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HILBERT MODULAR SURFACES ¹

by Friedrich E. P. HIRZEBRUCH

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§ 0. INTRODUCTION AND PREPARATORY MATERIAL

0.1. In my Tokyo IMU-lectures I began with a survey of the Hilbert modular group G of a totally real field of degree n over the rationals, or more generally of discontinuous groups Γ operating on \mathfrak{H}^n where \mathfrak{H} is the upper half plane. Then I concentrated on the case $n = 2$ and studied the non-singular algebraic surfaces (Hilbert modular surfaces) which arise by passing from \mathfrak{H}^2/G to the compactification $\overline{\mathfrak{H}^2/G}$ and by resolving all singular points of the normal complex space $\overline{\mathfrak{H}^2/G}$. I gave the proof for the resolution of the cusp singularities, a result announced in my Bourbaki lecture [39]. Then I talked about the calculation of numerical invariants (arithmetic genus, signature) of the Hilbert modular surfaces and on the problem of deciding which of these surfaces are rational. This problem is studied in the present paper with much more detail than in the lectures. We construct certain curves on the Hilbert modular surfaces (arising from imbeddings of \mathfrak{H} in \mathfrak{H}^2). Properties of the configuration of such curves

¹) International Mathematical Union lectures, Tokyo, February-March 1972.

together with the curves coming from the resolution of the cusp singularities imply in some cases that the surfaces are rational. In particular we take the field $K = \mathbf{Q}(\sqrt{p})$, where p is a prime $\equiv 1 \pmod{4}$ and investigate the corresponding compact non-singular Hilbert modular surface $Y(p)$ and the surface obtained by dividing $Y(p)$ by the involution T coming from the permutation of the factors of \mathfrak{S}^2 . The surface $Y(p)/T$ is rational for exactly 24 primes, a result which was not yet known completely when I lectured in Tokyo.

Up to now rationality of the Hilbert modular surface or of its quotient by the involution T was known only for the fields $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{3})$, $\mathbf{Q}(\sqrt{5})$, (H. Cohn, E. Freitag ([14], part II), Gundlach [22], Hammond [25], [26]).

In the following section I shall say a few words about further classification results which were mostly proved only after the time of the Tokyo lectures.

0.2. I have learnt a lot from van de Ven concerning the classification of algebraic surfaces; in fact, the rationality for many of the 24 primes was proved jointly using somewhat different methods. The surfaces $Y(p)$ and $Y(p)/T$ are regular, i.e. their first Betti number vanishes. Van de Ven and I (see [41]) used the above mentioned curves to decide how the surfaces $Y(p)$ ($p \equiv 1 \pmod{4}$, p prime) fit into the rough classification of algebraic surfaces (see Kodaira [46], part IV). The result is as follows: *the surface $Y(p)$ is rational for $p = 5, 13, 17$, a blown-up $K3$ -surface for $p = 29, 37, 41$, a blown-up elliptic surface (not rational, not $K3$) for $p = 53, 61, 73$, and of general type for $p > 73$.*

Also the surfaces $Y(p)/T$ ($p \equiv 1 \pmod{4}$, p prime) can be studied by the same methods, but here some refined estimates about certain numerical invariants are necessary.

A joint paper with D. Zagier [42] will show that the surfaces are blown-up $K3$ -surfaces for the seven primes $p = 193, 233, 257, 277, 349, 389, 397$ and blown-up elliptic surfaces (not rational, not $K3$) for $p = 241, 281$. We do not know what happens for the eleven primes $p = 353, 373, 421, 461, 509, 557, 653, 677, 701, 773, 797$. As indicated before, there are 24 primes for which the surface is rational. Except for these 44 primes (eleven of them undecided) the surface $Y(p)/T$ is of general type.

Unfortunately a report on these classification problems could not be included in this paper. It is already too long. We must refer to [41], [42].

0.3. Our standard reference for the study of discontinuous groups operating on \mathfrak{H}^n is Shimizu's paper [71] where other references are given.

For the general theory of compactification we refer to the paper of Baily and Borel [4] and the literature listed there. They mention in particular the earlier work on special cases by Baily, Pyatetskii-Shapiro [63], Satake and the Cartan seminar [67]. Compare also Christian [11], Gundlach [20]. Borel and Baily refer to similar general theorems found independently by Pyatetskii-Shapiro.

We cite from the introduction of the paper by Baily and Borel:

"This paper is chiefly concerned with a bounded symmetric domain X and an arithmetically defined discontinuous group Γ of automorphisms of X . Its main goals are to construct a compactification V^* of the quotient space $V = X/\Gamma$, in which V is open and everywhere dense, to show that V^* may be endowed with a structure of normal analytic space which extends the natural one on V , and to establish, using automorphic forms, an isomorphism of V^* onto a normally projective variety, which maps V onto a Zariski-open subset of the latter."

Of course, it suffices if X is equivalent to a bounded symmetric domain. We are concerned in this paper with the case $X = \mathfrak{H}^n$. We do not require that Γ be arithmetically defined, but assume that it satisfies Shimizu's condition (F), see 1.5 in the present paper. Also under this assumption the compactification of \mathfrak{H}^n/Γ (which we call $\overline{\mathfrak{H}^n/\Gamma}$) is well-defined and is a normally projective variety. The projective imbedding is given again by automorphic forms in the usual manner. (Compare also Gundlach [20] and H. Cartan ([9], [66] Exp. XV). For $n = 2$ we are able to resolve the singularities and obtain from $\overline{\mathfrak{H}^2/\Gamma}$ a non-singular (projective) algebraic surface.

0.4. As far as I know, the resolution (which exists according to Hironaka [34]) of the singularities of V^* (see the above quotation from the paper of Baily and Borel) has been explicitly constructed only in a very few cases: by Hemperly [33], if $X = \{z \in \mathbf{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$, in the present paper if $X = \mathfrak{H}^2$ (thus settling the only cases where the complex dimension of V^* equals 2) and by Igusa [43] for some groups Γ acting on the Siegel upper half plane of degree $g \leq 3$. Two days before writing this introduction (Jan. 27, 1973) I heard that Mumford is working on the general case (Lecture at the Tata Institute, January 1973).

0.5. It is assumed that the reader is familiar with some basic concepts and results of algebraic number theory ([6], [30], [52]), the theory of differentiable manifolds and characteristic classes [36], the theory of algebraic surfaces ([45], [46], [64]) and the resolution of singularities in the 2-dimensional case ([35], [49]). The definitions and theorems needed can be found, for example, in the literature as indicated.

0.6. The “adjunction formula” ([45], Part I) will be used very often. We therefore state it here.

Let X be a (non-singular) complex surface, not necessarily compact. By $e \cdot f$ we denote the intersection number of the integral 2-dimensional homology classes e, f (one of them may have non-compact support). For two divisors E, F (at least one of them compact), $E \cdot F$ denotes the intersection number of the homology classes represented by E and F . Let $c_1 \in H^2(X, \mathbb{Z})$ be the first Chern class of X . The value of c_1 on every 2-dimensional integral homology class of X (with compact support) is well-defined, and for a compact curve D on X we let $c_1[D]$ be the value of c_1 on the homology class represented by D . By \tilde{D} we denote the non-singular model of D and by $e(\tilde{D})$ its Euler number.

Adjunction formula.

Let D be a compact curve (not necessarily irreducible) on the complex surface X . Then

$$(1) \quad e(\tilde{D}) = c_1[D] - D \cdot D + \sum_{\mathfrak{p}} c_{\mathfrak{p}}$$

The sum extends over the singular points of D , and the summand $c_{\mathfrak{p}}$ is a positive even integer for every singular point \mathfrak{p} , depending only on the germ of D in \mathfrak{p} .

If K is a canonical divisor on X , then its cohomology class equals $-c_1$. We have

$$(2) \quad c_1[D] = -K \cdot D.$$

0.7. We shall use some basic facts on group actions [47].

Definition. A group G acts properly discontinuously on the locally compact Hausdorff space X if and only if for any $x, y \in X$ there exist neighborhoods U of x and V of y such that the set of all $g \in G$ with $gU \cap V \neq \emptyset$ is finite. An

equivalent condition is that, for any compact subsets K_1, K_2 of X , the set of all $g \in G$ with $g(K_1) \cap K_2 \neq \emptyset$ is finite.

For a properly discontinuous action, the orbit space X/G is a Hausdorff space. For any $x \in X$, there exists a neighborhood U of x such that the (finite) set of all $g \in G$ with $gU \cap U \neq \emptyset$ equals the isotropy group $G_x = \{g \mid g \in G, g(x) = x\}$. If X is a normal complex space and G acts properly discontinuously by biholomorphic maps, then X/G is a normal complex space.

THEOREM. (H. Cartan [8], and [66] Exp. I). *If X is a bounded domain in \mathbb{C}^n , then the group \mathfrak{A} of all biholomorphic maps $X \rightarrow X$ with the topology of compact convergence is a Lie group. For compact subsets K_1, K_2 of X , the set of all $g \in \mathfrak{A}$ such that $gK_1 \cap K_2 \neq \emptyset$ is a compact subset of \mathfrak{A} . A subgroup of \mathfrak{A} is discrete if and only if it acts properly discontinuously.*

If X is a bounded symmetric domain, then a discrete subgroup Γ of \mathfrak{A} operates freely if and only if it has no elements of finite order.

0.8. I wish to express my gratitude to M. Kreck and T. Yamazaki. Their notes of my lectures in Bonn (Summer 1971) and Tokyo (February-March 1972) were very useful when writing this paper. I should like to thank D. Zagier for mathematical and computational help. Conversations and correspondence with H. Cohn, E. Freitag, K.-B. Gundlach, W. F. Hammond, G. Harder, H. Helling, C. Meyer, W. Meyer, J.-P. Serre, A. V. Sokolovski, A. J. H. M. van de Ven (see 0.2) and A. Vinogradov were also of great help.

Last but not least, I have to thank Y. Kawada and K. Kodaira for inviting me to Japan. I am grateful to them and all the other Japanese colleagues for making my stay most enjoyable, mathematically stimulating, and profitable by many conversations and discussions.

§ 1. THE HILBERT MODULAR GROUP AND THE EULER NUMBER OF ITS ORBIT SPACE

1.1. Let \mathfrak{H} be the upper half plane of all complex numbers with positive imaginary part. \mathfrak{H} is embedded in the complex projective line $\mathbf{P}_1\mathbb{C}$. A complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$ operates on $\mathbf{P}_1\mathbb{C}$ by

$$z \mapsto \frac{az + b}{cz + d}$$

The matrices with real coefficients and $ad - bc > 0$ carry \mathfrak{H} over into itself and constitute a group $\mathbf{GL}_2^+(\mathbf{R})$. The group

$$(1) \quad \mathbf{PL}_2^+(\mathbf{R}) = \mathbf{GL}_2^+(\mathbf{R}) / \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \neq 0 \right\}$$

operates effectively on \mathfrak{H} . As is well known, this is the group of all biholomorphic maps of \mathfrak{H} to itself.

Writing $z = x + iy$ ($x, y \in \mathbf{R}, y > 0$) we have on \mathfrak{H} the Riemannian metric

$$\frac{(dx)^2 + (dy)^2}{y^2}$$

which is invariant under the action of $\mathbf{PL}_2^+(\mathbf{R})$. The volume element equals $y^{-2} dx \wedge dy$.

We introduce the Gauß-Bonnet form

$$(2) \quad \omega = -\frac{1}{2\pi} \cdot \frac{dx \wedge dy}{y^2}$$

If Γ is a discrete subgroup of $\mathbf{PL}_2^+(\mathbf{R})$ acting freely on \mathfrak{H} and such that \mathfrak{H}/Γ is compact, then \mathfrak{H}/Γ is a compact Riemann surface of a certain genus p whose Euler number $e(\mathfrak{H}/\Gamma) = 2 - 2p$ is given by the formula

$$(3) \quad e(\mathfrak{H}/\Gamma) = \int_{\mathfrak{H}/\Gamma} \omega$$

We recall that the discrete subgroup Γ acts freely if and only if Γ has no elements of finite order.

1.2. Consider the n -fold cartesian product $\mathfrak{H}^n = \mathfrak{H} \times \dots \times \mathfrak{H}$. Let \mathfrak{U} be the group of all biholomorphic maps $\mathfrak{H}^n \rightarrow \mathfrak{H}^n$. The connectedness component of the identity of \mathfrak{U} equals the n -fold direct product of $\mathbf{PL}_2^+(\mathbf{R})$ with itself. We have an exact sequence

$$(4) \quad 1 \rightarrow \mathbf{PL}_2^+(\mathbf{R}) \times \dots \times \mathbf{PL}_2^+(\mathbf{R}) \rightarrow \mathfrak{U} \rightarrow S_n \rightarrow 1,$$

where S_n is the group of permutations of n objects corresponding here to the permutations of the n factors of \mathfrak{H}^n . The sequence (4) presents \mathfrak{U} as a semi-direct product. On \mathfrak{H}^n we use coordinates z_1, z_2, \dots, z_n with $z_k = x_k + iy_k$ and $y_k > 0$. We have a metric invariant under \mathfrak{U} :

$$\sum_{j=1}^n \frac{(dx_j)^2 + (dy_j)^2}{y_j^2}$$

The corresponding Gauß-Bonnet form ω is obtained by multiplying the forms belonging to the individual factors; see (2). Therefore

$$(5) \quad \omega = (-1)^n \cdot \frac{1}{(2\pi)^n} \frac{dx_1 \wedge dy_1}{y_1^2} \wedge \dots \wedge \frac{dx_n \wedge dy_n}{y_n^2}$$

If Γ is a discrete subgroup of \mathfrak{A} acting freely on \mathfrak{H}^n and such that \mathfrak{H}^n/Γ is compact, then \mathfrak{H}^n/Γ is a compact complex manifold whose Euler number is given by

$$(6) \quad e(\mathfrak{H}^n/\Gamma) = \int_{\mathfrak{H}^n/\Gamma} \omega.$$

$e(\mathfrak{H}^n/\Gamma)$ is always divisible by 2^n : for a compact complex n -dimensional manifold X we denote by $[X]$ the corresponding element in the complex cobordism group [58]. We have

$$(7) \quad [\mathfrak{H}^n/\Gamma] = 2^{-n} e(\mathfrak{H}^n/\Gamma) \cdot [(\mathbf{P}_1\mathbf{C})^n].$$

This follows, because the Chern numbers of \mathfrak{H}^n/Γ are proportional [37] to those of $(\mathbf{P}_1\mathbf{C})^n$. In particular, the Euler number and the arithmetic genus (Todd genus) of $(\mathbf{P}_1\mathbf{C})^n$ are 2^n and 1 respectively and thus $2^{-n} \cdot e(\mathfrak{H}^n/\Gamma)$ is the arithmetic genus of \mathfrak{H}^n/Γ .

1.3. We shall study special subgroups of the group of biholomorphic automorphisms of \mathfrak{H}^n . They are in fact discrete subgroups of $(\mathbf{PL}_2^+(\mathbf{R}))^n$. Let K be an algebraic number field of degree n over the field \mathbf{Q} of rational numbers. We assume K to be totally real, i.e., there are n different embeddings of K into the reals. We denote them by

$$K \rightarrow \mathbf{R}, \quad x \mapsto x^{(j)}, \quad x \in K$$

We may assume $x = x^{(1)}$. The element x is called totally positive (in symbols, $x \gg 0$) if all $x^{(j)}$ are positive. The group

$$\mathbf{GL}_2^+(K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in K, ad - bc \gg 0 \right\}$$

acts on \mathfrak{H}^n as follows: for $z = (z_1, \dots, z_n) \in \mathfrak{H}^n$ we have

$$z_j \mapsto \frac{a^{(j)} z_j + b^{(j)}}{c^{(j)} z_j + d^{(j)}}.$$

The corresponding projective group

$$\mathbf{PL}_2^+(K) = \mathbf{GL}_2^+(K) / \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in K^* \right\}$$

acts effectively on \mathfrak{H}^n . Thus $\mathbf{PL}_2^+(K) \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$.

Let \mathfrak{o}_K be the ring of algebraic integers in K , then by considering only matrices with $a, b, c, d \in \mathfrak{o}_K$ and $ad - bc = 1$ we get the subgroup $\mathbf{SL}_2(\mathfrak{o}_K)$ of $\mathbf{GL}_2^+(K)$. The group $\mathbf{SL}_2(\mathfrak{o}_K) / \{1, -1\}$ is the famous Hilbert modular group. It is a discrete subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$. We shall denote it by $G(K)$ or simply by G , if no confusion can arise.

$$G = \mathbf{SL}_2(\mathfrak{o}_K) / \{1, -1\} \subset \mathbf{PL}_2^+(K) \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$$

The Hilbert modular group was studied by Blumenthal [5]. An error of Blumenthal concerning the number of cusps was corrected by Maaß [53].

The quotient space \mathfrak{H}^n/G is not compact, but it has a finite volume with respect to the invariant metric. It is natural to use the Euler volume given in (5). The quotient space \mathfrak{H}^n/G is a complex space and not a manifold (for $n > 1$). We shall return to this point later. But the volume of \mathfrak{H}^n/G is well-defined and was calculated by Siegel ([72], [74]). The ζ -function of the field K enters. It is defined by

$$\zeta_K(s) = \sum_{\substack{\mathfrak{a} \in \mathfrak{o}_K \\ \mathfrak{a} \text{ an ideal}}} \frac{1}{N(\mathfrak{a})^s}.$$

This sum extends over all ideals in \mathfrak{o}_K , and $N(\mathfrak{a})$ denotes the norm of \mathfrak{a} . The series converges if the real part of the complex number s is greater than 1. It converges absolutely uniformly on any compact set contained in the half plane $\operatorname{Re}(s) > 1$. The function ζ_K can be holomorphically extended to $\mathbf{C} - \{1\}$. It has a pole of order 1 for $s = 1$. Let D_K denote the discriminant of the field K .

Then

$$(8) \quad D_K^{\frac{s}{2}} \cdot \pi^{-\frac{sn}{2}} \cdot \Gamma(s/2)^n \cdot \zeta_K(s)$$

is invariant under the substitution $s \rightarrow 1 - s$.

This is the well-known functional equation of $\zeta_K(s)$. It can be found in most books on algebraic number theory. See, for example, [52].

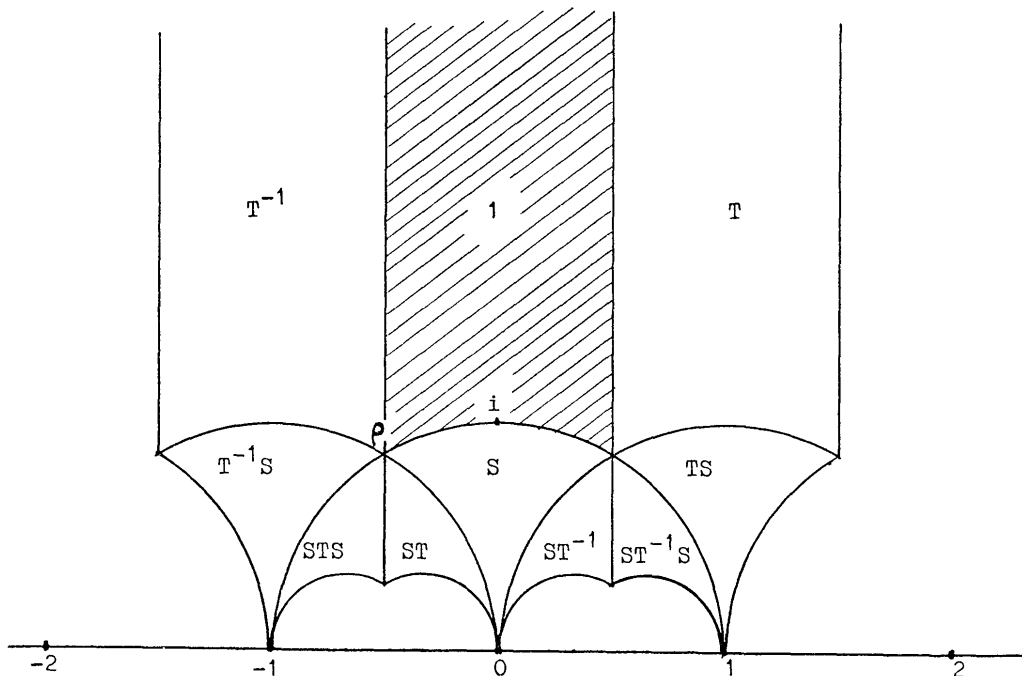
THEOREM (Siegel). *The Euler volume of \mathfrak{H}^n/G relates to the zeta-function as follows*

$$(9) \quad \int_{\mathfrak{H}^n/G} \omega = 2 \zeta_K(-1).$$

The formula (19) of [72] uses the volume element $\frac{dx_1 \wedge dy_1}{y_1^2} \wedge \dots \wedge \frac{dx_n \wedge dy_n}{y_n^2}$ and gives for the volume the value $2 \pi^{-n} \cdot D_K^{3/2} \zeta_K(2)$. If we multiply this value with $(-1)^n \cdot (2\pi)^{-n}$, we get $\int_{\mathfrak{H}^n/G} \omega$.

Formula (9) follows from the functional equation. It was pointed out by J. P. Serre [69] that such Euler volume formulas may be written more conveniently using values of the zeta functions at negative odd integers. $2\zeta_K(-1)$ is a rational number, a result going back to Hecke, see Siegel ([73] Ges. Abh. I, p. 546, [76]) and Klingen [44]. The rational number $2\zeta_K(-1)$ is in fact the rational Euler number of G in the sense of Wall [77], as we shall see later.

1.4. We shall write down explicit formulas for $2\zeta_K(-1)$ in some cases. For $K = \mathbf{Q}$, the group G is the ordinary modular group acting on \mathfrak{H} . A fundamental domain is described by the famous picture (see, for example, [68] p. 128).



The volume of \mathfrak{H}/G equals the volume of the shaded domain. By Siegel's general formula, the volume of the shaded domain with respect to $\frac{dx \wedge dy}{y^2}$ equals

$$2\pi^{-1} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{3}.$$

Therefore, we get for the Euler volume

$$(10) \quad \int_{\mathfrak{H}/G} \omega = -\frac{1}{6} = 2\zeta_{\mathbf{Q}}(-1).$$

We consider the real quadratic fields $K = \mathbf{Q}(\sqrt{d})$ where d is a square-free natural number > 1 . We recall that the discriminant D of K is given by

$$\begin{aligned} D &= 4d \quad \text{for } d \equiv 2, 3 \pmod{4} \\ D &= d \quad \text{for } d \equiv 1 \pmod{4}. \end{aligned}$$

The ring \mathfrak{o}_K has additively the following \mathbf{Z} -bases.

$$\begin{aligned} \mathfrak{o}_K &= \mathbf{Z} + \mathbf{Z}\sqrt{d} \quad \text{for } d \equiv 2, 3 \pmod{4} \\ \mathfrak{o}_K &= \mathbf{Z} + \mathbf{Z}\frac{1 + \sqrt{d}}{2} \quad \text{for } d \equiv 1 \pmod{4} \end{aligned}$$

THEOREM. Let $K = \mathbf{Q}(\sqrt{d})$ be as above. Then for $d \equiv 1 \pmod{4}$

$$(11) \quad 2\zeta_K(-1) = \frac{1}{15} \sum_{\substack{1 \leq b < \sqrt{d} \\ b \text{ odd}}} \sigma_1\left(\frac{d-b^2}{4}\right)$$

and for $d \equiv 2, 3 \pmod{4}$

$$(12) \quad 2\zeta_K(-1) = \frac{1}{30} (\sigma_1(d) + 2 \cdot \sum_{1 \leq b < \sqrt{d}} \sigma_1(d-b^2))$$

where $\sigma_1(a)$ equals the sum of the divisors of a .

This theorem, though not exactly in this form, can be found in Siegel [76]. Compare also Gundlach [22], Zagier [78]. The κ_2 of Gundlach equals $4/\zeta_K(-1)$.

1.5. A reference for the following discussion is [71].

We always assume that Γ is a discrete subgroup of $(\mathbf{PL}^+(\mathbf{R}))^n$ and that \mathfrak{H}^n/Γ has finite volume.

Γ is irreducible if it contains no element $\gamma = (\gamma^{(1)}, \dots, \gamma^{(n)})$ such that $\gamma^{(i)} = 1$ for some i and $\gamma^{(j)} \neq 1$ for some j . See [71], p. 40 Corollary.

An element of $\mathbf{PL}_2^+(\mathbf{R})$ is parabolic if and only if it has exactly one fixed point in $\mathbf{P}_1\mathbf{C}$. This point belongs to $\mathbf{P}_1\mathbf{R} = \mathbf{R} \cup \infty$. An element $\gamma = (\gamma^{(1)}, \dots, \gamma^{(n)})$ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ is called parabolic if and only if all $\gamma^{(i)}$ are parabolic. The parabolic element γ has exactly one fixed point in $(\mathbf{P}_1\mathbf{C})^n$. It belongs to $(\mathbf{P}_1\mathbf{R})^n$. The parabolic points of Γ are by definition fixed points of the parabolic elements of Γ .

The above notation, hopefully, will not confuse the reader. The $\gamma^{(i)}$ are simply the components of the element γ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$. If $\gamma \in \mathbf{PL}_2^+(K) \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ (compare 1.3), then, for γ represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the element $\gamma^{(i)}$ is represented by $\begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}$ where $x \mapsto x^{(i)}$ is the i -th embedding of K in \mathbf{R} . For any group $\Gamma \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ we consider the orbits of parabolic points under the action of Γ on $(\mathbf{P}_1\mathbf{R})^n$. They are called parabolic orbits. Each such orbit consists only of parabolic points.

If Γ is irreducible, then there are only finitely many parabolic orbits. ([71], p. 46 Theorem 5).

Hereafter we shall assume in addition that Γ is irreducible.

If $x \in (\mathbf{P}_1\mathbf{R})^n$ is a parabolic point of Γ , we transform it to $\infty = (\infty, \dots, \infty)$ by an element ρ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$, not necessarily belonging to Γ , of course. Thus $\rho x = \infty$.

Let Γ_x be the isotropy group of x .

$$\Gamma_x = \{ \gamma \mid \gamma \in \Gamma, \gamma x = x \}.$$

Then any element of $\rho \Gamma_x \rho^{-1}$ is of the form

$$(13) \quad z_j \mapsto \lambda^{(j)} z_j + \mu^{(j)}, \lambda^{(j)} > 0.$$

Consider the following multiplicative group

$$(14) \quad \Lambda = \{ t \mid t^{(i)} \in \mathbf{R}, t^{(i)} > 0, \prod_{j=1}^n t^{(j)} = 1 \}.$$

It is isomorphic to \mathbf{R}^{n-1} by taking logarithms. Each element of $\rho \Gamma_x \rho^{-1}$ (see (13)) satisfies $\lambda^{(1)} \cdot \lambda^{(2)} \dots \cdot \lambda^{(n)} = 1$, (compare [71], p. 43, Theorem 3). Therefore we have a natural homomorphism $\rho \Gamma_x \rho^{-1} \rightarrow \Lambda$ whose image is a discrete subgroup Λ_x of Λ of rank $n - 1$. The kernel consists of all the translations

$$z_j \mapsto z_j + \mu^{(j)}$$

where $\mu = (\mu^{(1)}, \dots, \mu^{(n)})$ belongs to a certain discrete subgroup M_x of \mathbf{R}^n of rank n . Thus we have an exact sequence

$$(15) \quad 0 \rightarrow M_x \rightarrow \rho \Gamma_x \rho^{-1} \rightarrow \Lambda_x \rightarrow 1.$$

Using the inner automorphisms of $\rho \Gamma_x \rho^{-1}$, the group Λ_x acts on M_x by componentwise multiplication. However, in the general case, (15) does not present $\rho \Gamma_x \rho^{-1}$ as a semi-direct product. For $n = 1$, the group Λ_x is trivial. For $n = 2$ it is infinite cyclic, $\rho \Gamma_x \rho^{-1}$ is a semi-direct product, and ρ can be chosen in such a way that $\rho \Gamma_x \rho^{-1}$ is exactly the group of all elements of the form (13) with $\lambda \in \Lambda_x$ and $\mu \in M_x$.

For any positive number d , the group $\rho \Gamma_x \rho^{-1}$ acts freely on

$$(16) \quad W = \{ z \mid z \in \mathfrak{H}^n, \prod_{j=1}^n \text{Im}(z_j) \geq d \}$$

where Im denotes the imaginary part. The orbit space $W / \rho \Gamma_x \rho^{-1}$ is a (non-compact) manifold with compact boundary

$$N = \partial W / \rho \Gamma_x \rho^{-1}.$$

Since ∂W is a principal homogeneous space for the semi-direct product $E = \mathbf{R}^n \rtimes \Lambda$ of all transformations

$$z_j \mapsto t^{(j)} z_j + a^{(j)}, t \in \Lambda, a \in \mathbf{R}^n$$

we can consider N as the quotient space of the group E (homeomorphic to \mathbf{R}^{2n-1}) by the discrete subgroup $\rho \Gamma_x \rho^{-1}$. Thus N is an Eilenberg-MacLane space. The $(2n-1)$ -dimensional manifold N is a torus bundle over the $(n-1)$ -dimensional torus Λ / Λ_x . The fibre is the n -dimensional torus \mathbf{R}^n / M_x , and N is obtained by the action of Λ_x on \mathbf{R}^n / M_x which is induced by the action $x_j \mapsto \lambda^{(j)} x_j + \mu^{(j)}$ of $\rho \Gamma_x \rho^{-1}$ on \mathbf{R}^n . Since, in general, $\mu^{(j)}$ is not necessarily an element of M_x , the action of Λ_x on \mathbf{R}^n / M_x need not be the one given by componentwise multiplication.

Definition ([71], p. 48). Let Γ be as before a discrete irreducible subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ such that \mathfrak{H}^n / Γ has finite volume. Let x_v ($1 \leq v \leq t$) be a complete set of Γ -inequivalent parabolic points of Γ . Choose elements $\rho_v \in (\mathbf{PL}_2^+(\mathbf{R}))^n$ with $\rho_v x_v = \infty$ and put $U_v = \rho_v^{-1}(W_v)$ where W_v is defined as in (16) with some positive number d_v instead of d . We say that Γ satisfies condition (F) if it admits (for some d_v) a fundamental domain F of the form

$$F = F_0 \cup V_1 \cup \dots \cup V_t \quad (\text{disjoint union})$$

where F_0 is relatively compact in \mathfrak{H}^n and V_v is a fundamental domain of Γ_{x_v} in U_v .

The fundamental domain $F \subset \mathfrak{H}^n$ is by definition in one-to-one correspondence with \mathfrak{H}^n/Γ and V_v is in one-to-one correspondence with U_v/Γ_{x_v} .

The Hilbert modular group G of any totally real field K is a discrete irreducible subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ with finite volume of \mathfrak{H}^n/G which satisfies condition (F). The existence of a fundamental domain with the required properties was shown by Blumenthal [5] as corrected by Maaß [53]. See Siegel [75] for a detailed exposition.

Two subgroups of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ are called commensurable if their intersection is of finite index in both of them.

Any subgroup Γ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ which is commensurable with the Hilbert modular group G also satisfies (F).

We define

$$(17) \quad [G : \Gamma] = [G : (G \cap \Gamma)] / [\Gamma : (G \cap \Gamma)]$$

Then we get for the Euler volume

$$(18) \quad \int_{\mathfrak{H}^n/\Gamma} \omega = [G : \Gamma] \cdot \int_{\mathfrak{H}^n/G} \omega = [G : \Gamma] \cdot 2\zeta_K(-1)$$

Remark. It is not known whether every discrete irreducible subgroup Γ of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ such that \mathfrak{H}^n/Γ has finite volume satisfies Shimizu's condition (F).

Selberg has conjectured that any Γ satisfying (F) and having at least one parabolic point ($t \geq 1$) is conjugate in the group \mathfrak{A} of all automorphism of \mathfrak{H}^n to a group commensurable with the Hilbert modular group G of some totally real field K with $[K : \mathbf{Q}] = n$.

1.6. Harder [28] has proved a general theorem on the Euler number of not necessarily compact quotient spaces of finite volume. For the following result a direct proof can be given by the method used in [40].

THEOREM (Harder). *Let $\Gamma \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ be a discrete irreducible group satisfying condition (F) of the definition in 1.5. Suppose moreover*

that Γ operates freely on \mathfrak{H}^n . Then \mathfrak{H}^n/Γ is a complex manifold whose Euler number is given by

$$(19) \quad e(\mathfrak{H}^n/\Gamma) = \int_{\mathfrak{H}^n/\Gamma} \omega.$$

If Γ is commensurable with the Hilbert modular group G of K , (where K is a totally real field of degree n over \mathbf{Q}) then

$$(20) \quad e(\mathfrak{H}^n/\Gamma) = [G : \Gamma] \cdot 2\zeta_K(-1).$$

Proof. It follows from 1.5 that \mathfrak{H}^n/Γ contains a compact manifold Y with t boundary components $B_v = \partial W_v / \rho \Gamma_x \rho^{-1}$ (which are T^n -bundles over T^{n-1}). We have to choose the numbers d_v sufficiently large. By the Gauß-Bonnet theorem of Allendoerfer-Weil-Chern [10]

$$e(\mathfrak{H}^n/\Gamma) = \int_Y \omega + \sum_{v=1}^t \int_{B_v} \prod$$

where \prod is a certain $(2n-1)$ -form. By the argument explained in [40], one can show easily that

$$\lim_{d_v \rightarrow \infty} \int_{B_v} \prod = 0. \text{ Q.E.D.}$$

Since the Hilbert modular group G always contains a subgroup Γ of finite index which operates freely and since \mathfrak{H}^n/Γ can be replaced up to homotopy by the compact manifold Y with boundary, $[G : \Gamma] \cdot 2\zeta_K(-1)$ is the Euler number of Γ in the sense of the rational cohomology theory of groups and thus $2\zeta_K(-1)$ is the Euler number of G in the sense of Wall [77].

THEOREM. Let $\Gamma \subset (\mathbf{PL}_2^+(\mathbf{R}))^n$ be a discrete irreducible group such that \mathfrak{H}^n/Γ has finite volume. Assume that Γ satisfies condition (F). The isotropy groups $\Gamma_z (z \in \mathfrak{H}^n)$ are finite cyclic and the set of those z with $|\Gamma_z| > 1$ projects down to a finite set in \mathfrak{H}^n/Γ . Thus \mathfrak{H}^n/Γ is a complex space with finitely many singularities. (For $n = 1$, these “branching points” are actually not singularities.)

Let $a_r(\Gamma)$ be the number of points in \mathfrak{H}^n/Γ which come from isotropy groups of order r . The Euler number of the space \mathfrak{H}^n/Γ is well-defined, and we have

$$(21) \quad e(\mathfrak{H}^n/\Gamma) = \int_{\mathfrak{H}^n/\Gamma} \omega + \sum_{r \geq 2} a_r(\Gamma) \frac{r-1}{r}.$$

The proof is an easy consequence of the Allendoerfer-Weil-Chern formula (compare [40], [65]).

The easiest example of (21) is of course the ordinary modular group $G = G(\mathbf{Q})$. We have $a_2(G) = a_3(G) = 1$ whereas the other $a_r(G)$ vanish. Thus

$$e(\mathfrak{H}/G) = -\frac{1}{6} + \frac{1}{2} + \frac{2}{3} = 1.$$

This checks, since \mathfrak{H}/G and \mathbf{C} are biholomorphically equivalent.

1.7. We shall apply (21) to the Hilbert modular group G and the extended Hilbert modular \hat{G} of a real quadratic field. \hat{G} is defined for any totally real field K . To define it we must say a few words about the units of K . They are the units of the ring \mathfrak{o}_K of algebraic integers. Let U be the group of these units. Its rank equals $n-1$ by Dirichlet's theorem [6]. Let U^+ be the group of all totally positive units (see 1.3). It also has rank $n-1$ because it contains $U^2 = \{\varepsilon^2 \mid \varepsilon \in U\}$.

The extended Hilbert modular group is defined as follows

$$\hat{G} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathfrak{o}_K, ad - bc \in U^+ \right\} / \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in U \right\}$$

We have an exact sequence

$$1 \rightarrow G \rightarrow \hat{G} \rightarrow U^+ / U^2 \rightarrow 1.$$

obtained by associating to each element of \hat{G} its determinant mod U^2 .

If $K = \mathbf{Q}(\sqrt{d})$ with d as in 1.4., then U^+ and U^2 are infinite cyclic groups and U^+ / U^2 is of order 2 or 1. The first case happens if and only if there is no unit in \mathfrak{o}_K with negative norm. If d is a prime p , then

$$U^+ \neq U^2 \Leftrightarrow p \equiv 3 \pmod{4}$$

$$U^+ = U^2 \Leftrightarrow p = 2 \text{ or } p \equiv 1 \pmod{4}.$$

Compare [30], Satz 133.

To apply (21) to the groups G and \hat{G} belonging to a real quadratic field we must know the numbers $a_r(G)$ and $a_r(\hat{G})$. They were determined by Gundlach [21] in some cases and in general by Prestel [61] using the idea that the isotropy groups G_z and \hat{G}_z respectively ($z \in \mathfrak{H}^2$) determine orders in imaginary extensions of K , which by an additional step relates

the $a_r(G)$ and $a_r(\hat{G})$ to ideal class numbers of quadratic imaginary fields over \mathbf{Q} . To write down Prestel's result we fix the following notation. A quadratic field k over \mathbf{Q} (real or imaginary) is completely given by its discriminant D . The class number of the field will be denoted by $h(D)$ or by $h(k)$.

Prestel has very explicit results for the Hilbert modular group G of any real quadratic field K and for the extended group \hat{G} in case the class number of K is odd. We shall indicate part of his result.

THEOREM. (Prestel). *Let d be squarefree, $d \geq 7$ and $(d, 6) = 1$. Let $K = \mathbf{Q}(\sqrt{d})$. Then for the Hilbert modular group $G(K)$ we have for*

$$d \equiv 1 \pmod{4}$$

$$a_2(G) = h(-4d), a_3(G) = h(-3d), a_r(G) = 0 \text{ for } r \neq 2, 3$$

and for $d \equiv 3 \pmod{8}$

$$a_2(G) = 10 \cdot h(-d), a_3(G) = h(-12d), a_r(G) = 0 \text{ for } r \neq 2, 3$$

and for $d \equiv 7 \pmod{8}$

$$a_2(G) = 4h(-d), a_3(G) = h(-12d), a_r(G) = 0 \text{ for } r \neq 2, 3$$

If d is a prime $\equiv 3 \pmod{4}$ and $d \neq 3$ we have for the extended group $\hat{G}(K)$ the following result:

If $d \equiv 3 \pmod{8}$, then

$$a_2(\hat{G}) = 3h(-d) + h(-8d), a_3(\hat{G}) = h(-12d)/2,$$

$$a_4(\hat{G}) = 4h(-d),$$

$$a_r(\hat{G}) = 0 \text{ for } r \neq 2, 3, 4.$$

If $d \equiv 7 \pmod{8}$, then

$$a_2(\hat{G}) = h(-d) + h(-8d), a_3(\hat{G}) = h(-12d)/2,$$

$$a_4(\hat{G}) = 2h(-d),$$

$$a_r(\hat{G}) = 0 \text{ for } r \neq 2, 3, 4.$$

Prestel gives the numbers $a_r(G)$ and $a_r(\hat{G})$ also for $d = 2, 3, 5$. For $d = 3$ we have

$$a_2(\hat{G}) = 3, a_3(\hat{G}) = 1, a_4(\hat{G}) = 1, a_{12}(\hat{G}) = 1,$$

all other $a_r(\hat{G}) = 0$.

We apply (12), (20) and (21) for $K = \mathbf{Q}(\sqrt{3})$ as an example

$$2\zeta_K(-1) = \frac{1}{30}(4 + 2\sigma_1(2)) = \frac{10}{30} = \frac{1}{3},$$

$$[G : \hat{G}] = \frac{1}{2},$$

$$e(H^2/\hat{G}) = \frac{1}{6} + 3 \cdot \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{11}{12} = 4.$$

We shall copy Prestel's table [61] of the $a_r(G)$ and the $a_r(\hat{G})$ (if known) for $K = \mathbf{Q}(\sqrt{d})$ up to $d = 41$. In [61] the table contains an error which was corrected in [62].

We also tabulate the values of $2\zeta_K(-1)$, $e(\mathfrak{H}^2/G)$, and of $e(\mathfrak{H}^2/\hat{G})$ if known. In the columns before $2\zeta_K(-1)$ we find the values of the $a_r(G)$; the values of the $a_r(\hat{G})$ are written behind $2\zeta_K(-1)$. If there is no entry, then the value is zero.

If the $a_r(\hat{G})$ and $e(\mathfrak{H}^2/\hat{G})$ are not given in the table, this means that either there exists a unit of negative norm and thus $G = \hat{G}$ or that the values are not known. This is indicated in the last column.

By Prestel $a_r(G) = 0$ for $r > 3$ and $K = \mathbf{Q}(\sqrt{d})$ with $d > 5$, and we have for $d > 5$

$$(22) \quad e(\mathfrak{H}^2/G) = 2\zeta_K(-1) + \frac{a_2(G)}{2} + a_3(G) \cdot \frac{2}{3}$$

Since the Euler number is an integer, we obtain by (11) and (12):

For $d > 5$, $d \equiv 1 \pmod{4}$, d square-free,

$$\sum_{\substack{1 \leq b < \sqrt{d} \\ b \text{ odd}}} \sigma_1\left(\frac{d-b^2}{4}\right) \equiv 0 \pmod{5}$$

For $d > 5$, $d \equiv 2, 3 \pmod{4}$, d square-free

$$\sigma_1(d) + 2 \sum_{1 \leq b < \sqrt{d}} \sigma_1(d-b^2) \equiv 0 \pmod{5}$$

Problem. Prove these congruences in the framework of elementary number theory.

d	2	3	4	5	6	$2\zeta_K(-1)$	2	3	4	6	12	$e(\mathfrak{H}^2/G)$	$e(\mathfrak{H}^2/\hat{G})$
2	2	2	2			1/6	—	—	—	—	—	4	$G = \hat{G}$
3	3	2			1	1/3	3	1	1		1	4	4
5	2	2		2		1/15	—	—	—	—	—	4	$G = \hat{G}$
6	6	3				1	5	1	2	1		6	6
7	4	4				4/3	5	2	2			6	6
10	6	4				7/3	—	—	—	—	—	8	$G = \hat{G}$
11	10	4				7/3	5	2	4			10	8
13	2	4				1/3	—	—	—	—	—	4	$G = \hat{G}$
14	12	4				10/3	8	2	4			12	10
15	8	6				4	—	—	—	—	—	12	?
17	4	2				2/3	—	—	—	—	—	4	$G = \hat{G}$
19	10	4				19/3	9	2	4			14	12
21	4	5				2/3	3	2		1		6	4
22	6	8				23/3	12	4	2			16	14
23	12	8				20/3	7	4	6			18	14
26	18	4				25/3	—	—	—	—	—	20	$G = \hat{G}$
29	6	6				1	—	—	—	—	—	8	$G = \hat{G}$
30	12	10				34/3	—	—	—	—	—	24	?
31	12	4				40/3	11	2	6			22	18
33	4	3				2	7	1		1		6	6
34	12	4				46/3	—	—	—	—	—	24	?
35	20	8				38/3	—	—	—	—	—	28	?
37	2	8				5/3	—	—	—	—	—	8	$G = \hat{G}$
38	18	8				41/3	16	4	6			28	22
39	16	10				52/3	—	—	—	—	—	40	?
41	8	2				8/3	—	—	—	—	—	8	$G = \hat{G}$

§ 2. THE CUSPS AND THEIR RESOLUTION FOR THE 2-DIMENSIONAL CASE

2.1. Let K be a totally real algebraic field of degree n over \mathbf{Q} and M an additive subgroup of K which is a free abelian group of rank n . Such a group M is called a complete \mathbf{Z} -module of K . Let U_M^+ be the group of those units ε of K which are totally positive and satisfy $\varepsilon M = M$. Any $\alpha \in K$ with $\alpha M = M$ is automatically an algebraic integer and a unit.

The group U_M^+ is free of rank $n - 1$ (compare [6]).

Two modules M_1, M_2 are called (strictly) equivalent if there exists a (totally positive) number $\lambda \in K$ with $\lambda M_1 = M_2$. Of course, $U_{M_1}^+ = U_{M_2}^+$ for equivalent modules.

According to [71] p. 45, Theorem 4, for any parabolic point x of an irreducible discrete subgroup Γ of $(\mathbf{PL}^+(\mathbf{R}))^n$ with \mathfrak{H}^n/Γ of finite volume the element $\rho \in (\mathbf{PL}^+(\mathbf{R}))^n$ with $\rho x = \infty$ can be chosen in such a way that the group $\rho \Gamma_x \rho^{-1}$ (see 1.5 (15)) is contained in $\mathbf{PL}^+(K) \subset (\mathbf{PL}^+(\mathbf{R}))^n$ where K is a suitable totally real field. Then we have an exact sequence

$$0 \rightarrow M \rightarrow \rho \Gamma_x \rho^{-1} \rightarrow V \rightarrow 1$$

where M is a complete \mathbf{Z} -module in K and V is a subgroup of U_M^+ of rank $n - 1$. The field K , the strict equivalence class of M and the group V are completely determined by the parabolic orbit and do not depend on the choice of ρ .

It can be shown more generally ([71] p. 45, footnote 3) that there exists a $\rho \in (\mathbf{PL}_2^+(\mathbf{R}))^n$ such that $\rho \Gamma \rho^{-1} \subset \mathbf{PL}_2^+(K)$, provided there is at least one parabolic orbit. Therefore, the field K is the same for all parabolic orbits. The conjecture of Selberg (1.5 Remark) remains unsettled, because, if we represent the elements of $\rho \Gamma \rho^{-1}$ by matrices with coefficients in \mathfrak{o}_K , we have no information on the determinants of these matrices.

A parabolic orbit will be called a *cusp*. We say that the cusp is of *type* (M, V) . If x is a point in the parabolic orbit, we often say that the cusp is at x . Sometimes the cusp will be denoted by x .

For a given pair (M, V) with $V \subset U_M^+$ (where V has rank $n - 1$) we define

$$G(M, V) = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix} \mid \varepsilon \in V, \mu \in M \right\} = M \rtimes V \quad (\text{semi-direct product})$$

For $n = 2$, the element $\rho \in \mathbf{PL}_2^+(\mathbf{R})^n$ can be chosen in a such a way that $\rho \Gamma_x \rho^{-1} = G(M, V)$.

Let K be a totally real field of degree n over \mathbf{Q} , let M be a complete \mathbf{Z} -module in K and V a subgroup of U_M^+ of finite index. Suppose \mathfrak{G} is a group of matrices $\begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix}$ (with $\varepsilon \in V$, $\mu \in K$, and $\mu \in M$ for $\varepsilon = 1$) such that the sequence

$$(1) \quad 0 \rightarrow M \rightarrow \mathfrak{G} \rightarrow V \rightarrow 1$$

is exact.

The group \mathfrak{G} operates freely and properly discontinuously on \mathfrak{H}^n . We add one additional point ∞ to the complex manifold $\mathfrak{H}^n/\mathfrak{G}$. A complete

system of open neighborhoods of ∞ in the new space $\overline{\mathfrak{H}^n/\mathfrak{G}} = \mathfrak{H}^n/\mathfrak{G} \cup \infty$ is given by the sets

$$(2) \quad (\overset{\circ}{W}(d) / \mathfrak{G}) \cup \infty$$

where, for any positive d ,

$$(3) \quad \overset{\circ}{W}(d) = \{z \mid z \in \mathfrak{H}^n, \prod_{j=1}^n \operatorname{Im}(z_j) > d\}$$

The local ring $\mathfrak{O}(\mathfrak{G})$ at ∞ is defined as the ring of functions holomorphic in some neighborhood of ∞ (except ∞) and continuous in ∞ . For $n > 1$ the condition "continuous in ∞ " can be dropped ([71], p. 50, lemma 7).

If $\mathfrak{G} = G(M, V)$ we put $\mathfrak{O}(\mathfrak{G}) = \mathfrak{O}(M, V)$. We shall only give the structure of $\mathfrak{O}(M, V)$ explicitly. *For $n = 2$ this is no loss of generality.* The ring $\mathfrak{O}(M, V)$ has the following structure:

Let M^* be the complete module in K which is dual to M : An element $x \in K$ belongs to M^* if and only if the trace $\operatorname{tr}(xa)$ is an integer for all $a \in M$. We recall that

$$\operatorname{tr}(xa) = \sum_{j=1}^n x^{(j)} a^{(j)}$$

Let M^{*+} be the set of all totally positive elements of M . The local ring $\mathfrak{O}(M, V)$ is the ring of all Fourier series

$$(4) \quad f = a_0 + \sum_{x \in M^{*+}} a_x \cdot e^{2\pi i(x^{(1)}z_1 + \dots + x^{(n)}z_n)},$$

for which the coefficients a_x satisfy $a_{\varepsilon x} = a_x$ for all $\varepsilon \in V$, and which converge on $\overset{\circ}{W}(d)$ for some positive d depending on f .

Proposition. The space $\mathfrak{H}^n/\mathfrak{G}$ with the local ring $\mathfrak{O}(\mathfrak{G})$ at ∞ is a normal complex space.

This is known for $n = 1$, of course. For $n \geq 2$ we have to check H. Cartan's condition ([67] Exposé 11, Théorème 1) that there is some neighbourhood U of ∞ such that for any two different points $p_1, p_2 \in U - \{\infty\}$ there exists a holomorphic function f in $U - \{\infty\}$ with $f(p_1) \neq f(p_2)$. If \mathfrak{G} occurs as group $\rho\Gamma_x\rho^{-1}$ for some cusp of a group Γ satisfying condition (F) of 1.5, Cartan's condition is proved in the theory

of compactification (0.3) by the use of Γ -automorphic forms. The group $G(M, U_M^+)$ occurs in such a way. Namely, M is strictly equivalent to an ideal in some order \mathfrak{o} of K (see [6]) where $\mathfrak{o} = \{x \in K \mid xM \subset M\}$. Therefore, we may assume that M is such an ideal. The cusp at ∞ of the arithmetic group (commensurable with the Hilbert modular group)

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in \mathfrak{o}, \beta \in M, \alpha\delta - \beta\gamma \in U_M^+ \right\}$$

has the isotropy group $G(M, U_M^+)$.

As W. Meyer pointed out to me, the group $H^2(V, M)$ —the set of all equivalence classes of extensions over V with kernel M and belonging to the action of V on M —is finite. (It vanishes for $n \leq 2$.) This implies the existence of a translation $\rho \in \mathbf{PL}_2^+(K)$ with $\rho z = z + a$ such that $\rho \mathfrak{G} \rho^{-1} \subset G(\tilde{M}, V)$ where $\tilde{M} = \frac{1}{k} M$ and k is the order of the extension \mathfrak{G} as element of $H^2(V, M)$. Therefore $\rho \mathfrak{G} \rho^{-1}$ is commensurable with $G(M, U_M^+)$, and it follows from general results on ramifications of complex spaces [18] that $\mathfrak{H}^n/\mathfrak{G}$ is a normal complex space. (See also 0.7 for quotients of normal complex spaces).

Remark. It would be interesting to check Cartan's condition directly using only the structure of the ring $\mathfrak{O}(\mathfrak{G})$. It seems to be unknown if every \mathfrak{G} occurs for a cusp of a group Γ of type (F) . We shall call the point ∞ of the normal complex space $\overline{\mathfrak{H}^n/\mathfrak{G}}$ a "cusp", even if it does not occur for a group Γ .

The point ∞ (with the local ring $\mathfrak{O}(\mathfrak{G})$) is non-singular for $n = 1$. Probably it is always singular for $n \geq 2$. This was shown by Christian [11] to be true for the cusps of the Hilbert modular group of a totally real field of degree $n \geq 2$. For $n = 2$, see [21].

Our aim is to resolve the point ∞ of $\overline{\mathfrak{H}^2/G(M, V)}$ in the sense of the theory of resolution of singularities in a normal complex space of dimension 2 (see, for example, [35], [49]). This will be done in 2.4 and 2.5. The resolution process shows that ∞ is always a singular point.

It remains an open problem to give explicit resolutions also for $n > 2$.

If Γ is a discrete irreducible subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ satisfying the condition (F) of the definition in 1.5, then \mathfrak{H}^n/Γ can be compactified by adding t points (cusps) where t is the number of Γ -inequivalent parabolic points of Γ . The resulting space is a compact normal complex space. It is even a projective algebraic variety (0.3).

2.2. In the next sections we shall consider the case $n = 2$, construct certain normal singularities of complex surfaces and show that they are cusps in the sense of 2.1. The construction will be very much related to continued fractions.

Consider a function $k \mapsto b_k$ from the integers to the natural numbers greater or equal 2. For each integer k take a copy R_k of \mathbf{C}^2 with coordinates u_k, v_k . We define R'_k to be the complement of the line $u_k = 0$ and R''_k to be the complement of $v_k = 0$. The equations

$$(5) \quad \begin{aligned} u_{k+1} &= u_k^{b_k} v_k \\ v_{k+1} &= 1/u_k \end{aligned}$$

give a biholomorphic map $\varphi_k : R'_k \rightarrow R''_{k+1}$

In the disjoint union $\cup R_k$ we make all the identifications (5). We get a set Y . We may now consider each R_k as a subset of Y . Each R_j is mapped by (u_j, v_j) bijectively onto \mathbf{C}^2 . This defines an atlas of Y . A subset of Y is open if and only if its intersection with each R_j is an open subset of R_j .

Lemma. *The topological space Y defined by (5) satisfies the Hausdorff separation axiom.*

Proof. Denote the map $R_j \rightarrow \mathbf{C}^2$ by ψ_j . Let k be an integer. According to Bourbaki [7] p. 36, we have to show that the graph of

$$(6) \quad \psi_{j+k} \circ \psi_j^{-1} : \psi_j(R_j \cap R_{j+k}) \rightarrow \psi_{j+k}(R_j \cap R_{j+k})$$

is closed in $\psi_j(R_j) \times \psi_{j+k}(R_{j+k}) = \mathbf{C}^2 \times \mathbf{C}^2$. Without loss of generality we may assume $j = 0$ and $k > 0$. The map $\psi_k \circ \psi_0^{-1}$ is given by

$$(7) \quad \begin{aligned} u_k &= u_0^{p_k} \cdot v_0^{q_k} \\ v_k &= u_0^{-p_{k-1}} \cdot v_0^{-q_{k-1}} \end{aligned}$$

where

$$\begin{pmatrix} p_k & q_k \\ -p_{k-1} & -q_{k-1} \end{pmatrix} = \begin{pmatrix} b_{k-1} & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} b_{k-2} & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\frac{p_k}{q_k} = b_0 - \frac{1}{b_1 - \frac{1}{b_2 - \frac{1}{\ddots - \frac{1}{b_{k-1}}}}}$$

p_k, q_k are coprime. We define $p_0 = 1, q_0 = 0$ and have

$$p_{k+1} = b_k p_k - p_{k-1} \quad \text{for } k \geq 1,$$

$$q_{k+1} = b_k p_k - p_{k+1} \quad \text{for } k \geq 1,$$

$$p_k > q_k, p_{k+1} > p_k \geq 1, q_{k+1} > q_k \geq 0, \quad \text{for } k \geq 0.$$

The intersection $R_0 \cap R_k$ as subset of R_0 is given by $u_0 \neq 0, v_0 \neq 0$ for $k \geq 2$ and by $u_0 \neq 0$ for $k = 1$. The graph of $\psi_k \cdot \psi_0^{-1}$ (see (6)) is given by

$$u_k = u_0^{p_k} \cdot v_0^{q_k}, v_k \cdot u_0^{p_{k-1}} \cdot v_0^{q_{k-1}} = 1$$

$$u_0 \neq 0, v_0 \neq 0 \quad (k \geq 2)$$

$$u_0 \neq 0 \quad (k = 1)$$

But the inequalities follow from the equations. Therefore the graph is closed in $\mathbf{C}^2 \times \mathbf{C}^2$. This finishes the proof of the lemma. The negative exponents in the second line of (7) were essential.

The argument would break down, for example, if $k = 6$ and $b_i = 1$ for $0 \leq i \leq 5$, because $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}^6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The topological space Y obviously has a countable basis. For any function $k \mapsto b_k \geq 2$ we have constructed a complex manifold Y of complex dimension 2. In Y we have a string of compact rational curves S_k non-singularly embedded ($k \in \mathbf{Z}$). The curve S_k is given by $u_{k+1} = 0$ in the $(k+1)$ -th coordinate system and by $v_k = 0$ in the k -th coordinate system. S_k, S_{k+1} intersect in just one point transversally, namely in the origin of the $(k+1)$ -th coordinate system. S_i, S_k ($i < k$) do not intersect, if $k - i \neq 1$. The union of all the S_k is a closed subset of Y .

Lemma. The self-intersection number of the curve S_k equals $-b_k$.

Proof. The coordinate function u_{k+1} extends to a meromorphic function on Y . Its divisor is an infinite integral linear combination of the S_j which because of (5) contains S_{k-1} with multiplicity b_k , the curve S_k with multiplicity 1 and the curve S_{k+1} with multiplicity 0. The intersection number of S_k with this divisor is zero. Since it is also equal to $b_k + S_k \cdot S_k$, the result follows.

Remark. The construction of Y is analogous to the resolution of a quotient singularity in [35], 3.4. For technical reasons we have changed

the notation by shifting the indices of S_k and b_k by 1. This should also be taken into account when comparing with [39], § 4.

2.3. Let us assume that the function $k \mapsto b_k \geq 2$ of 2.2 is periodic, i.e. there exists a natural number $r \geq 1$ such that

$$b_{k+r} = b_k.$$

Continued fractions of the form

$$a_0 - \frac{1}{a} - \cfrac{\cdot}{\cdot} - \cfrac{1}{a_s}$$

shall be denoted by $[[a_0, \dots, a_s]]$; similarly, $[[a_0, a_1, a_2, \dots]]$ stands for infinite continued fractions of this kind. For our given function $k \mapsto b_k \geq 2$ we consider the numbers

$$(8) \quad w_k = [[b_k, b_{k+1}, \dots]], \quad k \in \mathbf{Z}.$$

The w_k are all equal to 1 if $b_j = 2$ for all j . Therefore, we assume $b_j \geq 3$ for at least one j . Then all w_k are quadratic irrationalities which are greater than 1. They satisfy $w_{k+r} = w_k$ and all belong to the same real quadratic field K . We consider the complete \mathbf{Z} -module

$$M = \mathbf{Z}w_0 + \mathbf{Z} \cdot 1 \subset K$$

Let $x \mapsto x'$ be the non-trivial automorphism of K . Thus $x = x^{(1)}$ and $x' = x^{(2)}$ in the notation of 1.3. The module M acts freely on \mathbf{C}^2 by $(z_1, z_2) \mapsto (z_1 + a, z_2 + a')$ for $a \in M$. For our function $j \mapsto b_j \geq 2$ we have constructed in 2.2 a complex manifold Y . We now define a biholomorphic map

$$\Phi : Y - \bigcup_{j \in \mathbf{Z}} S_j \rightarrow \mathbf{C}^2/M$$

$$\Phi : (u_0, v_0) \mapsto (z_1, z_2)$$

by

$$(9) \quad \begin{aligned} 2\pi iz_1 &= w_0 \log u_0 + \log v_0 \\ 2\pi iz_2 &= w'_0 \log u_0 + \log v_0 \end{aligned}$$

The logarithms are defined modulo integral multiples of $2\pi i$, thus (z_1, z_2) is well-defined modulo M . Observe that

$$Y - \bigcup_{j \in \mathbb{Z}} S_j = \{ (u_0, v_0) \mid u_0 \neq 0, v_0 \neq 0 \}.$$

Since the determinant $\begin{vmatrix} w_0 & w'_0 \\ 1 & 1 \end{vmatrix} \neq 0$, we can solve (9) for $\log u_0$ and $\log v_0$ and obviously have a biholomorphic map. The map Φ can be written down with respect to the k -th coordinate system ($k \in \mathbb{Z}$). The result is as follows.

Put $A_0 = 1$ and $A_{k+1} = w_{k+1}^{-1} \cdot A_k$. This defines A_k inductively for any integer k :

$$A_k = (w_1 w_2 \dots w_k)^{-1} \text{ for } k \geq 1, \quad A_{-k} = w_0 w_{-1} \dots w_{-k+1} \text{ for } k \geq 1, \\ 0 < A_{k+1} < A_k \text{ for } k \in \mathbb{Z}, \quad A_k \neq 1 \text{ for } k \neq 0.$$

Formula (8) implies $w_k = b_k - \frac{1}{w_{k+1}}$ and

$$(10) \quad b_k A_k = A_{k-1} + A_{k+1}$$

For any integer k , the numbers A_{k-1}, A_k are a basis for M . From the coordinate transformations (5) we get the expression for the map Φ in the k -th coordinate system

$$(11) \quad \begin{aligned} 2\pi i z_1 &= A_{k-1} \cdot \log u_k + A_k \cdot \log v_k \\ 2\pi i z_2 &= A'_{k-1} \cdot \log u_k + A'_k \cdot \log v_k \end{aligned}$$

We had assumed that the b_j are periodic with period r which implies $w_{k+r} = w_k$ for any k . Therefore, A_r^{-1} equals the product of any r consecutive w_j which gives

$$(12) \quad \begin{aligned} A_{k+r} &= A_r A_k && \text{for any } k \in \mathbb{Z} \\ (A_r)^n &= A_{nr} && \text{for any } n \in \mathbb{Z}. \end{aligned}$$

This implies that $A_r M = M$. Therefore A_r is an algebraic integer and a unit $\neq 1$.

If we apply the non-trivial automorphism of K to the equation

$$w_k = b_k - \frac{1}{w_{k+1}} \text{ and use the periodicity we get}$$

$$(13) \quad \begin{aligned} w'_{k+1} &= b_k - \frac{1}{w'_k} \\ w'_{k+1} &= [[b_k, b_{k-1}, \dots]] > 1 \end{aligned}$$

Therefore,

$$(14) \quad 0 < w'_k < 1 < w_k \quad \text{for } k \in \mathbf{Z}.$$

Thus the w_k and the A_k are totally positive. Let V be the (infinite cyclic) subgroup of U_M^+ generated by A_r . Thus we have associated to our function $j \mapsto b_j \geq 2$ (at least one $b_j \geq 3$) and the given period r (which need not be the smallest one) a pair (M, V) and a group $G(M, V)$ (see 2.1) which determines a cusp singularity. We shall use the complex manifold Y constructed in 2.2 for a resolution of this cusp singularity.

We restrict Φ to the open subset $\Phi^{-1}(\mathfrak{H}^2/M)$ of Y . According to (11) this set is given by

$$\begin{aligned} A_{k-1} \cdot \log |u_k| + A_k \cdot \log |v_k| &< 0 \\ A'_{k-1} \cdot \log |u_k| + A'_k \cdot \log |v_k| &< 0 \end{aligned}$$

Since $v_k = 0$ or $u_{k+1} = 0$ for a point on S_k and the above inequalities do not depend on the coordinate system, it follows that

$$Y^+ = \Phi^{-1}(\mathfrak{H}^2/M) \cup \bigcup_{k \in \mathbf{Z}} S_k$$

is an open subset of Y . The group

$$V = \{ (A_r)^n \mid n \in \mathbf{Z} \}$$

acts on Y^+ as follows:

$(A_r)^n$ sends a point with coordinates u_k, v_k in the k -th coordinate system to the point with the same coordinates in the $(k+nr)$ -th coordinate system. Because of the periodicity $b_{j+r} = b_j$, this is compatible with the identifications (5). Therefore the action of the infinite cyclic group V on the complex manifold Y^+ is well-defined. We have the exact sequence

$$0 \rightarrow M \rightarrow G(M, V) \rightarrow V \rightarrow 1$$

Thus V acts on \mathfrak{H}^2/M . On the other hand we have a biholomorphic map

$$\Phi : Y^+ - \bigcup_{k \in \mathbf{Z}} S_k \rightarrow \mathfrak{H}^2/M$$

Lemma. The actions of V on Y^+ and \mathfrak{H}^2/M are compatible with Φ .

Proof. If a point p has coordinates u_k, v_k in the k -th system, its image point (z_1, z_2) under Φ is given by (11). If we let A_r act on p , its image point

is mapped under Φ (use formula (11) for the $(k+r)$ -th coordinate system and (12)) to $(A_r z_1, A'_r z_2)$.

Lemma. The action of V on Y^+ is free and properly discontinuous.

Proof. In view of the preceding lemma the action is free on $Y^+ - \bigcup_{k \in \mathbb{Z}} S_k$. By A_r^n (assume $n \neq 0$) a point on S_k is mapped to a point on S_{k+nr} . If it is fixed, it will be an intersection point $S_{j-1} \cap S_j$ of two consecutive curves, but this point is carried to $S_{j+nr-1} \cap S_{j+nr}$.

To prove that V is properly discontinuous we must show that for points p, q on Y^+ there exist neighborhoods U_1 and U_2 of p and q such that $gU_1 \cap U_2 \neq \emptyset$ only for finitely many $g \in V$. Since V acts properly discontinuously on \mathfrak{H}^2/M and $\bigcup_{k \in \mathbb{Z}} S_k$ is closed in Y^+ , this is clear if p and q both do not belong to $\bigcup_{k \in \mathbb{Z}} S_k$. If $p \in \bigcup_{k \in \mathbb{Z}} S_k$ and $q \notin \bigcup_{k \in \mathbb{Z}} S_k$ we use the function Φ .

For $(z_1, z_2) \in \mathfrak{H}^2$ put $\rho(z_1, z_2) = \text{Im } z_1 \cdot \text{Im } z_2$ and set

$$U_1 = \{u \mid u \in Y^+, \rho \Phi(u) < \rho \Phi(p) + 1\},$$

and let U_2 be the complement of $\overline{U_1}$ in Y^+ .

Then $U_1 \cap U_2 = \emptyset$ and $gU_1 = U_1$ for $g \in V$.

Now suppose both points p and q lie on $\bigcup_{k \in \mathbb{Z}} S_k$. It is sufficient to prove the existence of neighborhoods U_1 and U_2 of p and q such that $gU_1 \cap U_2 \neq \emptyset$ for only finitely many $g = (A_r)^n$ with $n < 0$. Recall that A_r generates V . If q lies on S_j and in the j -th coordinate system and p on S_k and in the k -th system, then a neighborhood U_2 of q is given by

$$0 \leq |u_j| < \frac{1}{\varepsilon}, |v_j| < \varepsilon \quad (\text{for } \varepsilon \text{ sufficiently small}).$$

A neighborhood U_1 of p is given by

$$0 \leq |u_k| < \frac{1}{\varepsilon}, |v_k| < \varepsilon \quad (\text{for } \varepsilon \text{ sufficiently small}).$$

Suppose that $|n| \geq k - j + 1$. Then a point (u_k, v_k) in the k -th system is mapped under $(A_r)^n$ ($n < 0$) to a point (u_j, v_j) in the j -th system if and only if

$$(15) \quad \begin{aligned} u_j &= u_k^a \cdot v_k^b \\ u_j \cdot u_k^c \cdot v_k^d &= 1 \end{aligned}$$

where a, b, c, d are non-negative integers and $c > d$. In fact $\begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$ is a matrix of type (7) depending on n , of course. If the points (u_j, v_j) and (u_k, v_k) lie in the chosen neighborhoods of p and q we obtain from (15) the inequality

$$\varepsilon^{d-c-1} < 1$$

which is not true for $\varepsilon \leq 1$. Therefore, the image of U_1 under $(A_r)^n$ does not intersect U_2 for $n < 0$ and $|n| \geq k - j + 1$.

Remark. The elements of $M = \mathbf{Z}w_0 + \mathbf{Z}$ can be written in the form $y - xw_0$ with $x, y \in \mathbf{Z}$. The number $y - xw_0$ is totally positive if and only if

$$y - xw_0 > 0 \quad \text{and} \quad y - xw'_0 > 0$$

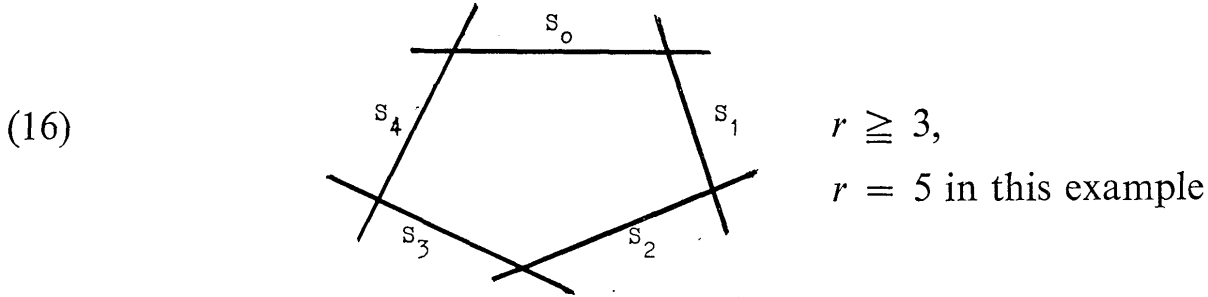
Since $w_0 > 1 > w'_0 > 0$, the totally positive elements of M correspond exactly to the integral points in the (x, y) -plane which lie in the quadrant (angle $< 180^\circ$) bounded by $y - xw_0 = 0$ ($x \geq 0$) and $y - xw'_0 = 0$ ($x \leq 0$). If we write $A_k = p_k - q_k w_0$, then for $k \geq 0$ these are the p_k, q_k of 2.2. We have

$$\lim_{k \rightarrow \infty} \frac{p_k}{q_k} = w_0, \quad \lim_{k \rightarrow -\infty} \frac{p_k}{q_k} = w'_0$$

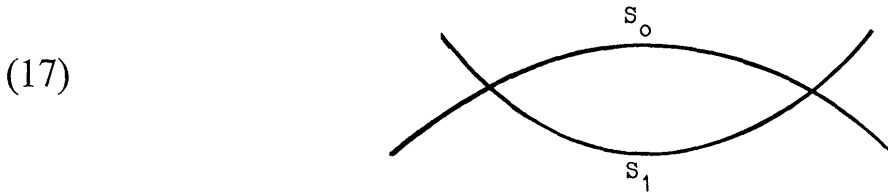
More precisely, it can be shown [12] that the A_k are exactly the lattice points of the support polygon, i.e. the polygon which bounds the convex hull of the lattice points in the above quadrant. It follows [12] that every totally positive number of M can be written uniquely as a linear combination of one or of two consecutive numbers A_k with positive integers as coefficients.

2.4. In section 2.3 we have constructed for a periodic function $k \mapsto b_k \geq 2$ (with $b_j \geq 3$ for at least one j) a complex manifold Y^+ together with a free properly discontinuous action of an infinite cyclic group V on Y^+ . The orbit space Y^+/V is a complex manifold. The curve S_k in Y^+ was mapped by the generator A_r of V onto the curve S_{k+r} where r was the period. Thus S_k and S_{k+r} become the same curve in Y^+/V . We shall denote the curves in Y^+/V again by S_k ($k \in \mathbf{Z}$) with the understanding

that we have $S_k = S_{k+r}$. We have in Y^+/V for $r \geq 3$ a cycle S_0, S_1, \dots, S_{r-1} of non-singular rational curves such that S_k and S_{k+1} intersect transversally in exactly one point ($k \in \mathbb{Z}/r\mathbb{Z}$) and the selfintersection number $S_k \cdot S_k$ equals $-b_k$. Otherwise there are no intersections. The configuration is illustrated by the diagram:

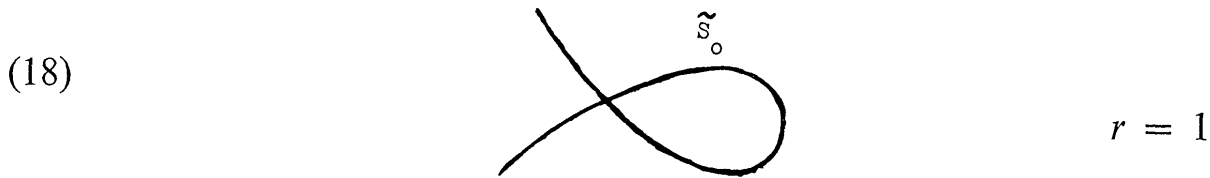


For $r = 2$ the configuration looks as follows:



There are two transversal intersections of S_0 and S_1 .

If $r = 1$, there is a special situation because the curves S_0 and S_1 of Y^+ intersect transversally in one point and S_0 and S_1 become identified under V . Thus under the map $Y^+ \rightarrow Y^+/V$ the string of rational curves S_k is mapped onto one rational curve \tilde{S}_0 in Y^+/V with one ordinary double point (which was previously also denoted by S_0 , but must here be distinguished).



Lemma. For $r = 1$ we have in Y^+/V

$$-\tilde{S}_0 \cdot \tilde{S}_0 = b_0 - 2$$

Proof. Let c_1 and \tilde{c}_1 denote the first Chern classes of Y^+ and Y^+/V respectively. Let π be the map $Y^+ \rightarrow Y^+/V$. Then $\pi^*\tilde{c}_1 = c_1$ and

$$\tilde{c}_1(\tilde{S}_0) = c_1(S_0)$$

where we evaluated the first Chern classes on the cycles \tilde{S}_0 and S_0 . By the adjunction formula (0.6)

$$\begin{aligned} c_1(S_0) - S_0 \cdot S_0 &= 2 \\ c_1(\tilde{S}_0) - \tilde{S}_0 \cdot \tilde{S}_0 + 2 &= 2. \end{aligned}$$

The summand 2 on the left side of the second formula is the contribution of the double point of \tilde{S}_0 in the adjunction formula. We get

$$\tilde{S}_0 \cdot \tilde{S}_0 = S_0 \cdot S_0 + 2 = -b_0 + 2$$

which completes the proof.

By $((b_0, b_1, \dots, b_{r-1}))$ we denote a *cycle* of numbers. (A cycle is given by an ordered set of r numbers. Two ordered sets are identified if they can be obtained from each other by a cyclic permutation.)

For any cycle $((b_0, b_1, \dots, b_{r-1}))$ of natural numbers ≥ 2 (at least one $b_j \geq 3$) we have constructed a complex manifold Y^+/V which we shall denote now by $Y((b_0, \dots, b_{r-1}))$.

In this complex manifold of complex dimension 2 (we shall often say “complex surface”) we have a configuration (16), (17) or (18) of rational curves. The corresponding matrices of intersection numbers are

$$\begin{bmatrix} -b_0 & 1 & 0 & \dots & 0 & 1 \\ 1 & -b_1 & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 & -b_{r-2} & 1 \\ 1 & 0 & \dots & 0 & 1 & -b_{r-1} \end{bmatrix} \quad \text{for } r \geq 3$$

and

$$\begin{pmatrix} -b_0 & 2 \\ 2 & -b_1 \end{pmatrix} \quad \text{for } r = 2$$

By the lemma we have for $r = 1$ the 1×1 -matrix $(-b_0 + 2)$. It is easy to show that these matrices are negative definite in all cases.

If all the b_i of a cycle equal 2, then the matrix is negative semi-definite with a null-space of dimension 1. Thus to get negative definiteness we do need the assumption $b_j \geq 3$ for at least one j .

The negative definiteness implies, according to Grauert [17], that the configurations (16), (17) or (18) can be blown down to give an isolated normal point P in a complex space $\bar{Y}((b_0, \dots, b_{r-1}))$. We have a holomorphic map

$$\sigma : Y((b_0, \dots, b_{r-1})) \rightarrow \bar{Y}((b_0, \dots, b_{r-1}))$$

with

$$\sigma\left(\bigcup_{k=0}^{r-1} S_k\right) = P$$

The map

$$\sigma : Y((b_0, \dots, b_{r-1})) - \bigcup_{k=0}^{r-1} S_k \rightarrow \bar{Y}((b_0, \dots, b_{r-1})) - \{P\}$$

is biholomorphic. The configurations (16), (17), (18) represent the unique minimal resolution of the point P , because they do not contain exceptional curves of the first kind, i.e. non-singular rational curves of selfintersection number -1 . *Thus the point P is singular.*

The first lemma of 2.3 shows that we have a natural map

$$Y((b_0, \dots, b_{r-1})) \rightarrow \overline{\mathfrak{H}^2/G(M, V)}$$

and a commutative diagram

$$\begin{array}{ccc} Y((b_0, \dots, b_{r-1})) & \rightarrow & \overline{\mathfrak{H}^2/G(M, V)} \\ \downarrow \sigma & & \nearrow \tilde{\sigma} \\ \bar{Y}((b_0, \dots, b_{r-1})) & & \end{array}$$

where $\tilde{\sigma}$ is biholomorphic and $\tilde{\sigma}(P) = \infty$ (in the notation of 2.1). The map $\tilde{\sigma}$ is biholomorphic also in P because one can introduce at most one normal complex structure in $\overline{\mathfrak{H}^2/G(M, V)}$ extending the complex structure of $\mathfrak{H}^2/G(M, V)$.

We have established the existence of the normal complex space $\overline{\mathfrak{H}^2/G(M, V)}$ directly without using the Proposition given in 2.1. We need only *define* $\tilde{\sigma}$ to be biholomorphic. Also we have given the resolution of the singular point ∞ which was added to $\mathfrak{H}^2/G(M, V)$. We summarize our results:

THEOREM. *Let $((b_0, b_1, \dots, b_{r-1}))$ be a cycle of natural numbers ≥ 2 (at least one $b_j \geq 3$). Put*

$$w_0 = [[b_0, \dots, b_{r-1}, b_0, \dots, b_{r-1}, \dots]] = [\overline{[b_0, \dots, b_{r-1}]}]$$

(infinite periodic continued fraction). Then $K = \mathbf{Q}(w_0)$ is a real quadratic field and $M = \mathbf{Z}w_0 + \mathbf{Z} \cdot 1$ a complete \mathbf{Z} -module of K . The cycle $((b_0, \dots, b_{r-1}))$ determines a totally positive unit A_r of K with $A_r M = M$. The unit A_r generates an infinite cyclic subgroup V of U_M^+ , the group of all totally positive units ε of K with $\varepsilon M = M$. The unique singular point ∞ of $\mathfrak{S}^2/G(M, V)$, where $G(M, V)$ is the natural semi-direct product of M and V , admits a cyclic resolution by rational curves S_k (configuration (16), (17) or (18)) with selfintersection numbers $S_k \cdot S_k = -b_k$ (for $r = 1$ we have $\tilde{S}_0 \cdot \tilde{S}_0 = -b_0 + 2$). This resolution is given by the complex surface $Y((b_0, \dots, b_{r-1}))$ which we canonically associated to a cycle.

Remark 1. Laufer [50] has shown that two normal singular points (in complex dimension 2) which admit a resolution with a given cyclic configuration of rational curves of type (16), (17) or (18) and given selfintersection numbers are isomorphic. Hence the singularity P of $Y((b_0, \dots, b_{r-1}))$ which we have constructed is up to isomorphism the unique singularity with the given cyclic configuration of rational curves and the given selfintersection numbers. (These singularities are called cyclic singularities.) Reversal of the cycle gives an isomorphic singularity.

Remark 2. The construction of Y in 2.2 applies also to the case where all b_k equal 2. Then we have $u_j \cdot v_j = u_k \cdot v_k$ (compare (5)) and hence obtain a holomorphic function $f: Y \rightarrow \mathbf{C}$. As in 2.3, we have a properly discontinuous action of an infinite cyclic group V on $Y^\varepsilon = \{p \mid p \in Y, |f(p)| < \varepsilon\}$, for ε positive and sufficiently small, whose generator maps the curve S_k to S_{k+r} . The period $r \geq 1$ can be chosen arbitrarily.

The function f is invariant under V ; thus we get a holomorphic map

$$f: Y^\varepsilon / V \rightarrow \{z \mid |z| < \varepsilon\}$$

All fibres of f are non-singular elliptic curves except $f^{-1}(0)$ which is a configuration of rational curves of type (16), (17), (18) where now all b_k equal 2. The fibring we have constructed is of type ${}_1I_r$ in the sense of Kodaira [45], Part II. We have seen:

Cycles $((2, \dots, 2))$ give an infinite continued fraction of value 1 and correspond to an elliptic fibring. Cycles $((b_0, \dots, b_{r-1}))$, $(b_k \geq 2, \text{ at least one } b_j \geq 3)$, give an infinite continued fraction whose value is a quadratic irrationality. These cycles determine singular points.

2.5. The theorem in 2.4 actually provides a *resolution* of the singular point of $\overline{\mathfrak{H}^2/G(M, V)}$ (see 2.1 with $n = 2$) for any complete \mathbf{Z} -module M of a real quadratic field K and an infinite cyclic subgroup V of U_M^+ of any given index $a = [U_M^+ : V]$. We need a lemma.

Lemma. Consider the \mathbf{Z} -module M defined by a periodic function $k \mapsto b_k \geq 2$ (with $b_j \geq 3$ for at least one j). Let $r \geq 1$ be the smallest period. Then A_r (see 2.3) is a generator of U_M^+ .

Proof. We shall denote ordinary continued fractions

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

by $[a_0, a_1, a_2, \dots]$. The relation between the two types of continued fractions is as follows:

$$(19) \quad [a_0, a_1, z] = [[a_0 + 1, \underbrace{2, \dots, 2}_{a_1 - 1}, z + 1]]$$

where z is an indeterminate and a_1 a natural number ≥ 1 . Using (19) the lemma can be derived from similar results for ordinary continued fractions (compare [6], Kap. II, § 7). A proof is also given in [12]. Another proof was communicated to the author by J. Rohlf.

Two complete \mathbf{Z} -modules M_1, M_2 of the same real quadratic field K are *strictly equivalent* (2.1) if there exists a totally positive number $\alpha \in K$ with $\alpha M_1 = M_2$. We have $U_{M_1}^+ = U_{M_2}^+$.

The actions of $G(M_1, V)$ and $G(M_2, V)$ on \mathfrak{H}^2 are equivalent under the automorphism $(z_1, z_2) \mapsto (\alpha z_1, \alpha z_2)$ of \mathfrak{H}^2 . Any module M_1 is strictly equivalent to a module of the form $M_2 = \mathbf{Z}w_0 + \mathbf{Z} \cdot 1$ where $w_0 \in K$ and $0 < w_0' < 1 < w_0$. (This is easy to prove, as was shown to me by H. Cohn.) Then the continued fraction $w_0 = [[b_0, b_1, \dots]]$ is purely periodic, i.e. periodicity starts with b_0 . This can be proved in the same way as an analogous result for ordinary continued fractions ([60], § 22). Let r be the smallest

period. We can resolve the singularity of $\overline{\mathfrak{H}^2/G(M_2, U_{M_2}^+)}$ by the method of 2.3 and 2.4, since by the preceding lemma $U_{M_2}^+ = \{ (A_r)^n \mid n \in \mathbb{Z} \}$. The resolution is described by the primitive cycle $((b_0, \dots, b_{r-1}))$ where primitive means that the cycle cannot be written as an “unramified covering” of degree > 1 . The cycle $((2, 3, 5, 2, 3, 5)) = ((2, 3, 5))^2$ is not primitive, for example.

For any primitive cycle $((b_0, \dots, b_{r-1}))$ we obtain a module $\mathbb{Z}w_0 + \mathbb{Z} \cdot 1$ with $w_0 = [[b_0, b_1, \dots]]$. In the cycle we must allow cyclic permutations. This changes the module to a module $\mathbb{Z}w_k + \mathbb{Z} \cdot 1$ (see 2.3). But $\mathbb{Z}w_0 + \mathbb{Z} \cdot 1 = \mathbb{Z}A_{k-1} + \mathbb{Z}A_k$ and $A_{k-1}/A_k = w_k$, where A_k is totally positive (see 2.3). Therefore, the strict equivalence class of the module only depends on the cycle. *If one reverses the order (orientation) of the cycle, the associated equivalence class of modules is replaced by the conjugate one* (see (13)).

If we start from a strict equivalence class of modules, it determines, as explained above, an isomorphism class of singularities (represented by the singularity of $\mathfrak{H}^2/G(M_2, U_{M_2}^+)$).

But isomorphic singularities must give the same unoriented cycle in their canonical minimal resolutions. “Unoriented” means that we cannot distinguish between $((b_0, \dots, b_{r-1}))$ and $((b_{r-1}, \dots, b_0))$. But, in fact, if we represent the class of modules as above by $M_2 = \mathbb{Z}w_0 + \mathbb{Z} \cdot 1$, then the cycle of w_0 is uniquely determined including the orientation. If this were not the case, it would follow that M_2 and M'_2 are strictly equivalent. Then the singularity and its resolution admit an involution showing that the cycles $((b_0, \dots, b_{r-1}))$ and $((b_{r-1}, \dots, b_0))$ are equal. (Details are left to the reader. The relation between strict equivalence classes of modules and primitive cycles can be derived, of course, also without using the resolution, compare 2.6.)

We have established a bijective map between primitive admissible cycles (all $b_k \geq 2$ and at least one $b_j \geq 3$) and the strict equivalence classes of complete \mathbb{Z} -modules (where the real quadratic field K varies).

The preceding discussion yields the following theorem.

THEOREM. *Let K be a real quadratic field and M a complete \mathbb{Z} -module in K . Let $((b_0, b_1, \dots, b_{r-1}))$ be the primitive cycle belonging to M . Let V be the subgroup of U_M^+ of index a . Then the resolution of the singular point of $\overline{\mathfrak{H}^2/G(M, V)}$ is given by the cycle $((b_0, b_1, \dots, b_{r-1}))^a$.*

Remark. The structure of the local ring $\mathfrak{O}(M, V)$ at the point ∞ of $\xi^2/G(M, V)$ was described in 2.1. For any admissible cycle $((b_0, \dots, b_{r-1}))$, not necessarily primitive, the functions $f \in \mathfrak{O}(M, V)$ can be written as power series' in u_0, v_0 where u_0, v_0 is the coordinate system of 2.3 (11) with $A_0 = 1$ and $A_{-1} = w_0 = [\overline{b_0, \dots, b_{r-1}}]$. We could use also any other coordinate system u_k, v_k .

Let $(u_0, v_0)^n = u_0^{n_1} \cdot v_0^{n_2}$ for $n = (n_1, n_2)$ and

$$Tn = \begin{pmatrix} -q_{r-1} & p_{r-1} \\ -q_r & p_r \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \text{ see 2.2 (7),}$$

then $\mathfrak{O}(M, V)$ is the ring of all power series'

$$f = a_0 + \sum_n a_n (u_0, v_0)^n$$

where the summation extends over all pairs $n = (n_1, n_2)$ of positive integers with $w'_0 \leq n_1/n_2 \leq w_0$, the coefficients satisfy $a_{Tn} = a_n$, and the power series converges for

$$\begin{aligned} w_0 \log |u_0| + \log |v_0| &< 0, \quad w'_0 \log |u_0| + \log |v_0| < 0 \\ (w_0 \log |u_0| + \log |v_0|) \cdot (w'_0 \log |u_0| + \log |v_0|) &> \frac{1}{c}, \end{aligned}$$

(the positive constant c depending on f).

Observe that T (as fractional linear transformation) maps the interval $[w'_0, w_0]$ bijectively onto itself ($Tw'_0 = w'_0$, $Tw_0 = w_0$). We have $Tx < x$ for $w'_0 < x < w_0$ and therefore

$$\lim_{k \rightarrow \infty} T^k x = w'_0 \text{ (for } w'_0 \leq x < w_0) \text{ and } \lim_{k \rightarrow -\infty} T^k x = w_0 \text{ (for } w'_0 < x \leq w_0)$$

Example. Consider the Fibonacci numbers

$$\dots, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13$$

where $F_0 = 0$, $F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$ ($k \in \mathbb{Z}$). The numbers $G_k = F_{2k+1}$ ($k \in \mathbb{Z}$) are all positive and satisfy $G_{k+1} = 3G_k - G_{k-1}$. The function

$$f(u_0, v_0) = \sum_{k=-\infty}^{\infty} u_0^{G_{k-1}} \cdot v_0^{G_k}$$

represents an element of $\mathfrak{D}(M, U_M^+)$ where

$$M = \mathbf{Z}w_0 + \mathbf{Z} \quad \text{and} \quad w_0 = [\bar{3}] = \frac{1}{2}(3 + \sqrt{5}).$$

2.6. The primitive cycle associated to a module M can be found also without using a base $w_0, 1$ of M with $0 < w'_0 < 1 < w_0$: Real numbers x, y are called strictly equivalent if there exists an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$ such that

$$y = \frac{ax + b}{cx + d}.$$

Any irrational number x has a unique infinite continued fraction development

$$x = [[a_0, a_1, a_2, \dots]]$$

where $a_i \in \mathbf{Z}$ and $a_i \geq 2$ for $i \geq 1$ and where $a_i \geq 3$ for infinitely many indices i . Two irrational numbers are strictly equivalent if and only if their continued fractions $[[a_0, a_1, \dots]]$ and $[[a'_0, a'_1, \dots]]$ coincide from certain points on, i.e. $a_{j+i} = a'_{k+i}$ for some j and k and for all $i \geq 0$. This is analogous to a classical result on ordinary continued fractions ([60], Satz 2.24).

A quadratic irrationality w admits a continued fraction which is periodic from a certain point on. It is purely periodic if and only if $0 < w' < 1 < w$ as mentioned before. The periodicity of the continued fraction of w determines a primitive cycle $((b_0, \dots, b_{r-1}))$ which is admissible (all $b_i \geq 2$, at least one $b_j \geq 3$). Thus two quadratic irrationalities are strictly equivalent if and only if their cycles agree, and *we have a bijection between strict equivalence classes of quadratic irrationalities and admissible primitive cycles. The admissible primitive cycles are in one-to-one correspondence with the strict equivalence classes of complete \mathbf{Z} -modules in real quadratic fields K where K varies* (see 2.5).

A complete \mathbf{Z} -module M of a real quadratic field K will be oriented by using the admissible bases (β_1, β_2) of M with $\beta_1\beta'_2 - \beta_2\beta'_1 > 0$. By restricting the norm function ($N(x) = xx'$ for $x \in K$) to M we obtain an indefinite quadratic form f on M with rational values. There exists a unique positive rational number m such that $m \cdot f$ is integral and with respect to an admissible base of M can be written as

$$au^2 + buv + cv^2$$

where $a, b, c \in \mathbf{Z}$ and $(a, b, c) = 1$. The pairs (u, v) are in $\mathbf{Z} \oplus \mathbf{Z} \cong M$. The discriminant $D_M = b^2 - 4ac$ is positive and not a square number.

In this way, we get a *bijection between strict equivalence classes of complete \mathbf{Z} -modules of real quadratic fields and the isomorphism classes under $\mathbf{SL}_2(\mathbf{Z})$ of integral indefinite primitive binary quadratic forms of non-square discriminant.*

Remark. The discriminant D of such a quadratic form can be written uniquely as

$$D = D_K : f^2, \quad f \geq 1,$$

where D_K is the discriminant of the real quadratic field $K = \mathbf{Q}(\sqrt{D})$. Then the corresponding strict equivalence class of modules can be represented by an ideal in the order (subring of \mathfrak{o}_K) which as an additive group has index f in \mathfrak{o}_K , and this is the smallest f such that the equivalence class of M can be represented in this way.

The strict equivalence class of the “first root”

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \text{where } \sqrt{b^2 - 4ac} > 0$$

depends only on the equivalence class of the quadratic form.

We obtain a bijection between $\mathbf{SL}_2(\mathbf{Z})$ -equivalence classes of integral indefinite primitive binary quadratic forms of non-square discriminant and strict equivalence classes of quadratic irrationalities.

All the bijections are compatible with each other as can be checked easily. Let us collect the bijections:

strict equivalence classes of complete \mathbf{Z} -modules in real quadratic fields	\leftrightarrow
admissible primitive cycles of natural numbers	\leftrightarrow
strict equivalence classes of quadratic irrationalities	\leftrightarrow
$\mathbf{SL}_2(\mathbf{Z})$ -equivalence classes of integral indefinite primitive binary quadratic forms of non-square discriminant	\leftrightarrow
isomorphism classes of cyclic singularities with a primitive cycle and as additional structure a preferred orientation of the cycle (compare 2.4, Remark 1).	

Example. Let d be a square-free number > 1 and suppose $d \equiv 2 \pmod{4}$ or $d \equiv 3 \pmod{4}$. The $(\sqrt{d}, 1)$ is an admissible \mathbf{Z} -base of the ideal (1) in \mathfrak{o}_K for $K = \mathbf{Q}(\sqrt{d})$. The quadratic form is given by

$$-u^2d + v^2$$

and has discriminant $4d$. The first root equals $\frac{-\sqrt{d}}{d} = -\frac{1}{\sqrt{d}}$ which is equivalent to \sqrt{d} . (Take always the positive square root). The admissible cycle of natural numbers is obtained by developing \sqrt{d} in a continued fraction.

§ 3. NUMERICAL INVARIANTS OF SINGULARITIES AND OF HILBERT MODULAR SURFACES

3.1. Let X be a compact oriented manifold of dimension $4k$ with or without boundary. Then $H^{2k}(X, \partial X; \mathbf{R})$ is a finite dimensional real vector space over which we have a bilinear symmetric form B with

$$B(x, y) = (x \cup y) [X, \partial X], \text{ for } x, y \in H^{2k}(X, \partial X; \mathbf{R}),$$

where $[X, \partial X]$ denotes the generator of $H_{4k}(X, \partial X; \mathbf{Z})$ defined by the orientation. The signature of B , i.e., the number of positive entries minus the number of negative entries in a diagonalized version, is called $\text{sign}(X)$. If X has no boundary and is differentiable, then according to the signature theorem ([36], p. 86)

$$(1) \quad \text{sign}(X) = L_k(p_1, \dots, p_k) [X],$$

where L_k is a certain polynomial of weight k in the Pontrjagin classes of X with rational coefficients ($p_j \in H^{4j}(X, \mathbf{Z})$).

Let N be a compact oriented differentiable manifold without boundary of dimension $4k - 1$ together with a given trivialization α of its stable tangent bundle. (Such a trivialization need not exist). We shall associate to the pair (N, α) a rational number $\delta(N, \alpha)$. Since N has a trivial stable tangent bundle, all its Pontrjagin and Stiefel-Whitney numbers vanish. Therefore N bounds a $4k$ -dimensional compact oriented differentiable manifold X . By the parallelization α we get from the stable tangent bundle of X an \mathbf{SO} -bundle over X/N . We denote its Pontrjagin classes by

$\tilde{p}_j \in H^{4j}(X/N, \mathbf{Z})$. Then the element $L_k(\tilde{p}_1, \dots, \tilde{p}_k) \in H^{4k}(X/N, \mathbf{Z}) = H^{4k}(X, \partial X; \mathbf{Z})$ is well-defined.

The number $\delta(N, \alpha)$ is defined by the following formula

$$(2) \quad \delta(N, \alpha) = L_k(\tilde{p}_1, \dots, \tilde{p}_k)[X, \partial X] - \text{sign}(X)$$

Thus $\delta(N, \alpha)$ is the deviation from the validity of the signature theorem. It follows from the Novikov additivity of the signature ([3], p. 588) that $\delta(N, \alpha)$ does not depend on the choice of X . If N is of dimension $2n - 1$ (n odd), then we put $\delta(N, \alpha) = 0$.

Remark. The invariant $\delta(N, \alpha)$ and similar invariants were studied also by other authors (Atiyah [1], Kreck [48], W. Meyer [57], S. Morita [59]). In [48] the invariant $\delta(N, \alpha)$ was calculated in several cases.

3.2. We now go back to 2.1. For a cusp of type (M, V) with isotropy group \mathfrak{G} (see 2.1. (1)) we have a $(2n-1)$ -dimensional manifold N which is a T^n -bundle over T^{n-1} (see 1.5). We can write (for a fixed positive d)

$$N = \partial X, \quad \text{where} \quad X = W(d) / \mathfrak{G}, \quad \text{and}$$

$$W(d) = \{z \mid z \in \mathfrak{H}^n, \prod_{j=1}^n \text{Im}(z_j) \geq d\}.$$

Here X is a (non-compact) complex manifold and is canonically parallelized. Namely, it inherits the standard parallelization of \mathfrak{H}^n given by the coordinates $x_1, y_1, \dots, x_n, y_n$ (with $z_k = x_k + iy_k$). This parallelization is respected by \mathfrak{G} if we use unit vectors with respect to the invariant metric of \mathfrak{H}^n . Thus the stable tangent bundle of N has a canonical parallelization α . We orient N by the orientation induced by the orientation of X . The rational number $\delta(N, \alpha)$ is now defined. We associate it to the cusp and call it $\delta(\mathfrak{G})$ or $\delta(M, V)$ if $\mathfrak{G} = G(M, V)$. Observe that X cannot be used for the calculation of δ according to (2) because it is not compact. If one compactifies X by adding the point ∞ , then one would get a compact manifold \tilde{X} with $\partial\tilde{X} = N$ after resolving the singularity at ∞ . This manifold \tilde{X} could be used to calculate δ .

We have associated a rational number $\delta(\mathfrak{G})$ to any "cusp" of type (M, V) with isotropy group \mathfrak{G} where M is a complete \mathbf{Z} -module of a totally real field K of degree n over \mathbf{Q} and V a subgroup of finite index of U_M^+ . If $V = U_M^+$, we write $\delta(M)$ instead of $\delta(M, U_M^+) = \delta(G(M, U_M^+))$.

By definition, $\delta(\mathfrak{G}) = 0$ if n is odd

If we multiply M by $\gamma \in K$, then

$$\delta(\gamma M, V) = \text{sign } N(\gamma) \cdot \delta(M, V)$$

where $N(\gamma) = \gamma^{(1)} \cdot \gamma^{(2)} \cdot \dots \cdot \gamma^{(n)}$. Namely, the map

$$z_j \mapsto \gamma^{(j)} z_j^{(\gamma)}$$

with $z_j^{(\gamma)} = z_j$ if $\gamma^{(j)} > 0$ and $z_j^{(\gamma)} = \overline{z_j}$ if $\gamma^{(j)} < 0$ induces a diffeomorphism of $W(d)/G(M, V)$ onto $W(|N(\gamma)| \cdot d)/G(\gamma M, V)$ of degree $\text{sign } N(\gamma)$ which is compatible with the parallelizations, and it follows from (2) that the invariant changes sign under orientation reversal.

In particular, $\delta(M, V) = 0$ if there exist a unit ε of K with $\varepsilon M = M$ and $N(\varepsilon) = -1$.

Problem. Give a number-theoretical formula for $\delta(M, V)$. This problem can be solved for $n = 2$:

THEOREM. *Let M be a complete \mathbf{Z} -module of a real quadratic field and $[U_M^+ : V] = a$, then*

$$(3) \quad \delta(M, V) = \frac{a}{3} [-(b_0 + b_1 + \dots + b_{r-1}) + 3r]$$

where $((b_0, \dots, b_{r-1}))$ is the primitive cycle associated to M , (see 2.5).

Proof. The torus bundle N bounds \tilde{X} which is obtained by resolving the singularity ∞ of $X \cup \infty$ where $X = W(d)/G(M, V)$. The boundary of $W(d)$ is a principal homogeneous space (1.5). Therefore the normal unit vector field of the boundary (defined using the orthogonal structure of the tangent bundle of \mathfrak{H}^2 given by the invariant metric of \mathfrak{H}^2) has constant coefficients with respect to the parallelization of \mathfrak{H}^2 . The same holds for the normal unit vector field of $N = \partial \tilde{X}$. By a classical result of H. Hopf we can extend the normal field to a section of the tangent bundle of \tilde{X} admitting finitely many singularities whose number counted with the proper multiplicities equals the Euler number $e(\tilde{X})$. Because this section is constant on the boundary with respect to the parallelization, it can be pushed down to a section of the complex vector bundle ξ (fibre \mathbf{C}^2) over

\tilde{X}/N induced from the parallelization of the tangent bundle of X . Therefore,

$$(4) \quad e(\tilde{X}) = c_2(\xi) [\tilde{X}, N]$$

where $c_i(\xi) \in H^{2i}(\tilde{X}/N, \mathbf{Z})$ are the Chern classes. The equation (4) follows from the definition of $c_2(\xi)$ by obstruction theory.

We have ([36], Theorem 4.5.1)

$$p_1(\xi) = c_1(\xi)^2 - 2c_2(\xi)$$

and, since $L_1 = p_1/3$,

$$(5) \quad \begin{aligned} \delta(M, V) &= \frac{1}{3} p_1(\xi) [\tilde{X}, N] - \text{sign}(\tilde{X}) \\ &= \frac{1}{3} (c_1(\xi)^2 [\tilde{X}, N] - 2e(\tilde{X})) - \text{sign}(\tilde{X}) \end{aligned}$$

By the theorem at the end of 2.5, the manifold \tilde{X} is obtained from $X \cup \infty$ by blowing up ∞ into a cycle of ar rational curves. \tilde{X} has the union of these curves as deformation retract. Thus

$$(6) \quad \begin{aligned} e(\tilde{X}) &= b_0(\tilde{X}) - b_1(\tilde{X}) + b_2(\tilde{X}) \\ &= 1 - 1 + ar = ar. \end{aligned}$$

The intersection matrix of the curves is negative-definite:

$$(7) \quad \text{sign}(\tilde{X}) = -ar.$$

The cohomology class $c_1(\xi) \in H^2(\tilde{X}, N; \mathbf{Z})$ corresponds by Poincaré duality to an element $z \in H_2(\tilde{X}, \mathbf{Z})$. Let us denote the rational curves of the cycle by S_j ($j \in \mathbf{Z}/ar\mathbf{Z}$). Then z must be an integral linear combination of the S_j which satisfies

$$(8) \quad z \cdot S_j - S_j \cdot S_j = 2 \quad (\text{for } ar \geq 2)$$

$$(8') \quad z \cdot S_0 - S_0 \cdot S_0 + 2 = 2 \quad (ar = 1).$$

This follows from the adjunction formula and the information given in 2.4. Since the intersection matrix of the curves of the resolution has non-vanishing determinant, the equations (8) are satisfied by exactly one element z . We obtain that *the first Chern class $c_1(\xi)$ corresponds by Poincaré duality to*

$$(9) \quad z = \sum_{j=0}^{ar-1} S_j$$

Since $c_1(\xi)^2[X, N] = z \cdot z = -a \sum_{j=0}^{r-1} b_j + 2ar$, formula (3) follows from (5), (6), (7).

3.3. We shall define an invariant φ for certain isolated normal singularities of a complex space of dimension n . In my Tokyo lectures the invariant φ was introduced for $n = 2$ and then generalized to arbitrary n by Morita [59]. Let us first recall that the signature theorem (3.1 (1)) for a compact complex manifold X can be written in terms of the Chern classes

$$(10) \quad \text{sign}(X) = \bar{L}_n(c_1, \dots, c_n)[X]$$

where \bar{L}_n is a certain polynomial of weight n with rational coefficients in the Chern classes of X , ($c_i \in H^{2i}(X, \mathbf{Z})$). It is identically zero if n is odd. Let β_n be the coefficient of c_n in \bar{L}_n . If n is even ($n = 2k$), then

$$(11) \quad \beta_{2k} = (-1)^k \frac{2^{2k+1} (2^{2k-1} - 1) B_k}{(2k)!}, \quad k \geq 1$$

where B_k is the k -th Bernoulli number ([36], 1.3 (7) and 1.5 (11)). For n odd, $\beta_n = 0$.

An isolated normal singularity P of a complex space of complex dimension n is called rationally parallelizable if there exists a compact neighborhood U of P containing no further singularities such that the Chern classes of $U - \{P\}$ are torsion classes, i.e. their images in the rational cohomology groups of $U - \{P\}$ vanish. We may assume that ∂U is a $(2n-1)$ -dimensional manifold and U the cone over ∂U with P as center. According to Hironaka [34a] the point P can be "blown-up". We obtain a compact complex manifold \tilde{U} which has a boundary as differentiable manifold, namely $\partial \tilde{U} = \partial U$. The Chern classes c_i of \tilde{U} have vanishing images in the rational cohomology of $\partial \tilde{U}$, thus can be pulled back to classes $\tilde{c}_i \in H^{2i}(\tilde{U}, \partial \tilde{U}; \mathbf{Q})$. The Chern numbers $\tilde{c}_{j_1} \cdot \tilde{c}_{j_2} \dots \tilde{c}_{j_s}[\tilde{U}, \partial \tilde{U}]$ where $j_1 + \dots + j_s = n$ and $s \geq 2$ are rational numbers not depending on the pull-back. Therefore, the rational number $\bar{L}_n(\tilde{c}_1, \dots, \tilde{c}_n)[\tilde{U}, \partial \tilde{U}]$ is well-defined if we replace in this expression $\tilde{c}_n[\tilde{U}, \partial \tilde{U}]$ by the Euler number of \tilde{U} . The invariant φ of the isolated normal singular point P is now defined by

$$(12) \quad \varphi(P) = \bar{L}_n(\tilde{c}_1, \dots, \tilde{c}_n)[U, \partial \tilde{U}] - \text{sign}(\tilde{U})$$

It can be shown (compare [59]) that $\varphi(P)$ does not depend on the resolution. By definition $\varphi(P) = 0$ for n odd.

For a cusp singularity of type (M, V) the invariants δ and φ coincide. This follows from (4) with 2 replaced by n . The proof of (4) remains unchanged for arbitrary n . Of course, X and \tilde{X} in 3.2 play the role of U and \tilde{U} here.

How to calculate φ for a quotient singularity? Let G be the group of p -th roots of unity where p is a natural number. Let q_1, \dots, q_n be integers which are all prime to p . Then G operates on \mathbb{C}^n by

$$(13) \quad (z_1, \dots, z_n) \mapsto (\zeta^{q_1} z_1, \dots, \zeta^{q_n} z_n), \quad \zeta^p = 1,$$

and \mathbb{C}^n/G is a normal complex space with exactly one singular point coming from the origin of \mathbb{C}^n .

THEOREM. *Let P be the quotient singularity defined by $(p; q_1, \dots, q_n)$ where $(p, q_j) = 1$ for all j , then*

$$(14) \quad \varphi(P) = \frac{\text{def}(p; q_1, \dots, q_n)}{p} + \frac{\beta_n}{p}$$

where

$$(15) \quad \text{def}(p; q_1, \dots, q_n) = i^n \sum_{j=1}^{p-1} \cot \frac{\pi q_1 j}{p} \dots \cot \frac{\pi q_n j}{p}$$

is the cotangent sum arising from the equivariant signature theorem of Atiyah-Bott-Singer ([2], [3]) and studied in [38], [79]. Recall that for n odd the cotangent sum (15), the number β_n and the invariant $\varphi(P)$ all vanish.

The proof of (14) was given by Don Zagier and the author for $n = 2$ using the explicit resolution of the singularity ([35], 3.4). For arbitrary n see Morita [59] whose proof uses the equivariant signature theorem and is similar to a proof in [1] concerning a related invariant. It would be interesting to check (14) also for $n > 2$ by an explicit resolution. But, unfortunately, these are not known.

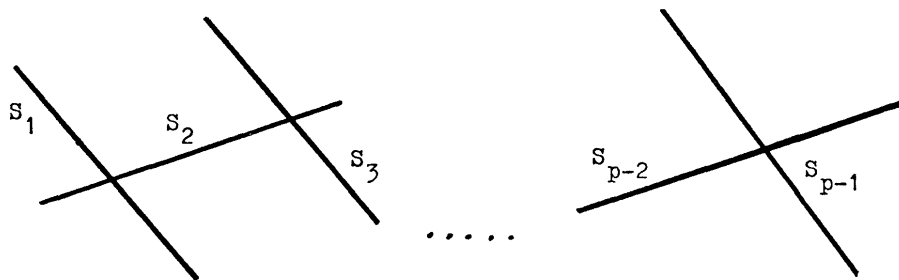
For a quotient singularity P we put

$$(16) \quad \delta(P) = \varphi(P) - \frac{\beta_n}{p} = \frac{\text{def}(p; q_1, \dots, q_n)}{p}$$

Observe that the δ -invariant in the sense of 3.1 (2) is not defined for a quotient singularity because the boundary N of a neighborhood of such a singularity is a lens space which in general does not admit a parallelization of its stable tangent bundle. However, Atiyah [1] has defined $\delta(N, \alpha)$ by (2) if N is an arbitrary compact oriented differentiable $(4k-1)$ -dimensional manifold without boundary and α an integrable connection of the stable tangent bundle of N :

The connection α is extended to a connection $\tilde{\alpha}$ for the stable tangent bundle of X (the extension being taken trivial in a collar of N). Then the Pontrjagin differential forms \tilde{p}_i of $\tilde{\alpha}$ vanish near N and in (2) the value $L_k(\tilde{p}_1, \dots, \tilde{p}_k)$ is an integral over a form with compact support in $\overset{\circ}{X}$. Again $\delta(N, \alpha)$ does not depend on the choice of X . If one takes in the special case of a quotient singularity for N the lens space and for α the connection inherited from the flat connection on the Euclidean space $\mathbf{R}^{4k} \supset \mathbf{S}^{4k-1}$ ($n = 2k$) then $\delta(N, \alpha)$ equals the number $\delta(P)$ in (16), (see [1]).

As an example, we calculate $\delta(P)$ if P is the quotient singularity given by $(p; 1, p-1)$. Since $p/(p-1) = [[2, \dots, 2]]$ with $p-1$ denominators 2 in the continued fraction, the resolution ([35], 3.4) looks as follows:



where $S_j \cdot S_j = -2$. The adjunction formula implies $\tilde{c}_1 = 0$.

Thus

$$\begin{aligned} \varphi(P) &= \frac{\tilde{c}_1^2 [\tilde{U}, \partial \tilde{U}] - 2e(\tilde{U})}{3} - \text{sign } U \\ &= \frac{-2p}{3} + p - 1 \\ \delta(P) &= \varphi(P) + \frac{2/3}{p} = \frac{(p-1) \cdot (p-2)}{3p} \end{aligned}$$

Therefore

$$(17) \quad \frac{\text{def}(p; 1, p-1)}{p} = \frac{(p-1) \cdot (p-2)}{3p}$$

Let us recall

$$(18) \quad \begin{aligned} \text{def}(p; 1, q) &= -\text{def}(p; 1, -q) \\ \text{def}(p; 1, q) &= \text{def}(p; 1, q') \text{ if } qq' \equiv 1 \pmod{p} \end{aligned}$$

To check the first equation (18) choose the quotient singularity $(p; 1, 1)$. The resolution consists of one curve S_1 with $S_1 \cdot S_1 = -p$. Therefore by the adjunction formula \tilde{c}_1 is represented by a homology class $a \cdot S_1$ with

$$a S_1 \cdot S_1 - S_1 \cdot S_1 = 2$$

$$\text{Thus } a = \frac{p-2}{p} \text{ and } \tilde{c}_1^2 [\tilde{U}, \partial \tilde{U}] = -\frac{(p-2)^2}{p}. \text{ We get}$$

$$\begin{aligned} \frac{1}{p} \text{def}(p; 1, 1) &= \frac{1}{3} \left(-\frac{(p-2)^2}{p} - 4 \right) + 1 + \frac{2/3}{p} \\ &= -\frac{1}{3p} (p-1)(p-2) \end{aligned}$$

which checks with (17) and the first equation of (18).

3.4. If Γ is a discrete irreducible subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ satisfying the condition (F) of the definition in 1.5, then \mathfrak{H}^n/Γ has finitely many quotient singularities and no other singularities. It is a rational homology manifold, i.e. every point has a neighborhood which is a cone over a rational homology sphere (in our case a lens space). For $n = 2k$ the signature of \mathfrak{H}^{2k}/Γ can be defined using the bilinear symmetric form over $H_{2k}(\mathfrak{H}^{2k}/\Gamma; \mathbf{R})$ given by the intersection number of two elements of this homology group.

In \mathfrak{H}^{2k} we choose around each point z with $|\Gamma_z| > 1$ a closed disk with radius ε measured in the invariant metric and sufficiently small. Then the image of these disks in \mathfrak{H}^{2k}/Γ is a finite disjoint union $\bigcup_{v=1}^s D_{z_v}$ where z_1, \dots, z_s are s points in \mathfrak{H}^{2k} representing the s quotient singularities of \mathfrak{H}^{2k}/Γ , each D_{z_v} can be identified with the quotient of the chosen disk around z_v by the isotropy group Γ_{z_v} .

Let x_1, \dots, x_t be a complete set of Γ -inequivalent parabolic points. Choose open sets U_v as in the definition of 1.5 and denote their images in \mathfrak{H}^{2k}/Γ by $D_{x_v} = U_v/\Gamma_{x_v}$. Then

$$(19) \quad X = \mathfrak{H}^{2k}/\Gamma - \bigcup_{v=1}^s \overset{\circ}{D}_{z_v} - \bigcup_{v=1}^t \overset{\circ}{D}_{x_v}$$

is a compact manifold with boundary whose signature (as defined in 3.1) equals the signature of \mathfrak{H}^{2k}/Γ .

THEOREM. *Let Γ be a group of type (F) acting on \mathfrak{H}^{2k} . Then*

$$(20) \quad \text{sign}(\mathfrak{H}^{2k}/\Gamma) = \sum_{v=1}^s \delta(z_v) + \sum_{v=1}^t \delta(x_v)$$

where z_1, \dots, z_s are points of \mathfrak{H}^{2k} representing the quotient singularities of \mathfrak{H}^{2k}/Γ and x_1, \dots, x_t is a complete set of Γ -inequivalent parabolic points. For the invariants $\delta(z_v)$ see (16). Recall that the structure of each cusp is determined by a group $\mathfrak{G} \cong \Gamma_{x_v}$ (see 2.1 (1)). The number $\delta(x_v)$ is defined as the number $\delta(\mathfrak{G})$ introduced in 3.2.

Proof. We first remark that $\text{sign}(\mathfrak{H}^{2k}/\Gamma) = 0$ if Γ operates freely and \mathfrak{H}^{2k}/Γ is compact. This is a special case of the proportionality of \mathfrak{H}^{2k}/Γ and $(\mathbf{P}_1\mathbf{C})^{2k}$, see 1.2, and explains already why (20) does not involve a volume contribution.

Let c_i be the Chern classes of X and \tilde{c}_i pull-backs to the rational cohomology of $X/\partial X$. Then the additivity of the signature and of the Euler number and the validity of the signature theorem for the manifold obtained by resolving all the singularities of the compactification of \mathfrak{H}^{2k}/Γ imply

$$(21) \quad \bar{L}_{2k}(\tilde{c}_1, \dots, \tilde{c}_{2k})[X/\partial X] - \text{sign } X + \sum_{v=1}^s \varphi(z_v) + \sum_{v=1}^t \varphi(x_v) = 0$$

where φ is defined as in 3.3. In $\bar{L}_{2k}(c_1, \dots, c_{2k})$ we have to interpret $c_{2k}[X/\partial X]$ as Euler number $e(X)$. By § 1 (21)

$$e(X) = \int_{\mathfrak{H}^{2k}/\Gamma} \omega - \sum_{r \geq 2} a_r(\Gamma)/r$$

The coefficient of c_{2k} in \bar{L}_{2k} equals β_{2k} . Therefore by (21), (16) and because $\varphi(x_v) = \delta(x_v)$, (see 3.3),

$$(22) \quad \begin{aligned} \text{sign } X &= \text{sign } \mathfrak{H}^{2k}/\Gamma \\ &= \bar{L}_{2k}(\tilde{c}_1, \dots, \tilde{c}_{2k-1}, \omega)[X/\partial X] + \sum_{v=1}^s \delta(z_v) + \sum_{v=1}^t \delta(x_v) \end{aligned}$$

where $\omega[X/\partial X]$ has to be interpreted as $\int_{\mathfrak{H}^{2k}/\Gamma} \omega$.

Let d_i be the invariant differential form on \mathfrak{H}^{2k} representing the i -th Chern class in terms of the invariant metric of \mathfrak{H}^{2k} . In fact d_i is the

i -th elementary symmetric function of the forms $\omega_j = -\frac{1}{2\pi} \frac{dx_j \wedge dy_j}{y_j^2}$

(see 1.2). The form $\bar{L}_{2k}(d_1, \dots, d_{2k})$ is identically 0, because it is a symmetric function in the ω_j^2 which vanish. Recall that $d_{2k} = \omega$. By (22) it remains to show that

$$(23) \quad \tilde{c}_{j_1} \dots \tilde{c}_{j_s} [X/\partial X] = \int_{\mathfrak{H}^{2k}/\Gamma} d_{j_1} \dots d_{j_s}$$

for $j_1 + \dots + j_s = 2k$ and $s \geq 2$. In the neighborhood of a parabolic point (transformed to ∞) we write

$$\omega_j = d\alpha_j \quad \text{with} \quad \alpha_j = -\frac{1}{2\pi} \frac{dx_j}{y_j}$$

The form α_j is invariant under the isotropy group of the cusp. In the neighborhood of $z_v \in \mathfrak{H}^{2k}$ we introduce in each factor of \mathfrak{H}^{2k} geodesic polar coordinates r_j, φ_j with

$$(24) \quad \omega_j = -\frac{1}{2\pi} \sinh(r_j) dr_j \wedge d\varphi_j$$

$$\omega_j = d\alpha_j, \quad \text{where} \quad \alpha_j = -\frac{1}{2\pi} (\cosh(r_j) - 1) d\varphi_j$$

The form α_j is invariant under the isotropy group Γ_{z_v} . Take compact manifolds $X''' \subset X'' \subset X' \subset X$ all defined as in (19) and each a compact subset of the interior of the next larger one. We may assume that all the α_j are defined in $\mathfrak{H}^{2k}/\Gamma - X'''$. Choose a C^∞ -function ρ which is 0 on X'' and 1 outside X' . Then $\rho\alpha_j$ is a form on \mathfrak{H}^{2k}/Γ minus singular points. The form $\omega_j - d(\rho\alpha_j)$ has compact support in X . Thus the elementary symmetric functions in the $\omega_j - d(\rho\alpha_j)$ represent the \tilde{c}_i and the left side of (23) becomes also an integral over \mathfrak{H}^{2k}/Γ . Recall that the d_i are the elementary symmetric function in the ω_j . By Stokes' theorem the difference of the two sides of (23) is a sum of expressions

$$(25) \quad \lim_{\partial D_{x_v}} \int \alpha_j \wedge \omega_1 \wedge \dots \wedge \hat{\omega}_j \wedge \dots \wedge \omega_{2k}$$

$$(26) \quad \lim_{\partial D_{z_v}} \int \alpha_j \wedge \omega_1 \wedge \dots \wedge \hat{\omega}_j \wedge \dots \wedge \omega_{2k}$$

where the limit means that the neighborhoods D_{x_v} and D_{z_v} become smaller and smaller, (the number d in 1.5 (16) converges to ∞ , the radii of the

discs converge to zero). The form in (25) is invariant under the isotropy group of x_v in the whole group $(\mathbf{PL}_2^+(\mathbf{R}))^{2k}$. Therefore, the integral equals a constant factor times the $(4k-1)$ -dimensional volume of ∂D_{x_v} .

But this volume converges to zero. In (26) for the limit process the integral can be extended over the boundary of a cartesian product of $2k$ discs of radius r divided by Γ_{z_v} . Let W_r be this cartesian product divided by Γ_{z_v} . Then

$$|\Gamma_{z_v}| \cdot \int_{\partial W_r} \alpha_j \wedge \omega_1 \wedge \dots \wedge \hat{\omega}_j \wedge \dots \wedge \omega_{2k} = (\cosh(r) - 1)^{2k}$$

which converges to zero for $r \rightarrow 0$.

3.5. Suppose a cusp is of type (M, V) , see 2.1. For $n > 1$ Shimizu ([71], p. 63) associates to the cusp a number $w(M, V)$ which depends only on the strict equivalence class M and the group $V \subset U_M^+$:

Let $(\beta_1, \dots, \beta_n)$ be a base of M . We define

$$d(M) = |\det(\beta_i^{(j)})|.$$

Consider the function

$$(27) \quad L(M, V, s) = \sum_{\mu \in M - \{0\}/V} \frac{\text{sign } N(\mu)}{|N(\mu)|^s}$$

where $N(\mu) = \mu^{(1)} \cdot \mu^{(2)} \cdot \dots \cdot \mu^{(n)}$. (The summand in (27) does not change if μ is multiplied with a totally-positive unit. Therefore, it makes sense to sum over the elements of $M - \{0\}/V$.) The function $L(M, V, s)$ can be extended to a holomorphic function in the whole s -plane \mathbf{C} . Shimizu defines

$$(28) \quad w(M, V) = \frac{(-1)^{n/2}}{(2\pi)^n} d(M) \cdot L(M, V, 1)$$

We conjecture that also the invariant $\delta(\mathfrak{G})$ (see 3.2) depends only on the pair (M, V) . This is clear for $n = 2$. In 3.2 we have defined $\delta(M, V) = \delta(\mathfrak{G})$ if $\mathfrak{G} = G(M, V)$.

The two invariants $\delta(M, V)$ and $w(M, V)$ have similar properties. For example, both vanish if there exists a unit ε of negative norm with $\varepsilon M = M$. Is there a relation between them? A guess would be, I hesitate to say conjecture,

$$(?) \quad 2^n w(M, V) = \delta(M, V)$$

This would imply that $w(M, V)$ is always rational. Even this is not known in full generality. However, if M is an ideal in the ring of integers of K , the number $w(M, V)$ is rational. (As Gundlach told me this can be deduced from his paper [24].)

The equation (?) is true for $n = 2$ as we shall see. This was the motivation for Atiyah and Singer to try to relate the invariant δ to L -functions of differential geometry (Lecture of Atiyah at the Arbeitstagung, Bonn 1972). Compare the recent results of Atiyah, Patodi and Singer.

THEOREM. *Let K be a real-quadratic field, M a complete \mathbf{Z} -module in K and $V \subset U_M^+$. Then*

$$(29) \quad 4 w(M, V) = \delta(M, V).$$

“Proof”. Curt Meyer [55] has already studied $w(M, V)$ in 1957. He expressed it in elementary number-theoretical terms using Dedekind sums. It turns out that $\delta(M, V)$ as given in (3) equals Meyer’s expression. This will be shown in [42]. Meyer’s formula can be found explicitly in [56] (see formulas (6) and (11)) and in Siegel [75] (see formula (120) on p. 183). For more information on the number theory involved we must refer to [42].

3.6. For a non-singular compact connected algebraic surface S the arithmetic genus is defined:

$$\chi(S) = 1 - g_1 + g_2,$$

where g_j is the dimension of the space of holomorphic differential forms of degree j on S . In classical notation $g_1 = q$ and $g_2 = p_g$. The first Betti number of S equals $2g_1$. The numbers g_j are birational invariants. Therefore we can speak of the invariants g_j and of the arithmetic genus of an arbitrary surface possibly with singularities meaning always the corresponding invariant of some non-singular model. We have ([36], 0.1, 0.3)

$$(30) \quad \chi(S) = \frac{1}{12} (c_1^2 + c_2) [S] \\ \frac{1}{4} (c_2 [S] + \frac{1}{3} (c_1^2 - 2c_2) [S]),$$

$$(31) \quad \chi(S) = \frac{1}{4} (e(S) + \text{sign}(S)),$$

where $e(S)$ is the Euler number and $\text{sign}(S)$ the signature of S . Thus the arithmetic genus is expressed in topological terms, a fact which does not hold in dimensions > 2 .

Let Γ be a discrete irreducible group of type (F) acting on \mathfrak{H}^2 (see 1.5). The compactification of \mathfrak{H}^2/Γ is an algebraic surface. A non-singular model S is obtained by resolving the quotient singularities and the cusp singularities. Then S is a union (glueing along the boundaries) of a manifold X like (19) and of suitable neighborhoods of the configurations of curves into which the singularities were blown up. For every manifold in this union we consider the expression $\frac{1}{4}$ (Euler number + signature). A quotient singularity has a linear resolution ([35], 3.4) and therefore for the neighborhood $\frac{1}{4}(e + \text{sign}) = \frac{1}{4}$, a cusp singularity has a cyclic resolution and therefore $\frac{1}{4}(e + \text{sign}) = 0$ by (6) and (7). The signature and the Euler number behave additively and thus in the notation of (19)

$$\chi(S) = \frac{1}{4} (e(X) + \text{sign}(X)) + \frac{s}{4}.$$

Since $e(\mathfrak{H}^2/\Gamma) = e(X) + s$, we get

$$(32) \quad \chi(S) = \frac{1}{4} (e(\mathfrak{H}^2/\Gamma) + \text{sign}(\mathfrak{H}^2/\Gamma))$$

Using the formulas for $e(\mathfrak{H}^2/\Gamma)$ (see § 1 (21)) and $\text{sign}(\mathfrak{H}^2/\Gamma)$ (see 20)) we obtain

$$(33) \quad \chi(S) = \frac{1}{4} \int_{\mathfrak{H}^2/\Gamma} \omega + \sum_{v=1}^s \frac{1}{4} (\delta(z_v) + (|\Gamma_{z_v}| - 1) : |\Gamma_{z_v}|) + \sum_{v=1}^t \frac{1}{4} \delta(x_v)$$

We have proved the following theorem.

THEOREM. *Let Γ be a discrete irreducible group of type (F) acting on \mathfrak{H}^2 . Then the arithmetic genus of the compactification $\overline{\mathfrak{H}^2/\Gamma}$ can be expressed by topological invariants of \mathfrak{H}^2/Γ : Four times the arithmetic genus equals the sum of the Euler number and the signature of \mathfrak{H}^2/Γ . The arithmetic genus is also given by (33) in terms of the Euler volume and contributions coming from the quotient singularities and the cusps.*

Instead of $\chi(S)$ where S is a non-singular model for \mathfrak{H}^2/Γ we shall write $\chi(\overline{\mathfrak{H}^2/\Gamma})$ or simply $\chi(\Gamma)$. Shimizu ([71], Theorem 11) calculated the dimension of the space $\mathfrak{S}_\Gamma(r)$ of cusp forms of weight r . A cusp form of weight r is defined on \mathfrak{H}^2 by a holomorphic form $a(z)(dz_1 \wedge dz_2)^r$ invariant under Γ which vanishes in the cusps. If r is a multiple of all $|\Gamma_{z_v}|$, then the Shimizu contributions of the quotient singularities are independent of r and are exactly the contributions which enter in (33).

By (29) Shimizu's cusp contributions are exactly the $\frac{\delta(x_v)}{4}$. Therefore, we can rewrite a special case of Shimizu's result in the following way.

THEOREM. *The assumptions are as in the preceding theorem. Let $r \geq 2$ be a multiple of all the orders of the isotropy groups of the elliptic fixed points (quotient singularities). Then*

$$(34) \quad \dim \mathfrak{S}_\Gamma(r) = (r^2 - r) \cdot \int_{\mathfrak{H}^2/\Gamma} \omega + \chi(\Gamma)$$

Hence the arithmetic genus of $\overline{\mathfrak{H}^2/\Gamma}$ appears as constant term of the Shimizu polynomial (compare [15], [26]).

Lemma. *Let Γ be a discrete irreducible group of type (F) acting on \mathfrak{H}^2 . The invariant g_1 of the algebraic surface $\overline{\mathfrak{H}^2/\Gamma}$ vanishes. The number $g_2(\overline{\mathfrak{H}^2/\Gamma})$ equals the dimension of the space $\mathfrak{S}_\Gamma(1)$ of cusp forms of weight 1.*

“Proof”. For g_1 , see ([14] Teil I, Satz 8) and [26]. For the result on g_2 , we have to show that any cusp form of weight 1 can be extended to a holomorphic form θ of degree 2 on the non-singular model obtained by resolving the singularities of $\overline{\mathfrak{H}^2/\Gamma}$. A priori, we have a holomorphic form θ of degree 2 only outside the singularities. It can be extended to the resolution of the quotient singularities ([14], Teil I, Satz 1).

For a cusp singularity the form $\frac{du_k \wedge dv_k}{u_k v_k}$ does not depend on the coordinate system (see 2.2 (5)). The form θ is a holomorphic function $f(u_k, v_k)$ multiplied with $\frac{du_k \wedge dv_k}{u_k v_k}$. This follows from 2.3 (9) and the remark in 2.5. It is a cusp form if and only if $f(u_k, v_k)$ is divisible by $u_k v_k$. Therefore, θ can be extended.

By the lemma we have

$$(35) \quad \chi(\Gamma) = 1 + g_2(\overline{\mathfrak{H}^2/\Gamma}) = 1 + \dim \mathfrak{S}_\Gamma(1)$$

The group Γ operates also on $\mathfrak{H} \times \mathfrak{H}^-$ where \mathfrak{H}^- is the lower half plane of all complex numbers with negative imaginary part. Since \mathfrak{H}^2 and $\mathfrak{H} \times \mathfrak{H}^-$ are equivalent domains, our results are applicable for the action of Γ on $\mathfrak{H} \times \mathfrak{H}^-$. The map $(z_1, z_2) \mapsto (z_1, \overline{z_2})$ induces a homeomorphism

$$(36) \quad \kappa : \mathfrak{H}^2/\Gamma \rightarrow (\mathfrak{H} \times \mathfrak{H}^-)/\Gamma$$

It follows that Γ (as a group acting on $\mathfrak{H} \times \mathfrak{H}^-$) is also of type (F). Because κ is a homeomorphism, the Euler numbers of $(\mathfrak{H} \times \mathfrak{H}^-)/\Gamma$ and \mathfrak{H}^2/Γ are equal. Since κ is orientation reversing, we have

$$(37) \quad \text{sign}(\mathfrak{H} \times \mathfrak{H}^-)/\Gamma = -\text{sign} \mathfrak{H}^2/\Gamma$$

We have denoted the arithmetic genus of $\overline{\mathfrak{H}^2/\Gamma}$ by $\chi(\Gamma)$ and shall put $\chi^-(\Gamma)$ for the arithmetic genus of $(\mathfrak{H} \times \mathfrak{H}^-)/\Gamma$. By (32), (35) and (37):

$$(38) \quad \chi(\Gamma) - \chi^-(\Gamma) = \dim \mathfrak{S}_\Gamma(1) - \dim \mathfrak{S}_\Gamma^-(1) = \frac{1}{2} \text{sign} \mathfrak{H}^2/\Gamma,$$

where $\mathfrak{S}_\Gamma^-(1)$ is the space of cusp forms of weight 1 for Γ on $\mathfrak{H} \times \mathfrak{H}^-$.

Remark. The quotient singularities of \mathfrak{H}^2/Γ are of the form $(r; 1, q)$. Any such singularity corresponds under κ to a singularity $(r; 1, -q)$. A cusp singularity of type (M, V) goes over into one of type $(\gamma M, V)$ where $N(\gamma) = -1$. Therefore (37) agrees with (20): all contributions coming from the singularities change their sign.

3.7. Let G be the Hilbert modular group for a totally real field K of degree n over \mathbf{Q} . The parabolic points are exactly the points of $\mathbf{P}_1 K$ where $\mathbf{P}_1 K$ is regarded as a subset of $(\mathbf{P}_1 \mathbf{R})^n$ by the embedding $x \mapsto (x^{(1)}, x^{(2)}, \dots, x^{(n)})$. The group G acts on $\mathbf{P}_1 K$. The orbits are in one-to-one correspondence with the wide ideal classes of \mathfrak{o}_K (two ideals $\mathfrak{a}, \mathfrak{b}$ are equivalent if there exists an element $\gamma \in K$ ($\gamma \neq 0$) such that $\gamma \mathfrak{a} = \mathfrak{b}$).

If $\frac{m}{n} \in \mathbf{P}_1 K$ (with $m, n \in \mathfrak{o}_K$) represents an orbit, then $\mathfrak{a} = (m, n)$ represents the corresponding ideal class. Thus the number of parabolic orbits (cusps) equals the class number h of K . As in ([75], p. 244) we choose a matrix

$$(39) \quad A = \begin{pmatrix} m & u \\ n & v \end{pmatrix}, \quad mv - nu = 1, \quad u, v \in \mathfrak{a}^{-1}.$$

A simple calculation shows that

$$(40) \quad A^{-1} \mathrm{SL}_2(\mathfrak{o}_K) A = \mathrm{SL}_2(\mathfrak{o}_K, \mathfrak{a}^2),$$

where, for any ideal $\mathfrak{b} \subset \mathfrak{o}_K$, we set (compare [31])

$$(41) \quad \mathrm{SL}_2(\mathfrak{o}_K, \mathfrak{b}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha\delta - \mathfrak{b}\gamma = 1, \alpha \in \mathfrak{o}_K, \delta \in \mathfrak{o}_K, \beta \in \mathfrak{b}^{-1}, \gamma \in \mathfrak{b} \right\}$$

Instead of studying the cusp of G at $\frac{m}{n}$, we can consider the cusp of $\mathrm{SL}_2(\mathfrak{o}_K, \mathfrak{a}^2)/\{1, -1\}$ at ∞ . Its isotropy group is

$$\begin{aligned} & \left\{ \begin{pmatrix} \varepsilon & w \\ 0 & 1/\varepsilon \end{pmatrix} \mid \varepsilon \in U, w \in \mathfrak{a}^{-2} \right\} / \{1, -1\} = \\ & \left\{ \begin{pmatrix} \varepsilon^2 & w \\ 0 & 1 \end{pmatrix} \mid \varepsilon \in U, w \in \mathfrak{a}^{-2} \right\} = G(\mathfrak{a}^{-2}, U^2), \quad \text{see 2.1.} \end{aligned}$$

Thus the cusp of G at $\frac{m}{n}$ with $m, n \in \mathfrak{o}_K$ and $(m, n) = \mathfrak{a}$ is given by the pair (\mathfrak{a}^{-2}, U^2) .

The extended Hilbert modular group \hat{G} (see 1.7) has the same number of cusps (we have $(\mathbf{P}_1 K)/G = (\mathbf{P}_1 K)/\hat{G}$). They are given by (\mathfrak{a}^{-2}, U^+) .

Let C be the ordinary ideal class group (i.e., the group of wide ideal classes of \mathfrak{o}_K) and C^+ the group of narrow ideal classes of \mathfrak{o}_K (with respect to strict equivalence: $\mathfrak{a}, \mathfrak{b}$ are strictly equivalent if there exists a totally positive $\gamma \in K$ with $\gamma\mathfrak{a} = \mathfrak{b}$). Then $\mathfrak{a} \mapsto \mathfrak{a}^{-2}$ induces a homomorphism

$$(42) \quad Sq : C \rightarrow C^+.$$

Both G and \hat{G} have h cusps ($h = |C| = h(K)$). The corresponding modules are the squares in C^+ , each module occurs k times where k is the order of the kernel of Sq and is a power of 2.

3.8. We consider the Hilbert modular group G and the extended group \hat{G} for $K = \mathbf{Q}(\sqrt{d})$ with d as in 1.4. The cusp singularities of $\overline{\mathfrak{H}^2}/G$ and $\overline{\mathfrak{H}^2}/\hat{G}$ are in one-to-one correspondence with the elements of C . They admit cyclic resolutions. To resolve the cusp belonging to $a \in C$ we take the primitive cycle $((b_0, b_1, \dots, b_{r-1}))$ associated to $Sq(a) \in C^+$ (see 2.5). This is already the cycle of the resolution if we consider the group \hat{G} . For G the cycle of the resolution is $((b_0, b_1, \dots, b_{r-1}))^c$ where $c = |U^+ : U^2|$.

The cusp at $\infty = \frac{1}{0} \in \mathbf{P}_1 K$ has the module \mathfrak{o}_K . For $d \equiv 2$ or $3 \pmod{4}$ the corresponding primitive cycle is the cycle of the continued fraction

for $\sqrt{d^-}$ (see 2.6). For $d \equiv 1 \pmod{4}$ it is the cycle of $\frac{1 + \sqrt{d}}{2}$. We list these primitive cycles for those d in the table of 1.7 for which K does not have a unit of negative norm. Also the values of $\delta(\mathfrak{o}_K)$ (see 3.2 (3)) and of the class numbers $h(K)$ are tabulated. If K has a unit of negative norm, then $\delta(\mathfrak{o}_K) = 0$.

d	cycle of \mathfrak{o}_K	$\delta(\mathfrak{o}_K)$	$h(K)$
3	((4))	$-\frac{1}{3}$	1
6	((2, 6))	$-\frac{2}{3}$	1
7	((3, 6))	-1	1
11	((2, 2, 8))	-1	1
14	((4, 8))	-2	1
15	((8))	$-\frac{5}{3}$	2
19	((2, 3, 2, 2, 3, 2, 10))	-1	1
21	((5))	$-\frac{2}{3}$	1
22	((4, 2, 2, 2, 4, 10))	-2	1
23	((5, 10))	-3	1
30	((2, 12))	$-\frac{8}{3}$	2
31	((3, 2, 2, 7, 2, 2, 3, 12))	-3	1
33	((2, 3, 2, 7))	$-\frac{2}{3}$	1
34	((6, 12))	-4	2
35	((12))	-3	2
38	((2, 2, 2, 2, 2, 14))	-2	1
39	((2, 2, 2, 14))	$-\frac{8}{3}$	2

3.9. In the next sections we study the signatures of \mathfrak{H}^2/G and \mathfrak{H}^2/\hat{G} . Because of (32) this gives also the arithmetic genera $\chi(G)$ and $\chi(\hat{G})$.

THEOREM. *If $K = \mathbf{Q}(\sqrt{d})$ has a unit ε of negative norm, then*

$$(43) \quad \text{sign } \mathfrak{H}^2/G = 0, \quad \chi(G) = \frac{1}{4} e(\mathfrak{H}^2/G).$$

Proof. The actions of G on \mathfrak{H}^2 and $\mathfrak{H} \times \mathfrak{H}^-$ are equivalent under $(z_1, z_2) \mapsto (\varepsilon z_1, \varepsilon' z_2)$, (we choose ε positive). The formula (43) follows from (37) and (32).

The following lemma is a corollary of the theorem in 3.4.

Lemma. *If K does not have a unit of negative norm, then*

$$(44) \quad \text{sign } \mathfrak{H}^2/G = \sum_{v=1}^s \delta(z_v) + 2 \sum_{a \in C} \delta(Sq(a)),$$

$$(45) \quad \text{sign } \mathfrak{H}^2/\hat{G} = \sum_{v=1}^{\hat{s}} \delta(\hat{z}_v) + \sum_{a \in C} \delta(Sq(a)),$$

Where the points z_v and \hat{z}_v represent the quotient singularities of \mathfrak{H}^2/G and \mathfrak{H}^2/\hat{G} respectively.

The contribution of the quotient singularities in (44) can be calculated using [61], (see 1.7). In [61] not only the orders of the quotient singularities of \mathfrak{H}^2/G are given, but also their types $(r; q_1, q_2)$, see (13). Since $\text{def}(2; 1, 1) = 0$ (see (17)), we only have to consider the quotient singularities of order $r \geq 3$. For $d \not\equiv 0 \pmod{3}$ the singularities of order 3 occur in pairs, one of type $(3; 1, 1)$ together with one of type $(3; 1, -1)$. Therefore, their contributions cancel out.

If d is divisible by 3, but $d \neq 3$, we have

$$(46) \quad \begin{aligned} a_3(G) &= 5h(\mathbf{Q}(\sqrt{-d/3})) & \text{for } d \equiv 3 \pmod{9} \\ a_3(G) &= 3h(\mathbf{Q}(\sqrt{-d/3})) & \text{for } d \equiv 6 \pmod{9} \end{aligned}$$

In the first case $\frac{4}{5}$ of the singularities are of type $(3; 1, 1)$, the others of type $(3; 1, -1)$, in the second case all are of type $(3; 1, 1)$. Therefore, in both cases their contribution in (44) equals (see (17)):

$$3h(\mathbf{Q}(\sqrt{-d/3})) \cdot \frac{1}{3} \text{def}(3; 1, 1) = -\frac{2}{3}h(\mathbf{Q}(\sqrt{-d/3}))$$

For $d = 3$ there are two singularities of type $(3; 1, 1)$ and one of type $(6; 1, -1)$:

$$d = 3 \Rightarrow \text{sign } \mathfrak{H}^2/G = -2 \cdot \frac{2}{9} + \frac{10}{9} - 2 \cdot \frac{1}{3} = 0$$

We have proved:

THEOREM. If $K = \mathbf{Q}(\sqrt{d})$ does not have a unit of negative norm, then

$$(47) \quad \text{sign } \mathfrak{H}^2/G = 2 \sum_{a \in C} \delta(Sq(a)) \quad \text{for } d \not\equiv 0 \pmod{3}$$

$$\text{sign } \mathfrak{H}^2/G = 0 \quad \text{for } d = 3$$

$$\text{sign } \mathfrak{H}^2/G = -\frac{2}{3}h(\mathbf{Q}(\sqrt{-d/3})) + 2 \sum_{a \in C} \delta(Sq(a))$$

for $d \equiv 0 \pmod{3}, d > 3$.

The group C^+ of narrow ideal classes contains the ideal class θ represented by the principal ideals (γ) with $N(\gamma) < 0$. If θ is a square, then

$$(48) \quad 2 \sum_{a \in C} \delta(Sq(a)) = \sum_{a \in C} \delta(Sq(a)) + \sum_{a \in C} \delta(Sq(a)\theta) = 0$$

θ is a square if and only if d is a sum of two squares [25] which happens if and only if d does not contain a prime $\equiv 3 \pmod{4}$.

In the contrary case, $\sum_{a \in C} \delta(Sq(a)) < 0$, see [27].

THEOREM. Let G be the Hilbert modular group for $K = \mathbf{Q}(\sqrt{d})$. Then $\text{sign } \mathfrak{H}^2/G = 0$ if and only if $d = 3$ or d does not contain a prime $\equiv 3 \pmod{4}$. In all other cases, $\text{sign } \mathfrak{H}^2/G < 0$.

If the class number of K equals 1, then $\sum_{a \in C} \delta(Sq(a)) = \delta(\mathfrak{o}_K)$. If the class number equals 2 and θ is not a square in C^+ , then C^+ is a product of two cyclic groups of order 2 and $\sum_{a \in C} \delta(Sq(a)) = 2\delta(\mathfrak{o}_K)$. Using the tables in 1.7 and 3.8 we have now enough information to calculate the arithmetic genera $\chi(G)$ for $d \leq 41$. The class numbers $h(\mathbf{Q}(\sqrt{-d/3}))$ which we need for $d = 3, 6, 15, 21, 30, 33, 39$ are 1, 1, 2, 1, 2, 1, 2.

d	$e(\mathfrak{S}^2/G)$	$\text{sign } \mathfrak{S}^2/G$	$\chi(G)$	d	$e(\mathfrak{S}^2/G)$	$\text{sign } (\mathfrak{S}^2/G)$	$\chi(G)$
2	4	0	1	22	16	-4	3
3	4	0	1	23	18	-6	3
5	4	0	1	26	20	0	5
6	6	-2	1	29	8	0	2
7	6	-2	1	30	24	-12	3
10	8	0	2	31	22	-6	4
11	10	-2	2	33	6	-2	1
13	4	0	1	34	24	0	6
14	12	-4	2	35	28	-12	4
15	12	-8	1	37	8	0	2
17	4	0	1	38	28	-4	6
19	14	-2	3	39	40	-12	7
21	6	-2	1	41	8	0	2

Estimates as in [40] and [42] show that $\chi(G) = 1$ only for finitely many d . Are those in the table the only ones? If d is a prime p , then $\chi(G) = 1$ if and only if $p = 2, 3, 5, 7, 13, 17$ (see 3.12).

The values for $\text{sign } \mathfrak{S}^2/G$ are also of interest because (see (38))

$$(49) \quad \dim \mathfrak{S}_G^-(1) - \dim \mathfrak{S}_G(1) = -\frac{1}{2} \text{sign } \mathfrak{S}^2/G$$

Thus $\dim \mathfrak{S}_G^-(1) \geq \dim \mathfrak{S}_G(1)$, where the inequality is true if and only if d is greater than 3 and divisible by a prime $p \equiv 3 \pmod{4}$.

3.10. In view of the preceding theorems it is interesting to calculate $\sum_{a \in C} \delta(Sq(a))$. This was done in [27] for any d using the relation to L -series as explained in 3.5. If d is a prime $\equiv 3 \pmod{4}$ the result is especially simple.

THEOREM. *Let p be a prime $\equiv 3 \pmod{4}$ and $p > 3$. Then, for $K = \mathbf{Q}(\sqrt{p})$, we have*

$$(50) \quad \sum_{a \in C} \delta(Sq(a)) = -h(-p)$$

Proof. The formulas (27), (28) and (29) imply ([71], p. 69)

$$(51) \quad \sum_{a \in C} \delta(Sq(a)) = \frac{-2}{\pi^2} \sqrt{4p} L(1, \chi).$$

Here χ is the unique character with values in $\{1, -1\}$ which is defined for all ideals in \mathfrak{o}_K , depends only on the narrow ideal class and satisfies $\chi((\alpha)) = \text{sign } N(\alpha)$ for principal ideals (α) .

The function

$$L(s, \chi) = \sum_{\substack{\alpha \text{ an ideal} \\ \text{in } \mathfrak{o}_K}} \frac{\chi(\alpha)}{|N(\alpha)|^s}$$

can be written as a product

$$(52) \quad L(s, \chi) = L_{-4}(s) L_{-p}(s),$$

where L_{-4} and L_{-p} are the L -functions of $\mathbf{Q}(\sqrt{-4})$ and $\mathbf{Q}(\sqrt{-p})$ over \mathbf{Q} . The product decomposition (52) belongs to a decomposition of the discriminant $4p$ of K , namely $4p = (-4)(-p)$, and χ is the genus character corresponding to it ([75], p. 79-80). Evaluating (52) for $s = 1$ implies by a classical formula ([6], V § 4, p. 369)

$$L(1, \chi) = \frac{2\pi}{4} 4^{-1/2} h(-4) \cdot \frac{2\pi}{2} p^{-1/2} h(-p),$$

and this gives (50).

The formula (50) establishes an amusing connection between continued fractions and class numbers. Ordinary continued fractions

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

will be denoted by $[a_0, a_1, a_2, \dots]$. Let p be a prime $\equiv 3 \pmod{4}$. Then ([60], §§ 24-26)

$$(53) \quad \sqrt{p} = [a_0, \overline{a_1, a_2, \dots, a_{2s}}], \quad a_i \geq 1,$$

where $a_0 = [\sqrt{p}]$ and $a_{2s} = 2a_0$. The bar over a_1, a_2, \dots, a_{2s} indicates here the primitive period. The continued fraction development for \sqrt{p} which we needed for the resolution is of the form

$$\sqrt{p} = a_0 + 1 - \frac{1}{b_0} - \frac{1}{b_1} \dots = [[a_0 + 1, \overline{b_0, \dots, b_{r-1}}]],$$

where the bar indicates again the primitive period. The primitive cycle $((b_0, \dots, b_{r-1}))$ looks as follows:

$$((\underbrace{2, \dots, 2}_{a_1-1}, a_2 + 2, \underbrace{2, \dots, 2}_{a_3-1}, a_4 + 2, \dots, \underbrace{2, \dots, 2}_{a_{2s-1}-1}, a_{2s} + 2))$$

This is shown by an easy calculation (see 2.5 (19)). For $K = \mathbf{Q}(\sqrt{p})$ the signature deviation invariant $\delta(\mathfrak{o}_K)$ is defined (see 3.2 (3)). We have

$$(54) \quad -3\delta(\mathfrak{o}_K) = \sum_{i=0}^{r-1} (b_i - 3) = \sum_{j=1}^{2s} (-1)^j a_j$$

By (50) and (53) we get:

Proposition. Let p be a prime $\equiv 3 \pmod{4}$ and $p > 3$. Suppose that the class number of $K = \mathbf{Q}(\sqrt{p})$ equals 1. Then

$$(55) \quad \sum_{j=1}^{2s} (-1)^j a_j = 3h(-p)$$

where $(a_1, a_2, \dots, a_{2s})$, with $a_{2s} = 2[\sqrt{p}]$, is the primitive period for the ordinary continued fraction development (53) of \sqrt{p} .

Example. $p = 163$, $h(K) = 1$

$$\sqrt{163} = [12, \overline{1, 3, 3, 2, 1, 1, 7, 1, 11, 1, 7, 1, 1, 2, 3, 3, 1, 24}]$$

$$3h(-163) = 3 \cdot 1 =$$

$$-1 + 3 - 3 + 2 - 1 + 1 - 7 + 1 - 11 + 1 - 7 + 1 - 1 + 2 - 3 + 3 - 1 + 24$$

For further information on these and more general number theoretical facts see [42].

3.11. The theorem in 3.10 enables us to give very explicit formulas for the signatures of \mathfrak{H}^2/G and \mathfrak{H}^2/\hat{G} in terms of class numbers of imaginary quadratic fields if $K = \mathbf{Q}(\sqrt{p})$ and p a prime $\equiv 3 \pmod{4}$. (For the other primes the signatures vanish).

THEOREM. Let p be a prime $\equiv 3 \pmod{4}$ and G the Hilbert modular group (\hat{G} the extended one) for $K = \mathbf{Q}(\sqrt{p})$. Then

$$(56) \quad \begin{aligned} \text{sign } \mathfrak{H}^2/G &= 0 & \text{for } p = 3 \\ \text{sign } \mathfrak{H}^2/G &= -2h(-p) & \text{for } p > 3 \\ \text{sign } \mathfrak{H}^2/\hat{G} &= 0 & \text{for } p \equiv 3 \pmod{8} \\ \text{sign } \mathfrak{H}^2/\hat{G} &= -2h(-p) & \text{for } p \equiv 7 \pmod{8} \end{aligned}$$

Proof. The first two equations follow from (47) and (50). For $p > 3$ the quotient singularities of order 3 in \mathfrak{H}^2/\hat{G} occur again in pairs $(3; 1, 1)$, $(3; 1, -1)$ and cancel out in (45). For $p > 3$ and $p \equiv 3 \pmod{8}$, there are $h(-p)$ singularities of type $(4; 1, 1)$ and $3h(-p)$ singularities of type $(4; 1, -1)$. For $p \equiv 7 \pmod{8}$ there are $2h(-p)$ singularities of type $(4, 1, 1)$, see [61].

The sum of their contributions in (45) equals (see (17))

$$\begin{aligned} 2h(-p) \frac{\text{def}(4; 1, -1)}{4} &= h(-p) & \text{for } p \equiv 3 \pmod{8} \\ 2h(-p) \frac{\text{def}(4; 1, 1)}{4} &= -h(-p) & \text{for } p \equiv 7 \pmod{8} \end{aligned}$$

By (45), $\text{sign } \mathfrak{H}^2/\hat{G} = \pm h(-p) - h(-p)$.

It remains to consider the case $p = 3$. We have 3 quotient singularities of order 2, there are 3 others of type $(4; 1, -1)$, $(3; 1, 1)$, $(12; 1, 5)$. By Dedekind-Rademacher reciprocity ([38], (36)) and because $\text{def}(5; 1, 12) = 0$ (see (18))

$$\frac{\text{def}(12; 1, 5)}{12} = 1 - \frac{144 + 1 + 25}{180} = \frac{1}{18}$$

Therefore (see (17) and 3.8):

$$p = 3 \Rightarrow \text{sign } \mathfrak{H}^2/\hat{G} = \frac{1}{2} - \frac{2}{9} + \frac{1}{18} - \frac{1}{3} = 0$$

3.12. For any prime p we know the Euler numbers and the signatures of \mathfrak{H}^2/G and \mathfrak{H}^2/\hat{G} . Using 1.6 (21), 3.6 (32) and the theorem of 3.11 we can write down explicit formulas for the arithmetic genera $\chi(G)$ and $\chi(\hat{G})$.

THEOREM. Let p be a prime $K = \mathbf{Q}(\sqrt{p})$. Let G be the Hilbert modular group for K and \hat{G} the extended one. Then

$$\chi(G) = 1 \quad \text{for } p = 2, 3, 5$$

$$\chi(\hat{G}) = 1 \quad \text{for } p = 3$$

For $p > 5$ we have

$$\chi(G) = \frac{1}{2}\zeta_K(-1) + \frac{h(-4p)}{8} + \frac{1}{6}h(-3p) \quad \text{for } p \equiv 1 \pmod{4}$$

$$\chi(G) = \frac{1}{2}\zeta_K(-1) + \frac{3}{4}h(-p) + \frac{1}{6}h(-12p) \quad \text{for } p \equiv 3 \pmod{8}$$

$$\chi(G) = \frac{1}{2}\zeta_K(-1) + \frac{1}{6}h(-12p) \quad \text{for } p \equiv 7 \pmod{8}$$

$$\chi(\hat{G}) = \frac{1}{4}\zeta_K(-1) + \frac{9}{8}h(-p) + \frac{1}{8}h(-8p) + \frac{1}{12}h(-12p) \quad \text{for } p \equiv 3 \pmod{8}$$

$$\chi(\hat{G}) = \frac{1}{4}\zeta_K(-1) + \frac{1}{8}h(-8p) + \frac{1}{12}h(-12p) \quad \text{for } p \equiv 7 \pmod{8}$$

The formulas at the end of 1.3 imply

$$2\zeta_K(-1) = \frac{1}{2} \cdot \pi^{-4} D_K^{3/2} \zeta_K(2) > \frac{1}{2} \pi^{-4} D_K^{3/2} \zeta(4) = \frac{D_K^{3/2}}{180}.$$

It is easy to deduce from this estimate that $\chi(G) = 1$ if and only if $p = 2, 3, 5, 7, 13, 17$ and (for $p \equiv 3 \pmod{4}$) $\chi(\hat{G}) = 1$ if and only if $p = 3, 7$. Because of (38) and (56) we also know the arithmetic genera of $(\mathfrak{H} \times \mathfrak{H}^-)/G$ and $(\mathfrak{H} \times \mathfrak{H}^-)/\hat{G}$ ($p \equiv 3 \pmod{4}$). They are equal to 1 if $p = 3$, and both different from 1 if $p > 3$.

§ 4. CURVES ON THE HILBERT MODULAR SURFACES AND PROOFS OF RATIONALITY

We shall construct curves in the Hilbert modular surfaces. They can be used to show that these surfaces are rational in some cases and also for further investigations of the surfaces ([41], [42]). Such curves were studied earlier by Gundlach [23] and Hammond [25]. We need information

about the decomposition of numbers into prime ideals in quadratic fields. (See [6], [30].)

4.1. Let K be a real quadratic field and \mathfrak{o}_K its ring of integers. We often write \mathfrak{o} instead of \mathfrak{o}_K . Let \mathfrak{b} be an ideal in \mathfrak{o} which is not divisible by any natural number > 1 . We consider the group $\mathbf{SL}_2(\mathfrak{o}, \mathfrak{b})$, see § 3 (41). Let $\Gamma_{\mathfrak{b}}$ be the subgroup of those elements of $\mathbf{SL}_2(\mathfrak{o}, \mathfrak{b})$ which when acting on \mathfrak{H}^2 carry the diagonal $z_1 = z_2$ over into itself. An element $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of $\mathbf{SL}_2(\mathfrak{o}, \mathfrak{b})$ belongs to $\Gamma_{\mathfrak{b}}$ if and only if

$$(1) \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} -\alpha' & -\beta' \\ -\gamma' & -\delta' \end{pmatrix}$$

The matrices satisfying the first condition of (1) are in $\mathbf{SL}_2(\mathbf{Q})$ with $\alpha, \delta \in \mathfrak{o}$, $\beta \in \mathfrak{b}^{-1}$, $\gamma \in \mathfrak{b}$. Thus α, δ, γ are integers. The ideal \mathfrak{b} is not divisible by any natural number > 1 . Therefore β is also an integer. A rational integer γ is contained in \mathfrak{b} if and only if $\gamma \equiv 0 \pmod{N(\mathfrak{b})}$ where $N(\mathfrak{b})$ is the norm of the ideal \mathfrak{b} .

For any natural number N we let $\Gamma_0(N)$ be the group of those elements $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$ for which $\gamma \equiv 0 \pmod{N}$. This group was studied by Klein and Fricke ([16], p. 349; see [70], p. 24).

We have proved the following lemma:

Lemma. Let \mathfrak{b} be an ideal in \mathfrak{o} which is not divisible by any natural number > 1 . Then $\Gamma_0(N(\mathfrak{b}))$ is the subgroup of those elements of $\Gamma_{\mathfrak{b}}$ which satisfy the first condition of (1). The group $\Gamma_{\mathfrak{b}}$ equals $\Gamma_0(N(\mathfrak{b}))$ or is an extension of index 2 of $\Gamma_0(N(\mathfrak{b}))$.

If $K = \mathbf{Q}(\sqrt{d})$ where d is square free, then a matrix of $\Gamma_{\mathfrak{b}}$ satisfying the second condition of (1) is of the form $\sqrt{d} \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix}$ where $\alpha_0, \gamma_0, \delta_0$ are rational integers, β_0 is a rational number, $\gamma_0 \sqrt{d} \in \mathfrak{b}$, and $\beta_0 \sqrt{d} \in \mathfrak{b}^{-1}$.

If \mathfrak{b} is not divisible by (\sqrt{d}) , then the fractional ideal $(\beta_0 \sqrt{d})$ has in its numerator a prime ideal dividing the ideal (\sqrt{d}) and the determinant of our matrix would be divisible by this prime ideal, this is a contradiction. Thus a matrix satisfying the second condition of (1) does not exist in this case. If \mathfrak{b} is divisible by (\sqrt{d}) , then $[\Gamma_{\mathfrak{b}} : \Gamma_0(N(\mathfrak{b}))] = 2$.

In fact, \mathfrak{b} is divisible by (\sqrt{d}) if and only if $N(\mathfrak{b})$ is divisible by d and the matrices satisfying the second condition of (1) are of the form

$$\begin{pmatrix} \alpha_0 \sqrt{d} & \beta_1 / \sqrt{d} \\ \gamma_0 \sqrt{d} & \delta_0 \sqrt{d} \end{pmatrix}$$

where $\alpha_0, \beta_1, \gamma_0, \delta_0$ are rational integers and $\gamma_0 \equiv 0 \pmod{N(\mathfrak{b})/d}$. Such matrices exist, because $(d, N(\mathfrak{b})/d) = 1$. If $\mathfrak{b} = (\sqrt{d})$, then $\Gamma_{\mathfrak{b}}$ is the extension of index 2 of $\Gamma_0(d)$ by the matrix

$$\begin{pmatrix} 0 & -1/\sqrt{d} \\ \sqrt{d} & 0 \end{pmatrix}$$

This group will be denoted by $\Gamma^*(d)$, see Fricke ([16], p. 357). We have proved:

Proposition: Let $K = \mathbf{Q}(\sqrt{d})$ be a real quadratic field (d square free). Let \mathfrak{b} be an ideal in \mathfrak{o}_K which is not divisible by any natural number > 1 . If $N(\mathfrak{b})$ is not divisible by d , then the group $\Gamma_{\mathfrak{b}}$ of those elements of $\mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b})$ which carry the diagonal of \mathfrak{H}^2 into itself equals $\Gamma_0(N(\mathfrak{b}))$. If $N(\mathfrak{b})$ is divisible by d , then $\Gamma_{\mathfrak{b}}$ is an extension of index 2 of $\Gamma_0(N(\mathfrak{b}))$. In particular, if $\mathfrak{b} = (\sqrt{d})$, then $\Gamma_{\mathfrak{b}} = \Gamma^*(d)$.

We also consider the group $\widehat{\mathbf{SL}}_2(\mathfrak{o}_K, \mathfrak{b})$ of matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\alpha, \delta \in \mathfrak{o}_K$, $\beta \in \mathfrak{b}^{-1}$, $\gamma \in \mathfrak{b}$ and $\alpha\delta - \beta\gamma$ a totally positive unit.

The groups $\mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b})$ and $\widehat{\mathbf{SL}}_2(\mathfrak{o}_K, \mathfrak{b})$ do not act effectively on \mathfrak{H}^2 . If we divide them by their subgroups of diagonal matrices, we get the groups $G(\mathfrak{o}_K, \mathfrak{b})$ and $\hat{G}(\mathfrak{o}_K, \mathfrak{b})$ which act effectively and generalize the Hilbert modular groups G and \hat{G} (see 1.7). As in 1.7 we have an exact sequence

$$(2) \quad 0 \rightarrow G(\mathfrak{o}_K, \mathfrak{b}) \rightarrow \hat{G}(\mathfrak{o}_K, \mathfrak{b}) \rightarrow U^+/U^2 \rightarrow 0$$

The subgroup of those elements of $G(\mathfrak{o}_K, \mathfrak{b})$ which carry the diagonal over into itself is $\Gamma_{\mathfrak{b}}/\{1, -1\}$ which acts effectively on \mathfrak{H} . The subgroup of the elements of $\hat{G}(\mathfrak{o}_K, \mathfrak{b})$ which keep the diagonal invariant is an extension of index 1 or 2 of $\Gamma_{\mathfrak{b}}/\{1, -1\}$. We can write it in the form $\hat{\Gamma}_{\mathfrak{b}}/\{1, -1\}$ where $\hat{\Gamma}_{\mathfrak{b}} \subset \mathbf{SL}_2(\mathbf{R})$ is an extension of index 1 or 2 of $\Gamma_{\mathfrak{b}}$.

The embedding of the diagonal in \mathfrak{H}^2 induces maps π and $\hat{\pi}$ of $\mathfrak{H}/\Gamma_{\mathfrak{b}}$ and $\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}$ in $\mathfrak{H}^2/G(\mathfrak{o}_K, \mathfrak{b})$ and $\mathfrak{H}^2/\hat{G}(\mathfrak{o}_K, \mathfrak{b})$ respectively. The maps π and $\hat{\pi}$ need not be injective. We have a commutative diagram

$$(3) \quad \begin{array}{ccc} \mathfrak{H}/\Gamma_b & \xrightarrow{\pi} & \mathfrak{H}^2/G(\mathfrak{o}_K, b) \\ \sigma \downarrow & & \downarrow \rho \\ \mathfrak{H}/\hat{\Gamma}_b & \xrightarrow{\hat{\pi}} & \mathfrak{H}^2/\hat{G}(\mathfrak{o}_K, b) \end{array}$$

The maps π and $\hat{\pi}$ map \mathfrak{H}/Γ_b and $\mathfrak{H}/\hat{\Gamma}_b$ with degree 1 onto their images.

If K has a unit of negative norm, then the two lines of diagram (3) can be identified. If there does not exist a unit of negative norm in K , then ρ has degree 2 and σ is bijective or has degree 2, depending on whether

$$\hat{\Gamma}_b = \Gamma_b \quad \text{or} \quad [\hat{\Gamma}_b : \Gamma_b] = 2.$$

If we compactify $\mathfrak{H}^2/G(\mathfrak{o}_K, b)$ and $\mathfrak{H}^2/\hat{G}(\mathfrak{o}_K, b)$ and resolve all quotient and cusp singularities by their minimal resolutions, then we get non-singular algebraic surfaces $Y(\mathfrak{o}_K, b)$ and $\hat{Y}(\mathfrak{o}_K, b)$. On $Y(\mathfrak{o}_K, b)$ we have an involution α induced by $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$ where ε is a generator of U^+ . We have a rational map $\rho : Y(\mathfrak{o}_K, b) \rightarrow \hat{Y}(\mathfrak{o}_K, b)$ compatible with α . The map ρ is regular outside the isolated fixed points of α . The maps π and $\hat{\pi}$ induce maps of the compactifications $\overline{\mathfrak{H}/\Gamma_b}$ and $\overline{\mathfrak{H}/\hat{\Gamma}_b}$ into the non-singular algebraic surfaces. We have a commutative diagram

$$(4) \quad \begin{array}{ccc} \overline{\mathfrak{H}/\Gamma_b} & \xrightarrow{\pi} & Y(\mathfrak{o}_K, b) \\ \sigma \downarrow & & \downarrow \rho \\ \overline{\mathfrak{H}/\hat{\Gamma}_b} & \xrightarrow{\hat{\pi}} & \hat{Y}(\mathfrak{o}_K, b) \end{array}$$

If K has a unit of negative norm, then the two lines of (4) can be identified, the vertical maps are bijective.

We denote the irreducible curve $\hat{\pi}(\overline{\mathfrak{H}/\hat{\Gamma}_b})$ by $C(b)$. It may have singularities. $\overline{\mathfrak{H}/\hat{\Gamma}_b}$ is its non-singular model which is mapped by $\hat{\pi}$ with degree 1 on $C(b)$.

We put $D(b) = \rho^{-1} C(b)$. If degree $(\rho) = 2$ and degree $(\sigma) = 1$ then $D(b)$ is the union of two irreducible curves $D_1(b)$, $D_2(b)$. If degree $(\sigma) = 2$, then $D(b)$ is irreducible. The involution α carries $D(b)$ into itself, mapping $D_1(b)$ to $D_2(b)$ if $D(b)$ is reducible.

The resolution of the cusp at ∞ of $\widehat{\mathfrak{H}}^2/\widehat{G}(\mathfrak{o}_K, \mathfrak{b})$ is described by the primitive cycle $((b_0, \dots, b_{r-1}))$ of \mathfrak{b}^{-1} (see 2.6) It determines a (narrow) ideal class with respect to strict equivalence whose inverse we denote by \mathfrak{B} . A quadratic irrationality w is called reduced if $0 < w' < 1 < w$. The quadratic irrationality w is reduced if and only if its continued fraction is purely periodic. There are exactly r reduced quadratic irrationalities belonging to the cycle, namely the numbers

$$(5) \quad w_k = [[b_k, b_{k+1}, \dots]], \quad (\text{see 2.3 (8)}).$$

After calling one of them w_0 , the notation for the others is fixed. Then they correspond bijectively to $\mathbb{Z}/r\mathbb{Z}$.

If we speak of the curve S_k of the resolution (where $k \in \mathbb{Z}/r\mathbb{Z}$), this has an invariant meaning. It is the curve associated to the quadratic irrationality w_k .

The fractional ideals $\mathfrak{b}^{-1} \in \mathfrak{B}^{-1}$ (where $\mathfrak{b} \subset \mathfrak{o}_K$ and \mathfrak{b} is not divisible by any natural number > 1) are exactly the \mathbb{Z} -modules $\mathbb{Z}w + \mathbb{Z} \cdot 1$ where w is a quadratic irrationality having the given primitive cycle in its continued fraction. (If we require that $0 < w' < 1$ and $w' < w$, then w is uniquely determined by \mathfrak{b}^{-1} .)

Since the module $\mathfrak{b}^{-1} = \mathbb{Z}w + \mathbb{Z} \cdot 1$ is strictly equivalent to $M = \mathbb{Z}w_0 + \mathbb{Z} \cdot 1$ (see 2.3), there exists a totally positive number λ in M (uniquely determined up to multiplication by a totally positive unit) such that

$$(6) \quad \mathfrak{b}^{-1} = \mathbb{Z}w + \mathbb{Z} \cdot 1 = \frac{1}{\lambda} M = \frac{1}{\lambda} \mathfrak{b}_0^{-1}$$

where we defined the ideal $\mathfrak{b}_0 \in \mathfrak{B}$ by $\mathfrak{b}_0^{-1} = M$. We have

$$(7) \quad \widehat{SL}_2(\mathfrak{o}_K, \mathfrak{b}) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \widehat{SL}_2(\mathfrak{o}_K, \mathfrak{b}_0) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$$

Instead of looking at the diagonal and at the action of $\widehat{SL}_2(\mathfrak{o}_K, \mathfrak{b})$ on \mathfrak{H}^2 , we can consider the action of $\widehat{SL}_2(\mathfrak{o}_K, \mathfrak{b}_0)$ on \mathfrak{H}^2 and the curve $z_1 = \lambda\zeta, z_2 = \lambda'\zeta$ in \mathfrak{H}^2 , where $\zeta \in \mathfrak{H}$. Any totally positive number $\lambda \in M$ can be written uniquely as a linear combination of two consecutive numbers A_{k-1}, A_k with non-negative integers p and q as coefficients (see 2.3, Remark):

$$(8) \quad \lambda = p \cdot A_{k-1} + q \cdot A_k.$$

If we multiply λ by a totally positive unit, then p, q do not change and k only changes modulo r . See the lemma in 2.5 and 2.3 (12). The equation 2.3 (11) shows that the curve $C(\mathfrak{b})$ has in the k -th coordinate system (u_k, v_k) of the resolved cusp the equation

$$(9) \quad u_k = t^p, v_k = t^q,$$

where t can be restricted to some neighborhood of 0. Namely, we just want to study locally the intersection of our curve with the curves of the resolution. Observe, that p, q are relatively prime because λ is an element of a \mathbf{Z} -base of M . The fractional ideals $\mathfrak{b}^{-1} \in \mathfrak{B}^{-1}$ which satisfy our conditions ($\mathfrak{b} \subset \mathfrak{o}_K$ and \mathfrak{b} not divisible by any natural number > 1) are in one-to-one correspondence with the triples $(k \mid p, q)$ where $k \in \mathbf{Z}/r\mathbf{Z}$ and p, q are relatively prime natural numbers and where $(k \mid 0, 1)$ is to be identified with $(k+1 \mid 1, 0)$.

We call $(k \mid p, q)$ the characteristic of the ideal $\mathfrak{b} \in \mathfrak{B}$. Actually, k does not stand for an element $k \in \mathbf{Z}/r\mathbf{Z}$, but rather for the corresponding quadratic irrationality w_k which has an invariant meaning. If as in (6)

$$(10) \quad \mathfrak{b}^{-1} = \mathbf{Z}w + \mathbf{Z} \cdot 1,$$

then (see (8))

$$(11) \quad w = \frac{\bar{p} A_{k-1} + \bar{q} A_k}{p A_{k-1} + q A_k} = \frac{\bar{p} w_k + \bar{q}}{p w_k + q}$$

where $\begin{pmatrix} \bar{p} & \bar{q} \\ p & q \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$ and $p \geq 0, q \geq 0$. Therefore, we can determine the characteristic of \mathfrak{b} by writing w in the form (11).

In view of (7) the algebraic surface $\hat{Y}(\mathfrak{o}_K, \mathfrak{b})$ depends only on the ideal class \mathfrak{B} . The identification of $\hat{Y}(\mathfrak{o}_K, \mathfrak{b})$ and $\hat{Y}(\mathfrak{o}_K, \mathfrak{b}_0)$ is uniquely defined by (7). We shall denote the surface by $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$. In a similar way the algebraic surface $Y(\mathfrak{o}_K, \mathfrak{B})$ is defined. The preceding discussions (see in particular (9)) yields the following theorem.

THEOREM. *Let K be a real quadratic field and \mathfrak{B} a narrow ideal class of \mathfrak{o}_K . For every ideal $\mathfrak{b} \subset \mathfrak{o}_K$ with $\mathfrak{b} \in \mathfrak{B}$ such that \mathfrak{b} is not divisible by any natural number > 1 , we have defined an irreducible curve $C(\mathfrak{b}) = \hat{\pi}(\mathfrak{S}/\hat{\Gamma}_{\mathfrak{b}})$ in the non-singular algebraic surface $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$. The cusp at ∞ of $\mathfrak{S}/\hat{\Gamma}_{\mathfrak{b}}$ is mapped by $\hat{\pi}$ to a point on the union of the curves S_0, \dots, S_{r-1} in $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$ which were obtained by the resolution of the cusp at ∞ of $\mathfrak{S}^2/\hat{G}(\mathfrak{o}_K, \mathfrak{b})$.*

If \mathfrak{b} has the characteristic $(k \mid 0, 1)$, then $\mathfrak{b}^{-1} = \mathbf{Z}w_k + \mathbf{Z} \cdot 1$ where w_k is the reduced quadratic irrationality belonging to k , and the curve $C(\mathfrak{b})$ intersects S_k transversally in P which is not a double point of $\cup S_j$. The curve S_k is given in the local coordinate system (u_k, v_k) by $v_k = 0$ and $C(\mathfrak{b})$ by $u_k = 1$. If \mathfrak{b} has the characteristic $(k \mid p, q)$ where $p > 0$ and $q > 0$, then P is given in the k -th coordinate system by $u_k = v_k = 0$, the curve S_k by $v_k = 0$, the curve S_{k-1} by $u_k = 0$, and $C(\mathfrak{b})$ has the local equation $u_k^q = v_k^p$.

If K has a unit of negative norm, then $Y(\mathfrak{o}_K, \mathfrak{B}) = \hat{Y}(\mathfrak{o}_K, \mathfrak{B})$. If K does not have such a unit, then in the non-singular algebraic surface $Y(\mathfrak{o}_K, \mathfrak{B})$ we have a curve $D(\mathfrak{b})$ which in the neighborhood of the resolved cusp at ∞ is just the inverse image of $C(\mathfrak{b})$, the resolution of the cusp at ∞ being an unbranched double cover of the cycle of curves S_0, \dots, S_{r-1} . (The fundamental group of a neighborhood of $S_0 \cup \dots \cup S_{r-1}$ is infinite cyclic and we have to take the corresponding covering of degree 2.) The curve $D(\mathfrak{b})$ is irreducible or the union of two irreducible curves $D_1(\mathfrak{b}), D_2(\mathfrak{b})$.

Remark. For different $\mathfrak{b}, \hat{\mathfrak{b}} \in \mathfrak{B}$ the curves $C(\mathfrak{b}), C(\hat{\mathfrak{b}})$ may coincide. The curve $C(\mathfrak{b}) = \hat{\pi}(\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}})$ may intersect $\cup S_j$ in other points than P which correspond to other cusps of $\mathfrak{H}/\hat{\Gamma}_{\mathfrak{b}}$.

4.2. In view of the preceding proposition and the theorem it is important to have a simple method to calculate $N(\mathfrak{b})$ if $\mathfrak{b}^{-1} = \mathbf{Z}w + \mathbf{Z} \cdot 1$. Let D be the discriminant of K (see 1.4), then w can be written uniquely in the form

$$(12) \quad w = \frac{M + \sqrt{D}}{2N} \quad (\text{see 2.6})$$

where $N > 0$ and $M^2 - D \equiv 0 \pmod{4N}$. Then we have

$$(13) \quad N(\mathfrak{b}) = N$$

To prove (13), one checks

$$N \cdot \mathfrak{b}^{-1} \cdot (\mathfrak{b}^{-1})' = (1)$$

If we start with a reduced quadratic irrationality w_0 of the form (12), then the formula

$$w_k = b_k - \frac{1}{w_{k+1}}$$

where $b_k \in \mathbf{Z}$ and $w_{k+1} > 1$, determines inductively for $k \geq 0$ the b_k and the w_k . We put

$$(14) \quad w_k = \frac{M_k + \sqrt{D}}{2N_k}$$

This is the process of calculating the continued fraction for w_0 . If \mathfrak{b} is the ideal of characteristic $(k \mid p, q)$, see (11), then

$$(15) \quad N(\mathfrak{b}) = p^2 N_{k-1} + pqM_k + q^2 N_k, \quad \text{where} \quad M_k^2 - 4N_{k-1}N_k = D,$$

as follows from (11), (13) and (14).

We shall tabulate the values of b_k, M_k, N_k for some w_0 , namely for those quadratic irrationalities which are needed later to show that the Hilbert modular surfaces $\overline{\mathfrak{H}^2}/G$ are rational for $d = 2, 3, 5, 6, 7, 13, 15, 17,$

21, 33 (compare the table in 3.9). We also include $w_0 = \frac{3 + \sqrt{3}}{3}$ which

is needed for $(\mathfrak{H} \times \mathfrak{H}^-)/G$ in the case $d = 3$ (see 3.12).

If r is the length of the cycle of the quadratic irrationality, we tabulate b_k, M_k, N_k only for $0 \leq k \leq r-1$, because they are periodic with period r .

$w_0 = 2 + \sqrt{2}$	$w_0 = 3 + \sqrt{7}$	$w_0 = \frac{5 + \sqrt{21}}{2}$																																												
$D = 8$	$D = 28$	$D = 21$																																												
<table><tr><td>k</td><td>0</td><td>1</td></tr><tr><td>b_k</td><td>4</td><td>2</td></tr><tr><td>M_k</td><td>4</td><td>4</td></tr><tr><td>N_k</td><td>1</td><td>2</td></tr></table>	k	0	1	b_k	4	2	M_k	4	4	N_k	1	2	<table><tr><td>k</td><td>0</td><td>1</td></tr><tr><td>b_k</td><td>6</td><td>3</td></tr><tr><td>M_k</td><td>6</td><td>6</td></tr><tr><td>N_k</td><td>1</td><td>2</td></tr></table>	k	0	1	b_k	6	3	M_k	6	6	N_k	1	2	<table><tr><td>k</td><td>0</td></tr><tr><td>b_k</td><td>5</td></tr><tr><td>M_k</td><td>5</td></tr><tr><td>N_k</td><td>1</td></tr></table>	k	0	b_k	5	M_k	5	N_k	1												
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k	0																																													
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b_k	7	2	3	2																																										
M_k	7	7	9	9																																										
N_k	1	4	3	4																																										

$$w_0 = \frac{3 + \sqrt{5}}{2}$$

$$D = 5$$

k	0
b_k	3
M_k	3
N_k	1

$$w_0 = 4 + \sqrt{15}$$

$$D = 60$$

k	0
b_k	8
M_k	8
N_k	1

$$w_0 = \frac{3 + \sqrt{3}}{3}$$

$$D = 12$$

k	0	1
b_k	2	3
M_k	6	6
N_k	3	2

$$w_0 = 3 + \sqrt{6}$$

$$D = 24$$

k	0	1
b_k	6	2
M_k	6	6
N_k	1	3

$$w_0 = \frac{5 + \sqrt{17}}{2}$$

$$D = 17$$

k	0	1	2	3	4
b_k	5	3	2	2	3
M_k	5	5	7	9	7
N_k	1	2	4	4	2

4.3. We consider the situation of the theorem in 4.1. Let F be one of the irreducible curves $C(\mathfrak{b})$, $D(\mathfrak{b})$, $D_1(\mathfrak{b})$ or $D_2(\mathfrak{b})$. The curve F has \mathfrak{S}/Γ as non-singular model where Γ acts effectively on \mathfrak{S} and equals $\hat{\Gamma}_{\mathfrak{b}}/\{1, -1\}$ or $\Gamma_{\mathfrak{b}}/\{1, -1\}$. The curve F lies in a non-singular algebraic surface Y , namely $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$ or $Y(\mathfrak{o}_K, \mathfrak{B})$. We shall calculate the value of the first Chern class c_1 of Y on F which is up to sign the intersection number of a canonical divisor K of Y with F :

$$(16) \quad c_1[F] = -K \cdot F.$$

The surface Y is a disjoint union of a complex surface (4-dimensional manifold) X with boundary as in 3.4 (19) and open neighborhoods $N_v (1 \leq v \leq s+t)$ of the configurations of curves into which the s quotient singularities and the t cusp singularities were blown up (minimal resolutions). The first Chern class of X can be represented as in 3.4 by a differential form $\tilde{\gamma}_1$ with compact support in the interior of X and it follows as in 3.4 (25), (26) that

$$(17) \quad \int_F \tilde{\gamma} = \int_F (\omega_1 + \omega_2)$$

where $\omega_j = -\frac{1}{2\pi} y_j^{-2} dx_j \wedge dy_j$. Since F comes from the diagonal

$z_1 = z_2$ of \mathfrak{H}^2 , we obtain that $\int_F \tilde{\gamma}_1$ equals twice the Euler volume of \mathfrak{H}/Γ . Thus by 1.4 (10)

$$(18) \quad \int_F \tilde{\gamma}_1 = 2 \int_{\mathfrak{H}/\Gamma} \omega = -\frac{1}{3} [G : \Gamma],$$

where $G = \mathbf{SL}_2(\mathbf{Z})/\{1, -1\}$.

We have denoted the open neighborhoods of the resolved quotient singularities and cusps singularities by N_v ($1 \leq v \leq s+t$) where s is the number of quotient singularities and t the number of cusp singularities in the surface $\mathfrak{H}^2/\hat{G}(\mathfrak{o}_K, \mathfrak{B})$ or $\mathfrak{H}^2/G(\mathfrak{o}_K, \mathfrak{B})$ which has Y as non-singular model. The first Chern class of N_v can be represented by a differential form $\tilde{\gamma}_1^{(v)}$ with compact support in N_v in such a way that $\tilde{\gamma}_1 + \sum \tilde{\gamma}_1^{(v)}$ represents the first Chern class of Y . By Poincaré duality in N_v each $\tilde{\gamma}_1^{(v)}$ corresponds to a linear combination with rational coefficients of the curves into which the singularity was blown up. This linear combination will be called the Chern divisor of the singularity and denoted by $c_1^{(v)}$. It follows that

$$(19) \quad c_1[F] = 2 \int_{\mathfrak{H}/\Gamma} \omega + \sum_{v=1}^{s+t} c_1^{(v)} \cdot F$$

We denote the curves of the minimal resolution of a singularity by S_j .

For a quotient singularity the Chern divisor equals $\sum a_j S_j$ where the rational numbers a_j are determined by the linear equations

$$(20) \quad \sum_j (S_i \cdot S_j) a_j = 2 + S_i \cdot S_i$$

This follows by the adjunction formula, since all the S_i are rational and non-singular. In some cases we have calculated the numbers a_j at the end of 3.3. For any quotient singularity of type $(p; 1, q)$ the matrix $(S_i \cdot S_j)$ equals

$$\begin{bmatrix} -b_1 & 1 & 0 & \dots & 0 \\ & & & & \vdots \\ 1 & -b_2 & 1 & 0 & \dots & \vdots \\ & & & & & \vdots \\ 0 & 1 & & & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & & 1 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & 1 & -b_r \end{bmatrix} \quad \begin{aligned} b_i &\geq 2 \\ b_i &= -S_i \cdot S_i \end{aligned}$$

where $p/q = [[b_1, \dots, b_r]]$, (see [35], 3.4).

The inverse of this matrix has only non-positive entries. Since $2 + S_i \cdot S_i = 2 - b_i \leq 0$, we have $a_j \geq 0$.

For a cusp singularity the Chern divisor equals $\sum S_j$, (see 3.2 (9)).

Therefore, in (19) all the terms $c_1^{(v)} \cdot F$ are non-negative.

Every cusp of $\overline{\mathfrak{H}/\Gamma}$, the non-singular model of F , maps under $\overline{\mathfrak{H}/\Gamma} \rightarrow Y$ to a point on some curve in the Chern divisor of a cusp singularity. This intersection point gives at least the contribution 1 in (19).

Let $a_r(\Gamma)$ be defined as in 1.6. If an element γ of Γ has order r , then (since $\Gamma \subset \hat{G}(\mathfrak{o}_K, \mathfrak{b})$ or $\Gamma \subset G(\mathfrak{o}_K, \mathfrak{b})$) we have a quotient singularity of type $(r; 1, 1)$ whose Chern divisor intersects F in a point coming by $\mathfrak{H} \rightarrow \mathfrak{H}/\Gamma \rightarrow F$ from a point z of \mathfrak{H} whose isotropy group is generated by γ . The Chern divisor contains in this case just one curve S and equals $\frac{r-2}{r} S$, (see the end of 3.3).

If we denote by $\sigma(\Gamma)$ the number of cusps of $\overline{\mathfrak{H}/\Gamma}$ we get by (19) the estimate

$$(20) \quad c_1[F] \geq 2 \int_{\mathfrak{H}/\Gamma} \omega + \sum_{r \geq 2} \frac{r-2}{r} a_r(\Gamma) + \sigma(\Gamma)$$

The Euler number of the non-singular model $\overline{\mathfrak{H}/\Gamma}$ of F is given by the classical formula

$$(21) \quad e(\overline{\mathfrak{H}/\Gamma}) = \int_{\mathfrak{H}/\Gamma} \omega + \sum_{r \geq 2} \frac{r-1}{r} a_r(\Gamma) + \sigma(\Gamma),$$

which follows from 1.6 (21), because $\sigma(\Gamma)$ points are attached to \mathfrak{H}/Γ by the compactification. By (20) and (21)

$$(22) \quad c_1[F] \geq 2e(\overline{\mathfrak{H}/\Gamma}) - \sum_{r \geq 2} a_r(\Gamma) - \sigma(\Gamma)$$

The right side of (20) is defined for any discrete subgroup of type (F) which is equivalent in this case to \mathfrak{H}/Γ having a finite volume.

Definition :

$$\begin{aligned} c_1(\Gamma) &= 2 \int_{\mathfrak{H}/\Gamma} \omega + \sum_{r \geq 2} \frac{r-2}{r} a_r(\Gamma) + \sigma(\Gamma) \\ &= 2e(\overline{\mathfrak{H}/\Gamma}) - \sum_{r \geq 2} a_r(\Gamma) - \sigma(\Gamma). \end{aligned}$$

If Γ is the Klein-Fricke group $\Gamma_0(N)$ divided by $\{1, -1\}$ we shall write $c_1(N)$ for $c_1(\Gamma)$ and also $a_r(N)$ for $a_r(\Gamma)$ and $\sigma_0(N)$ for $\sigma(\Gamma)$. The numbers $a_r(N)$ vanish for $r > 3$. There are well-known formulas for $[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)]$, for $a_r(N)$ and $\sigma_0(N)$, (see, for example, [70], p. 24). The Euler number $e(\mathfrak{H}/\Gamma_0(N))$ will be written as $2 - 2g_0(N)$. By (21) there is a formula for $g_0(N)$ which implies (as Helling has shown recently [32])

$$(23) \quad g_0(N) = 0 \Leftrightarrow N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25$$

$$g_0(N) = 1 \Leftrightarrow N = 11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49$$

Compare [13] where the values of $g_0(N)$, $a_r(N)$ and $\sigma_0(N)$ are tabulated for $N \leq 1\,000$. Therefore, we can write down easily a list of $c_1(N)$ for the *rational and elliptic curves* $\mathfrak{H}/\Gamma_0(N)$ (see (23)):

$$(24) \quad g_0(N) = 0$$

N	1	2	3	4	5	6	7	8	9	10	12	13	16	18	25
$c_1(N)$	1	1	1	1	0	0	0	0	0	-2	-2	-2	-2	-4	-4

$$g_0(N) = 1$$

N	11	14	15	17	19	20	21	24	27	32	36	49
$c_1(N)$	-2	-4	-4	-4	-4	-6	-6	-8	-6	-8	-12	-10

4.4. We want to prove that the Hilbert modular surfaces are rational in some cases. An algebraic surface is rational if and only if it is birationally equivalent to the complex projective plane, or equivalently if the field of meromorphic functions on the surface is a purely transcendental extension of the field of complex numbers of degree 2.

Let S be a non-singular algebraic surface and K a canonical divisor of S . The “complete linear system” $|mK|$ of all non-negative divisors D which are linearly equivalent to mK is a complex projective space whose dimension is denoted by $P_m - 1$. The numbers $P_m (m \geq 1)$ are the pluri-genera of the surface S (see, for example, [64] and [36], p. 151).

We have $P_1 = g_2$ (see 3.6). The equality $P_m = 0$ means, that $|mK|$ is empty. The numbers $P_m (m \geq 1)$ are birational invariants. They vanish for rational surfaces.

Castelnuovo's criterion ([46], Part IV):

A non-singular connected algebraic surface S is rational if and only if $g_1 = P_2 = 0$.

Remark. Clearly, $P_2 = 0$ implies $g_2 = 0$. There are algebraic surfaces with $g_1 = g_2 = 0$ which are not rational (Enriques' surfaces with $g_1 = g_2 = 0$ and $P_2 = 1$, see [64]). The condition $g_1 = g_2 = 0$ is equivalent to $g_1 = 0$ and $\chi(S) = 1$ (see 3.6). For Hilbert modular surfaces $g_1 = 0$ (see the lemma in 3.6). Up to now all Hilbert modular surfaces and similar surfaces (see § 5) with $\chi(S) = 1$ have turned out to be rational. The number P_m of a non-singular model of $\overline{\mathfrak{H}^2}/\Gamma$ equals the dimension of the vector space of those cusp forms of weight m which can be extended holomorphically to the non-singular model. Therefore $P_m \leq \dim \mathfrak{S}_\Gamma(m)$. The calculation of P_m seems to be a very difficult problem.

We shall base everything on Castelnuovo's criterion, not worrying whether in a systematic exposition of the theory of algebraic surfaces some results would have to be presented before this criterion. The following theorem is an immediate consequence of Castelnuovo's criterion.

THEOREM. *Let S be a non-singular connected algebraic surface with $g_1 = 0$. Let c_1 be the first Chern class of S and K a canonical divisor of S . If D is an irreducible curve in S with $c_1[D] = -K \cdot D > 0$ and $D \cdot D \geq 0$, then S is rational.*

Proof. We show that $P_m = 0$ for $m \geq 1$.

If $A \in |mK|$, then

$$A = aD + R, \quad \text{where } a \geq 0, R \cdot D \geq 0.$$

Therefore,

$$mK \cdot D = aD \cdot D + R \cdot D \geq 0$$

which is a contradiction. Thus mK is empty.

COROLLARY I. *Let S be a non-singular connected algebraic surface with $g_1 = 0$. Let c_1 be the first Chern class of S and K a canonical divisor of S . If D is an irreducible curve on S with $c_1[D] \geq 2$, then S is rational. If D is an irreducible curve on S with $c_1[D] \geq 1$ which has at least one singular point or which is not a rational curve, then S is rational.*

Proof. By the adjunction formula (0.6)

$$c_1 [D] - D \cdot D = -K \cdot D - D \cdot D$$

equals the Euler number $e(\tilde{D})$ of the non-singular model \tilde{D} of D minus contributions coming from the singular points of D which are positive and even for each singular point. Thus

$$c_1 [D] - D \cdot D \leq e(\tilde{D}) \leq 2$$

and

$$c_1 [D] - D \cdot D \leq 0,$$

if D has a singular point or is not rational. Therefore, the assumptions in the corollary imply $D \cdot D \geq 0$.

COROLLARY II. *Let S be a non-singular connected algebraic surface with $g_1 = 0$. Let c_1 be the first Chern class of S . Suppose that S is not a rational surface. If D is an irreducible curve on S with $c_1 [D] = -K \cdot D = 1$, then D is rational and does not have a singular point. Furthermore, $D \cdot D = -1$.*

A non-singular rational curve E on a non-singular surface S which satisfies $E \cdot E = -1$ (or equivalently $c_1 [E] = 1$) is called an exceptional curve (of the first kind). It can be blown down to a point:

In a natural way, S/E is again an algebraic surface ([64], p. 32). The surfaces S and S/E are birationally equivalent.

If c_1 is the first Chern class of S and \tilde{c}_1 the first Chern class of S/E , then for any irreducible curve D in S and the image curve \tilde{D} in S/E we have

$$(25 \text{ a}) \quad \tilde{c}_1 [\tilde{D}] = c_1 [D] + D \cdot E$$

This is true because $c_1 = \pi^* \tilde{c}_1 - e$, where $\pi : S \rightarrow S/E$ is the natural map and $e \in H^2(S, \mathbf{Z})$ the cohomology class corresponding to E under Poincaré duality.

If D is non-singular and $D \cdot E = 1$, then \tilde{D} is also non-singular and by (25 a) and the adjunction formula

$$(25 \text{ b}) \quad D \cdot D = \tilde{D} \cdot \tilde{D} - 1$$

COROLLARY III. *Let S be a non-singular algebraic surface with $g_1 = 0$ which is not rational. If E_1, E_2 are two different exceptional curves of the first kind, then E_1, E_2 do not intersect.*

Proof. We have $c_1[E_1] = 1$. If we blow down E_2 , then in S/E_2 (first Chern class \tilde{c}_1)

$$\tilde{c}_1(\tilde{E}_1) = 1 + E_1 \cdot E_2$$

Therefore, by Corollary I, $E_1 \cdot E_2 = 0$ and thus $E_1 \cap E_2 = \emptyset$.

4.5. Let G be the Hilbert modular group for $K = \mathbf{Q}(\sqrt{d})$, d square free. If we resolve all singularities in $\overline{\mathfrak{H}^2}/G$ (minimal resolutions) we get a non-singular algebraic surface $Y(d)$ which in 4.1 was denoted by $Y(\mathfrak{o}_K, \mathfrak{B})$ where \mathfrak{B} is here the ideal class of principal ideals $(\lambda) \subset \mathfrak{o}_K$ with $\lambda > 0$, $\lambda' > 0$. If λ is not divisible by a natural number > 1 , we can consider the curve

$$(26) \quad z_1 = \lambda\zeta, \quad z_2 = \lambda'\zeta \quad (\zeta \in \mathfrak{H})$$

which according to (7) gives one of the (one or two) irreducible components of the curve $D((\lambda))$ in $Y(d)$. If we replace λ by λ' we get the same curve. Namely, our curve can also be written as

$$(27) \quad z_1 = \lambda \left(-\frac{1}{\lambda\lambda'\zeta} \right), \quad z_2 = \lambda' \left(-\frac{1}{\lambda\lambda'\zeta} \right), \quad \zeta \in \mathfrak{H},$$

because $\zeta \mapsto -\frac{1}{\lambda\lambda'\zeta}$ is an automorphism of \mathfrak{H} .

If we apply the element $z_j \mapsto -\frac{1}{z_j}$ of G to (27) we get

$$(28) \quad z_1 = \lambda'\zeta, \quad z_2 = \lambda\zeta, \quad \zeta \in \mathfrak{H}.$$

We consider the involution $(z_1, z_2) \mapsto (z_2, z_1)$ on \mathfrak{H}^2 which induces an involution T on $\overline{\mathfrak{H}^2}/G$ and hence on $Y(d)$, because the minimal resolutions are canonical. (26) and (28) show that our curve is carried over to itself by T .

The cusp at ∞ of $\overline{\mathfrak{H}^2}/G$ admits a resolution:

We have to take the primitive cycle or twice the primitive cycle of the reduced quadratic irrationality w_0 such that $\mathbf{Z}w_0 + \mathbf{Z} \cdot 1 = \mathfrak{o}_K$:

$$(29) \quad \begin{aligned} w_0 &= [\sqrt{d}] + 1 + \sqrt{d} && \text{for } d \equiv 2, 3 \pmod{4} \\ w_0 &= \frac{1}{2} (\{\sqrt{d}\} + \sqrt{d}) && \text{for } d \equiv 1 \pmod{4} \end{aligned}$$

where $\{\sqrt{d}\}$ denotes the smallest odd number greater than \sqrt{d} . We have $(w_1^{-1})' = w_0$, (see 4.2) and therefore (2.3 (13)) for the continued fraction of w_0 :

$$(30) \quad w_{-k} = w_k, \quad b_k = b_{-k}, \quad M_k = M_{-k}, \quad N_k = N_{-k}, \quad (w_k^{-1})' = w_{-k+1}$$

If r is the length of the cycle, then for w_k, b_k, M_k, N_k the index k can be taken mod r . However, for the curves S_k we have to consider k modulo r or modulo $2r$.

We note

$$(31) \quad b_0 = M_0 = 2([\sqrt{d}] + 1), \quad N_0 = 1, \quad N_1 = ([\sqrt{d}] + 1)^2 - d \\ \text{for } d \equiv 2, 3 \pmod{4}$$

$$(32) \quad b_0 = M_0 = \{\sqrt{d}\}, \quad N_0 = 1, \quad N_1 = \frac{1}{4}(\{\sqrt{d}\}^2 - d) \\ \text{for } d \equiv 1 \pmod{4}$$

For any characteristic $(k \mid p, q)$ we have one or two curves (26) in the Hilbert modular surface $Y(d)$. Compare the theorem in 4.1. Let D be such a curve. Suppose

$$(33) \quad N = N(\lambda) = p^2 N_{k-1} + pq M_k + q^2 N_k \not\equiv 0 \pmod{d},$$

then the non-singular model of D is $\mathfrak{H}/\Gamma_0(N)$. Suppose also $N > 1$. Then the curve D intersects the Chern divisor $\cup S_j$ of the resolution at least twice, the intersection points correspond to the cusp at ∞ and at 0 of $\mathfrak{H}/\Gamma_0(N)$ which are different cusps for $N > 1$. By applying the theorem in 4.1 to the curves (26) and (28) which both represent D we see by (11) that the two intersection points are of characteristic $(k \mid p, q)$ and $(-k+1 \mid q, p)$. The involution T maps S_k to S_{-k} and interchanges the two intersection points. If the characteristic is $(k \mid 0, 1)$, then the symmetric one is $(-k+1 \mid 1, 0) = (-k \mid 0, 1)$. If (33) is satisfied, then

$$c_1[D] \geq c_1(N), \quad \text{see (24),}$$

because the non-singular model of D is $\overline{\mathfrak{H}/\Gamma_0(N)}$.

Since the intersection number of $u^q - v^p = 0$ and $uv = 0$ equals $p + q$, the intersection number of D and the Chern divisor $\sum S_j$ is $\geq p + q$ in each of the two intersection points and therefore

$$(34) \quad c_1 [D] \geq c_1 (N) + 2(p + q - 1)$$

Because of 4.4 (Corollary I) we have the following theorem.

THEOREM. *Let $K = \mathbf{Q}(\sqrt{d})$, d square free, and G the Hilbert modular group. Consider the continued fraction for w_0 (see (29)) and the corresponding numbers M_k, N_k (see 4.2). We look at the following representations of natural numbers N :*

$$(35) \quad N = p^2 N_{k-1} + pq M_k + q^2 N_k$$

(for some k and for relative prime natural numbers p, q).

If N is represented as in (35), if $N \not\equiv 0 \pmod{d}$ and $N > 1$, then

$$c_1 (N) + 2(p + q - 1) < 2$$

or the Hilbert modular surface \mathfrak{H}^2/G is rational.

$N_{-1} + M_0 + N_0$ equals 7 for $d = 2, 21$, it equals 8 for $d = 17$, it equals 9 for $d = 7, 13$ (see 4.2 or recall that $N_{-1} = N_1$ and use (31), (32)). For $d = 3$, we have $13 = 4N_1 + 2M_0 + N_0$. For $d = 5$, we have $11 = 4N_1 + 2M_0 + N_0$. For these d we get $c_1 (N) + 2(p + q - 1) = 2$ (see (24)). Thus the Hilbert modular surface is rational in these cases.

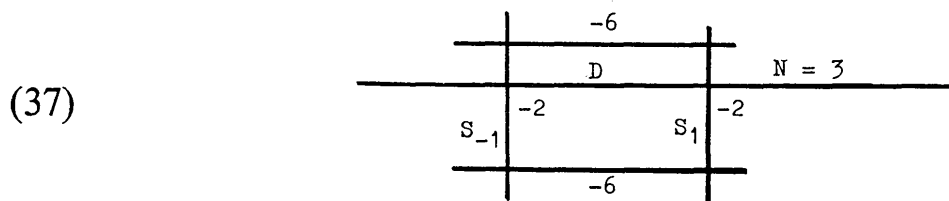
For $d = 6, 15, 33$ a more refined argument is needed. Actually, the theorem throws away some information, because we have only used two cusps of $\mathfrak{H}/\Gamma_0(N)$, ($N > 1$). If N is not a prime, then $\mathfrak{H}/\Gamma_0(N)$ has more cusps. This is relevant for $d = 15$: There are two cusps of the Hilbert modular surface which are of equal type (3.9). We have $10 = N_{-1} + M_0 + N_0$. The curve $\mathfrak{H}/\Gamma_0(10)$ has 4 cusps. One can prove that the intersection of D with the Chern cycles of the two cusps of the Hilbert modular surface looks as follows (in this case the curve D (b) of theorem 4.1 is irreducible)

$$(35) \quad \begin{array}{c} \text{Diagram showing two cusps of the Hilbert modular surface. Each cusp is represented by a horizontal line with two loops. The top loop is labeled -8 and the bottom loop is labeled -8. Between the two cusps, the text 'D, N = 10' is written. The entire diagram is labeled (35) on the left.$$

Therefore

$$c_1 [D] \geq c_1 (10) + 4 = 2.$$

For $d = 6$ we have a diagram



Again the curve $D = D(b)$ of the theorem in 4.1 is irreducible.

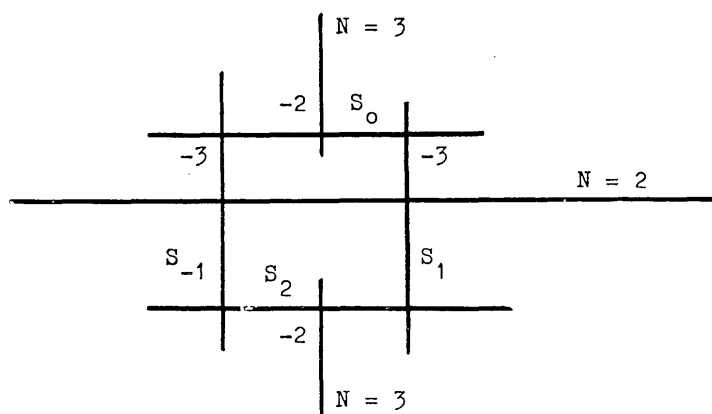
For the curve D we have $c_1[D] \geq c_1(N) = 1$. Thus the surface $Y(6)$ is rational or D is an exceptional curve of the first kind. If D is exceptional, then we blow it down. The images of S_1 and S_{-1} become exceptional curves which intersect each other. Thus $Y(6)$ is rational by Corollary III in 4.4. We could have also used $N = 10$. The corresponding curve goes through the 4 corners of diagram (37).

For $d = 33$, the same argument works using $N = 4$.

We have proved

THEOREM. Let $K = \mathbf{Q}(\sqrt{d})$, d square free, and G the Hilbert modular group, then $\overline{\mathfrak{H}^2/G}$ is rational for $d = 2, 3, 5, 6, 7, 13, 15, 17, 21, 33$.

For $d = 3$ we consider also $\overline{(\mathfrak{H} \times \mathfrak{H}^-)/G}$. The non-singular model is $Y(\mathfrak{o}_K, \mathfrak{B})$ where \mathfrak{B} is now the ideal class of all ideals (λ) with $\lambda\lambda' < 0$. The resolution of the cusp at infinity is



We have one curve with $N = 2$ (non-singular model $\mathfrak{H}/\Gamma_0(2)$) and two curves with $N = 3$ (non-singular model $\mathfrak{H}/\Gamma^*(3)$).

If $\Gamma = \Gamma^*(3)/\{1, -1\}$, then $e(\overline{\mathfrak{H}^2/\Gamma}) = 2$, $a_2(\Gamma) = a_6(\Gamma) = 1$, all other $a_r(\Gamma) = 0$, $\sigma(\Gamma) = 1$. Thus

$$c_1(\Gamma^*(3)) = 4 - 2 - 1 = 1$$

Either the surface is rational, or the three curves with $N = 2, 3$ can be blown down. Then S_0 can be blown down and S_1 and S_{-1} give two exceptional curves which intersect in two points. Thus the surface is rational.

Observe that in general the rationality of $Y(\mathfrak{o}_K, \mathfrak{B})$ implies the rationality of $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$ (Lüroth's theorem [64], Chap. III, § 2). We could show this directly by using our curves in $\hat{Y}(\mathfrak{o}_K, \mathfrak{B})$.

Exercise. Let $K = \mathbf{Q}(\sqrt{69})$. Calculate the arithmetic genera of $\overline{\mathfrak{S}^2/G}$ and $\overline{\mathfrak{S}^2/\hat{G}}$. Prove that the surface $\overline{\mathfrak{S}^2/\hat{G}}$ is rational !

In all cases where we know that the arithmetic genus equals 1 we have proved rationality.

§ 5. THE SYMMETRIC HILBERT MODULAR GROUP FOR PRIMES $p \equiv 1 \pmod{4}$

5.1. Let S be a compact connected non-singular algebraic surface. The fixed point set D of a holomorphic involution T of S (different from the identity) consist of finitely many isolated fixed points P_1, \dots, P_r and a disjoint union of connected non-singular curves D_1, \dots, D_s .

If there are no isolated fixed points P_j , then S/T is non-singular and the arithmetic genera of S and S/T are related by the formula

$$(1) \quad \chi(S/T) = \frac{1}{2} \left(\chi(S) + \frac{1}{4} c_1[D] \right)$$

where $D = \sum D_i$ and c_1 is the first Chern class of S (see [40], § 3).

Furthermore, if F is a curve on S (not necessarily irreducible) with $T(F) = F$ and F not contained in D and if \tilde{F} is the image curve on S/T , then

$$(2) \quad \tilde{c}_1[\tilde{F}] = \frac{1}{2} (c_1[F] + F \cdot D), \quad \text{where } c_1 = \text{first Chern class of } S/T.$$

Proof. If $\pi : S \rightarrow S/T$ is the natural projection, then $c_1 = \pi^* \tilde{c}_1 - d$ where $d \in H^2(S, \mathbf{Z})$ is the Poincaré dual of the branching divisor D . Thus

$$(c_1 + d)[F] = \tilde{c}_1[2\tilde{F}].$$

5.2. Let p be a prime $\equiv 1 \pmod{4}$. We consider the field $K = \mathbf{Q}(\sqrt{p})$ and its Hilbert modular group G . We make these restrictions throughout § 5 though some of our results are valid more generally.

The involution $(z_1, z_2) \mapsto (z_2, z_1)$ induces an involution T of \mathfrak{H}^2/G and of $\overline{\mathfrak{H}^2/G}$. As mentioned before (4.5), it can be lifted to an involution T of our non-singular model $Y(p)$ because this was obtained by the canonical minimal resolution of all singularities in $\overline{\mathfrak{H}^2/G}$.

We shall study the algebraic surface $Y(p)/T$ (the isolated fixed points of T give rise to quotient singularities of type $(2; 1, 1)$ of this surface), calculate its arithmetic genus and determine for which p the surface is rational (see [39], [40]).

Equivalently we can consider the *symmetric Hilbert modular* group G_T which is an extension of index 2 of G by the involution $(z_1, z_2) \mapsto (z_2, z_1)$ and study \mathfrak{H}^2/G_T :

The surface $Y(p)/T$ (with the quotient singularities resolved) is a non-singular model of the compactification of \mathfrak{H}^2/G_T .

5.3. The field K has a unit of negative norm. Therefore, the groups G and \hat{G} coincide (1.7). The class number of K is odd. The ideal class groups C and C^+ are equal and the homomorphism Sq in 3.7 (42) is an isomorphism. Therefore for any ideal $\mathfrak{b} \subset \mathfrak{o}_K$ we can find a matrix $A \in \mathbf{GL}_2^+(K)$ (see 1.3) such that

$$(3) \quad A^{-1} \mathbf{SL}_2(\mathfrak{o}_K) A = \mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b})$$

(see 3.7 (40) and 4.1 (7)). If A_1, A_2 are matrices satisfying (3), then, for $B = A_1 A_2^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $B \mathbf{SL}_2(\mathfrak{o}_K) B^{-1} = \mathbf{SL}_2(\mathfrak{o}_K)$.

Proposition.

If $B \in \mathbf{GL}_2^+(K)$ and $B \mathbf{SL}_2(\mathfrak{o}_K) B^{-1} = \mathbf{SL}_2(\mathfrak{o}_K)$, then

$$(4) \quad \sqrt{\det B} \in K, \frac{1}{\sqrt{\det B}} \cdot B \in \mathbf{SL}_2(\mathfrak{o}_K)$$

Proof (compare Maaß [54]). Put $h^2 = \det B$. We may assume that B is an integral matrix. Since

$$\mathbf{SL}_2(\mathfrak{o}_K) \ni B \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} B^{-1} = \begin{pmatrix} 1 - ac/h^2 & a^2/h^2 \\ -c^2/h^2 & 1 + ac/h^2 \end{pmatrix}$$

and a similar formula holds for $B \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} B^{-1}$, we see that $\frac{1}{h} B$ has coefficients which are algebraic integers. Thus the ideal (a, c) of \mathfrak{o}_K consists exactly of those elements x in \mathfrak{o}_K such that x/h is an algebraic integer. This implies that $(a, c)^2$ equals the principal ideal $(\det B)$. In our case, the ideal class group has odd order. Thus (a, c) is principal and $\det B$ multiplied with a totally positive unit is a square in \mathfrak{o}_K . But every totally positive unit is a square of a unit. Therefore $h \in \mathfrak{o}_K$. For the algebraic number theory needed, see [30], § 37.

An ideal is called *admissible* if it is not divisible by any natural number > 1 . For any admissible ideal $\mathfrak{b} \subset \mathfrak{o}_K$ we have (4.1) a curve $C(\mathfrak{b})$ on $Y(\mathfrak{o}_K, \mathfrak{b}) = \hat{Y}(\mathfrak{o}_K, \mathfrak{b})$. In view of (3) we have a curve (which we also call $C(\mathfrak{b})$) on our Hilbert modular surface $Y(p)$. The curve is given in \mathfrak{H}^2/G by

$$(5) \quad z_1 = A\zeta, \quad z_2 = A'\zeta, \quad \zeta \in \mathfrak{H}.$$

Because of (4) it does not depend on the choice of A . (Multiplication of A from the left by an element of $\mathbf{SL}_2(\mathfrak{o}_K)$ does not change the curve.)

We can also say that the surfaces $\mathfrak{H}^2/\mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b})$ are canonically identified and the curves $C(\mathfrak{b})$ are the diagonals in the different representations of \mathfrak{H}^2/G as $\mathfrak{H}^2/\mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b})$. If we change A by multiplying from the right by a rational matrix with positive determinant, we get the same curve, because we make just a change of the parameter $\zeta \in \mathfrak{H}$. This implies that $C(\mathfrak{b}_1) = C(\mathfrak{b}_2)$ if there exists a matrix $A_0 \in \mathbf{GL}_2^+(\mathbf{Q})$ such that $A_0 \mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b}_1) A_0^{-1} = \mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b}_2)$.

Lemma I. If $\mathfrak{b}_1, \mathfrak{b}_2$ are admissible ideals in \mathfrak{o}_K , then the curves $C(\mathfrak{b}_1), C(\mathfrak{b}_2)$ coincide if and only if $N(\mathfrak{b}_1) = N(\mathfrak{b}_2)$.

Proof. If $N(\mathfrak{b}_1) = N(\mathfrak{b}_2) = N$, then put $d = N/N((\mathfrak{b}_1, \mathfrak{b}_2))$. We have $(d, N/d) = 1$, because the ideals are admissible. Thus there exists a rational matrix of determinant d of the form

$$(6) \quad A_0 = \begin{pmatrix} \alpha_0 d & \beta_1 \\ \gamma_0 & \delta_0 d \end{pmatrix}, \quad \gamma_0 \equiv 0 \pmod{N}$$

where $\alpha_0, \beta_1, \gamma_0, \delta_0$ are integers. (Such a matrix occurred in a related context in 4.1). Then, for any A_0 with these properties,

$$A_0 \mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b}_1) A_0^{-1} = \mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b}_2)$$

which shows that the curves coincide. If the curves coincide, then the norms are equal. (We leave the proof to the reader.)

A natural number $N \geq 1$ is called admissible (with respect to p) if it is the norm of an admissible ideal. The prime ideal theory of quadratic fields which we always have used tacitly yields the following lemma.

Lemma II. The natural number $N \geq 1$ is admissible with respect to p if and only if N is not divisible by p^2 and not by any prime $q \neq p$ with $\left(\frac{q}{p}\right) = -1$.

Definition.

In view of Lemma I we have a well-defined curve for any admissible natural number N . This curve on the surface $Y(p)$ will be called F_N .

Lemma III. For the involution T of $Y(p)$ and any admissible N we have $T(F_N) = F_N$.

Proof. If $N = N(\mathfrak{b})$, then $F_N = C(\mathfrak{b})$ is given in \mathfrak{H}^2/G by (5) where A is as in (3). Therefore $T(F_N)$ is the curve $z_1 = A'\zeta$, $z_2 = A\zeta$. But this is $C(\mathfrak{b}')$ which equals $C(\mathfrak{b})$ by lemma I.

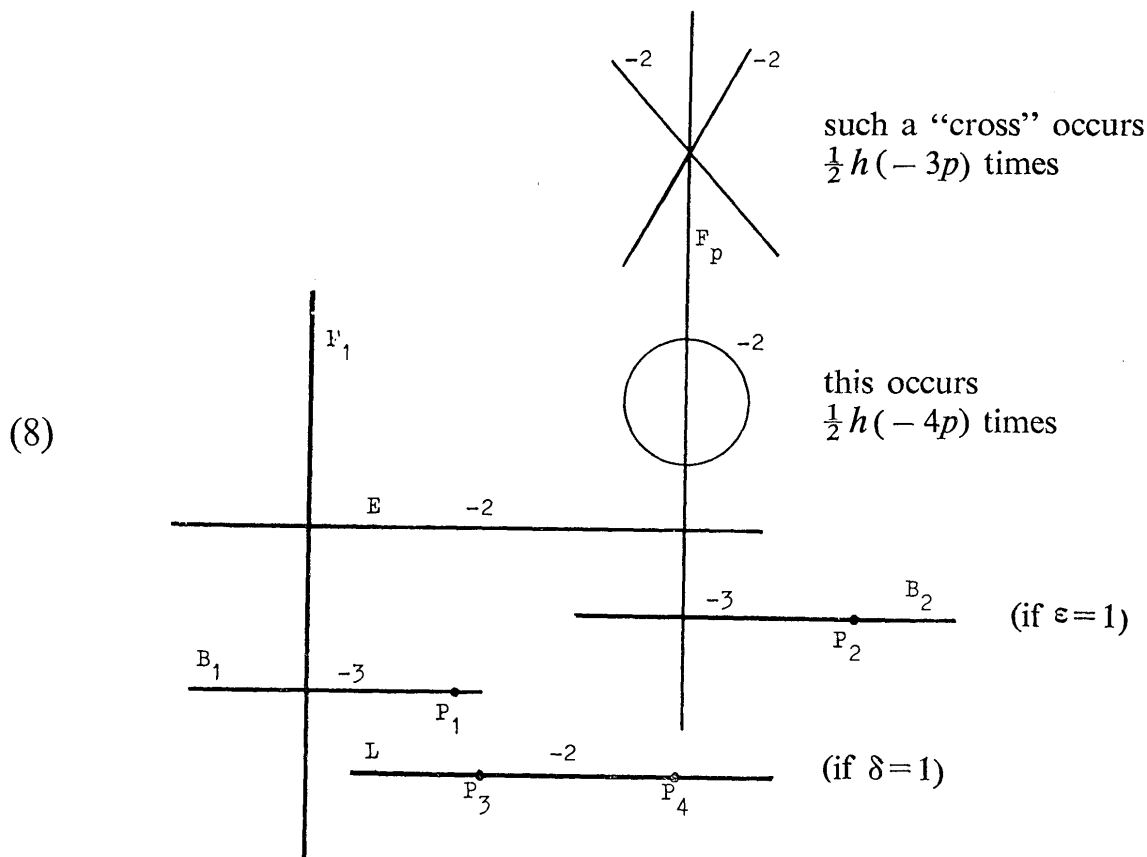
Remark. If $N \not\equiv 0 \pmod{p}$, then $N((\mathfrak{b}, \mathfrak{b}')) = 1$ and the involution T on F_N can be given by the matrix $A_0 = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ (see (6)) if we lift T to the non-singular model $\overline{\mathfrak{H}/\Gamma_0(N)}$ of F_N . Thus $\overline{\mathfrak{H}/\Gamma_*(N)}$ is the non-singular model of F_N/T . (see 4.1). In particular, T is not the identity on F_N if $N \not\equiv 0 \pmod{p}$ and $N > 1$.

5.4. The curves F_1 and F_p (considered as curves in \mathfrak{H}^2/G) are the only curves which are fixed pointwise under T , (see [14] Part II, [62]). The curve F_p belongs to the ideal $(\sqrt{p} e_0)$ where e_0 is a unit of negative norm and can be given by $z_1 = \sqrt{p} e_0 \zeta$, $z_2 = -\sqrt{p} e_0' \zeta$ or by $z_1 = e_0^2 z_2$.

The involution T acts on the quotient singularities of \mathfrak{H}^2/G . The description of this action [62] depends on the residue class of $p \pmod{24}$. Therefore we define

$$(7) \quad \begin{aligned} \varepsilon &= 1 \text{ for } p \equiv 1 \pmod{3}, \quad \varepsilon = 0 \text{ for } p \equiv 2 \pmod{3} \\ \delta &= 1 \text{ for } p \equiv 1 \pmod{8}, \quad \delta = 0 \text{ for } p \equiv 5 \pmod{8} \end{aligned}$$

In \mathfrak{H}^2/G the following holds [62]: Of the $h(-4p)$ quotient singularities of order 2, half of them lie on F_p and not on F_1 , and one of them lies on F_1 and F_p and is the only intersection point of F_1 and F_p in \mathfrak{H}^2/G . There are in addition δ quotient singularities of order 2 which are fixed under T . “They” lie neither on F_1 nor on F_p . The remaining order 2 singularities are interchanged pairwise under T . Of the $h(-3p)$ quotient singularities of order 3, exactly half of them are of type $(3; 1, 2)$. They lie on F_p . There is one singularity of type $(3; 1, 1)$ which lies on F_1 whereas ε such singularities lie on F_p . The remaining singularities of type $(3; 1, 1)$ are interchanged pairwise. For $p = 5$, the two singularities of order 5 are interchanged under T . The involution T acts freely outside F_1, F_p and the quotient singularities. If we pass to the non-singular model $Y(p)$ of $\overline{\mathfrak{H}^2/G}$, we get the following configuration of curves. We omit the curves coming from the quotient singularities which are pairwise interchanged and only show the intersection behaviour outside of the resolved cusp singularities.



The curves F_1, F_p are pointwise fixed under the involution T of $Y(p)$, therefore they are non-singular curves on $Y(p)$. All curves in the diagram are non-singular and (except F_p) rational. F_p is rational if and only if $p = 5, 13, 17, 29, 41$ (see 5.7). The points P_1 , and P_2 if $\varepsilon = 1$, and P_3, P_4

if $\delta = 1$ are the only isolated fixed points of T on $Y(p)$ outside the resolved cusp singularities.

The following lemma is easy to prove and very useful for deducing from Prestel's results [62] that the configuration on $Y(p)$ is as indicated in (8).

Lemma. If S is a compact complex manifold of dimension 2 and T an involution on S which carries the non-singular rational curve C over into itself, then T is the identity on C or T has exactly two fixed points P and Q on C . In the latter case the following holds:

If $C \cdot C$ is odd, then one of the points P, Q is an isolated fixed point of T , the other one is a transversal intersection point of C with one of the (non-singular) curves which are pointwise fixed under T . If $C \cdot C$ is even, then P and Q both are isolated fixed points of T or both are such transversal intersection points with a curve pointwise fixed under T .

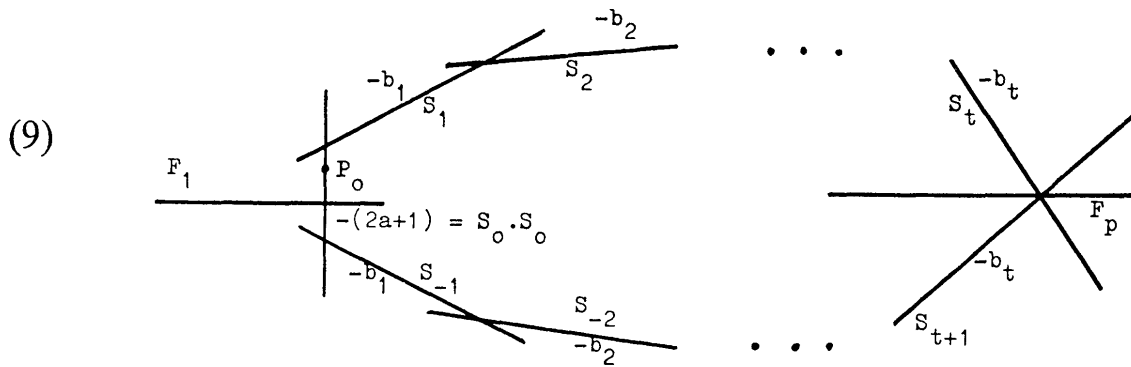
The class number h of $K = \mathbf{Q}(\sqrt{p})$ is odd. There are h cusp singularities corresponding to the h ideal classes (see 3.7). The involution T on \mathfrak{S}^2/G leaves one cusp fixed and interchanges the others pairwise. T maps the cusp of type (M, U^2) where M is a fractional ideal representing an ideal class to the cusp of type (M', U^2) . If M is the \mathbf{Z} -module $\mathbf{Z} \cdot w + \mathbf{Z} \cdot 1$ (with $0 < w' < 1 < w$), then M' is strictly equivalent to $\mathbf{Z} \frac{1}{w'} + \mathbf{Z} \cdot 1$.

The resolution of (M, U^2) is given by the primitive cycle of the purely periodic continued fraction of w , the resolution of (M', U^2) by the primitive cycle of $\frac{1}{w'}$, which is the same cycle in opposite order. The involu-

tion on $Y(p)$ maps the cycle of curves in the resolution of (M, U^2) onto the cycle of curves in the resolution of (M', U^2) . The fixed cusp is of type (M, U^2) where $M = \mathbf{Z}w_0 + \mathbf{Z} \cdot 1$ and where $w_0 = \frac{1}{2}(\{\sqrt{p}\} + \sqrt{p})$, see 4.5 (29). It is the cusp at ∞ .

THEOREM. *The length r of the cycle of $w_0 = \frac{1}{2}(\{\sqrt{p}\} + \sqrt{p})$ is an odd number $r = 2t + 1$. The involution T on $Y(p)$ maps the curve S_k to the curve S_{-k} (see 4.5). The curve F_1 intersects S_0 transversally. It has the characteristic $(0|0, 1)$. The curve F_p intersects S_{-t} and S_t , it has the characteristic $(-t|1, 1)$. We put $\{\sqrt{p}\} = 2a + 1$. The intersection*

behaviour of the cycle of curves with F_1 and F_p is illustrated by the following diagram:



$$w_0 = [[2a + 1, b_1, \dots, b_t, b_t, \dots, b_1]]$$

The point P_0 indicates an isolated fixed point of T . The points P_0, P_1 , and P_2 (if $\varepsilon = 1$), and P_3, P_4 (if $\delta = 1$) are all the isolated fixed points of T . The curves F_1, F_p are the only one-dimensional components of the fixed point set.

Proof. As in 2.5 and 3.10 we denote ordinary continued fractions by

$[a_0, a_1, a_2, \dots]$. Then, since $a = \left[\frac{1 + \sqrt{p}}{2} \right]$,

$$(10) \quad w_0 = \frac{2a + 1 + \sqrt{p}}{2} = [2a, \overline{a_1, \dots, a_m, a_m, \dots, a_1}, 2a - 1]$$

(See [60], § 30. Because there exists a unit of negative norm, the length of the primitive period in (10) is odd.)

If one applies the formula which transforms the continued fraction (10) into a continued fraction of our type (see 2.5 (19)) one has to go twice over the period in (10). We have

$$(11) \quad w_0 = [[\overline{2a+1, \underbrace{2, \dots, 2}_{a_1-1}, \dots, a_1+2, \underbrace{2, \dots, 2}_{2a-2}, a_1+2, \dots, \underbrace{2, \dots, 2}_{a_1-1}}]]$$

Thus the length r of the primitive cycle of w_0 is odd ($r = 2t + 1$). In fact, $t = a_1 + \dots + a_m + a - 1$. Under the involution T only S_0 (self-intersection number $-(2a+1)$) is carried over into itself. *The only symmetric characteristics are $(0|0, 1)$ and $(-t|1, 1)$.* The existence of the isolated fixed point P_0 follows from the preceding lemma. Q.E.D.

For the number w_0 in (11) we wish to calculate w_{t+k} (where $k = 1, \dots, a$), see 4.2. The continued fraction $[[\dots]]$ of w_{t+k} begins with $a - k$ two's. Using again formula 2.5 (19) we obtain

$$\begin{aligned} 0 - \frac{1}{w_{t+k}} &= [-1, a-k+1, a_1, a_2, \dots] \\ &= -1 + \frac{1}{w_0 - a - k + 1} \end{aligned}$$

which yields

$$w_{t+k} = \frac{\sqrt{p} - (2k-3)}{\sqrt{p} - (2k-1)} = \frac{M_{t+k} + \sqrt{p}}{2N_{t+k}}$$

where

$$(12) \quad N_{t+k} = \frac{1}{4}(p - (2k-1)^2), \quad M_{t+k} = 2N_{t+k} + (2k-1)$$

F_p has the characteristic $(-t | 1, 1) = (t+1 | 1, 1)$ which was obtained in the above proof by a symmetry argument.

It follows also from the theorem in 4.1, because

$$N_{t+1} + N_t + M_t = N_{t+1} + N_{t+1} + M_{t+1} = 4N_{t+1} + 1 = p.$$

In view of (12) and the theorem in 4.1 we have the following proposition.

Proposition. On the Hilbert modular surface $Y(p)$ the cusp at ∞ gives the following configuration of curves $\left(a = \left[\frac{1 + \sqrt{p}}{2}\right]\right)$

(13)

$F_{\frac{1}{4}}(p-(2a-1)^2) \quad F_{\frac{1}{4}}(p-(2a-3)^2) \quad \dots \quad F_{\frac{1}{4}}(p-9) \quad F_{\frac{1}{4}}(p-1)$

$S_{-(t+a)} \quad S_{-(t+a-1)} \quad \dots \quad S_{-(t+2)} \quad S_{-(t+1)} \quad S_{t+1}$

$S_{t+a} \quad S_{t+a-1} \quad \dots \quad S_{t+2} \quad S_{t+1}$

$S_{-(t+k)} = S_{t-k+1}$

We have $S_{t+k} \cdot S_{t+k} = S_{-(t+k)} \cdot S_{-(t+k)} = -2$ for $1 \leq k \leq a-1$ and $S_{t+a} \cdot S_{-(t+a)} = -(a_1+2)$. If $p = (2a-1)^2 + 4$, then $S_{-(t+a)} = S_{t+a} = S_0$, the curve $F_{\frac{1}{2}(p-(2a-1)^2)}$ equals F_1 and the diagram has to be changed accordingly. In this case we have

$$w_0 = [2a, \overline{2a-1}] = [[2a+1, \underbrace{2, \dots, 2}_{2a-2}]]$$

and

$$S_{t+a} \cdot S_{t+a} = S_0 \cdot S_0 = -(2a+1).$$

We do not claim that the F_N are non-singular and do not indicate their mutual intersections nor their intersections with F_p . The intersections indicated are transversal.

5.5. The curve F_1 on $Y(p)$ is non-singular. It follows from (8) and 4.3 that it is exceptional. In general, we do not know whether F_N is non-singular. In view of 4.3 (24) the curves F_2, F_3, F_4 are candidates for exceptional curves. In fact, it follows from Corollaries I, II in 4.4 that they are exceptional if $Y(p)$ is not rational. $Y(p)$ is rational if and only if $p = 5, 13, 17$. Thus we have

Lemma. If p is a prime $\equiv 1 \pmod{4}$ and > 17 , then the curves F_N on the Hilbert modular surface $Y(p)$ are exceptional for $N = 1, 2, 3, 4$ provided N is admissible (see 5.3):

We always have the curve F_1 . The curves F_2, F_4 exist for $p \equiv 1 \pmod{8}$. The curve F_3 exists for $p \equiv 1 \pmod{3}$.

For the following discussion we assume $p > 17$. The curves F_1, E, B_1 in diagram (8) can be blown down successively. In view of corollary III in 4.4, the curves F_2, F_3, F_4 are disjoint and do not intersect any of the curves F_1, E, B_1 . According to the lemma in 5.4 the curves F_2, F_3, F_4 pass through exactly one of the isolated fixed points of the involution T .

For F_3 the value $c_1[F_3]$ equals 1, therefore by 4.3 it meets in \mathfrak{H}^2/G exactly one quotient singularity of type $(3; 1, 1)$, thus it must be the one which is fixed under T . It intersects B_2 (see (8)) only in P_2 and transversally because otherwise we would have $c_1[F_3] > 1$. The curve F_4 has the model $\overline{\mathfrak{H}/\Gamma_0(4)}$ which has three cusps. Therefore F_4 must intersect the curves of the resolved cusps of \mathfrak{H}^2/G in three points. One of them is fixed under T . Thus F_4 passes through P_0 .

The curve F_2 passes through P_3 or P_4 in diagram (8), say P_3 . It intersects L transversally in P_3 and does not intersect L in any other point, because otherwise L would give in the surface with F_2 blown down a curve \hat{L} with $c_1[\hat{L}] \geq 2$. The curves F_2, L can be blown down successively. Therefore L is disjoint to any exceptional curve different from F_2 .

We have found an exceptional curve passing through P_0 only for $p \equiv 1 \pmod{8}$. But there exists such a curve F for any $p > 17$.

For the cusp at ∞ we put as before $w_0 = \frac{1}{2}(\{\sqrt{p}\} + \sqrt{p}) = \frac{1}{2}(2a+1 + \sqrt{p})$. The involution T is given in the coordinate system (u_0, v_0) by

$$(14) \quad (u_0, v_0) \mapsto (u_0^{-1}, u_0^{-(2a+1)} \cdot v_0),$$

as follows from 2.3 (9). The isolated fixed point P_0 of T has the coordinates $(-1, 0)$. Thus it lies on the curve $F \subset Y(p)$ given by $u_0 = -1$ which can be presented in $\mathfrak{H} \times \mathfrak{H}$ by

$$(15) \quad z_1 = \zeta + \frac{w_0}{2}, \quad z_2 = \zeta + \frac{w'_0}{2}, \quad (\zeta \in \mathfrak{H}).$$

Let Γ be the subgroup of those matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of

$$\begin{pmatrix} 1 & -w_0/2 \\ 0 & 1 \end{pmatrix} \mathbf{SL}_2(\mathfrak{o}_K) \begin{pmatrix} 1 & w_0/2 \\ 0 & 1 \end{pmatrix}$$

which, when acting on \mathfrak{H}^2 carry the diagonal into itself. The curve $\overline{\mathfrak{H}/\Gamma}$ is a non-singular model of F . The group Γ is characterized by 4.1 (1), but the second condition is impossible. Thus Γ is the subgroup of $\mathbf{SL}_2(\mathbf{Q})$ of matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ for which

$$\begin{pmatrix} \alpha + \gamma w_0/2 & -\alpha w_0/2 + \beta - \gamma w_0^2/4 + \delta w_0/2 \\ \gamma & \delta - \gamma w_0/2 \end{pmatrix}$$

is integral. Since $w_0, 1$ is a \mathbf{Z} -base of \mathfrak{o}_K , we get that α, δ are integers and γ is an even integer. We have

$$(16) \quad -\alpha w_0/2 + \beta - \gamma w_0^2/4 + \delta w_0/2 = \\ (-\alpha/2 - \gamma(2a+1)/4 + \delta/2)w_0 + \beta + \gamma((2a+1)^2 - p)/16$$

If $p \equiv 1 \pmod{8}$, then β is an integer and $\alpha\delta - \beta\gamma = 1$ implies $\alpha \equiv \delta \pmod{2}$ and $\gamma \equiv 0 \pmod{4}$, because the coefficient of w_0 in (16) must be integral. Thus $\Gamma = \Gamma_0(4)$ in this case.

If $p \equiv 5 \pmod{8}$, then $\Gamma_0(4) \subset \Gamma$. We put $\gamma = 2\gamma^*$ and $\beta = \beta^*/2$. Then γ^*, β^* are integers which are congruent modulo 2. We have $\alpha + \delta \equiv \gamma^* \pmod{2}$.

The matrix $\begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}$, whose third power is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, satisfies these conditions. Γ is a normal extension of index 3 of $\Gamma_0(4)$. The three cusp of $\mathfrak{H}/\Gamma_0(4)$ are identified. $\overline{\mathfrak{H}/\Gamma}$ is a rational curve. Put $\tilde{\Gamma} = \Gamma/\{1, -1\}$. We have $a_3(\tilde{\Gamma}) = 2$ ($a_r(\tilde{\Gamma}) = 0$ otherwise) and $\sigma(\tilde{\Gamma}) = 1$.

Therefore $c_1(\tilde{\Gamma}) = 1$ (see the definition in 4.3), and the curve F is exceptional. It passes through the isolated fixed point P_0 of T . For $p \equiv 1 \pmod{8}$, the curve F equals F_4 because two different exceptional curves do not intersect. We have $T(F) = F$.

We can now state the following proposition.

Proposition. If we blow down the curves F_1, E, B_1, F , and F_2, L (for $\delta = 1$), and F_3 (for $\varepsilon = 1$) on the surface $Y(p)$ for $p > 17$, then we obtain a non-singular algebraic surface $Y^0(p)$. The involution T is also defined on $Y^0(p)$. It does not have any isolated fixed point. The curve F_p has a non-singular image F_p^0 in $Y^0(p)$ which is the complete fixed point set of T .

5.6. If c_1 is again the first Chern class of $Y(p)$, then

$$(17) \quad c_1[F_p] = -\frac{p+1}{6} + \frac{\varepsilon}{3} + 2$$

This follows from 4.3 (19), because $[\mathbf{SL}_2(\mathbf{Z}) : \Gamma_0(p)] = p+1$ and $[\Gamma^*(p) : \Gamma_0(p)] = 2$. We further use (8) and (9).

Let us now assume that $Y(p)$ is not rational which is the case for $p > 17$. In $Y(p)$ we have blown down $3 + 1 + 2\delta + \varepsilon$ curves and obtained the surface $Y^0(p)$ on which T has the fixed point set F_p^0 . Let c_1^0 be the first Chern class of $Y^0(p)$. Then

$$(18) \quad c_1^0[F_p^0] = -\frac{p+1}{6} + \frac{\varepsilon}{3} + 2 + 2 + 1 + 2\delta + \varepsilon.$$

This follows from 4.4 (25a) using that F, F_2, F_3 intersect F_p transversally in exactly one point (see the lemma in 5.4). By 5.1 (1) the number $c_1^0[F_p^0]$ must be divisible by 4. We have

$$(19) \quad \frac{1}{4}c_1^0[F_p^0] = -\left[\frac{p-29}{24}\right],$$

since $\frac{1}{4} \left(\frac{\varepsilon}{3} + 2\delta + \varepsilon \right) < 1$. The surface $Y^0(p)/T$ is a non singular model for the compactification of \mathfrak{H}^2/G_T (see 5.2). The arithmetic genus of $Y^0(p)/T$ will be denoted by $\chi_T(p)$. In 3.12 we have given a formula for the arithmetic genus of $Y(p)$ which we shall call here $\chi(p)$. Then

$$(20) \quad \chi(p) = \frac{1}{2} \zeta_K(-1) + \frac{h(-4p)}{8} + \frac{1}{6} h(-3p),$$

where $K = \mathbf{Q}(\sqrt{p})$. By 5.1 (1) and (19) the arithmetic genera $\chi(p)$ and $\chi_T(p)$ are related by the formula [40]

$$(21) \quad \chi_T(p) = \frac{1}{2} \left(\chi(p) - \left[\frac{p-29}{24} \right] \right),$$

(compare [14], Part II, Satz 2).

This formula is also valid for $p = 5, 13, 17$. In these cases the surface $Y(p)$ and therefore also $Y^0(p)/T$ are rational and (21) reduces to $1 = \frac{1}{2}(1+1)$. It was shown in [40] that

$$\chi_T(p) > \frac{p^{3/2}}{1440} - \frac{p+1}{48} \quad (\text{compare 3.12}),$$

and explicit calculations gave the result that $\chi_T(p) = 1$ for exactly 24 primes, namely for all primes $(\equiv 1 \pmod{4})$ smaller than the prime 193 and for $p = 197, 229, 269, 293, 317$.

We wish to show in the next sections that the surfaces $Y^0(p)/T$ are rational for these primes. Since the rationality is already known for $p = 5, 13, 17$ it remains to consider 21 primes. Since the first Betti number of $Y(p)$ vanishes (3.6), the same holds for $Y^0(p)/T$. Thus the rationality criteria of 4.4 (Corollaries I, II, III) can be applied.

5.7. The curve F_N in $Y(p)$ (for an admissible natural number $N > 4$) projects down to a curve F_N^0 in $Y^0(p)$ and to a curve $F_N^* = F_N^0/T$ in $Y^0(p)/T$. If N is not divisible by p , then F_N^* has $\mathfrak{H}/\Gamma^*(N)$ as non-singular model (see the remark in 5.3). We have a commutative diagram:

$$\begin{array}{ccccc} \overline{H / \Gamma_0(N)} & \rightarrow & F_N^0 & \subset & Y_0(p) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{H / \Gamma_*(N)} & \rightarrow & F_N^* & \subset & Y_0(p) / T \end{array}$$

There is an involution τ on $\overline{\mathfrak{H}/\Gamma_0(N)}$ compatible with T and having $\overline{\mathfrak{H}/\Gamma_*(N)}$ as orbit space. Recall that F_p^0 is the fixed point set of T on $Y^0(p)$. Thus the intersection number $F_N^0 \cdot F_p^0$ is greater or equal to the number $\text{fix}(\tau)$ of fixed points of τ on $\overline{\mathfrak{H}/\Gamma_0(N)}$:

$$(22) \quad F_N^0 \cdot F_p^0 \geq \text{fix}(\tau) = 2e(\overline{\mathfrak{H}/\Gamma_*(N)}) - e(\overline{\mathfrak{H}/\Gamma_0(N)})$$

Let c_1^* be the first Chern class of $Y^0(p)/T$. By 5.1 (2) we get

$$c_1^*[F_N] = \frac{1}{2}(c_1^0[F_N^0] + F_N^0 \cdot F_p^0)$$

Since $c_1^0[F_N^0] \geq c_1[F_N] \geq c_1(N)$, see 4.3 and 4.4 (25a), the following estimate is obtained:

$$(23) \quad c_1^*[F_N^*] \geq \frac{1}{2}c_1(N) + e(\overline{H/\Gamma_*(N)}) - \frac{1}{2}e(\overline{H/\Gamma_0(N)})$$

The right side of (23) only depends on N . We shall denote it by $c_1^*(N)$ and have

$$(24) \quad c_1^*[F_N^*] \geq c_1^*(N).$$

There are explicit formulas for the Euler numbers or equivalently the genera of the curves $\overline{\mathfrak{H}/\Gamma_*(N)}$, see [16], p. 357, and [13]. Helling [32] has shown that there are exactly 37 values $N \geq 2$ for which $\overline{\mathfrak{H}/\Gamma_*(N)}$ is rational. (In [16], p. 367, Fricke omits the value $N = 59$). We shall give a list of the $c_1^*(N)$ for the 34 values ≥ 5 .

By the definition of $c_1(N)$ we get:

If $\overline{\mathfrak{H}/\Gamma_*(N)}$ is rational, then (for $N \geq 5$)

$$(25) \quad c_1^*(N) = 3 - g_0(N) - \frac{1}{2}(a_2(N) + a_3(N) + \sigma_0(N)).$$

Using [13] we obtain the following list:

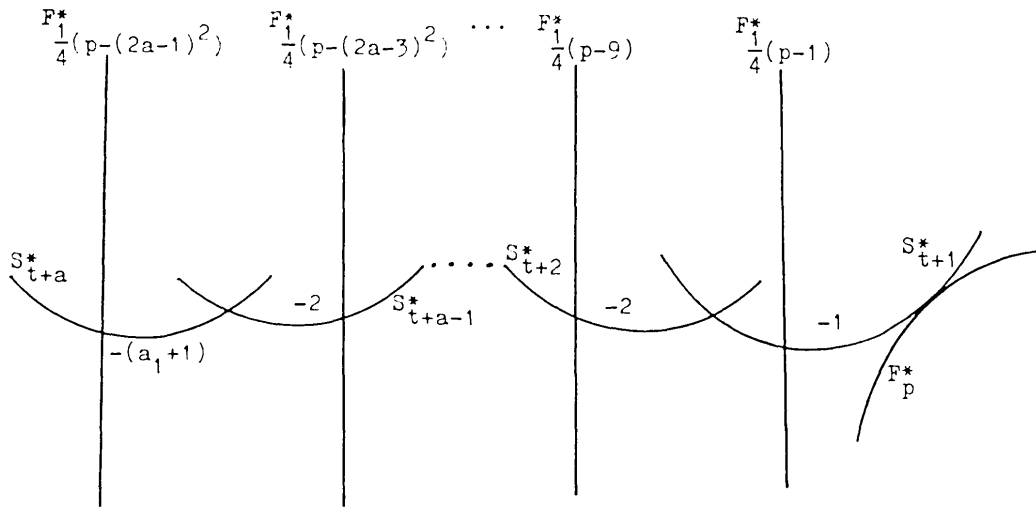
$$e(\overline{\mathfrak{H}/\Gamma_*(N)}) = 2$$

$$(26) \quad \begin{array}{c|cccccccccccccccccccccccc} N & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ \hline c_1^*(N) & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \end{array}$$

$$(26) \quad \begin{array}{c|cccccccccccccccccccccccc} N & 23 & 24 & 25 & 26 & 27 & 29 & 31 & 32 & 35 & 36 & 39 & 41 & 47 & 49 & 50 & 59 & 71 \\ \hline c_1^*(N) & 0 & -2 & -1 & -2 & -1 & -1 & -1 & -2 & -2 & -4 & -3 & -2 & -2 & -3 & -6 & -3 & -4 \end{array}$$

5.8. The curves F_N will be used for rationality proofs. Consider the diagram (13) for $p > 17$. We have $\frac{1}{4}(p - (2a-3)^2) \geq 5$. It follows from 4.2 (15) that the exceptional curves F_1, F, F_2, F_3 do not intersect S_{t+k} and $S_{-(t+k)}$ for $1 \leq k \leq a-1$. These exceptional curves also do not meet S_{t+a} and $S_{-(t+a)}$ if $\frac{1}{4}(p - (2a-1)^2) \geq 5$. In this case, the configuration (13) is not changed by passing to $Y^0(p)$. If we apply the involution T we get the following configuration on $Y^0(p)/T$.

(27)



If $\frac{1}{4}(p - (2a-1)^2) < 5$, the diagram has to be changed. But the sub-diagram of (27) obtained by not showing $F_{\frac{1}{4}(p-(2a-1)^2)}^*$ and S_{t+a}^* exists on the surface $Y^0(p)/T$ for any $p > 17$.

We do not know whether the curves $F_{\frac{1}{4}(p-(2k-1)^2)}^*$ are non-singular and do not claim anything about their mutual intersection behaviour. The S_{t+k}^* are the image of S_{t+k} and $S_{-(t+k)}$. They are non-singular. The equation $S_{t+1}^* \cdot S_{t+1}^* = -1$ or equivalently $c_1^*[S_{t+1}^*] = 1$ follows from 5.1 (2). The curves S_{t+k}^* ($1 \leq k \leq a-1$) can be blown down successively. Then $F_{(p-(2k-1)^2)/4}^*$ gives in the resulting surface a curve for which the value of the first Chern class of the new surface on this curve is greater or equal to $c_1^*(p - (2k-1)^2)/4 + a - k$.

Proposition. Let p be a prime $\equiv 1 \pmod{4}$ (and $p > 17$). The non-singular model $Y^0(p)/T$ for the symmetric Hilbert modular group is rational if there exists a natural number k with $1 \leq k \leq a-1 = \left\lfloor \frac{\sqrt{p}-1}{2} \right\rfloor$ such that

$$c_1^*((p - (2k-1)^2)/4) + a - k \geq 2$$

This is a consequence of corollary I in 4.4. For the above proposition one does not need any assumption about the genus of F_N where $N = \frac{1}{4}(p - (2k-1)^2)$. However, we shall try to get through using the N listed in 5.7 for which the curves F_N are rational.

The tables in 5.7 give immediately

$$c_1^* \left(\frac{p-1}{4} \right) + a - 1 \geq 2$$

for $p = 29, 37, 41, 53, 61, 73, 97, 101, 109, 197$.

We find

$$c_1^* \left(\frac{p-9}{4} \right) + a - 2 \geq 2$$

for $p = 89, 137, 293$.

For $p = 173$ we have

$$c_1^* \left(\frac{p-81}{4} \right) + a - 5 = c_1^*(23) + 7 - 5 = 2$$

For the remaining 7 primes 113, 149, 157, 181, 229, 269, 317 we shall try to use the following lemma.

Lemma. We keep the notations of the preceding proposition. Suppose there exist two natural numbers k_1, k_2 with $1 \leq k_1 < k_2 \leq a-1$ such that

$$c_1^* \left(\frac{p - (2k_i - 1)^2}{4} \right) + a - k_i = 1 \quad \text{for } i = 1, 2$$

Then $Y^0(p)/T$ is rational.

Proof. Blowing down $S_{i+1}^*, \dots, S_{i+a-1}^*$ in $Y^0(p)/T$ gives a surface in which the images of $F_{N_i}^*$ ($N_i = (p - (2k_i - 1)^2)/4$, $i = 1, 2$) are exceptional curves or the surface is rational (4.4, Corollary II). If we have the two exceptional curves, then they intersect and the surface is rational by Corollary III in 4.4.

The assumptions of the lemma are true for $p = 113$ and $k_1 = 2$, $k_2 = 4$, for $p = 149$ and $k_1 = 4$, $k_2 = 5$, for $p = 157$ and $k_1 = 4$ and $k_2 = 5$, for $p = 181$ and $k_1 = 5$, $k_2 = 6$, for $p = 229$ and $k_1 = 6$, $k_2 = 7$, for $p = 317$ and $k_1 = 5$, $k_2 = 8$.

For $p = 269$ we have $a = 8$. The curve S_{t+8}^* has self-intersection number -3 . It intersects F_{11}^* , since $11 = \frac{269 - 15^2}{4}$. Either the surface is rational or F_{11}^* is exceptional. If F_{11}^* is exceptional, then we blow down $F_{11}^*, S_{t+1}^*, \dots, S_{t+8}^*$. The curve F_{47}^* ($k = 5$) gives in the resulting surface \tilde{Y} a curve \tilde{D} with $\tilde{c}_1 [D] \geq 2$ where \tilde{c}_1 is the first Chern class of \tilde{Y} .

We have proved the desired result.

THEOREM. *Let p be a prime $\equiv a \pmod{4}$. Let G_T be the symmetric Hilbert modular group for $K = \mathbf{Q}(\sqrt{p})$. Then the surface $\overline{\mathfrak{H}^2}/G_T$ is rational, (or equivalently the field of meromorphic automorphic functions with respect to G_T is a purely transcendental extension of \mathbf{C}), if and only if $p < 193$ or $p = 197, 229, 269, 293, 317$.*

5.9. *Example.* If the prime $p \equiv 1 \pmod{4}$ is of the form

$$p = (2a - 1)^2 + 4,$$

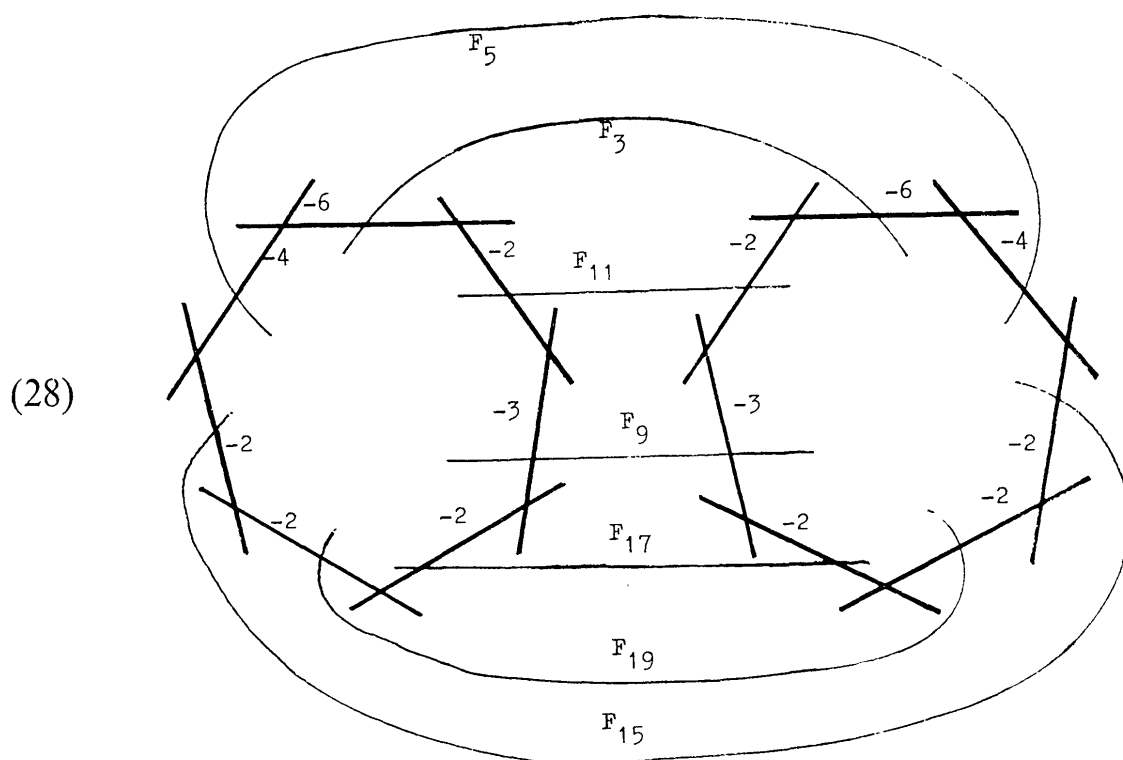
then
$$w_0 = \frac{2a + 1 + \sqrt{p}}{2} = [[\overline{2a + 1}, \underbrace{2, \dots, 2}_{2a-2}]]$$

and we have in diagram (13) that $S_{t+a} = S_{-(t+a)} = S_0$. Since $(p - (2a - 3)^2)/4 = 2a - 1$, the smallest admissible $N > 1$ which can be written in the form $x^2 N_k + xy M_k + y^2 N_{k-1}$ (with integers $x, y \geq 0$) equals $2a - 1$ (see 4.2 and 5.4 (12)). Any divisor d of $2a - 1$ is admissible. If d is a prime dividing $2a - 1$ and $1 < d < 2a - 1$, then the curve F_d has two cusps and does not pass through the cusp at ∞ of $\overline{\mathfrak{H}^2}/G$. Thus there must be other cusps of $\overline{\mathfrak{H}^2}/G$. We have proved

Proposition. *If $p = (2a - 1)^2 + 4$ (p prime) and if $2a - 1$ is not a prime, then $h(p) > 1$. (See [29], [51]).*

The first example is $p = 229 = 15^2 + 4$. We have $h(p) = 3$. The number 229 is the only one of our 24 primes in the preceding theorem with class number greater than one. (If $(2a - 1) \equiv \pm 2 \pmod{7}$, then 7 is admissible for p . Thus, also in this case $h(p) > 1$ provided $2a - 1 > 7$. Example: $p = 1373 = 37^2 + 4$, $h(p) = 3$.)

The cycles for the 2 cusps not at ∞ of $Y(229)$ look as follows



We also have drawn some curves. The curve F_{15} has $\mathfrak{H}/\Gamma_0(15)$ as non-singular model. This has 4 cusps corresponding to the fact that F_{15} also passes through the cusp at ∞ of $Y(p)$, namely through the curves S_1 and S_{-1} of this cusp. One can show that F_9 passes through S_0 of the cusp at ∞ in two points ($\mathfrak{H}/\Gamma_0(9)$ has 4 cusps).

If N is admissible and is a product of k different primes ($\neq p$), then $\mathfrak{H}/\Gamma_0(N)$ has 2^k cusps. The 2^k intersections of F_N with the resolved cusps in $Y(p)$ correspond to 2^k admissible ideals \mathfrak{b} with $N(\mathfrak{b}) = N$ (see 5.3).

In general, it is possible to give a complete description of the intersection of F_N with the resolved cusps of $Y(p)$. The corresponding theory can be developed for any Hilbert modular surface.

ADDED IN PROOF:

A. Selberg has informed me that he has proved the following result.

If Γ is a discrete irreducible subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ such that \mathfrak{H}^n/Γ has finite volume, but is not compact, then Γ is conjugate in $(\mathbf{PL}_2^+(\mathbf{R}))^n$ to a group commensurable with the Hilbert modular group of some totally real field K with $[K:\mathbf{Q}] = n$.

Thus Selberg's conjecture mentioned in the remark at the end of 1.5 is true. Actually, Selberg's results are more general. The proof has not been published yet. There is a sketch (still involving additional assumptions which could be eliminated later) in the Proceedings of the 15th Scandinavian

congress, Oslo 1968, Lecture Notes in Mathematics, Springer Verlag, vol. 118; in particular pp. 106-113.

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