth except for a corner along $\partial(N \cap \partial M)$.

Using the techniques of [7], it is easy to prove

LEMMA 2.1. Let N' be an SRN of $K \cap \partial M$ in ∂M . There is an SRN of K in M, say N, such that $N \cap \partial M = N'$.

In order to state the main results of this section we use the notation " $M_0 \stackrel{h}{\sim} M_1$ " to indicate that smooth manifolds are *h*-cobordant. This means that there is a smooth manifold W whose boundary can be expressed as the disjoint union $M'_0 \cup M'_1$ such that $M'_i \approx M_i$, and M'_i is a deformation retraction

of W(i = 0, 1).

THEOREM 2.2. Let K_i (i=0, 1) be subcomplexes of compact unbounded smooth manifolds M_i . Let $N_i \subset M_i$ be smooth regular neighborhoods of K_i . If $M_0 - K_0 \approx M_1 - K_1$, then $\partial N_0 \stackrel{h}{\sim} \partial N_1$.

PROOF. We may assume K_i and M_i are connected. Let $f: M_0 - K_0 \rightarrow M_1 - K_1$ be a diffeomorphism. From the compactness of M_0 it is easy to see that $N \subset M_0$ is a neighborhood of K_0 if and only if the closure $cl(M_1 - f(M_0 - N))$ is a neighborhood of K_1 . Therefore if N_0 is an SRN of K_0 , there is an SRN N_1 of K_1 such that $N_1 - K_1 \subset int f(N_0 - K_0)$, and similarly for f^{-1} . Hence we can find SRN's X_i , Y_i , Z_i of K_i such that:

(a) $Z_i \subset \text{int } Y_i \text{ and } Y_i \subset \text{int } X_i$;

(b) $f(X_0 - K_0) \subset \text{int } X_1 - K_1$, and similarly for Y_i and Z_i .

Let V be the connected submanifold of X_1 bounded by $\partial Y_1 \cup f(\partial Y_0)$.



We shall show that $f(\partial Y_0)$ is a deformation retract of V, and the reader will then be able to show that ∂Y_1 is a deformation retract of V; together we shall have proved Theorem 2.2.

Define A and B to be the connected submanifolds of X_1 bounded respectively by $f(\partial X_0) \cup f(\partial Y_0)$ and $\partial X_1 \cup \partial Y_1$. Since $f(\partial Y_0)$ is a deformation retract of A, we obtain

(2.3) There is a deformation of V in A onto $f(\partial Y_0)$, leaving $f(\partial Y_0)$ fixed. Clearly ∂Y_1 is a retract of B, and hence also of cl(A - V). Since $V \cap cl(A - V) = \partial Y_1$, we have

(2.4) A retracts onto V.

Combining 2.3 and 2.4 proves that ∂Y_0 is a deformation retract of V.

The Z_i are used to show that $F(\partial Y_0)$ is a deformation retract of V, thus proving Theorem 2.2.

Let K_0 and K_1 be locally flat PR submanifolds of a smooth submanifold M. We say that K_0 and K_1 are *weakly h-cobordant* (as submanifolds of M) if there exists a subcomplex $C \subset M \times I$ such that:

(a) C is a topological manifold;

(b) C is topologically locally flat in $M \times I$;

(c) $C \cap M \times i = K_i \times i \ (i = 0, 1);$

(d) C admits $K_0 imes 0$ and $K_1 imes 1$ as deformation retracts. We call C a weak h-cobordism.

THEOREM 2.5. Let K_0 and K_1 be locally flat PR submanifolds of a smooth manifold M, and N_i smooth regular neighborhoods of K_i (i = 0, 1). If K_0 and K_1 are weakly h-cobordant, then $\partial N_0 \stackrel{h}{\sim} \partial N_1$.

The proof depends on E. Fadell's concept of normal fibre space [13], which we discuss first.

Let G be a topological manifold and $F \subset G$ a locally flat submanifold of codimension k. Let G^{I} be the space of paths in G, with the compact open topology. Fadell defines the normal fibre space $\gamma = \nu(G, F)$ of F in G to be the fibre space $\pi: Q \to F$ where $Q = Q(G, F) = \{\omega \in G^{I} | \omega^{-1}(F) = 0\}$ and $\pi: Q \to F$ is given by $\pi(\omega) = \omega(0)$. (What we call Q, Fadell calls N_{0} .) Fadell assumes $\partial F = \partial G = \varphi$, and proves that $\nu(G, P)$ is a fibre bundle whose fibre has the homotopy type of S^{k-1} . It is easily seen that this is also true if F has a boundary.

LEMMA 2.6. The natural fibre map

$$\nu(\partial G, \partial F) \longrightarrow \nu(G, F) | \partial F$$

is a fibre homotopy equivalence.

PROOF. The details are left to the reader. The idea is to shrink each element ω of $Q(G, F) | \partial F$ until $\omega(I)$ lies in a neighborhood W of ∂F in G such that $(W, W \cap F, \partial F)$ is homeomorphic to $(\partial F \times R^2, \partial F \times R^1 \times 0, \partial F \times 0 \times 0)$, and then deform $\omega(I)$ into ∂G using the product representation, in such a way that ω stays in Q(G, F) during the deformation.

Let (X, A) be a topological pair. A neighborhood U of A in X is *clean* if there is a strong deformation retraction of U onto A given by a homotopy $f_t: U \to U$ ($0 \le t \le 1$) such that $f_t^{-1}(A) = A$ for $0 \le t < 1$. We call such a deformation f a *clean* deformation retraction. If X is a complex, A a subcomplex, and U the second barycentric regular neighborhood, then U is clean. If X is a smooth manifold, A a smooth unbounded submanifold, and U a tubular neighborhood, then again U is clean. More generally, a mapping cylinder neighborhood [14] is clean.

LEMMA 2.7. Let G be a topological manifold, F a locally flat submanifold, and U a clean neighborhood of F in G. Let $\nu = (\pi, Q, F)$ be the normal fibre space of F in U. Let $\mu: Q \to U - F$ be the "exponential" map $\mu(\omega) = \omega(1)$. There is a map $\lambda: U - F \to Q$ such that

$$\mu \circ \lambda = 1_{U-F}$$
 and $\lambda \circ \mu \cong 1_Q$.

PROOF. Let $f_t: U \to U$ be a clean deformation retraction of U onto F. For each $x \in U$ let $\lambda(x)$ be the path $\lambda(x)(t) = f_{1-t}(x)$. From the cleanliness of f we see that for all $x \in U - F$, $\lambda(x) \in Q$. It is clear that $\mu \circ \lambda = 1_{U-F}$. To prove that $\lambda \circ \mu \cong 1_q$, we define a homotopy $g_t: Q \to Q$ such that $g_0 = \lambda \circ \mu$ and $g_1 = 1_q$ as follows. For s > 0, let $h_s: R \to R$ be the linear map such that $h_s(0) = 1$ and $h_s(s) = 0$. Now define $g_s(\omega) = \omega_s: I \to U$ by

$$\omega_s(t) = egin{pmatrix} \omega(t) & ext{if } s = 0 \ \lambda(\omega(s))(h_s(t)) & ext{if } 0 \leq t \leq s ext{ and } s > 0 \ \omega(t) & ext{if } s \leq t \leq 1 \ . \end{cases}$$

It is easy to prove the continuity of ω_s , and of the map $s \to \omega_s$. It is clear that $\omega_s \in Q$, that $\omega_0 = \omega$ and $\omega_1 = \lambda \mu(\omega)$, completing the proof of Lemma 2.7.

PROOF OF THEOREM 2.5. The notation being that of 2.4, let $C \subset M \times I$ be a weak *h*-cobordism between K_0 and K_1 . Let W be a smooth regular neighborhood of C in $M \times I$, such that $W \cap (M \times i) = N_i \times i$ (i = 0, 1). (See Lemma 2.1.) Put $V = cl(\partial W - W \cap \partial(M \times I))$, so that V is a smooth submanifold of $M \times I$, and $\partial V = \partial N_0 \times 0 \cup \partial N_1 \times 1$. We shall prove that V is an *h*-cobordism.

Let Q be the normal fibre space of C in $M \times I$, and Q_i that of $K_i \times i$ in $M \times i$. According to Lemma 2.6, the injection $Q_i \rightarrow Q \mid K_i \times i$ is a homotopy equivalence.

Since N_0 , N_1 , W are clean neighborhoods of $K_0 \times 0$, $K_1 \times 1$, and C, respectively, there are homotopy equivalences

by Lemma 2.7.

Moreover, we can choose the three clean deformation retractions so that the following diagram commutes:

The unlabelled maps are injections.

Now observe that since K_i is a deformation retract of C, it follows that the injections $Q | K_i \times i \rightarrow Q$ are homotopy equivalences. Therefore the injections $(N_i - K_i) \times i \rightarrow W - C$ are homotopy equivalences. It is clear that the injections

$$\partial N_i \longrightarrow (N_i - K_i) \text{ and } V \longrightarrow W - C$$

are homotopy equivalences (since N_i and W are regular neighborhoods K_i and C). Therefore, from the commuting diagram

$$\partial N_i imes i \longrightarrow (N_i - K_i) imes i igg v \ igc v igc v igc v igc v igc v$$

we infer that the injections $\partial N_i \times i \to V$ are homotopy equivalences. Therefore $\partial N_0 \stackrel{h}{\sim} \partial N_1$, q.e.d.

We end this section with a well known fact about h-cobordism.

LEMMA 2.8. Assume $n \geq 5$. Let A and B be compact, connected, smooth, unbounded, oriented n-manifolds, such that $\pi_i(A) = 0$. If $A \times S^1 \stackrel{h}{\sim} B \times S^1$, then $A \approx B$.

PROOF. From the theorems of Whitehead [15, 16] and Higman [23] we see that every homotopy equivalence between complexes whose fundamental groups are either trivial or infinite cyclic is simple. Therefore $A \times S^1$ and $B \times S^1$ are *s*-cobordant. Applying the *s*-cobordism theorem of Mazur [9] or Stallings, we conclude that $A \times S^1 \approx B \times S^1$. Therefore $A \times R^1 \approx B \times R^1$, and it is easy to see that this means $A \stackrel{h}{\sim} B$. Another application of the *s*cobordism theorem (or the *h*-cobordism theorem of Smale [17]) proves $A \approx B$. One can verify that orientations are preserved.

3. *n*-knots

Let S^k denote the unit sphere in euclidean k+1 space, with its usual differential structure. An *n*-knot is the image $A = f(S^n)$ of a locally flat PR embedding $f: S^n \to S^{n+2}$. If A happens to be a smooth submanifold, we say that A is a smooth *n*-knot.

A natural way of defining equivalence between *n*-knots A and B is to require the existence of a PR homeomorphism of S^{n+2} carrying A onto B. That

the relation so defined actually is transitive depends on the uniqueness Theorem 1.2 for smooth triangulations. We call this relation simply equivalence. If A and B are oriented n-knots, we require the homeomorphism to carry the orientation of A into that of B.

Another possibility is to insist that A and B be PR *isotopic*. This is the same as asking for an orientation preserving homeomorphism of S^{n+2} carrying A onto B, by virtue of V. K. A. M. Gugenheim's theorem [22] that an orientation preserving PL homeomorphism of a combinatorial sphere is PL isotopic to the identity.

In many of the theorems in this section we conclude that two homotopy n-spheres are diffeomorphic for $n \ge 5$. In fact, any two homeomorphic 3-manifolds are diffeomorphic [20, 21] and it has been announced by Cerf that there is only one compatible differential structure on a combinatorial 4-sphere. Nevertheless we prefer to state our theorems for $n \ge 5$, since the proofs given are only valid under that restriction.

THEOREM 3.1. Assume $n \ge 5$. Let A and B be oriented smooth n-knots. If A and B are equivalent, then $A \approx B$.

PROOF. If A and B are equivalent, then obviously $S^{n+2} - A$ and $S^{n+2} - B$ are PR homeomorphic. The Munkres obstruction [4] to smoothing a PR homeomorphism lies in

$$H_i(S^{\,n+2}-A;\,\Gamma_{n+2-i})$$
 , $0 \leq i \leq n+2$

based on infinite chains. Since A is a homology n-sphere, this group vanishes for homology reasons except possibly for i = n + 1, when it vanishes anyhow because $\Gamma_1 = 0$. Therefore A and B have diffeomorphic complements. If U, V are smooth regular neighborhoods of A, B respectively, then by Theorem 2.2, $\partial U \stackrel{h}{\sim} \partial V$. Since A and B are smooth, U and V are tubular neighborhoods. Because A and B have trivial normal bundles, $\partial U \approx A \times S^1$ and $\partial V \approx B \times S^1$. Lemma 2.8 now shows that $A \approx B$. It can be shown that orientations are preserved if A and B are oriented.

Observe that in the proof, the application of Theorem 2.2 depends only on the hypothesis that $S^{n+2} - A$ and $S^{n+2} - B$ are PR homeomorphic. Hence we obtain

THEOREM 3.2. Assume $n \ge 5$. Let A and B be smooth (unoriented) nknots such that $S^{n+2} - A$ and $S^{n+2} - B$ are PR homeomorphic. Then $A \approx B$. In the same vein we have

THEOREM 3.3. Assume $n \ge 5$. Let A be a smooth n-knot such that $S^{n+2}-A$ has the homotopy type of the circle S^1 . Then $A \approx S^n$.

PROOF. We refer the reader to Kervaire [18], where the proof is outlined.

Problem. Is the diffeomorphism type of a smooth *n*-knot A determined by the homotopy type of $S^{n+2} - A$? Perhaps one should first consider the case where $\pi_i(S^{n+2} - A) = 0$ for i > 1.

The set of equivalence classes of ordinary knots forms a semi-group under the connected sum operation, but not a group. Fox and Milnor [19] introduced a weaker equivalence relation which does lead to a group. We now describe a generalization of Fox-Milnor equivalence.

Two n-knots A, B are concordant if there is a locally flat PR embedding $f: S^n \times I \to S^{n+2} \times I$ such that $f(S^n \times 0) = A \times 0$ and $f(S^n \times 1) = B \times 1$. If A and B are oriented, there must be an orientation of S^n such that the maps $g_0: S^n \to A$ and $g_1: S^n \to B$ given respectively by $g_i(x) = f(x, i)$ are orientation preserving. We refer to $f(S^n \times I)$ as a concordance from A to B.

Warning. If A and B are smooth, we do not assume that $f(S^n \times I)$ is smooth!

It is easy to prove that concordance is indeed an equivalence relation.

We have seen in Theorem 3.1 that the diffeomorphism type of a smooth n-knot is determined by its equivalence class for $n \ge 5$. The analogous result is true for its concordance class.

THEOREM 3.4. Assume $n \ge 5$. Let A and B be smooth oriented n-knots which are concordant (or more generally, weakly h-cobordant, as defined in § 2). Then $A \approx B$.

PROOF. By Theorem 2.5, the boundaries of smooth regular neighborhoods of A and B are h-cobordant. Since A and B have trivial normal bundles, this means that $A \times S^1 \stackrel{h}{\sim} B \times S^1$. Now apply Lemma 2.8. The reader can verify that orientations are preserved if A and B are oriented.

Observe that every *n*-knot A bounds an (n + 1)-disk E in the n + 3 disk D^{n+3} , simply by taking the cone on A from the center 0 of D^{n+3} . However, E might not be locally flat at 0. The last theorem shows that there are smooth *n*-knots not concordant to the trivial one (S^n) , since there exist exotic *n*-spheres smoothly embeddable in S^{n+2} . In Part II we shall study such embeddings in more detail. Here we merely state a well known fact.

THEOREM 3.5. Let M be a smooth homotopy n-spheres.

(a) If M bounds a smooth compact parallelizable manifold, then M is smoothly embeddable in S^{n+2} .

(b) If M is a smooth submanifold of S^{n+2} , then M bounds a smooth compact parallelizable submanifold of S^{n+2} .

PROOF. See Kervaire [18]; cf. also Hirsch [12].

For part (b), it is unnecessary to assume that M is a homotopy sphere; $H_{i}(M) = 0$ suffices.

Problem. Does there exist an *n*-knot A which does not bound a locally flat contractible topological manifold in D^{n+3} ? From Theorem 3.4 we deduce that there exists a smooth *n*-knot A not bounding a contractible PR subcomplex of D^{n+3} which is locally euclidean and locally flat. One can of course replace A by a flat PL *n*-knot with the same property.

4. Groups of *n*-knots

If A is an oriented n-knot, we denote by [A] the concordance class of A; i.e., the set of all oriented n-knots concordant to A. Let \mathcal{C}_n be the set of all concordance classes of n-knots. We make \mathcal{C}_n into an abelian group in the usual way, using the connected sum. To define the sum $[A_0] + [A_1]$, first replace A_0 and A_1 by n-knots B_0 , B_1 concordant respectively to A_0 , A_1 , and contained in opposite hemispheres of S^{n+2} . Let $D_i \subset B_i$ (i = 0, 1) be a small n-disk, PR embedded in S^{n+2} . Let $f: S^{n-1} \times I \to S^{n+2}$ be a PR locally flat embedding such that, putting $f(S^{n-1} \times I) = C$, we have $C \cap B_i = \partial D_i = f(S^{n-1} \times i)$, and in such a way that the homeomorphism $f: S^{n-1} \times i \to \partial D_i$ preserves orientation, where $S^{n-1} \times i$ inherits its orientation from $S^{n-1} \times I$, while ∂D_i inherits its orientation from B_i via D_i . (We assume $S^{n-1} \times I$ is given an orientation first.)

It is easy to verify that with this definition $\{A_0\} + \{A_1\}$ is well defined, and for $n \ge 1$ makes \mathcal{C}_n into an abelian group. The standard S^n represents $0 \in \mathcal{C}_n$, and the inverse of [A] is [-A], where -A is the reflection of A in a hyperplane.

Now let $\mathfrak{S}_n \subset \mathfrak{C}_n$ be the subset made up of concordance classes of smooth *n*-knots. It is easy to see that \mathfrak{S}_n is a subgroup, since the construction of $\{A_0\} + \{A_1\}$ can be carried out smoothly if A_0 and A_1 are smooth.

THEOREM 4.1. For n > 3 there is a homomorphism of S_n onto $\theta_n(\partial \pi)$, the group of diffeomorphism classes of oriented smooth homotopy n-spheres bounding compact parallelizable manifolds.

PROOF. For each smooth *n*-knot *A*, let $\Phi(A)$ be the diffeomorphism class of *A* in θ_n (the group of diffeomorphism classes of oriented smooth homotopy *n*-spheres). Then $\Phi(A) \in \theta_n(\partial \pi)$ by Theorem 3.5. From Theorem 3.4, we see that Φ is constant on concordance classes. Denoting the induced map $\delta_n \rightarrow \theta_n(\partial \pi)$ also by Φ , Theorem 3.5 proves that Φ is onto. Since the group operation in θ_n is defined by connected sum, Φ is the desired group homomorphism of δ_n onto $\theta_n(\partial \pi)$.

REMARKS. 1. If n is even or n = 5, then $\theta_n(\partial \pi) = 0$ (Milnor-Kervaire [11]).

If n = 3 or 4, it is more reasonable to map \mathfrak{S}_n into Γ_n , and $\Gamma_3 = 0$ by Smale [20] and Munkres [21], while Cerf has announced that $\Gamma_4 = 0$. Moreover, $\theta_4 = 0$. On the other hand, for $n \geq 7$ and odd, $\theta_n(\partial \pi)$ is frequently non-trivial. For instance, $\theta_7(\partial \pi) = \theta_7 = Z_{28}$.

2. From Theorem 4.1, one can get some vague information about the group S_n . For example, it follows that if $x \in S_n$ has finite order ν , then ν is divisible by the order of $\Phi(x)$ in θ_n . One can draw conclusions such as: if $[A] \in S_7$ has order 3, then $A \approx S^7$.

Kervaire and Milnor have proved that for n > 3, $\theta_n(\partial \pi)$ is finite [11]. Combining this fact with Theorem 4.1 yields

THEOREM 4.2. Let $\mathfrak{S}'_n \subset \mathfrak{S}_n$ be the subgroup whose elements are represented by smooth embeddings of S^n . If n > 3, then \mathfrak{S}'_n has finite index.

This theorem is also true for n = 3. In this case $\mathfrak{S}'_{\mathfrak{s}} = \mathfrak{S}_{\mathfrak{s}}$ because there is only one diffeomorphism class of differential structures on $S^{\mathfrak{s}}$ [20, 21].

Problem. Is $S_n = C_n$? Equivalently, is every locally flat combinatorial *n*-sphere in \mathbb{R}^{n+2} actually flat?

Problem. If two smooth *n*-knots are concordant, are they smoothly concordant? That is, can one find a concordance which is a smooth submanifold of $S^{n+2} \times I$?

PART II

5. Preliminaries to some computations

In the usual theory of knotted S^1 in S^3 a significant object of study is the Alexander matrix (see for example [24]). Fox showed in his classical paper [25], how the group of the knot (i.e., $\pi_1(S^3 - S^1)$) determines the equivalence class¹ of this matrix. One may, however, look upon the Alexander matrix as presenting a certain module. More specifically, if G denotes $\pi_1(S^3 - S^1)$, and G' the commutator subgroup of G, then H_1 of the covering of $S^3 - S^1$ corresponding to G' is presented as a module over the integral group ring of G/G'(the group of covering translations) by the Alexander matrix. See for example [26] the forthcoming [27], or various papers of Reidemeister on homotopy chains.

If now one has a PR $S^n \subset S^{n+2}$, then as in the classical case the homology groups of the universal abelian covering of $S^{n+2} - S^n$ are modules over the group ring of the abelianized fundamental group G/G'. These might be called the Alexander modules of the knotted S^n in S^{n+2} . Since the G/G'-free modules of the chains in the universal abelian covering are finitely generated, (qua

¹ The particular definition of equivalence used may be found in [25].

modules) it follows that the Alexander modules may be presented by matrices (see [28]) whose elementary ideals are invariants of these modules, and hence of the knot type of S^n in S^{n+2} . These Alexander matrices and Alexander ideals generalize the ideals described in [24] and [25].

Seifert showed in [29] how the Alexander matrix for an S^1 in S^3 may be computed by means of an orientable surface whose boundary is the S^1 in question. We will show how this process may frequently be generalized.

Applying the method to a particular sort of $S^n \subset S^{n+2}$, we find a higher dimensional counterpart to the Alexander polynomial.

6. Computational methods

Suppose we have an orientable PR manifold $M^{n+1} \subset S^{n+2}$, and $\partial M^{n+1} = S^n$. We shall construct the universal abelian covering of $S^{n+2} - S^n$.

First split² S^{n+2} along the interior of M^{n+1} . This creates a manifold Y^{n+2} whose boundary W^{n+1} contains S^n . Notice that since M^{n+1} is orientable $W^{n+1} - S^n$ consists of two components $_1W$ and $_2W$. Each $_iW$ is homeomorphic to $M^{n+1} - S^n$ by a natural map $_i\omega$, and $S^{n+2} - M^{n+1}$ is homeomorphic by a natural map φ^{-1} to the interior of Y^{n+2} .³

Now form the union of \aleph_0 disjoint copies of $Y^{n+2} - S^n$, indexed by the integers. Denote these copies Y_j^{n+2} $(j = \pm 1, \pm 2, \cdots)$ and the corresponding maps of the copies $_i W_j$ of $_i W$ to be found in the boundary of each Y_j^{n+2} by $_i \omega_j$. Denote by φ_j the map φ on the copy j.

Finally, form the quotient space, X, of $\bigcup_{j} Y_{j}^{n+2}$ by the identification mappings $({}_{1}\omega_{j+1}^{-1})({}_{2}\omega_{j})$ for all j. The following picture is provided as a visual aid.

$$\begin{vmatrix} Y_{j-1}^{n+2} & & Y_{j}^{n+2} \\ & & & Y_{j-1}^{n+2} \\ & & & & \\ & & & \\ & & & & \\ & &$$

There is clearly a regular covering map from X to $S^{n+2} - S^n$ defined by φ_j on points in the interior of the Y_j^{n+2} and $_i\omega_j$ on points in $_iW_j$. Thus X is a regular covering of $S^{n+2} - S^n$, and admits an infinite cyclic group of covering translations. It follows that X is the universal abelian covering of $S^{n+2} - S^n$.

This construction is described for the case n = 1 in [30].

² This may be done for example by triangulating (S^{n+2}, M^{n+1}) and then forming a simplicial complex with the same n+2 simplices as S^{n+2} , and the same incidence relations between them except for those incidences arising from n+1 simplices in $M^{n+1} - S^n$, these incidences become zero; alternatively one may remove a regular neighborhood of the interior of M^{n+1} although this complicates matters a bit.

³ These natural homeomorphisms arise from the construction described in footnote 2.

In the next section frequent use will be made of this covering; for the present we give the following theorem as an application. Unless otherwise indicated Σ^n denotes a PL *n*-sphere.

THEOREM 6.1. A. $\pi_1(S^{n+2} - \Sigma^n) \approx Z$ if and only if Σ^n bounds a simply connected n + 1 manifold in S^{n+2} .

B. If Σ^n bounds an orientable n + 1 manifold in S^{n+2} whose complement is simply connected, then $\pi_1(S^{n+2} - \Sigma^n) \approx Z$.

PROOF OF PART A. Denote $\pi_1(S^{n+2} - \Sigma^n)$ by G. If Σ^n bounds a simply connected n + 1 manifold, then this manifold must be orientable, and the construction of X, the universal abelian covering of $S^{n+2} - \Sigma^n$, may be used to compute $\pi_1(X) = G'$.

Applying the van Kampen theorem to the manifolds Y_j^{n+2} yields $G' = {}_{j}*\pi_1(Y_j^{n+2})$, and the action of a generator of G/G' pulled back to G on G' sends $\pi_1(Y_j^{n+2})$ to $\pi_1(Y_{j+1}^{n+2})$. Thus G may be presented as follows: $(x_j^1, x_j^2 \cdots x_j^n, t: r_j^1, \cdots, r_j^s, tx_j^it^{-1} = x_{j+1}^i), j \in Z$ where $(x_j^1, x_j^2 \cdots x_j^n: r_j^1, \cdots, r_j^s)$ is a presentation of $\pi_1(Y_j^{n+2})$. But this implies $G \approx \pi_1(Y^{n+2}) * (t:)$ which is impossible if $\pi_1(Y^{n+2})$ is non-trivial since the smallest normal subgroup of G containing t must be G itself [18]. Thus $\pi_1(Y^{n+2}) = 0$, and so $\pi_1(X)$ is trivial which means G' is trivial which means $G \approx Z$.

We now prove the other half of part A. According to Theorem 3.5, Σ^n bounds some orientable manifold $M^{n+1} \subset S^{n+2}$. By the construction of X any loop in M^{n+1} represents an element of G', but since G' is trivial by assumption every loop in M^{n+1} is contractible in $S^{n+2} - \Sigma^n$. Since n > 2 and $\pi_1(M^{n+1})$ is finitely generated, we may find non-singular disjoint 2-discs D_1, \dots, D_k such that $\partial D_i \subset M^{n+1}$, $D_i \cap \Sigma^n = \emptyset$, and $\pi_1(M^{n+1} \cup D_i) = 0$. Starting with an innermost component \hat{D}_i (which is a disc) in some $D_i - (D_i \cap M^{n+1})$ we add handles to M^{n+1} to kill π_1 by thickening this disc component \hat{D}_i to some $D^n \times \hat{D}_i$ so that $(D^n \times \hat{D}_i) \cap M^{n+1}$ is a regular neighborhood of the loop $\hat{D}_i \cap M^{n+1}$. Then $M^{n+1} \cup (\partial D^n \times \hat{D}_i)$ -interior $(D^n \times \hat{D}_i \cap M^{n+1})$ is a new orientable manifold, \hat{M}^{n+1} , with boundary S^n , and \hat{D}_i bounds a disc in $S^{n+2} - \hat{M}^{n+1}$ thus reducing the number of intersections of D_i with an orientable manifold bounded by Σ^n . Furthermore $\pi_1(\hat{M}^{n+1})$ is a homomorph of $\pi_1(M^{n+1})$. Continuing in this fashion we eventually will kill off π_1 of M^{n+1} and obtain some other simply connected orientable manifold bounded by Σ^n .

This completes the proof of part A.

Part B follows trivially from an application of van Kampen's theorem to the construction of X. One sees immediately that $\pi_1(x) = 0$, so that $\pi_1(S^{n+2} - \Sigma^n)$ is abelian.

We are grateful to the referee for pointing out the following result which answers a question raised by J. Stallings in his 1963 Annals paper On topologically unknotted spheres. Similar theorems have been obtained by J. Levine, and by C. T. C. Wall.

THEOREM 6.2. Let M be a smooth homotopy n-sphere which bounds a compact parallelizable manifold, and which is not diffeomorphic to S^n . There exists a smooth embedding $f: M \to S^{n+2}$ such that

(a) $\pi_1(S^{n+2} - f(M)) = Z$,

(b) $S^{n+2} - f(M)$ does not have the homotopy type of S^1 .

PROOF. Let $f: W \to S^{n+2}$ be a smooth embedding of a compact simply connected parallelizable n + 1 manifold bounded by M. Such an embedding exists, by Hirsch [12], or the classification [33] of homotopy spheres of the type of M. Theorems 6.1 and 3.3 imply that $f \mid M$ has the required properties.

7. Some applications

Let us compute some invariants of a certain knotted $\Sigma^7 \subset S^9$ and see where we are led.

The particular $\Sigma^7 \subset S^9$ we pick is described⁴ in [32] as the boundary of a thickened wedge of eight 4-spheres. Each S^4 is thickened so as to be imbedded in its thickening just as S^4 is imbedded in a closed neighborhood of the diagonal of $S^4 \times S^4$. Thus we are given 8 copies of a 4-disc bundle over S^4 . Call the total space of these bundles b_1, \dots, b_8 . Now these copies are plumbed together schematically as follows:



Two copies say 1 and 2 are plumbed together according to the following process. The pre-image of a small disc neighborhood; B^4 of a point of one copy of S^4 in the space b_1 is diffeomorphic to $D_1^4 \times B_1^4$, in b_2 take a similar subset $D_2^4 \times B_2^4$ and identify D^4 with B_1^4 and D_1^4 with B^4 by some diffeomorphism. Schematically



⁴ The Σ^7 here is very slightly different from that in [32], and was described to us by Milnor and Kervaire.

The pairs (1, 2) (2, 3) (3, 4) (4, 5) (5, 6) (6, 7) and (5, 8) are hooked up in this manner. This entire process may of course take place in S^9 . We denote the resulting 8-manifold imbedded in S^9 by M^8 .

As we will be interested in certain linking numbers we find it convenient to describe the situation for one of the thickened S^4 's as follows.

Consider S^{9} as the union of $D_{1}^{5} \times S_{1}^{4}$ with $S_{2}^{4} \times D_{2}^{5}$ with the boundaries identified so that ∂D_{1}^{5} is identified with S_{2}^{4} and S_{1}^{4} is identified with ∂D_{2}^{5} (all the identifications being performed by diffeomorphisms).

Denote the diagonal in $(D_1^5 \times S_1^4) \cap (S_2^4 \times D_2^5) = T$ by S_d and let N denote a regular neighborhood of S_d in $T \approx S^4 \times S^4$. Now one of the two normal vector fields to T in S^9 defines a map of S_d into $D_1^5 \times S_1^4$ by pushing S_d a small distance along this field (or its negative) into $D_1^5 \times S_1^4$, and there is a similar map pushing S_d into $S_2^4 \times D_2^5$. Denote the image of S_d under the first map by S_1 , and the image under the second map by S_2 . We wish to know the linking number of S_1 with S_d , $\mathcal{L}(S_1, S_d)$. Similarly we wish to know $\mathcal{L}(S_2, S_d)$. In the first case we may orient S_d and assign the linking number as plus or minus 1 according to the orientation of S^9 . Suppose we orient S^9 so that $\mathcal{L}(S_1, S_d) = +1$, then $\mathcal{L}(S_2, S_d) = -1$. (This follows since $\mathcal{L}(S_1, S_d) = \mathcal{L}(S_1, S_2) = -\mathcal{L}(S_2, S_1) =$ $-\mathcal{L}(S_2, S_d)$.)

Now S_2 lies in $S_2^4 \times D_2^5$, and is homologous (even isotopic) in $S_2^4 \times D_2^5$ to a 4-sphere σ lying on T homologous to 0 in $D_1^5 \times S_1^4$ and having intersection number 1 with S_d . Thus we may consider the wedge of two 4-spheres $\sigma \cup S_d$ as lying on T and compute the linking numbers of S_d with σ when S_d is pushed off T in each of two ways. From the previous remarks $\mathfrak{L}(\sigma, S_2) = 0$, $\mathfrak{L}(\sigma, S_1) =$ $\mathfrak{L}(S_2, S_1) = -1$. This permits the construction of the following table which describes the linking numbers we will need. As it is necessary to push a 4cycle represented by a 4-sphere off the 8-manifold, M^8 , in two distinct ways, (represented by each of the two normal vector fields) we distinguish these directions by + and -, and the image of the 4-sphere in question by a superscript + or -. Consider two contiguous 4-spheres in M^8 , and let them without loss of generality be σ and S_d , and suppose, again without loss of generality, N is a neighborhood of S_d in M^8 , then our computations yield the following table:

$$egin{array}{lll} {\mathfrak L}(S_1,\,S_d) = {\mathfrak L}(S_d^-,\,S_d) = +1 & {\mathfrak L}(S_2,\,S_d) = {\mathfrak L}(S_d^+,\,S_d) = -1 \ {\mathfrak L}(S_2,\,S_2) = {\mathfrak L}(S_d^+,\,\sigma) &= 0 & {\mathfrak L}(S_1,\,\sigma) &= {\mathfrak L}(S_d^-,\,\sigma) &= +1 \ . \end{array}$$

The information in this table is sufficient to compute a presentation of $H_4(X)$ as a module over $\pi_1(S^9 - \Sigma^7)$.

Recalling the construction of X, we may apply a Mayer-Vietoris sequence

to compute $H_4(X)$:

$$\cdots o H_4(Y_j^{\mathfrak{g}} \cap Y_{j+1}^{\mathfrak{g}}) \stackrel{
u}{\longrightarrow} H_4(Y_j^{\mathfrak{g}}) \oplus H_4(Y_{j+1}^{\mathfrak{g}}) \stackrel{\eta}{\longrightarrow} H_4(Y_j^{\mathfrak{g}} \cup Y_{j+1}^{\mathfrak{g}}) \ H_4(M^{\mathfrak{g}}) \stackrel{\mathcal{V}}{\longrightarrow} H_3(Y_j^{\mathfrak{g}} \cap Y_{j+1}^{\mathfrak{g}}) \stackrel{\mathcal{V}}{\longrightarrow} \cdots \cdot H_3(M^{\mathfrak{g}}) = 0$$

Thus $\eta(H_4(Y_j^{\mathfrak{g}}) \bigoplus H_4(Y_{j+1}^{\mathfrak{g}}))$ generates $H_4(Y_j^{\mathfrak{g}} \cup Y_{j+1}^{\mathfrak{g}})$. Since any copy of $S^{\mathfrak{g}} - M^{\mathfrak{g}}$ is a translate of $Y_0^{\mathfrak{g}}$, we conclude that $H_4(X)$ is generated as a module over the integral group ring of $G/G' \approx (t:)$ by $H_4(Y_0^{\mathfrak{g}})$.

By Alexander duality, $H_4(Y_0^9)$ is isomorphic to $H_4(M^8)$ which is free abelian of rank 8, and is generated by the fundamental cycles of each of eight 4spheres wedged in M^8 . The relations in the JZ-module⁵ $H_4(X)$ arise from the Mayer-Vietoris sequence for (Y_0^9, Y_1^9, M^8) , and clearly any relation is a consequence of some collection of covering translates of these relations (compare [29]).

The relations arising from the maps of M^{s} into Y_{0}^{s} and Y_{1}^{s} may be computed from our earlier calculations of linking numbers. Let us denote by x_{1}, \dots, x_{s} the previously described free generators of $H_{4}(Y_{0}^{s})$, then tx_{1}, \dots, tx_{s} will denote the covering translates of these generators which freely generate $H_{4}(Y_{1}^{s})$. Identifying the direction + with the location of the translate $tY_{0}^{s} =$ Y_{1}^{s} with respect to Y_{0}^{s} , recalling the arrangement of the 4-spheres in M^{s} , and our linking number calculations, we may write the relation for each generator of $H_{4}(M^{s})$ as it is pushed into Y_{0}^{s} and Y_{1}^{s} , viz.,

$$x_1 = t(-x_1) + t(x_2)$$
, $x_2 - x_1 = t(-x_2) + t(x_3)$ etc.

Arranging these data in matrix form we arrive at a presentation for $H_4(X)$ as a module over the group of covering translations of X.

	x_{i}	$x_{\scriptscriptstyle 2}$	$x_{\scriptscriptstyle 3}$	$x_{\scriptscriptstyle 4}$	$x_{\scriptscriptstyle 5}$	$x_{\scriptscriptstyle 6}$	x_{7}	$x_{\scriptscriptstyle 8}$
$\mathfrak{A} =$	$\overline{\sqrt{1+t}}$	-t						
	-1	1 + t	$\cdot - t$)
		- 1	1 + t	-t				
	ĺ			1+t	-t			
				- 1	1 + t	-t		-t
					- 1	1 + t	-t	
						- 1	1+t	
	,				-1			1 + t /

With the help of W. H. Mills, the determinant of this matrix was computed and the result, which we may look upon as a generalized Alexander polynomial for $\Sigma^7 \subset S^9$ is $1 + t - t^3 - t^4 - t^5 + t^7 + t^8$. This polynomial is an invariant

⁵ JZ denotes the integral group ring of the infinite cyclic group generated by t.

of the $J\pi_1(S^9 - \Sigma^7)$ module $H_4(X)$ [30] and so is an invariant of the knot type of $\Sigma^7 \subset S^9$.

The $\Sigma^{\tau} \subset S^{\vartheta}$ described above bounds no PR 3-connected 8-manifold in S^{ϑ} whose four dimensional homology has rank <8.

PROOF. The construction of X can be applied with any orientable 8-manifold bounded by Σ^7 , and if the rank of H_4 of such a manifold is less than 8, then the computation described above may be made since H_4 is torsion free, and the Alexander polynomial for $H_4(X)$ must have degree less than 8.

Any homology sphere $\Sigma^n \subset S^{n+2}$ has an Alexander polynomial in dimension [n + 1/2] if Σ^n bounds an orientable manifold $M^{n+1} \subset S^{n+2}$ such that $H_{[n+1/2]}(M^{n+1})$ is torsion free.

PROOF. Suppose n + 1 is even. By assumption, $H_{n+1/2}(M^{n+1})$ is torsion free. By Alexander and Poincaré duality $H_{n+1/2}(S^{n+2} - M^{n+1}) \approx H^{n+1/2}(M^{n+1}) \approx$ $H_{n+1/2}(M^{n+1})$, so that the computation leading to a square matrix \mathfrak{A} may be made in this case.⁶

If n + 1 is odd then our assumption means $H_{n/2}(M^{n+1})$ is torsion free. Again applying Alexander and Poincaré duality $H_{n/2}(S^{n+2} - M^{n+1}) \approx H^{n+2/2}(M^{n+1}) \approx H_{n/2}(M^{n+1})$, and a square matrix of the form of \mathfrak{A} will result when $H_{n/2}(X)$ is computed in the manner indicated.

We remark that setting t = 1 in \mathfrak{A} , we obtain a presentation for $H_{[n+1/2]}(S^{n+2} - \Sigma^n) = 0$. Thus the polynomial p(t) we have defined shares the property p(1) = 1 with the classical polynomial.

Returning to the example we have worked out, the matrix \mathfrak{A} may be used to compute $H_4(X)$. As a group $H_4(X) \approx \pi_4(X)$ since X is 3-connected. Since X is a covering of $S^9 - \Sigma^7 \pi_4(X) \approx \pi_4(S^9 - \Sigma^7)$. Thus we state and prove:

THEOREM 6.3. $\pi_4(S^9 - \Sigma^7)$ is free abelian of rank 8.

PROOF. $H_4(X) \approx \pi_4(S^9 - \Sigma^7)$ by the lines above. $H_4(X)$ may be computed from \mathfrak{A} , since \mathfrak{A} describes the maps in the Mayer-Vietoris sequence we have displayed. First notice that $H_4(Y_0^9 \cap Y_1^9)$ is mapped onto both $H_4(Y_0^9)$ and $H_4(Y_1^9)$ by the inclusion maps since the image of $H_4(Y_0^9 \cap Y_1^9)$ in $H_4(Y_0^9)$ is generated by

 $egin{array}{c} x_1 & & & \ & -x_1+x_2 & & \ & -x_2+x_3 & & \ & -x_3+x_4 & & \ & -x_4+x_5 & & \ & -x_5+x_6 & & \ & -x_6+x_7 & & \ & -x_5 & +x_8 \end{array}$

⁶ If there is torsion in $H_{[n+1/2]}(M^{n+1})$ then the construction may not lead to a square matrix, and we must define an Alexander ideal [23] rather than a polynomial.

and the image of $H_4(Y_0^9 \cap Y_1^9)$ in $H_4(Y_1^9)$ is generated by

and these sets of generators are easily seen to generate $H_4(Y_0^9)$ and $H_4(Y_1^9)$ respectively.

Since $H_3(Y_0^{\mathfrak{g}} \cap Y_1^{\mathfrak{g}}) = 0$, η is onto, and we see $H_4(Y_0^{\mathfrak{g}}) \bigoplus H_4(Y_1^{\mathfrak{g}})/\operatorname{im} \nu \approx H_4(Y_0^{\mathfrak{g}} \cup Y_1^{\mathfrak{g}})$ is free of rank 8. Adding copies $Y_i^{\mathfrak{g}}$ we obtain the desired result.

8. Some questions

A. When n is odd, is the generalized Alexander polynomial always defined, that is, does every $\Sigma^{2n+1} \subset S^{2n+3}$ bound an orientable manifold without torsion in $H_{[2n+1/2]}$? (When n = 2, one can find $S^2 \subset S^4$ such that $S^2 \subset M^3 \subset S^4 \Rightarrow H_1(M^3)$ has torsion.)

B. When the generalized Alexander polynomial is defined, is it always symmetric?

C. If an *n*-knot Σ^n and an unknotted S^n are concordant, then is there a polynomial for $H_{[n+1/2]}$ of Σ^n , and is this polynomial of the form $P(t)P(t^{-1})$? (Compare [19].)

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