39.

Singularities and exotic spheres

Séminaire Bourbaki, 1966/67, Exp. 314, Textes des conférences, o. S., Paris: Institut Henri Poincaré 1967

* Brieskorn has proved [4] that the n-dimensional affine algebraic variety $z_0^3+z_1^2+\ldots+z_n^2=0$ (n *odd*, $n \geqq 1$) is a topological manifold though the variety has an isolated singular point (which is normal for $n \geqq 2$). Such a phenomenon cannot occur for normal singularities of 2-dimensional varieties, as was shown by Mumford ([12], [6]). Brieskorn's result stimulated further research on the topology of isolated singularities (Brieskorn [5], Milnor [11] and the speaker [6], [7]). Brieskorn [5] uses the paper of F. Pham [14], whereas the speaker studied certain singularities from the point of view of transformation groups using results of Bredon ([2], [3]), W. C. Hsiang and W. Y. Hsiang [8] and Jänich [9].

§ 1. The integral homology of some affine hypersurfaces

Pham [14] studies the non-singular subvariety $V_a = V(a_0, a_1, \ldots, a_n)$ of $\mathbb{C}^{n+1}$ given by

$$z_0^{a_0}+z_1^{a_1}+\ldots+z_n^{a_n}=1 \qquad (n \geqq 0),$$

where $a=(a_0,\ldots,a_n)$ consists of integers $a_j \geqq 2$.

Let G_{a_j} be the cyclic group of order a_j multiplicatively written and generated by w_j. Define the group $G_a = G_{a_0} \times G_{a_1} \times \ldots \times G_{a_n}$ and put $\varepsilon_j = \exp(2\pi i/a_j)$.

Then $w_0^{k_0} w_1^{k_1} \ldots w_n^{k_n}$ is an element of G_a whereas $\varepsilon_0^{k_0}\varepsilon_1^{k_1}\ldots\varepsilon_n^{k_n}$ is a complex number. G_a operates on V_a by

$$w_0^{k_0}\ldots w_n^{k_n}(z_0,\ldots,z_n) = (\varepsilon_0^{k_0} z_0, \ldots, \varepsilon_n^{k_n} z_n).$$

Let $\hat{G}_{a_j}$ be the group of a_j-th roots of unity and $x \mapsto \hat{x}$ the isomorphism $G_{a_j} \to \hat{G}_{a_j}$ given by $w_j \mapsto \varepsilon_j = \hat{w}_j$.

Pham considers the following subspace U_a of V_a

$$U_a = \{z \mid z \in V_a \text{ and } z_j^{a_j} \text{ real } \geqq 0 \text{ for } j=0,\ldots,n\}.$$

Lemma. *The subspace U_a is a deformation retract of V_a by a deformation compatible with the operations of G_a.*

For the proof see Pham [14], p. 338.

U_a can also be described by the conditions

$$z_j = u_j |z_j| \quad \text{with} \quad u_j \in \hat{G}_{a_j} \qquad (j=0,\ldots,n).$$

Put $|z_j|^{a_j} = t_j$. Then U_a becomes the space of $(n+1)$-tuples of complex numbers

$$t_0 u_0 \oplus t_1 u_1 \oplus \ldots \oplus t_n u_n$$

with

$$u_j \in \hat{G}_{a_j}, \qquad t_j \geqq 0, \qquad \sum_{j=0}^{n} t_j = 1.$$

Thus U_a can be identified with the join $G_{a_0} * G_{a_1} * \ldots * G_{a_n}$ of the finite sets G_{a_j} (see Milnor [10]).

Lemma 2.1 in [10] states in particular that the reduced integral homology groups of the join $A * B$ of two spaces A, B without torsion are given by a canonical isomorphism

$$\tilde{H}_{r+1}(A*B) \cong \sum_{i+j=r} \tilde{H}_i(A) \otimes \tilde{H}_j(B),$$

whereas Lemma 2.2 in [10] shows that $A*B$ is simply connected provided B is arcwise connected and A is any non-vacuous space. These properties of the join together with its associativity imply

Theorem. *The subvariety V_a of $\mathbb{C}^{n+1}$ is $(n-1)$-connected. Moreover*

(1) $$\tilde{H}_n(V_a) \cong \tilde{H}_0(G_{a_0}) \otimes \tilde{H}_0(G_{a_1}) \otimes \ldots \otimes \tilde{H}_0(G_{a_n}).$$

This is a free abelian group of rank $r = \prod (a_j - 1)$.
The isomorphism (1) is compatible with the operations of G_a.
All other reduced integral homology groups of V_a vanish.

It can be shown that V_a has the homotopy type of a connected union $S^n \vee \ldots \vee S^n$ of r spheres of dimension n.

The identification of U_a with a join was explained to the speaker by Milnor.

$U_a = G_{a_0} * G_{a_1} * \ldots * G_{a_n}$ is an n-dimensional simplicial complex which has an n-simplex for each element of G_a. The n-simplex belonging to the unit of G_a is denoted by e. All other n-simplices are obtained from e by operations of G_a. Thus we have for the n-dimensional simplicial chain group

(2) $$C_n(U_a) = J_a e$$

where J_a is the group ring of G_a. The homology group $\tilde{H}_n(U_a) = \tilde{H}_n(V_a)$ is an additive subgroup of $J_a e = C_n(U_a) \cong J_a$.

The face operator ∂_j commutes with all operations of G_a on $C_n(U_a)$ and furthermore satisfies $\partial_j = w_j \partial_j$. Therefore

(3) $$h = (1-w_0)(1-w_1)\ldots(1-w_n)\, e$$

is a cycle. Thus $h \in \tilde{H}_n(U_a)$. It follows easily that $\tilde{H}_n(V_a) = J_a h$. This yields the

Theorem. *The map $w \mapsto w\,h$ ($w \in G_a$) induces an isomorphism*

$$J_a / I_a \cong \tilde{H}_n(V_a) = J_a h$$

where $I_a \subset J_a$ is the annihilator ideal of h which is generated by the elements

$$1 + w_j + w_j^2 + \ldots + w_j^{a_j - 1} \qquad (j=0,\ldots,n).$$

Therefore $w_0^{k_0} w_1^{k_1} \ldots w_n^{k_n} h$ (where $0 \leqq k_j \leqq a_j - 2$, $j=0,\ldots,n$) *is a basis of* $\tilde{H}_n(V_a)$.

We recall that $\tilde{H}_n(V_a)$ is the integral singular homology group (of course with *compact* support). V_a is a $2n$-dimensional oriented manifold without boundary (non-compact for $n \geqq 1$). Therefore the bilinear intersection form S is well defined over $\tilde{H}_n(V_a)$. It is symmetric for n even, skew-symmetric for n odd. It is compatible with the operations of G_a.

Pham ([14], p. 358) constructs an n-dimensional cycle $\tilde{h}$ in V_a which is homologous to h and intersects U_a exactly in two interior points of the simplices e and $w_0 w_1 \dots w_n e$ (sign questions have to be observed). In this way he obtains (using the G_a-invariance of S) the following result, reformulated somewhat for our purposes.

Theorem. *Put* $\eta = (1-w_0)\dots(1-w_n)$. *The bilinear form* S *over* $J_a \eta \cong \tilde{H}_n(V_a)$ *is given by*

$$S(x\eta, y\eta) = f(\bar{y} x \eta) \qquad (x, y \in J_a),$$

where $f\colon J_a \to \mathbb{Z}$ *is the additive homomorphism with*

$$f(1) = -f(w_0 \dots w_n) = (-1)^{\frac{n(n-1)}{2}},$$

$$f(w) = 0 \text{ for } w \in G_a, \quad w \neq 1, \quad w \neq w_0 \dots w_n,$$

and where $y \mapsto \bar{y}$ *is the ring automorphism of the group ring* J_a *induced by* $w \mapsto w^{-1}$ $(w \in G_a)$.

§ 2. The quadratic form of V_a

Let G be a finite abelian group, $J(G)$ its group ring. The ring automorphism of $J(G)$ induced by $g \mapsto g^{-1}$ $(g \in G)$ is denoted by $x \mapsto \bar{x}$ $(x \in J(G))$. Give an element $\eta \in J(G)$ and a function $f\colon G \to \mathbb{Z}$. The additive homomorphism $J(G) \to \mathbb{Z}$ induced by f is also called f. Put $\hat{f} = \sum_{w \in G} f(w)\, w$. We assume

a) $f(\bar{x}\eta) = f(x\eta)$ for all $x \in J(G)$, [equivalently $\hat{f}\bar{\eta} = \bar{\hat{f}}\eta$]

or

b) $f(\bar{x}\eta) = -f(x\eta)$ for all $x \in J(G)$, [equivalently $\hat{f}\bar{\eta} = -\bar{\hat{f}}\eta$].

The bilinear form S over the lattice $J(G)\eta$ defined by

$$S(x\eta, y\eta) = f(\bar{y} x \eta) \qquad (x, y \in J(G)),$$

is symmetric in case a), skew symmetric in case b). Since S is a form with *integral* coefficients, its determinant is well-defined. The signature

$$\tau(S) = \tau^+(S) - \tau^-(S), \quad \text{case a)},$$

is the number $\tau^+(S)$ of positive minus the number $\tau^-(S)$ of negative diagonal entries in a diagonalisation of S over $\mathbb{R}$. Let χ run through the characters of G.

Lemma. *With the preceding assumptions*

$$\pm \det S = \Bigl(\prod_{\chi(\eta) \neq 0} \chi(\hat{f})\Bigr) \cdot \text{order of the torsion subgroup of } J(G)/J(G)\eta$$

and in case a)

$$\tau^+(S) = \text{number of characters } \chi \text{ with } \chi(\hat{f}\bar{\eta}) > 0$$

$$\tau^-(S) = \text{number of characters } \chi \text{ with } \chi(\hat{f}\bar{\eta}) < 0.$$

The proof is an exercise as in [1], p. 444.

The lemma and the last theorem of § 1 imply for the affine hypersurface $V_a = V(a_0, \dots, a_n)$ the

Theorem. *Let* S *be the intersection form of* V_a. *Then*

$$(1) \qquad \pm \det S = \prod_{1 \leqq k_j \leqq a_j - 1} (1 - \varepsilon_0^{k_0} \varepsilon_1^{k_1} \dots \varepsilon_n^{k_n})$$

where $\varepsilon_j = \exp(2\pi i / a_j)$. *For* n *even, we have*

$$(2) \qquad \begin{aligned} \tau^+(S) &= \text{number of } (n+1)\text{-tuples of integers } (x_0, \dots, x_n),\ 0 < x_j < a_j, \\ &\quad \text{with } 0 < \sum_{j=0}^{n} \frac{x_j}{a_j} < 1 \bmod 2\mathbb{Z}, \\ \tau^-(S) &= \text{number of } (n+1)\text{-tuples of integers } (x_0, \dots, x_n),\ 0 < x_j < a_j, \\ &\quad \text{with } -1 < \sum_{j=0}^{n} \frac{x_j}{a_j} < 0 \bmod 2\mathbb{Z}. \end{aligned}$$

See [5] for details.

Remark. The intersection form S of $V(a_0, \dots, a_n)$ with $n \equiv 0 \bmod 2$ is even, i.e. $S(x,x) \equiv 0 \bmod 2$ for $x \in \tilde{H}_n(V_n)$. Therefore, by a well-known theorem, $\det S = \pm 1$ implies $\tau^+(S) - \tau^-(S) = \tau(S) \equiv 0 \bmod 8$.

Following Milnor we introduce for $a = (a_0, \dots, a_n)$ the graph $\Gamma(a)$: $\Gamma(a)$ has the $(n+1)$ vertices $a_0, \dots, a_n$. Two of them (say a_i, a_j) are joined by an edge if and only if the greatest common divisor (a_i, a_j) is greater than 1. Then we have [5]

Lemma. $\det S$ *as given in the preceding theorem equals* ± 1 *if and only if* $\Gamma(a)$ *satisfies*

a) $\Gamma(a)$ *has at least two isolated points, or,*

b) *it has one isolated point and at least one connectedness component* K *with an odd number of vertices such that* $(a_i, a_j) = 2$ *for* $a_i, a_j \in K$ $(i \neq j)$.

Now suppose n even and $a = (a_0, \dots, a_n) = (p, q, 2, \dots, 2)$ with p, q odd and $(p,q) = 1$. Then $\det S = \pm 1$ and

$$(3) \qquad (-1)^{n/2} \cdot \tau(S) = \frac{(p-1)(q-1)}{2} + 2(N_{p,q} + N_{q,p}),$$

where $N_{p,q}$ is the number of $q \cdot x$ $\left(1 \leqq x \leqq \frac{p-1}{2}\right)$ whose remainder mod p of smallest absolute value is negative. This follows from the preceding theorem. Observe that by the above remark $\tau(S)$ is divisible by 4 (even by 8) and that this is related to one of the proofs of the quadratic reciprocity law ([1], p. 450).

In particular, for n even and $(a_0, \dots, a_n) = (3, 6k-1, 2, \dots, 2)$ the signature S) equals $(-1)^{n/2} \cdot 8k$.

§ 3. Exotic spheres

A k-dimensional compact oriented differentiable manifold is called a -sphere if it is homeomorphic to the k-dimensional standard sphere. k-sphere not diffeomorphic to the standard k-sphere is said to be exotic. The rst exotic sphere was discovered by Milnor in 1956. Two k-spheres are alled equivalent if there exists an orientation preserving diffeomorphism etween them. The equivalence classes of k-spheres constitute for $k \geqq 5$ a nite abelian group Θ_k under the connected sum operation. Θ_k contains the ıbgroup bP_{k+1} of those k-spheres which bound a parallelizable manifold. P_{4m} ($m \geqq 2$) is cyclic of order

$$2^{2m-2}(2^{2m-1}-1) \text{ numerator}\left(\frac{4B_m}{m}\right),$$

'here B_m is the m-th Bernoulli number. Let g_m be the Milnor generator of P_{4m}, see § 5. If a $(4m-1)$-sphere Σ bounds a parallelizable manifold B of imension $4m$, then the signature $\tau(B)$ of the intersection form of B is divis- ɔle by 8 and

$$\Sigma = + \frac{\tau(B)}{8} g_m \qquad)$$

g_m should be chosen in such a way that we have always the plus-sign in (1)). or $m = 2$ and 4 we have

$$bP_8 = \Theta_7 = \mathbb{Z}_{28}, \qquad bP_{12} = \Theta_{11} = \mathbb{Z}_{992}.$$

.ll these results are due to Milnor-Kervaire. The group bP_{2n} (n odd, $n \geqq 3$) either 0 or $\mathbb{Z}_2$. It contains only the standard sphere and the Kervaire sphere ɔbtained by plumbing two copies of the tangent bundle of S^n). It is known ıat bP_{2n} is $\mathbb{Z}_2$ (equivalently that the Kervaire sphere is exotic) if $\equiv 1 \bmod 4$ and $n \geqq 5$ (E. Brown-F. Peterson).

Let $V_a^0 = V^0(a_0, a_1, \dots, a_n) \subset \mathbb{C}^{n+1}$ (where $a_j \geqq 2$) be defined by

$$z_0^{a_0} + z_1^{a_1} + \dots + z_n^{a_n} = 0.$$

'his affine variety has exactly one singular point, namely the origin of $\mathbb{C}^{n+1}$. et

$$S^{2n+1} = \left\{ z \mid z \in \mathbb{C}^{n+1}, \sum_{j=0}^{n} z_j \bar{z}_j = 1 \right\}.$$

'hen $\Sigma_a = \Sigma(a_0, \dots, a_n) = V_a^0 \cap S^{2n+1}$ is a compact oriented differentiable ıanifold (without boundary) of dimension $2n-1$.

'heorem. *Let $n \geqq 3$. Then Σ_a is $(n-2)$-connected. It is a $(2n-1)$-sphere if and nly if the graph $\Gamma(a)$ defined in § 2 satisfies the condition a) or b). If Σ_a is a*

$(2n-1)$-sphere, then it belongs to bP_{2n}. If, moreover, $n = 2m$, then

$$\Sigma_a = \frac{\tau}{8} g_m,$$

where $\tau = \tau^+ - \tau^-$ and τ^+, τ^- are as in § 2 (2). In particular,

$$\sum_{i=0}^{2m} z_i \bar{z}_i = 1, \qquad z_0^3 + z_1^{6k-1} + z_2^2 + \dots + z_{2m}^2 = 0$$

is a $(4m-1)$-sphere embedded in $S^{4m+1} \subset \mathbb{C}^{2m+1}$ which represents the element $(-1)^m k \cdot g_m \in bP_{4m}$. Example: For $m = 2$ and $k = 1, \dots, 28$ we get the 28 classes of 7-spheres, for $m = 3$ and $k = 1, \dots, 992$ the 992 classes of 11-spheres.

Corollary. *The affine variety $V^0(a_0, \dots, a_n)$, $n \geqq 3$, is a topological manifold if and only if the graph $\Gamma(a)$ satisfies a) or b) of § 2.*

For this theorem and for the case n odd see Brieskorn [5].

Proof. If we remove from V_a^0 the points with $z_n = 0$, we get a space $\tilde{V}_a$ whose fundamental group has $\pi_1(V_a^0 - \{0\}) \cong \pi_1(\Sigma_a)$ as homomorphic image. $\tilde{V}_a$ is fibered over $\mathbb{C}^*$ with $V(a_0, \dots, a_{n-1})$ as fibre which is simply-connected. Thus $\pi_1(\tilde{V}_a) \cong \mathbb{Z}$ and $\pi_1(\Sigma_a)$ is commutative. Because of this and by Smale-Poincaré we have to study only the homology of Σ_a.

Let $V_a^\varepsilon \subset \mathbb{C}^{n+1}$ be the affine variety

$$z_0^{a_0} + z_1^{a_1} + \dots + z_n^{a_n} = \varepsilon$$

$(V_a = V_a^1)$. Let D^{2n+2} be the full ball in $\mathbb{C}^{n+1}$ with center 0 and radius 1 and S^{2n+1}, as before, its boundary. Σ_a is diffeomorphic to $\Sigma_a^\varepsilon = S^{2n+1} \cap V_a^\varepsilon$ for $\varepsilon > 0$ and small. It is the boundary of $B_a^\varepsilon = D^{2n+2} \cap V_a^\varepsilon$ whose interior (for ε small) is diffeomorphic to V_a^ε and V_a. The exact homology sequence of the pair $(B_a^\varepsilon, V_a^\varepsilon)$ shows that Σ_a is $(n-2)$-connected. Using Poincaré duality we get the exact sequence

$$0 \to H_n(\Sigma_a) \to H_n(V_a) \xrightarrow{\sigma} \operatorname{Hom}(H_n(V_a), \mathbb{Z}) \to H_{n-1}(\Sigma_a) \to 0$$

where the homomorphism σ is given by the bilinear intersection form S of V_a (see § 2). This determines $H^*(\Sigma_a)$ completely: $H_n(\Sigma_a) = 0$ if and only if $\det S \neq 0$. If $\det S \neq 0$, then $|\det S|$ equals the order of $H_{n-1}(\Sigma_a)$.

The manifold B_a^ε is parallelizable since its normal bundle is trivial. This finishes the proof in view of § 2.

§ 4. Manifolds with actions of the orthogonal group

$O(n)$ denotes the real orthogonal group with $O(m) \subset O(n)$, $m < n$, by

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \qquad (A \in O(m),\ 1 = \text{unit of } O(n-m)).$$

Let X be a compact differentiable manifold of dimension $2n-1$ on which $O(n)$ acts differentiably ($n \geqq 2$). Suppose each isotropy group is conjugate to $O(n-2)$ or $O(n-1)$. Then the orbits are either Stiefel manifolds $O(n)/O(n-2)$

·f dimension $2n-3$) or spheres $\boldsymbol{O}(n)/\boldsymbol{O}(n-1)$ (of dimension $n-1$). Sup-)se that the 2-dimensional representation of an isotropy group of type $(n-2)$ normal to the orbit is trivial, whereas the n-dimensional representa-)n of an isotropy group of type $\boldsymbol{O}(n-1)$ normal to the orbit is the 1-dimen-)nal trivial representation plus the standard representation of $\boldsymbol{O}(n-1)$. nder these assumptions the orbit space is a compact 2-dimensional manifold ' with boundary, the interior points of X' corresponding to orbits of type $(n)/\boldsymbol{O}(n-2)$, the boundary points of X' to the orbits of type $\boldsymbol{O}(n)/\boldsymbol{O}(n-1)$. ıppose finally that X' is the 2-dimensional disk.

It follows from the classification theorems of [8] and [9] that the classes of anifolds X with the above properties under equivariant diffeomorphisms are one-to-one correspondence with the non-negative integers. We let $W^{2n-1}(d)$ the $(2n-1)$-dimensional $\boldsymbol{O}(n)$-manifold corresponding to the integer $\geqq 0$. The fixed point set of $\boldsymbol{O}(n-2)$ in $W^{2n-1}(d)$ is a 3-dimensional $\boldsymbol{O}(2)$- anifold, namely $W^3(d)$, which by ([9], § 5, Korollar 6) is the lens space $(d,1)$. Thus in order to determine the d associated to a given $\boldsymbol{O}(n)$-manifold our type we just have to look at the integral homology group H_1 of the fixed int set of $\boldsymbol{O}(n-2)$. The manifold $W^{2n-1}(0)$ is $S^n\times S^{n-1}$, whereas $W^{2n-1}(1)$ S^{2n-1}, the actions of $\boldsymbol{O}(n)$ are easily constructed. Consider for $d\geqq 2$ the anifold $\Sigma(d,2,\ldots,2)$ in $\mathbb{C}^{n+1}$ given by

$$z_0^d+z_1^2+\ldots+z_n^2=0\,,\qquad \sum_{i=0} z_i\bar z_i=1 \qquad)$$

;e § 3). It is easy to check that this is an $\boldsymbol{O}(n)$-manifold satisfying all our sumptions. The operation of $A\in\boldsymbol{O}(n)$ on $(z_0,z_1,\ldots,z_n)$ is, of course, given applying the real orthogonal matrix $A\in\boldsymbol{O}(n)$ on the complex vector $,\ldots,z_n)$ leaving z_0 untouched. The fixed point set of $\boldsymbol{O}(n-2)$ is $\Sigma(d,2,2)$ ıich is $L(d,1)$, see [6].

eorem. *The $\boldsymbol{O}(n)$-manifold $\Sigma(d,2,\ldots,2)$ given by (1) is equivariantly diffeo- ›rphic with $W^{2n-1}(d)$, $n\geqq 2$. It can also be obtained by equivariant plumbing $d-1$ copies of the tangent bundle of S^n along the graph A_{d-1}*


$d-1$ vertices.

For the proof it suffices to establish the $\boldsymbol{O}(n)$-action on the manifold tained by plumbing and check all properties: $\boldsymbol{O}(n)$ acts on S^n and on the it tangent bundle of S^n. Since the action of $\boldsymbol{O}(n)$ on S^n has two fixed points : plumbing can be done equivariantly. The fixed point set of $\boldsymbol{O}(n-2)$ is the anifold obtained by plumbing $d-1$ tangent bundles of S^2 which is well- own to be $L(d,1)$ (see [6], resolution of the singularity of $z_0^d+z_1^2+z_2^2=0$).

The above theorem gives another method to calculate the homology of $d,2,\ldots,2)$ and to prove that $\Sigma(d,2,\ldots,2)$ for d odd and an odd number 2's is a sphere. In particular, $\Sigma(3,2,2,2,2,2)$ is the exotic 9-dimensional rvaire sphere (see § 3). The calculation of the Arf invariant of the A_{d-1}- ımbing shows more generally that

$$\Sigma(d,2,\ldots,2)\qquad (d \text{ odd, an odd number of 2's})$$

is the standard sphere for $d\equiv\pm 1 \bmod 8$ and the Kervaire sphere for $d\equiv\pm 3 \bmod 8$, in agreement with a more general result in [5].

Remarks. The $\boldsymbol{O}(n)$-manifold $W^{2n-1}(d)$ coincides with Bredon's manifolds M_k^{2n-1} for $d=2k+1$, see Bredon [3]. $\Sigma(3,2,2,2)$ is the standard 5-sphere (since $\Theta_5=0$). Therefore S^5 admits a differentiable involution α with the lens space $L(3,1)$ as fixed point set and a diffeomorphism β of period 3 with the real projective 3-space as fixed point set. Compare [3]. α and β are defined on $\Sigma(3,2,2,2)$ given by (1) as follows

$$\alpha(z_0,z_1,z_2,z_3)=(z_0,z_1,z_2,-z_3)$$

$$\beta(z_0,z_1,z_2,z_3)=(\varepsilon z_0,z_1,z_2,z_3)\,,\quad \text{where}\quad \varepsilon=\exp(2\pi i/3)\,.$$

Many more such examples of "exotic" involutions etc. which are not differentiably equivalent to orthogonal involutions etc. can be constructed.

§ 5. Manifolds associated to knots

Let X be a compact differentiable manifold of dimension $2n-1$ on which $\boldsymbol{O}(n-1)$ acts differentiably ($n\geqq 3$). Suppose each isotropy group is conjugate to $\boldsymbol{O}(n-3)$ or $\boldsymbol{O}(n-2)$ or is $\boldsymbol{O}(n-1)$. Then the orbits are either Stiefel manifolds $\boldsymbol{O}(n-1)/\boldsymbol{O}(n-3)$ (of dimension $2n-5$) or spheres $\boldsymbol{O}(n-1)/\boldsymbol{O}(n-2)$ (of dimension $n-2$) or points (fixed points of the whole action). The representations of the isotropy groups $\boldsymbol{O}(n-3)$, $\boldsymbol{O}(n-2)$ and $\boldsymbol{O}(n-1)$ respectively normal to the orbit are supposed to be the 4-dimensional trivial representation, the 3-dimensional trivial plus the standard representation of $\boldsymbol{O}(n-2)$, the 1-dimensional trivial plus the sum of two copies of the standard representation of $\boldsymbol{O}(n-1)$. The orbit space X' is then a 4-dimensional manifold with boundary. We suppose that X' is the 4-dimensional disk D^4.

Then the points of the interior of D^4 correspond to Stiefel-manifold-orbits, the points of $\partial D^4=S^3$ to the other orbits. The set F of fixed points corresponds to a 1-dimensional submanifold of S^3, also called F.

We suppose F non-empty and connected, it is then a knot in S^3. We shall call an $\boldsymbol{O}(n-1)$-manifold of dimension $2n-1$ a "knot manifold" if all the above conditions are satisfied.

Let K be the set of isomorphism classes of differentiable knots (i.e. isomorphism classes of pairs (S^3,F) – F a compact connected 1-dimensional submanifold – under diffeomorphisms of S^3). For the following theorem see Jänich ([9], § 6), compare also W. C. Hsiang and W. Y. Hsiang [8].

Theorem. *For any $n\geqq 3$ there is a one-to-one correspondence*

$$\varkappa_n\colon\ K\ \to\ \Phi_{2n-1}\,,$$

where Φ_{2n-1} is the set of isomorphism classes of $(2n-1)$-dimensional knot manifolds under equivariant diffeomorphisms. $\varkappa_n^{-1}$ associates to a knot manifold the knot F considered above.

Remark. The 2-fold branched covering of S^3 along a knot F is an $\boldsymbol{O}(1)$-manifold which will be denoted by $\varkappa_2(F)$.

If we plumb 8 copies of the tangent bundles of S^n $(n \geqq 1)$ according to the tree E_8

we get a $(2n-1)$-dimensional manifold $M^{2n-1}(E_8)$. For $n=2$ this is S^3/G, where G is the binary pentagondodecahedral group [6]. For n odd, $M^{2n-1}(E_8)$ is the standard sphere, as the Arf invariant shows. For $n=2m \geqq 4$, the manifold $M^{4m-1}(E_8)$ is an exotic sphere, it is the famous Milnor sphere which represents the generator $\pm g_m$ of bP_{4m} (see § 3).

$M^{2n-1}(E_8)$ admits an action of $\boldsymbol{O}(n-1)$ as follows: $\boldsymbol{O}(n-1)$ operates as subgroup of $\boldsymbol{O}(n+1)$ on S^n and thus on the unit tangent bundle of S^n. The action on S^n leaves a great circle fixed.

When plumbing the eight copies of the tangent bundle, we put the center of the plumbing operation always on this great circle (for one copy, corresponding to the central vertex of the E_8-tree, we need three such centers, therefore, we cannot have an action of $\boldsymbol{O}(n)$ which has only 2 fixed points on S^n). Then the action of $\boldsymbol{O}(n-1)$ on each copy of the tangent bundle is compatible with the plumbing and extends to an action of $\boldsymbol{O}(n-1)$ on $M^{2n-1}(E_8)$ which, for $n \geqq 3$, becomes a knot manifold as can be checked. The resulting knot can be seen on a picture attached at the end of this lecture. The speaker had convinced himself that this is the torus knot $t(3,5)$, but Zieschang and Vogt showed him a better proof. This implies the

Theorem. *Suppose* $n \geqq 3$. *Then* $\varkappa_n(t(3,5))$ *is equivariantly diffeomorphic to* $M^{2n-1}(E_8)$ *with the* $\boldsymbol{O}(n-1)$*-action defined by equivariant plumbing.* (This is still true for $n=2$, see Remark above.)

We now consider the manifold $\Sigma(p,q,2,2,\ldots,2) \subset \mathbb{C}^{n+1}$ given by the equations (see § 3)

$$z_0^p + z_1^q + z_2^2 + \ldots + z_n^2 = 0$$

$$\sum_{i=0}^{n} z_i \bar{z}_i = 1 \qquad (n \geqq 3)\,.$$

This is an $\boldsymbol{O}(n-1)$-manifold, the action being defined similarly as in § 4. Suppose $(p,q)=1$. Then it can be shown that $\Sigma(p,q,2,2,\ldots,2)$ is a knot manifold: It is $\varkappa_n(t(p,q))$ where $t(p,q)$ is the torus knot. Therefore, by the preceding theorem we have an equivariant diffeomorphism

$$M^{2n-1}(E_8) \cong \Sigma(3,5,\underbrace{2,\ldots,2}_{n-1})\,.$$

This gives a different proof (based on the classification of knot manifolds) that $\Sigma(3,5,\underbrace{2,\ldots,2}_{2m-1})$ represents for $m \geqq 2$ a generator of bP_{4m} (compare § 3).

§ 6. A theorem on knot manifolds

Let F be a knot in S^3. Then the signature $\tau(F)$ can be defined in the following way which Milnor explained to the speaker in a letter. Milnor also considers higher dimensional cases. We cite from his letter, but restrict to classical knots:

Let X be the complement of an open tubular neighbourhood of F in S^3.

Then the cohomology

$$H^* = H^*(\hat{X}, \partial\hat{X}; \mathbb{R})$$

where $\hat{X}$ is the infinite cyclic covering of X, satisfies Poincaré duality just as if $\hat{X}$ were a 2-dimensional manifold bounded by F.

In particular, the pairing

$$\cup\colon H^1 \otimes H^1 \to H^2 \cong \mathbb{R}$$

is non-degenerate. Let t denote a generator for the group of covering transformations of $\hat{X}$. Then for $a, b \in H^1$ the pairing

$$\langle a,b\rangle = a \cup t^* b + b \cup t^* a$$

is symmetric and non-degenerate. Hence, the signature $\tau^+(F) - \tau^-(F) = \tau(F)$ is defined.

There exist earlier definitions of the signature by Murasugi [13] and Trotter [17]. The signature is a cobordism invariant of the knot. A cobordism invariant mod 2 was introduced by Robertello [15] inspired by an earlier paper of Kervaire-Milnor. Let F be a knot and Δ its Alexander polynomial, then the Robertello invariant $c(F)$ is an integer mod 2, namely

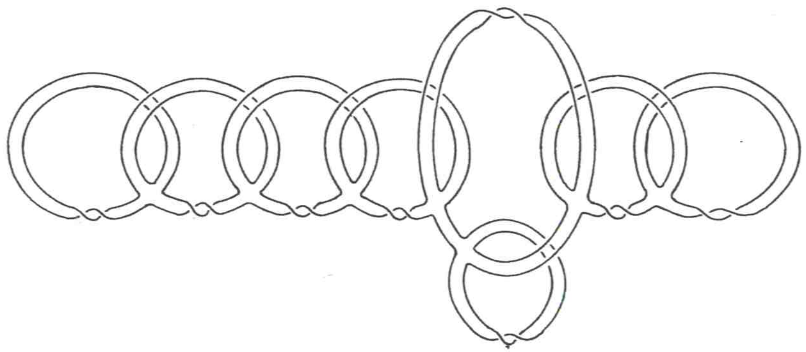
$$c(F) = 0\,, \quad \text{if} \quad \Delta(-1) \equiv \pm 1 \bmod 8\,,$$
$$c(F) = 1\,, \quad \text{if} \quad \Delta(-1) \equiv \pm 3 \bmod 8\,.$$

We recall that the first integral homology group of $\varkappa_2(F)$, the 2-fold branched covering of the knot F (see a remark in § 5), is always finite, its order is odd, and equals up to sign the determinant of F. We have $\pm \det F = \Delta(-1)$.

Theorem. *Let F be a knot, then $\varkappa_n(F)$, $n \geqq 2$, is the boundary of a parallelizable manifold. For n odd, $\varkappa_n(F)$ is homeomorphic to S^{2n-1} and thus represents an element of bP_{2n}, it is the standard sphere if $c(F)=0$, the Kervaire sphere if $c(F)=1$. If $n=2m$, then $\varkappa_{2m}(F)$ is $(2m-2)$-connected and $H_{2m-1}(\varkappa_{2m}(F),\mathbb{Z}) \cong H_1(\varkappa_2(F),\mathbb{Z})$. For $m \geqq 2$ it is homeomorphic to S^{4m-1} if and only if $\det F = \pm 1$. Then $\varkappa_{2m}(F)$ represents (up to sign) an element of bP_{4m} which is* $\pm \dfrac{\tau(F)}{8} \cdot g_m$ (see § 3).

The proof uses an equivariant handlebody construction starting out from a Seifert surface [16] spanned in the knot F. For simplicity, not out of necessity, we have disregarded orientation questions in § 5 and § 6.

Remark. § 2(3) gives up to sign a formula for the signature of the torus knot $t(p,q)$ (p,q odd with $(p,q)=1$).



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40.

Involutionen auf Mannigfaltigkeiten

In: Proceedings of the Conference on Transformation groups, New Orleans 1967, S. 148–166, Berlin-Heidelberg-New York: Springer 1968

Heinrich Behnke zum 70. Geburtstag gewidmet

* An der Konferenz über Transformationsgruppen (Tulane University, New Orleans) konnte ich leider wider Erwarten nicht teilnehmen. Der Aufforderung der Veranstalter, trotzdem einen Bericht für die Proceedings zu schreiben, komme ich gern nach. In New Orleans wollte ich über equivariant plumbing, equivariant handle body constructions und knot manifolds im Sinne von W. C. Hsiang, W. Y. Hsiang und Jänich vortragen. Der vorliegende Bericht hängt zwar hiermit sehr zusammen, legt jedoch das Schwergewicht auf Untersuchungen, mit denen ich mich in Berkeley (August und September 1967) beschäftigt habe. In Berkeley hatte ich zahlreiche Anregungen durch Gespräche mit D. Sullivan und C. T. C. Wall. Da es schwierig ist, diesen beiden Mathematikern stets an den in Frage kommenden Stellen des Berichtes zu danken, möchte ich dies hier in der Einleitung ganz herzlich tun. Die ursprünglich für New Orleans vorgesehenen Dinge kann man in den Lecture Notes von K. H. Mayer und dem Verf. [18] und in der Bonner Dissertation von D. Erle nachlesen. Der vorliegende Bericht entspricht im wesentlichen Kolloquiumsvorträgen, die der Verfasser im Oktober 1967 in Haverford, Princeton, New York und Boston gehalten hat; der Bericht ist manchmal ausführlicher als die Vorträge, muß sich aber an manchen Stellen trotzdem auf Beweisandeutungen beschränken.

Viele Dinge dieser Arbeit können verallgemeinert werden auf G-Mannigfaltigkeiten, wo G eine kompakte Liesche Gruppe ist (vgl. Atiyah and Singer [3]).

1. Beispiele von Involutionen

Wir betrachten die Gleichung

$$z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 0 \qquad (n \geqq 1). \tag{1}$$

Die Exponenten a_j sollen ganze Zahlen $\geqq 2$ sein. Wir setzen $a = (a_0, \ldots, a_n)$. Nach A. Weil [31] ist die Anzahl der Lösungen von (1) über jedem