45.

Free involutions on manifolds and some elementary number theory

In: Symposia Mathematica Vol. V, Istituto Nazionale di Alta Matematica Roma, 1970, S. 411–419, London-New York: Academic Press 1971

* 1. – Let G be a finite group. We shall work in the category of G-manifolds. A G-manifold is a compact oriented differentiable manifold with or without boundary, on which G acts by orientation preserving diffeomorphisms.

Let Y be a G-manifold of even dimension. Then the equivariant signature of Y is an element of the representation ring $R(G)$ of G; see [2], p. 578. Taking its character, for any $g \in G$ the complex number $\operatorname{Sign}(g, Y)$ is defined. It is a real number if $\dim Y \equiv 0$ (4), it is purely imaginary if $\dim Y \equiv 2$ (4). For $g = 1$, we have the ordinary signature which is 0 if $\dim Y \equiv 2$ (4).

Let X be a *free* G-manifold without boundary and of odd dimension. « Free » means that no element of G except the identity has a fixed point in X. To a free G-manifold X there is associated a function

$$\alpha(g, X)\colon G - \{1\} \to \boldsymbol{C}$$

which has been used by C.T.C. Wall and others for the classification of free G-manifolds. The definition of α is as follows ([2], p. 590).

According to equivariant cobordism theory some multiple NX bounds a *free* G-manifold Y.

We define

$$\alpha(g, X) = \frac{1}{N} \operatorname{Sign}(g, Y) \qquad \text{for } g \neq 1. \tag{1}$$

(Observe $\alpha(g, X) = -\sigma(g, X)$ as defined in [2]).

By the equivariant G-signature theorem ([2], p. 582) we have $\operatorname{Sign}(g, Z) = 0$ for an even-dimensional G-manifold Z *without boundary* and an element $g \in G$ acting without fixed points. This together with the Novikov additivity of the equivariant signature ([2], p. 588) ensures that the definition of $\alpha(g, X)$ in (1) is independent of the choice of Y.

If G is of order 2, i.e. $G = \{1, T\}$ with $T^2 = 1$, then $\alpha(T, X)$ is the invariant for free involutions studied in [4] and [6]. It is zero for $\dim X = 1$ (4). If $\dim X = 4k - 1$, then $\alpha(T, X)$ is the Browder-Livesay invariant which is always an integer.

We wish to calculate $\alpha(T, X)$ in special cases. Let G_q be the group of q-th roots of unity and p a natural number prime to q.

Then

$$\zeta(z_1, z_2) = (\zeta z_1, \zeta^p z_2) , \qquad \zeta \in G_q$$

is a free action of G_q on

$$\boldsymbol{S}^3 = \{(z_1, z_2) \in \boldsymbol{C}^2 |\ |z_1|^2 + |z_2|^2 = 1\} .$$

The orbit space $\boldsymbol{S}^3/G_q$ is the lens space $L(q, p)$. Since $G_q \subset G_{2q}$ we have for p prime to $2q$ a natural map

$$L(q, p) \to L(2q, p) .$$

This is a double covering whose covering translation is a free orientation preserving involution T on $L(q, p)$. We define for p prime to $2q$

$$c_{p,q} = \alpha\big(T, L(q, p)\big) . \tag{2}$$

2. – The numbers $c_{p,q}$ were introduced by W. D. Neumann ([7], § 17.2) and used for his calculation of the Browder-Livesay invariant for free involutions on Seifert manifolds. We shall relate the $c_{p,q}$ to well-known numbers $N_{p,q}$ occuring in elementary number theory in connection with the quadratic reciprocity law.

DEFINITION: *Let p, q be relatively prime natural numbers and q odd. $N_{p,q}$ is the number of integers x with*

$$1 \leq x \leq \frac{q-1}{2} ,$$

for which xp has modulo q a remainder of smallest absolute value which is negative.

LEMMA: *If p, q are relatively prime and q odd, then for the quadratic reciprocity symbol (Jacobi-Legendre symbol)*

$$\left(\frac{p}{q}\right) = (-1)^{N_{p,q}}$$

This is called «Lemma of Gauß» if q is a prime, and found in most books on elementary number theory. For the general case, see Frobenius, Gesammelte Abhandlungen, Springer Verlag, Band III, p. 630.

We can also define $N_{p,q}$ by

$$N_{p,q} = \# \left\{ x \,\middle|\, 1 \leqq x \leqq \frac{q-1}{2} \quad \text{and} \quad \frac{1}{2} < \frac{xp}{q} - \left[\frac{xp}{q}\right] \right\}. \tag{3}$$

For real numbers we shall use the function

$$((x)) = x - [x] - \tfrac{1}{2}, \qquad \text{if } x \text{ is not an integer,}$$

$$((x)) = 0, \qquad \text{if } x \text{ is an integer.}$$

Compare [8], p. 254. The function $((\ ))$ has period 1 and is odd.

It is convenient to introduce the function

$$f(x) = ((x + \tfrac{1}{2})) - ((x)).$$

We have

$$f(x) = \tfrac{1}{2}, \qquad \text{if } 0 < x - [x] < \tfrac{1}{2},$$

$$f(x) = -\tfrac{1}{2}, \qquad \text{if } \tfrac{1}{2} < x - [x] < 1,$$

$$f(x) = 0, \qquad \text{otherwise.}$$

Formula (3) implies

$$N_{p,q} = \frac{q-1}{4} - \sum_{\nu=1}^{(q-1)/2} f\left(\frac{\nu p}{q}\right). \tag{4}$$

Lemma: *If p, q are relatively prime and both odd, then*

$$N_{p,q} = \frac{q-1}{4} + \sum_{\nu=1}^{q-1} \left(\left(\frac{\nu p}{2q}\right)\right). \tag{5}$$

Proof: Since $pq \equiv -q \bmod 2q$ we have by (4)

$$N_{p,q} = \frac{q-1}{4} + \sum_{\nu=1}^{(q-1)/2} \left(\left(\frac{(q-2\nu)p}{2q}\right)\right) + \left(\left(\frac{2\nu p}{2q}\right)\right), \text{ q.e.d.}$$

The expression $\sum_{\nu=1}^{q-1} ((\nu p/2q))$ will also be useful for p, $2q$ relatively prime where q may be even or odd. We have

$$\sum_{\nu=1}^{q-1}\left(\left(\frac{\nu p}{2q}\right)\right)=S\left(\frac{p}{2q}\right)$$

where S is a function introduced by Dedekind ([3], formula (39)). Observe that $((\ ))$ in [3] is our function $((\ ))$ with a shift of $\frac{1}{2}$ in the independent variable. According to [3], formula (42), we have for p and $2q$ relatively prime

$$\text{(6)}\qquad -4S\left(\frac{p}{2q}\right)=$$
$$\#\left\{1\leqq\nu\leqq q-1 \mid 0<\frac{\nu p}{q}<1 \mod 2\right\}-$$
$$\#\left\{1\leqq\nu\leqq q-1 \mid 1<\frac{\nu p}{q}<2 \mod 2\right\}.$$

For the proof of (6) we write the right side of (6) as

$$2\sum_{\nu=1}^{q-1} f\left(\frac{\nu p}{2q}\right)=$$
$$2\sum_{\nu=1}^{q-1}\left\{\left(\left(\frac{\nu p+q}{2q}\right)\right)-\left(\left(\frac{\nu p}{2q}\right)\right)\right\}=$$
$$-2\sum_{\nu=1}^{q-1}\left\{\left(\left(\frac{p(q-\nu)}{2q}\right)\right)+\left(\left(\frac{\nu p}{2q}\right)\right)\right\}.$$

Now we state the relation between the Dedekind number $S(p/2q)$, the Gauß number $N_{p,q}$ and the Browder-Livesay invariant of lens spaces.

THEOREM: *Consider the lens space $L(q,p)$ for p and $2q$ relatively prime. It admits a free involution T whose orbit space is $L(2q,p)$. The Browder-Livesay invariant $c_{p,q}$ of T is given by*

$$c_{p,q}=-4S\left(\frac{p}{2q}\right),\quad \textit{see}\ (6).$$

If p and q are relatively prime and both odd, then

$$c_{p,q}=-4N_{p,q}+q-1.$$

This will be proved in § 5 as a special case of a more general theorem.

We shall have to prove the first formula in the above theorem only, the second one being a consequence of (5).

REMARK: The formula

$$-4S\left(\frac{p}{2q}\right) = -4N_{p,q} + q - 1 \qquad \text{for } p, q \text{ odd}$$

(see (5) and (6)) is easily seen to be equivalent to: « Von den absolut kleinsten Resten der Zahlen $p, 2p, \ldots, \frac{1}{2}(q-1)p \pmod q$ sind ebenso viele negativ, wie von ihren absolut kleinsten Resten $(\bmod 2q)$ ». (Frobenius, Gesammelte Abhandlungen, Springer Verlag, Band III, S. 646).

3. – The invariant α of free G-actions has the following property

PROPOSITION: *Let X be a free G-manifold without boundary of odd dimension. Let U be a normal subgroup of G. Then X/U is a free G/U-manifold. If $p\colon G \to G/U$ is the natural homomorphism, then for $\xi \in G/U$ $(\xi \neq 1)$*

$$\alpha(\xi, X/U) = \frac{1}{|U|} \sum_{g \in p^{-1}(\xi)} \alpha(g, X) \tag{7}$$

PROOF: Let W be a finite-dimensional vector space over $\boldsymbol{R}$ or $\boldsymbol{C}$ which is a representation space of G.

The group G/U acts on

$$W^U = \{x \in W \mid ux = x \text{ for all } u \in U\}\,.$$

We obtain in this way a linear map

$$\varrho\colon R(G) \to R(G/U)$$

where R denotes the representation ring.

The endomorphism $1/|U| \sum_{u \in U} gu$ of W has W^U as image and equals g when restricted to W^U. Therefore, its trace equals the trace of the restriction of g to W^U and, for $h \in R(G)$, the character of h and ϱh satisfy $(\xi \in G/U)$

$$\chi_{\varrho h}(\xi) = \frac{1}{|U|} \sum_{g \in p^{-1}\xi} \chi_h(g)\,. \tag{8}$$

If $NX = \partial Y$ where Y is a free G-manifold, then $N(X/U) = \partial(Y/U)$ where Y/U is a free G/U-manifold. The cohomology of Y/U with real (or complex) coefficients can be identified with the U-invariant part of the cohomology of Y. It follows easily that ϱ of the equivariant signature of Y is the equivariant signature of Y/U. Formula (8) implies (7).

4. – Suppose the *free* G-manifold X (without boundary and of odd dimension) bounds a G-manifold Y *not necessarily free*, then for any $g \in G (g \neq 1)$

$$\alpha(g, X) = \operatorname{Sign}(g, Y) - L(g, Y)\,. \tag{9}$$

$L(g, Y)$ is a number associated to the fixed-point set Y^g of g and to the action of g in the neighbourhood of Y^g. For $g \neq 1$ we have $Y^g \cap X = \emptyset$ and Y^g is a submanifold of Y without boundary. $L(g, Y)$ is defined in [2], p. 582. For the proof of (9) we have to use again the equivariant G-signature theorem

$$\operatorname{Sign}(g, Z) = L(g, Z)$$

for an even-dimensional G-manifold Z without boundary, see [2], p. 582 and p. 589.

Let G_q be as before the group of q-th roots of unity. Consider natural numbers $p_1, \ldots, p_n$, each p_j prime to q. Then G_q acts freely on

$$\boldsymbol{S}^{2n-1} = \left\{(z_1, \ldots, z_n) \in \boldsymbol{C}^n \,\middle|\, |z_1|^2 + \ldots + |z_n|^2 = 1\right\}$$

by

$$\zeta(z_1, \ldots, z_n) = (\zeta^{p_1} z_1, \ldots, \zeta^{p_n} z_n)\,, \qquad \zeta \in G_q\,.$$

The orbit space is the $(2n-1)$-dimensional lens space $\mathfrak{L}(q; p_1, \ldots, p_n)$. For $n = 2$

$$\mathfrak{L}(q; 1, p) = L(q, p)$$

as introduced in § 1.

The invariant α of the above G_q-manifolds $\boldsymbol{S}^{2n-1}$ can be calculated using (9), since $\boldsymbol{S}^{2n-1}$ bounds the disk $\boldsymbol{D}^{2n}$ on which G_q operates. Any $\zeta \neq 1$ has only the origin as fixed point, $\operatorname{Sign}(\zeta, \boldsymbol{D}^{2n})$ vanishes and

$$L(\zeta, \boldsymbol{D}^{2n}) = \prod_{j=1}^{n} \frac{\zeta^{p_j} + 1}{\zeta^{p_j} - 1} = -\alpha(\zeta, \boldsymbol{S}^{2n-1}) \tag{10}$$

(see for example, [1], Theorem 6.27).

If all p_j are prime to $2q$ we have the map

$$\mathfrak{L}(q; p_1, \ldots, p_n) \to \mathfrak{L}(2q; p_1, \ldots, p_n)$$

which is a covering of degree 2. The covering translation is an involution T of $\mathfrak{L}(q; p_1, \ldots, p_n)$.

PROPOSITION: *The Browder-Livesay invariant of the involution T on $\mathfrak{L}(q; p_1, \ldots, p_n)$ where all p_j are prime to $2q$ is given by the formula*

$$\alpha(T, \mathfrak{L}(q; p_1, \ldots, p_n)) = \frac{-1}{q} \sum_{\substack{\zeta^{2q}=1 \\ \zeta^q \neq 1}} \prod_{j=1}^{n} \frac{\zeta^{p_j}+1}{\zeta^{p_j}-1}. \tag{11}$$

If q is odd, then the involution T is induced by the antipodal map on S^{2n-1} and

$$\alpha(T, \mathfrak{L}(q; ,p_1, \ldots, p_n)) = \frac{-1}{q} \sum_{\zeta^q=1} \prod_{j=1}^{n} \frac{\zeta^{p_j}-1}{\zeta^{p_j}+1}. \tag{12}$$

PROOF: (11) is a consequence of (7) and (10). Formula (12) holds because $\{-\zeta | \zeta^q = 1\} = G_{2q} - G_q$ for q odd.

For n even we rewrite (11)

$$\alpha(T, \mathfrak{L}(q, p_1, \ldots, p_n)) = \tag{13}$$
$$= \frac{(-1)^{(n/2)+1}}{q} \sum_{\substack{j=1 \\ j \text{ odd}}}^{2q-1} \cot\left(\frac{jp_1}{2q}\pi\right) \cdot \cot\left(\frac{jp_2}{2q}\pi\right) \ldots \cot\left(\frac{jp_n}{2q}\pi\right).$$

5. – Using (13) we shall derive a different expression for $\alpha(T, \mathfrak{L}(q; p_1, \ldots, p_n))$.

For any $2m$—row of natural numbers

$$(a_1, \ldots, a_m; b_1, \ldots, b_m) \qquad \text{with } b_j \text{ and } 2a_j$$

relatively prime, we introduce the integer

$$t(a_1, \ldots, a_m; b_1, \ldots, b_m) = \tag{14}$$
$$\# \left\{ x \in \mathbf{Z}^m | 0 < x_k < a_k \text{ and } 0 < \sum_{k=1}^{m} \frac{x_k b_k}{a_k} < 1 \mod 2 \right\}$$
$$- \# \left\{ x \in \mathbf{Z}^m | 0 < x_k < a_k \text{ and } 1 < \sum_{k=1}^{m} \frac{x_k b_k}{a_k} < 2 \mod 2 \right\}.$$

This number is always 0 if m is even. For m odd we have the following proposition due to Zagier (who stated and proved it for $b_j = 1$).

PROPOSITION (Zagier): *Let m be odd and $(a_1, \dots, a_m; b_1, \dots, b_m)$ as above. Let N be any common multiple of $a_1, \dots, a_m$, then*

$$t(a_1, \dots, a_m; b_1, \dots, b_m) = \frac{(-1)^{(m-1)/2}}{N} \sum_{\substack{j=1 \\ j \text{ odd}}}^{2N-1} \cot\left(\frac{j}{2N}\pi\right) \cot\left(\frac{jb_1}{2a_1}\pi\right) \dots \cot\left(\frac{jb_m}{2a_m}\pi\right). \tag{15}$$

PROOF: For any positive rational number $r = a/b$ where a, b are natural numbers not necessarily relatively prime we have the formula of Eisenstein

$$((r)) = \frac{i}{2b} \sum_{j=1}^{b-1} \cot\left(\frac{j}{b}\pi\right) \exp[2\pi i \cdot jr] \tag{16}$$

(see [8], p. 276). This implies for the function f introduced in § 2

$$\begin{aligned} f(r) &= \left(\left(r + \frac{1}{2}\right)\right) - ((r)) \\ &= \frac{-i}{b} \sum_{\substack{j=1 \\ j \text{ odd}}}^{b-1} \cot\left(\frac{j}{b}\pi\right) \cdot \exp[2\pi i \cdot jr], \text{ for } b \text{ even.} \end{aligned} \tag{17}$$

Therefore

$$\begin{aligned} &t(a_1, \dots, a_m; b_1, \dots, b_m) = \\ &2 \sum_{0<x_k<a_k} f\left(\frac{x_1 b_1}{2a_1} + \frac{x_2 b_2}{2a_2} + \dots + \frac{x_m b_m}{2a_m}\right) = \\ &\frac{-i}{N} \sum_{\substack{j=1 \\ j \text{ odd}}}^{2N-1} \cot\left(\frac{j}{2N}\pi\right) \sum_{0<x_k<a_k} \exp\left(2\pi i \cdot j \left(\frac{x_1 b_1}{2a_1} + \dots + \frac{x_m b_m}{2a_m}\right)\right). \end{aligned}$$

Since for b and $2a$ relatively prime and j odd,

$$\sum_{\nu=1}^{a-1} \exp\left[2\pi i j \frac{\nu b}{2a}\right] = i \cot\left(\frac{jb}{2a}\pi\right),$$

formula (15) follows.

The number $t(a_1, \dots, a_m; 1, \dots, 1)$ is the signature of the Brieskorn variety $z_1^{a_1} + \dots + z_m^{a_m} = 1$. (Compare [5]). This is the reason why Zagier studied formulas like (15). We can also express the Browder-Livesay invariant of the involution on the lens space $\mathfrak{L}(q; p_1, \dots, p_n)$ in terms of t as defined in (14). We may assume $p_1 = 1$ without loss of generality.

THEOREM: *Consider the lens space* $\mathfrak{L}(q; 1, p_2, \ldots, p_n)$ *for* p_i *and* $2q$ *relatively prime and* n *even. It admits a free involution* T *whose orbit space is* $\mathfrak{L}(2q; 1, p_2, \ldots, p_n)$. *The Browder-Livesay invariant of* T *is given by*

$$\alpha(T, \mathfrak{L}(q; 1, p_2, \ldots, p_n)) = t(q, \ldots, q; p_2, \ldots, p_n)\,, \tag{18}$$

as defined in (14) *for* $m = n - 1$.

Formula (18) is a consequence of (13) and (15). The theorem in § 2 is the above theorem for $n = 2$.

Testo pervenuto il 21 settembre 1970.
Bozze licenziate l'11 novembre 1970.

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