41.

(mit K. Jänich)

Involutions and Singularities

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Heinrich Behnke zum 70. Geburtstag gewidmet.

1. Introduction. Let X be a compact oriented differentiable manifold without boundary of dimension $4k-1$ with $k \geqslant 1$. Let $T: X \to X$ be an orientation preserving fixed point free differentiable involution. In [7] an invariant $\alpha(X, T)$ was defined using a special case of the Atiyah-Bott-Singer fixed point theorem. If the disjoint union mX of m copies of X bounds a $4k$-dimensional compact oriented differentiable manifold N in such a way that T can be extended to an orientation preserving involution T_1 on N which may have fixed points, then

$$\alpha(X, T) = \frac{1}{m}\,(\tau(N, T_1) - \tau(\text{Fix } T_1 \circ \text{Fix } T_1)). \tag{1}$$

Here $\tau(N, T_1)$ is the signature of the quadratic form f_{T_1} defined over $H_{2k}(N, \mathbf{Q})$ by

$$f_{T_1}(x, y) = x \circ T_1 y$$

where "$\circ$" denotes the intersection number. $\tau(\text{Fix } T_1 \circ \text{Fix } T_1)$ is the signature of the "oriented self-intersection cobordism class" $\text{Fix } T_1 \circ \text{Fix } T_1$. According to Burdick [4] there exist N and T_1 with $m = 2$.

In § 2 we shall study a compact oriented manifold $\mathscr{D}$ whose boundary is $X - 2(X/T)$. This manifold $\mathscr{D}$ was first constructed by Dold [5]; we give a different description of it. Namely, $\mathscr{D}$ is a branched covering of degree 2 of $(X/T) \times I$, where I is the unit interval. The covering transformation is an orientation preserving involution T_1 of $\mathscr{D}$ which restricted to the boundary is T on X and the trivial involution on $2(X/T)$, and $\text{Fix } T_1$ is the branching locus.

We show that

$$\alpha(X, T) = \tau(\mathcal{D}, T_1) = -\tau(\mathcal{D}),$$

where $\tau(\mathcal{D})$ is the signature of the $4k$-dimensional manifold $\mathcal{D}$. Thus $\alpha(X, T)$ is always an integer. The construction of $\mathcal{D}$ is closely related to Burdick's result on the oriented bordism group of $B_{\mathbf{Z}_2}$ and can in fact be used to prove it.

In [7] it was claimed that if X^{4k-1} is an integral homology sphere then $\tau(\mathcal{D}) = \pm\beta(X,T)$, where $\beta(X, T)$ is the Browder-Livesay invariant [3]. The proof was not carried through. It turns out that the definition of Browder-Livesay is also meaningful without assumptions on the homology of X. In §3 we shall prove

$$\beta(X, T) = -\tau(\mathcal{D}). \tag{3}$$

By (2), we obtain

$$\alpha(X, T) = \beta(X, T). \tag{4}$$

Looking at $\mathcal{D}$ as a branched covering of $(X/T)\times I$ has thus simplified considerably the proof of (4) envisaged in [7].

If $a = (a_0, a_1, \ldots, a_{2k}) \in \mathbf{Z}^{2k+1}$ with $a_j \geqslant 2$, then the affine algebraic variety

$$z_0^{a_0} + z_1^{a_1} + \ldots + z_{2k}^{a_{2k}} = 0 \tag{5}$$

has an isolated singularity at the origin whose "neighborhood boundary" is the Brieskorn manifold [1]

$$\Sigma_a^{4k-1} \subset C^{2k+1}$$

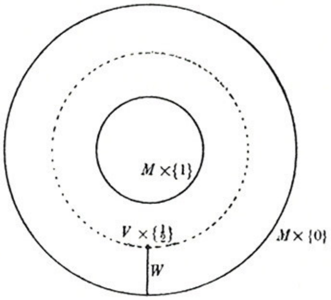
given by the equation (5) and

$$\sum_{i=0}^{2k} z_i \bar{z}_i = 1. \tag{6}$$

If all the a_j are odd, then $Tz = -z$ induces an orientation preserving fixed point free involution T_a on Σ_a. The calculation of $\alpha(\Sigma_a, T_a)$ is an open problem (compare [7]). This problem on isolated singularities is the justification for presenting our paper to a colloquium on algebraic geometry. In §4 we give the recipe for calculating $\alpha(\Sigma_a, T_a)$ for $k=1$ in the case where the exponents a_0, a_1, a_2 are pairwise prime and odd.

2. The Dold construction. Let Y be a compact differentiable manifold without boundary and W a 1-codimensional compact submanifold with boundary ∂W. Then, as it is well known, one can construct a double covering of Y, branched at ∂W, by taking two copies of Y, "cutting" them along W and then identifying each boundary point of the cut in copy one with its opposite point in copy two. The same can be done if Y is a manifold with boundary and W intersects ∂Y transversally in a union of connected components of ∂W. The covering will then be branched at $\partial W - \partial W \cap \partial Y$.

We are interested in a very special case of this general situation. Let M be a compact differentiable manifold without boundary and V a closed submanifold without boundary of codimension 1 in M. Then we define $Y = M \times [0, 1]$ and $W = V \times [0, \frac{1}{2}]$.



For the following we will need a detailed description of the double covering corresponding to $(M \times [0,1], V \times [0, \frac{1}{2}])$. The normal bundle of V in M defines a $\mathbf{Z}_2$ -principal bundle $\widetilde{V}$ over V. If we "cut" M along V, we obtain a compact differentiable manifold C with boundary $\partial C = \widetilde{V}$. As a set, C is the disjoint union of $M - V$ and $\widetilde{V}$, and there is an obvious canonical way to introduce topology and differentiable structure in $(M - V) \cup \widetilde{V}$. Similarly, let C' be the disjoint union of $M \times [0, 1] - V \times [0, \frac{1}{2})$ and $\widetilde{V} \times [0, \frac{1}{2})$, topologized in the canonical way. Then we consider two copies C_1' and C_2' of C' and identify in their disjoint union each $x \in V \times \{\frac{1}{2}\} \subset C_1'$ with the corresponding point $x \in V \times \{\frac{1}{2}\} \subset C_2'$ and for $0 \leqslant t < \frac{1}{2}$ each point $v \in \widetilde{V} \times \{t\} \subset C_1'$ with the opposite point $-v \in \widetilde{V} \times \{t\} \subset C_2'$. Let $\mathscr{D}$

denote the resulting topological space and $\pi: \mathscr{D} \to M \times [0, 1]$ the projection. Then C_1', C_2' and $V \times \{\frac{1}{2}\}$ are subspaces, and $\mathscr{D} - V \times \{\frac{1}{2}\}$ has a canonical structure as a differentiable manifold with boundary.

To introduce a differentiable structure on all of $\mathscr{D}$, we use a tubular neighbourhood of V in M. This may be given as a diffeomorphism

$$\kappa: \tilde{V} \times_{Z_2} D^1 \to M$$

onto a closed neighbourhood of V in M, such that the restriction of κ to $\tilde{V} \times_{Z_2} \{0\} = V$ is the inclusion $V \subset M$. Let Z_2 act on $D^2 \subset \mathbf{C}$ by complex conjugation. Then we get a tubular neighbourhood of $V \times \{\frac{1}{2}\}$ in $M \times [0, 1]$

$$\begin{gathered} \lambda: \tilde{V} \times_{Z_2} D^2 \longrightarrow M \times [0, 1] \text{ by} \\ [v, x+iy] \longmapsto (\kappa[v, y], \tfrac{1}{2} + \tfrac{1}{4}x). \end{gathered} \tag{1}$$

Let the "projection" $p: \tilde{V} \times_{Z_2} D^2 \to \tilde{V} \times_{Z_2} D^2$ be given on each fibre by $z \to z^2/|z|$. Then λp can be lifted to $\mathscr{D}$, which means that we can choose a map $\lambda_1: \tilde{V} \times_{Z_2} D^2 \to \mathscr{D}$ such that

$$\begin{array}{ccc} \tilde{V} \times_{Z_2} D^2 & \xrightarrow{\lambda_1} & \mathscr{D} \\ \downarrow p & & \downarrow \pi \\ \tilde{V} \times_{Z_2} D^2 & \xrightarrow{\lambda} & M \times [0, 1] \end{array} \tag{2}$$

is commutative. Then there is exactly one differentiable structure on $\mathscr{D}$ for which λ_1 is a diffeomorphism onto a neighborhood of $V \times \{\frac{1}{2}\}$ in $\mathscr{D}$ and which coincides on $\mathscr{D} - V \times \{\frac{1}{2}\}$ with the canonical structure. Up to diffeomorphism, of course, this structure does not depend on κ.

$\mathscr{D}$, then, is a double covering of $M \times [0, 1]$, branched at $V \times \{\frac{1}{2}\}$. The covering transformation on $\mathscr{D}$ shall be denoted by T_1. Note that on $\tilde{V} \times_{Z_2} D^2$ (identified by λ_1 with a subset of $\mathscr{D}$) the transformation T_1 is given by $[v, z] \to [v, -z]$.

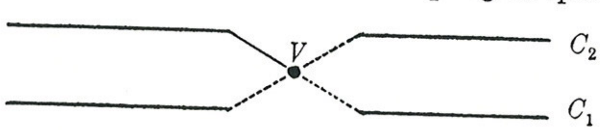
As a differentiable manifold, $\mathscr{D}$ is the same as the manifold constructed by Dold in his note [5].

Now consider once more the differentiable manifold C with boundary $\partial C = \widetilde{V}$, which we obtained from M by cutting along V. Let $C_1 \cup C_2$ be the disjoint union of two copies of C. If we identify $x \in \widetilde{V}_1 \subset C_1$ with $-x \in \widetilde{V}_1 \subset C_1$ and $x \in \widetilde{V}_2$ with $-x \in \widetilde{V}_2$, we obtain from $C_1 \cup C_2$ the disjoint union of two copies of M :

If we identify $x \in \widetilde{V}_1 \subset C_1$ with $-x \in \widetilde{V}_2 \subset C_2$, we get a differentiable manifold which we denote by $\widetilde{M}$:

If we identify $x \in \widetilde{V}_1 \subset C_1$ with $x \in \widetilde{V}_2 \subset C_2$, then $C_1 \cup C_2$ becomes a closed manifold B (the usual "double" of C), and we use κ to introduce the differentiable structure on B :

If we, finally, identify for each $x \in \widetilde{V}$ all four points $x \in \widetilde{V}_1$, $-x \in \widetilde{V}_1$, $x \in \widetilde{V}_2$, $-x \in \widetilde{V}_2$ to one, then we obtain a topological space A :



(3)

Now obviously we have $2M = \pi^{-1}(M \times \{1\})$, $A = \pi^{-1}(M \times \{\frac{1}{2}\})$ and $\widetilde{M} = \pi^{-1}(M \times \{0\})$, and by our choice of the differentiable structures of B and $\mathscr{D}$ (p in (2) is given by $z \to z^2/|z|$ instead of $z \to z^2$), the canonical map $B \to A$ defines an *immersion*

$$f : B \to \mathscr{D}.$$

It should be mentioned, perhaps, that for the same reason π: $\mathscr{D} \to M \times [0, 1]$ is not differentiable at $V \times \{\frac{1}{2}\}$.

Up to this point, we did not make any orientability assumptions. Considering now the "orientable case", we shall use the following convention : for orientable manifolds with boundary, we will always choose the orientations of the manifold and its boundary in such a way, that the orientation of the boundary, followed by the inwards pointing normal vector, gives the orientation of the manifold.

Now if X is any compact differentiable manifold without boundary and T a fixed point free involution on X with $X/T \cong M$, then (X, T) is equivariantly diffeomorphic to $(\widetilde{M}, T_1|\widetilde{M})$ for a suitably chosen $V \subset M$, and in fact our $(\widetilde{M}, T_1|\widetilde{M})$ plays the role of the (X, T) in §1. Therefore we will assume from now on, that $\widetilde{M}$ is *oriented* and $T_1|\widetilde{M}$ is *orientation preserving*. Let us also write T for $T_1|\widetilde{M}$.

Then the orientation of $\widetilde{M}$ defines an orientation of M and hence of C, and since $\widetilde{V} = \partial C$, an orientation of $\widetilde{V}$ is thus determined. Furthermore, the orientation of $\widetilde{M} \subset \partial\mathscr{D}$ induces an orientation of $\mathscr{D}$, relative to which

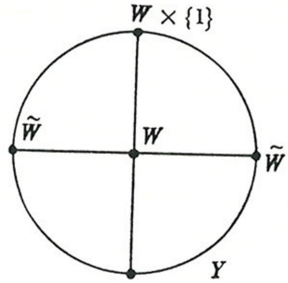
$$\partial\mathscr{D} = \widetilde{M} - 2M. \tag{4}$$

Clearly T_1 on $\mathscr{D}$ is orientation preserving. V may not be orientable, and $\widetilde{V} \to V$ is the orientation covering of V, because M is orientable.

The relation of the construction of $\mathscr{D}$ to the result of Burdick is the following. Let $\Omega_*(\mathbf{Z}_2)$ denote the bordism group of oriented manifolds with fixed point free orientation preserving involutions. Then we have homomorphisms

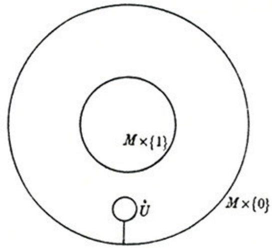
$$\Omega_n \oplus \mathfrak{N}_{n-1} \underset{j}{\overset{i}{\rightleftarrows}} \Omega_n(\mathbf{Z}_2)$$

as follows: if $[M] \in \Omega_n$ is represented by an oriented n-dimensional manifold M, the $i[M] \in \Omega_n(\mathbf{Z}_2)$ is simply given by $2M$ with the trivial involution. Now let $[W]_2 \in \mathfrak{N}_{n-1}$ be represented by an $(n-1)$-dimensional manifold W, let $\widetilde{W} \to W$ denote the orientation covering, and let Y be the sphere bundle of the Whitney sum of the real line bundle over W associated to $\widetilde{W}$ and the trivial line bundle $W \times \mathbf{R}$:



The manifold Y is orientable, and we may orient Y at $\widetilde{W}$ by the canonical orientation of $\widetilde{W}$ followed by the normal vector pointing toward $W \times \{1\}$. Then we denote by $i(W)$ the oriented double covering of Y corresponding to $(Y, W \times \{1\})$, and we define $i[W] \in \Omega_n(\mathbf{Z}_2)$ to be represented by $i(W)$.

As already mentioned, any element of $\Omega_n(\mathbf{Z}_2)$ can be represented by the (unbranched) double covering $\widetilde{M}$ corresponding to some (M, V), and we define $j[\widetilde{M}, T] = [M] \oplus [V]_2$. Then i and j are well defined homomorphisms and clearly $j \circ i = \mathrm{Id}$, so i is injective. To show that i is also surjective, we have to construct for given (M, V) an $(n+1)$-dimensional oriented manifold $\mathscr{B}$ with boundary and with an orientation preserving fixed point free involution, such that equivariantly $\partial\mathscr{B} = \widetilde{M} - 2M - i(V)$. But such a manifold is given by $\mathscr{B} = \pi^{-1}(M \times [0, 1] - U)$,



where U is the interior of the tubular neighborhood (1) of $V \times \{\frac{1}{2}\}$ in $M \times [0, 1]$:

$$\partial\mathscr{B} = \pi^{-1}(M \times \{0\}) \cup \pi^{-1}(M \times \{1\}) \cup \pi^{-1}(\dot{U})$$
$$= \widetilde{M} - 2M - i(V).$$

Thus $i\colon \Omega_n \oplus \mathfrak{N}_{n-1} \to \Omega_n(\mathbf{Z}_2)$ is an isomorphism. Burdick uses in [4] essentially the same manifold $\mathscr{B}$ to prove the surjectivity of i.

We will now consider the invariant α and therefore assume that $\dim \widetilde{M} = 4k - 1$ with $k \geqslant 1$. First we remark, that for the trivial involution T on $2M$ the invariant α vanishes: since the nontrivial elements of Ω_{4k-1} are all of order two, there is an oriented X with $\partial X = 2M$. Let T' be the trivial involution on $2X$. Then $2\alpha(2M, T) = \tau(2X, T') = 0$, because it is the signature of a quadratic form which can be given by a matrix of the form

$$\begin{array}{|c|c|} \hline O & E \\ \hline E & O \\ \hline \end{array}$$

where E is a symmetric matrix. Hence it follows, that $\alpha(\partial\mathscr{D}, T_1|\partial\mathscr{D}) = \alpha(\widetilde{M}, T)$ and therefore by (1) of §1 we have

$$\alpha(\widetilde{M}, T) = \tau(\mathscr{D}, T_1) - \tau(\mathrm{Fix}\, T_1 \circ \mathrm{Fix}\, T_1). \tag{5}$$

Notice that here we apply the definition (1) of §1 of α in a case, where Fix T_1 is not necessarily orientable, so that we have to use the Atiyah-Bott-Singer fixed point theorem also for the case of non-orientable fixed point sets.

PROPOSITION. $\alpha(\widetilde{M}, T) = \tau(\mathscr{D}, T_1) = -\tau(\mathscr{D})$.

PROOF. $\mathrm{Fix}\, T_1 \circ \mathrm{Fix}\, T_1 = 0 \in \Omega_*$, since the normal bundle of $\mathrm{Fix}\, T_1 = V \times \{\frac{1}{2}\}$ in $\mathscr{D}$ has a one-dimensional trivial subbundle. Therefore by (5), $\alpha(\widetilde{M}, T) = \tau(\mathscr{D}, T_1)$. To show that $\tau(\mathscr{D}, T_1) = -\tau(\mathscr{D})$, let again U denote our open tubular neighbourhood of $V \times \{\frac{1}{2}\}$ in $M \times [0, 1]$, $\bar{U}$ its closure in $M \times [0, 1]$ and correspondingly $U_1 = \pi^{-1}(U)$, $\bar{U}_1 = \pi^{-1}(\bar{U})$. Then $\tau(\bar{U}) = \tau(\bar{U}_1) = \tau(\bar{U}_1, T_1) = 0$, because $\bar{U}$ and $\bar{U}_1$ are disc bundles of vector bundles with a trivial summand and hence the zero section, which carries all the homology, can be deformed into a section which is everywhere different from zero.

Because of the additivity of the signature (compare (8) of [7]), we therefore have

$$\tau(\mathscr{D}, T_1) = \tau(\mathscr{D} - U_1, T_1).$$

But T_1 is fixed point free on $\mathscr{D} - U_1$, and hence we can apply formula (7) of [7], which is easy to prove and which relates the signature $\tau(M^{4k}, T)$ of a fixed point free involution with the signatures of M^{4k} and M^{4k}/T and we obtain

$$\begin{aligned}\tau(\mathscr{D}, T_1) = \tau(\mathscr{D} - U_1, T_1) &= 2\tau(M \times [0, 1] - U) - \tau(\mathscr{D} - U_1) \\ &= 2\tau(M \times [0, 1]) - \tau(\mathscr{D}).\end{aligned}$$

3. The Browder-Livesay invariant. The involution on $\tilde{V}$ which is given by $x \to -x$ shall also be denoted by T, because it is the restriction of T on $\tilde{M}$ to $\tilde{V}$, if we regard $\tilde{V}$ via $\tilde{V}_1 \subset C_1 \subset \tilde{M}$ as a submanifold of $\tilde{M}$. T is orientation reversing on $\tilde{V}$, and since the intersection form $(x, y) \to x \circ y$ on $H_{2k-1}(\tilde{V}, \mathbf{Q})$ is skew-symmetric, the quadratic form $(x, y) \to x \circ Ty$ is symmetric on $H_{2k-1}(\tilde{V}, \mathbf{Q})$. Now we restrict this form to

$$L = \text{kernel of } H_{2k-1}(\tilde{V}, \mathbf{Q}) \to H_{2k-1}(C, \mathbf{Q}), \tag{1}$$

where the homomorphism is induced by the inclusion $\tilde{V} = \partial C \subset C$, and we denote by $\beta(\tilde{M}, \tilde{V}, T)$ the signature of this quadratic form on L. (If $\tilde{M} = \Sigma^{4k-1}$ is a homotopy sphere, then $\beta(\tilde{M}, \tilde{V}, T)$ is by definition the *Browder-Livesay invariant* [3] $\sigma(\Sigma^{4k-1}, T)$ of the involution T on Σ^{4k-1}.)

THEOREM. $\alpha(\tilde{M}, T) = \beta(\tilde{M}, \tilde{V}, T)$.

Thus $\beta(\tilde{M}, T) = \beta(\tilde{M}, \tilde{V}, T)$ *is a well defined invariant of the oriented equivariant diffeomorphism class of* $(\tilde{M}, T)$.

PROOF OF THE THEOREM. First notice, that the canonical deformation retraction of $M \times [0, 1]$ to $M \times \{\frac{1}{2}\}$ induces a deformation retraction of $\mathscr{D} = \pi^{-1}(M \times [0, 1])$ to $A = \pi^{-1}(M \times \{\frac{1}{2}\})$. To study $H_{2k}(A, Q)$, we consider the following part of a Mayer-Vietoris sequence for A (all homology with coefficients in Q):

$$H_{2k}(V) \oplus H_{2k}(C_1 \cup C_2) \xrightarrow{\phi} H_{2k}(A) \xrightarrow{\chi}$$

$$H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) \xrightarrow{\psi} H_{2k-1}(V) \oplus H_{2k-1}(C_1 \cup C_2)$$

where $\widetilde{V}_1 \cup \widetilde{V}_2$ and $C_1 \cup C_2$ denote the disjoint unions, see figure (3) of §2.

In $H_{2k}(A)$ we have to consider the quadratic forms given by $(x, y) \to x \circ y$ and $(x, y) \to x \circ Ty$, where $\circ$ denotes the intersection number in $\mathscr{D}$. Now, the maps $V = V \times \{\frac{1}{2}\} \subset A$ and $C_1 \cup C_2 \to A$, which induce the homomorphism ϕ, are homotopic in $\mathscr{D}$ to maps into $\mathscr{D} - A$. Therefore if $x \in \operatorname{Im} \phi \subset H_{2k}(A)$ and y is any element of $H_{2k}(A)$, then $x \circ y = 0$. Thus if we denote

$$L' = H_{2k}(A)/\operatorname{Im} \phi, \tag{2}$$

then the quadratic forms $(x, y) \to x \circ y$ and $(x, y) \to x \circ Ty$ are well defined on L', and their signatures as forms on L' are $\tau(\mathscr{D})$ and $\tau(\mathscr{D}, T_1)$ respectively.

L' is isomorphic to the kernel of ψ, and hence we shall now take a closer look at ker ψ. For this purpose we consider the Mayer-Vietoris sequences for $\widetilde{M}$ and B:

$$H_{2k}(\widetilde{M}) \xrightarrow{\widetilde{\chi}} H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) \xrightarrow{\widetilde{\psi}} H_{2k-1}(\widetilde{V}) \oplus H_{2k-1}(C_1 \cup C_2)$$

$$H_{2k}(B) \xrightarrow{\chi^B} H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) \xrightarrow{\psi^B} H_{2k-1}(\widetilde{V}) \oplus H_{2k-1}(C_1 \cup C_2).$$

In the sequence for $\widetilde{M}$, the homomorphism $H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) \longrightarrow H_{2k-1}(\widetilde{V})$ is induced by the identity $\widetilde{V}_1 \to \widetilde{V}$ on $\widetilde{V}_1$ and by the involution $T : \widetilde{V}_2 \to \widetilde{V}$ on $\widetilde{V}_2$, in the sequence for B however by the identity on both components. If we write $H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2)$ as $H_{2k-1}(\widetilde{V}) \oplus H_{2k-1}(\widetilde{V})$, the kernel of $H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) \to H_{2k-1}(C_1 \cup C_2)$ is $L \oplus L$, and so we get:

$$\ker \widetilde{\psi} = \{(a,b) \in L \oplus L \mid a + Tb = 0\},$$

$$\ker \psi^B = \{(a,b) \in L \oplus L \mid a + b = 0\}.$$

Let a be an element of $H_*(\widetilde{V}, \mathbf{Q})$. Then $a + Ta$ vanishes if and only if a is in the kernel of $H_*(\widetilde{V}, \mathbf{Q}) \to H_*(V, \mathbf{Q})$. Thus the kernel of ψ is

$$\ker \psi = \{(a,b) \in L \oplus L \mid a + Ta + b + Tb = 0\}.$$

$\ker \psi^B$ and $\ker \widetilde{\psi}$ are subspaces of $\ker \psi$, and if we write (a,b) as $\left(\frac{a-b}{2}, \frac{b-a}{2}\right) + \left(\frac{a+b}{2}, \frac{a+b}{2}\right)$, we see that in fact

$$\ker \psi = \ker \psi^B + \ker \widetilde{\psi}. \tag{3}$$

By the isomorphism $L' \cong \ker \psi$, which is induced by χ, (3) becomes

$$L' = L^B + \widetilde{L},$$

where L^B denotes the subspace of L' corresponding to $\ker \psi^B$ under this isomorphism, and $\widetilde{L}$ corresponds to $\ker \widetilde{\psi}$.

Let us first consider $\widetilde{L}$. Any element in $\widetilde{L}$ can be represented by an element $\widetilde{f}_*(x)$, where $x \in H_{2k}(\widetilde{M})$ and $\widetilde{f}\colon \widetilde{M} \to A$ is the canonical map:

$$\begin{array}{ccccc}
H_{2k}(\widetilde{M}) & \xrightarrow{\ \widetilde{\chi}\ } & H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) & \xrightarrow{\ \widetilde{\psi}\ } & H_{2k-1}(\widetilde{V}) \oplus H_{2k-1}(C_1 \cup C_2) \\
\downarrow{\scriptstyle \widetilde{f}_*} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow \\
H_{2k}(A) & \xrightarrow{\ \chi\ } & H_{2k-1}(\widetilde{V}_1 \cup \widetilde{V}_2) & \xrightarrow{\ \psi\ } & H_{2k-1}(V) \oplus H_{2k-1}(C_1 \cup C_2).
\end{array}$$

But $\widetilde{f}$ is homotopic in $\mathscr{D}$ to the inclusion $\widetilde{M} = \pi^{-1}(M \times \{0\}) \subset \mathscr{D}$, hence we have $L^B \circ L' = 0$. Therefore the quadratic forms on L' given by $(x,y) \to x \circ y$ and $(x,y) \to x \circ Ty$ can be restricted to L^B and their signatures will still be $\tau(\mathscr{D})$ and $\tau(\mathscr{D}, T_1)$.

Now, any element in $L^B \subset H_{2k}(A)/\mathrm{Im}\ \phi$ can be represented by an element $f^B_*(x)$, where $x \in H_{2k}(B)$ and $f^B\colon B \to A$ is the canonical map. Furthermore, χ induces an isomorphism between L^B and the "Browder-Livesay Module" L, because

$$L^B \xrightarrow{\cong} \ker \psi^B = \{(a, -a) \mid a \in L\} \cong L.$$

Hence in view of the proposition in §2, our theorem would be proved if we can show that the following lemma is true.

LEMMA. *Let* $x, y \in H_{2k}(B)$ *and* $\bar{x} = f_*(x), \bar{y} = f_*(y)$ *the corresponding elements under the homomorphism* $f_*: H_{2k}(B) \to H_{2k}(\mathscr{D})$ *induced by the canonical map* $f: B \to A \subset \mathscr{D}$. *By* (3), *we have* $\chi^B(x) = \chi(\bar{x}) = (a, -a)$ *and* $\chi^B(y) = \chi(\bar{y}) = (b, -b)$ *for some* $a, b \in L$. *We claim:*

$$-\bar{x} \circ \bar{y} = a \circ Tb \tag{4}$$

PROOF OF THE LEMMA. First we note that we can make some simplifying assumptions on x and y. By a theorem of Thom ([9], p. 55), up to multiplication by an integer $\neq 0$, any integral homology class of a differentiable manifold can be realized by an oriented submanifold, and hence we may assume that x and y are given by oriented $2k$-dimensional submanifolds of B, which we will again denote by x and y. Of course x and y may be assumed to be transversal at $\tilde{V} \subset B$. Then $\tilde{V} \cap x$ and $\tilde{V} \cap y$ are differentiable $(2k-1)$-dimensional orientable submanifolds of $\tilde{V}$. We orient $\tilde{V} \cap x$ (and similarly $\tilde{V} \cap y$) as the boundary of the oriented manifold $C_1 \cap x$. Then $\tilde{V} \cap x$ and $\tilde{V} \cap y$ represent a and b in $H_{2k-1}(\tilde{V})$,† and we shall now denote $\tilde{V} \cap x$ by a and $\tilde{V} \cap y$ by b.

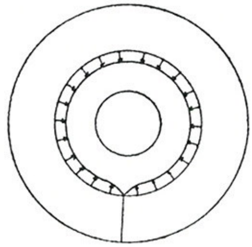
Since in a neighborhood of $\tilde{V}$, B is simply $\tilde{V} \times \mathbf{R}$ and x is $a \times \mathbf{R}$, it is clear that any isotopy of a in $\tilde{V}$ can be extended to an isotopy of x in B which is the identity outside a given neighborhood of $\tilde{V}$ in B, such that x remains transversal to $\tilde{V}$ during the isotopy. Therefore we may assume that the submanifold a of $\tilde{V}$ is transversal to b and Tb.

These are all the preparations we have to make in B. Now let us immerse B into $\mathscr{D}$ and thus get immersions of x and y into $\mathscr{D}$ which will represent $\bar{x}$ and $\bar{y} \in H_{2k}(\mathscr{D})$. To obtain transversality of these immersions however, we immerse x into $\mathscr{D}$ by the standard

†Or $-a$ and $-b$, but we may replace χ and χ^B in the Mayer-Vietoris sequences for A and B by $-\chi$ and $-\chi^B$, so let us assume that they represent a and b.

immersion $f: B \to A \subset \mathscr{D}$ and y by an immersion $f': B \to \mathscr{D}$, which is different, but isotopic to f.

To define f', let $0 < \epsilon < \frac{1}{4}$ and choose a real-valued C^∞ - function h on the interval $[0, 1]$ with $h(t) = t$ for $t < \frac{1}{2}\epsilon$, $h(t) = \epsilon$ for $t > \frac{1}{2}$ and $0 < h(t) \leqslant \epsilon$ for all other t. Using $\kappa: \tilde{V} \times_{Z_2} D^1 \to M$, we get a function on $\kappa(\tilde{V} \times_{Z_2} D^1) \subset M$ by $[v, t] \to h(|t|)$, which we now extend to a function $\bar{h}$ on M by defining $\bar{h}(p) = \epsilon$ for all $p \notin \kappa(\tilde{V} \times_{Z_2} D^1)$. Then $M \to M \times [0, 1]$, given by $p \to (p, \frac{1}{2} + \bar{h}(p))$



is obviously covered by an immersion $f': B \to \mathscr{D}$ which is isotopic to f.

Then in fact the immersions $f: x \to \mathscr{D}$ and $f': y \to \mathscr{D}$ are transversal to each other, and for $p \in x$, $q \in y$ we have

$$f(p) = f'(q) \Longleftrightarrow p = q \in a \cap b \text{ or } p = Tq \in a \cap Tb.$$

Looking now very carefully at all orientations involved, we obtain

$$-\bar{x} \circ \bar{y} = a \circ Tb + a \circ b. \tag{5}$$

Recall that $\tilde{V}$ is the boundary ∂C of the oriented manifold C and that a and b are in the kernel of $H_{2k-1}(\partial C) \to H_{2k-1}(C)$. Then the intersection *homology class* $s(a,b) \in H_0(\partial C)$ is in the kernel of $H_0(\partial C) \to H_0(C)$ (see Thom [8], Corollaire V.6, p. 173), and therefore the intersection *number* $a \circ b$ is zero, hence (5) becomes $-\bar{x} \circ \bar{y} = a \circ Tb$, and the lemma is proved.

4. Resolution of some singularities. For a tripel $a = (a_0, a_1, a_2)$ of pairwise prime integers with $a_j \geqslant 2$ consider the variety $V_a \subset \mathbf{C}^3$ given by

$$z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0. \qquad (1)$$

The origin is the only singularity of V_a. We shall describe a resolution of this singularity.

THEOREM. *There exist a complex surface (complex manifold of complex dimension* 2) *and a proper holomorphic map*

$$\phi: M_a \to V_a$$

such that the following is true:

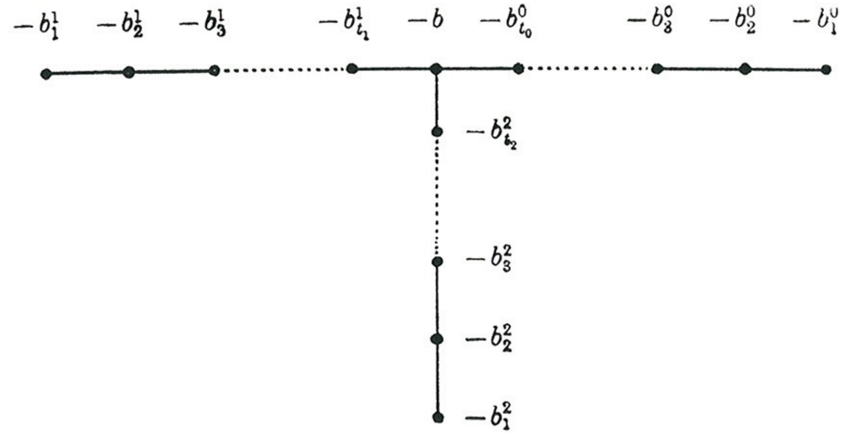
(i) $\phi: M_a - \phi^{-1}(0) \to V_a - \{0\}$ *is biholomorphic.*

(ii) $\phi^{-1}(0)$ *is a union of finitely many rational curves which are non-singularly imbedded in* M_a.

(iii) *The intersection of three of these curves is always empty. Two of these curves do not intersect or intersect transversally in exactly one point.*

(iv) *We introduce a finite graph* $\mathfrak{g}_a$ *in which the vertices correspond to the curves and in which two vertices are joined by an edge if and only if the corresponding curves intersect.* $\mathfrak{g}_a$ *is star-shaped with three rays.*

(v) *The graph* $\mathfrak{g}_a$ *will be weighted by attaching to each vertex the self-intersection number of the corresponding curve. This number is always negative. Thus* $\mathfrak{g}_a$ *looks as follows.*



(vi) $b = 1$ *or* $b = 2$; $b_i^j \geqslant 2$. *Let* q_0 *be determined by*

$$0 < q_0 < a_0 \text{ and } q_0 \equiv -a_1 a_2 \bmod a_0$$

and define q_1, q_2 *correspondingly. Let* q_j' *be given by*

$$0 < q_j' < a_j \text{ and } q_j\, q_j' \equiv 1 \bmod a_j.$$

Then the numbers b_i^j *in the graph* $\mathfrak{g}_a$ *are obtained from the continued fractions*

$$\frac{a_j}{q_j} = b_1^j - \cfrac{1}{b_2^j - \cfrac{1}{b_3^j - \cfrac{\cdots}{\cdots - \cfrac{1}{b_{t_j}^j}}}}.$$

(vii) *If the exponents* a_0, a_1, a_2 *are all odd, then*

$$b = 1 \Leftrightarrow q_0' + q_1' + q_2' \equiv 0 \bmod 2,$$

$$b = 2 \Leftrightarrow q_0' + q_1' + q_2' \equiv 1 \bmod 2.$$

Before proving (i)–(vii) we study as an example

$$z_0^3 + z_1^{6j-1} + z_2^{18j-1} = 0. \tag{2}$$

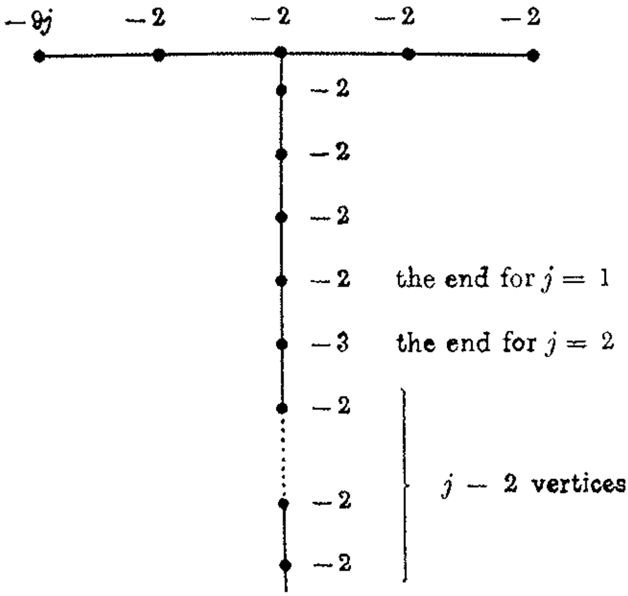
We have $q_0 = q_0' = 2$

$q_1 = 4$ for $j = 1$ and $q_1 = 6j - 7$ for $j \geqslant 2$

$q_1' = 5j - 1$

$q_2 = 2,\ q_2' = 9j.$

By (vii) we get $b = 2$. The continued fractions for $\frac{3}{2}, \frac{5}{4}$ resp. $\frac{6j-1}{6j-7}, \frac{18j-1}{2}$ lead then to the graph



(3)

PROOF OF THE PRECEDING THEOREM. We use the methods of [6]. The algebroid function

$$f = (-x_1^{a_1a_2} - x_2^{a_1a_2})^{1/a_0}$$

defines a branched covering $V_a^{(1)}$ of $\mathbf{C}^2$ (coordinates x_1, x_2 in $\mathbf{C}^2$). Blowing up the origin of $\mathbf{C}^2$ (compare [6], §1.3) gives a complex surface W with a non-singular rational curve $K \subset W$ of self-intersection number -1 and an algebroid function $\tilde{f}$ on W branched along K and along a_1a_2 lines which intersect K in the a_1a_2 points of K satisfying

$$-x_1^{a_1a_2} - x_2^{a_1a_2} = 0 \tag{4}$$

where x_1, x_2 are now regarded as homogeneous coordinates of K. The algebroid function $\tilde{f}$ defines a complex space $V_a^{(2)}$ lying branched over W with a_1a_2 singular points lying over the points of K defined by (4). In a neighborhood of such a point we have

$$\tilde{f} = (\zeta_1 \zeta_2^{a_1a_2})^{1/a_0} \tag{5}$$

where $\zeta_2 = 0$ is a suitable local equation for K and $\zeta_1 = 0$ for the line passing through the point and along which $V_a^{(2)}$ is branched over

W. The singularity of type (5) can be resolved according to [6], §3.4, where

$$w = (z_1 z_2^{n-q})^{1/n}, \; (0 < q < n, \; (q, n) = 1) \tag{6}$$

was studied. In our case, we have

$$n = a_0 \text{ and } q = q_0, \text{ see (vi) above,}$$

for all the $a_1 a_2$ singular points of $V_a^{(2)}$. The resolution gives a complex surface $V_a^{(3)}$ with the following property. The singularity of $V_a^{(1)}$ was blown up in a system of rational curves satisfying (iii) and represented by a star-shaped graph with $a_1 a_2$ rays of the same kind. The following diagram shows only one ray where the unweighted vertex represents the central curve $\tilde{K}$ which under the natural projection $V_a^{(3)} \to W$ has K as bijective image

(7)

$V_a^{(1)}$ is of course just the affine variety

$$x_0^{a_0} + x_1^{a_1 a_2} + x_2^{a_1 a_2} = 0$$

which can be mapped onto V_a (see (1)) by

$$(x_0, x_1, x_2) \to (z_0, z_1, z_2) = (x_0, x_1^{a_2}, x_2^{a_1}).$$

Denote by G the finite group of linear transformations

$$(x_1, x_2) \to (\epsilon_2 x_1, \epsilon_1 x_2) \text{ with } \epsilon_2^{a_2} = \epsilon_1^{a_1} = 1. \tag{8}$$

$$V_a = V_a^{(1)}/G.$$

Then the group G operates also on $V_a^{(3)}$. There are two fixed points, namely the points $0 = (0, 1)$ and $\infty = (1, 0)$ of $\tilde{K} = K$ (with respect to the homogeneous coordinates x_1, x_2 on K). The $a_1 a_2$ points of $\tilde{K}$ in which the curves with self-intersection number $-b_{t_0}^0$ of the $a_1 a_2$ rays intersect $\tilde{K}$ are an orbit under G. The $a_1 a_2$ rays are all identified in $V_a^{(3)}/G$. Thus $V_a^{(3)}/G$ is a complex space with two singular points P_0, P_∞ corresponding to the fixed points. $V_a^{(3)}/G$ is thus obtained from V_a by blowing up the singular point in a system of $t_0 + 1$ rational curves showing the following intersection behaviour:

(9)

but where the vertex without weight represents a rational curve passing through the singular points P_0, P_∞.

We must find the representation of G in the tangent spaces of the fixed points $0 = (0, 1)$ and $\infty = (1, 0)$. In the neighborhood of 0 we have local coordinates such that

$$y_1 = \frac{x_1}{x_2} \quad \text{and} \quad x_2 = y_2^{a_0}. \tag{10}$$

We consider G as the multiplicative group of all pairs (δ_2, δ_1) with $\delta_2^{a_2} = \delta_1^{a_1} = 1$ and put $\delta_1^{a_0} = \epsilon_1$ and $\delta_2 = \epsilon_2$ (see (8)). Then G operates in the neighborhood of the fixed point 0 as follows:

$$(y_1, y_2) \to (\delta_2 \delta_1^{-a_0} y_1, \delta_1 y_2). \tag{11}$$

Thus P_0 is the quotient singularity with respect to the action (11). If we first take the quotient with respect to the subgroup G_0 of G given by $\delta_1 = 1$ we obtain a non-singular point which admits local coordinates (t_1, t_2) with

$$t_1 = y_1^{a_2} \quad \text{and} \quad t_2 = y_2. \tag{12}$$

Thus P_0 is the quotient singularity with respect to the action of G/G_0 which is the group of a_1-th roots of unity. By (11) and (12) for $\delta_1^{a_1} = 1$ the action is

$$(t_1, t_2) \to (\delta_1^{-a_0 a_2} t_1, \delta_1 t_2) = (\delta_1^{q_1} t_1, \delta_1 t_2). \tag{13}$$

Looking at the invariants $\zeta_1 = t_1^{a_1}$, $\zeta_2 = t_2^{a_1}$ and $w = t_1 t_2^{a_1 - q_1}$ for which

$$w^{a_1} = \zeta_1 \zeta_2^{a_1 - q_1}$$

we see that P_0 is a singularity of type (6). We use [6], § 3.4 (or [2], Satz 2.10) for P_0 and in the same way for P_∞ and have finished the proof except for the statements on b in (vi) and (vii). The surface M_a of the theorem is $V_a^{(3)}/G$ with P_0 and P_∞ resolved. The function f we started from gives rise to a holomorphic function on M_a. Using the formulas of [6], § 3.4, we see that f has on the central curve of M_a the multiplicity $a_1 a_2$, and on the three curves intersecting the central curve the multiplicities

$$(a_1a_2q_0' + 1)/a_0,\ a_2q_1',\ a_1q_2'.$$

By [6], § 1.4 (1), we obtain

$$a_0a_1a_2b = q_0'a_1a_2 + q_1'a_0a_2 + q_2'a_0a_1 + 1.$$

Therefore

$$a_0a_1a_2b < 3a_0a_1a_2 \text{ and } b = 1 \text{ or } 2.$$

The congruence in (vii) also follows. This completes the proof.

1 REMARK. Originally the theorem was proved by using the $\mathbf{C}^*$-action on the singularity (1) and deducing abstractly from this that the resolution must look as described. Brieskorn constructed the resolution explicitly by starting from $x_0^n + x_1^n + x_2^n$ $(n = a_0a_1a_2)$ and then passing to a quotient. This is more symmetric. The method used in this paper has the advantage to give the theorem also for some other equations $z_0^{a_0} + h(z_1, z_2) = 0$ as was pointed out by Abhyankar in Bombay.

Now suppose moreover that the exponents a_0, a_1, a_2 are all odd. The explicit resolution shows that the involution $Tz = -z$ of $\mathbf{C}^3$ can be lifted to M_a. The lifted involution is also called T. It has no fixed points outside $\phi^{-1}(0)$. It carries all the rational curves of the graph $\mathfrak{g}_a$ over into themselves [7]. Thus T has the intersection points of two curves as fixed points. Let Fix T be the union of those curves which are pointwise fixed. Then Fix T is given by the following recipe.

THEOREM. *For the involution T on M_a (a_0, a_1, a_2 odd) we have: The central curve belongs to* Fix T. *If a curve is in* Fix T, *then the curves intersecting it are not in* Fix T. *If the curve C is not in* Fix T *and not an end curve of one of the three rays, then the following holds: If the self-intersection number $C \circ C$ is even, then the two curves intersecting C are both in* Fix T *or both not. If $C \circ C$ is odd, then one of the two curves intersecting C is in* Fix T *and one not. If C is an end curve of one of the three rays and if C is not in* Fix T, *then $C \circ C$ is odd if and only if the curve intersecting C belongs to* Fix T.

PROOF. The involution can be followed through the whole resolution. It is the identity on the curve K. On the three singularities of type (6) the involution is given by $(z_1, z_2) \to (z_1, -z_2)$. Here z_1 and z_2 are not coordinate functions of C^3 as used in (1), but have the same meaning as in [6], § 3.4. The theorem now follows from formula (8) in [6], § 3.4. Compare also the lemma at the end of §6 of [7].

For a_0, a_1, a_2 pairwise prime and odd, we can now calculate the invariant α of the involution T_a on $\Sigma^3_{(a_0,a_1,a_2)}$ (see the Introduction). The quadratic form of the graph $\mathfrak{g}_a$ is negative-definite. Therefore ([7], §6)

$$\alpha(\Sigma^3_{(a_0,a_1,a_2)}, T_a) = -(t_0 + t_1 + t_2 + 1) - \text{Fix } T \circ \text{Fix } T. \tag{14}$$

Here $t_0 + t_1 + t_2 + 1$ is the number of vertices of $\mathfrak{g}_a$ whereas Fix $T \circ$ Fix T is of course the sum of the self-intersection numbers of the curves belonging to Fix T. The calculation of α is a purely mechanical process by the two theorems of this §. The number α in (14) is always divisible by 8 (compare [7]) and for $(a_0, a_1, a_2) = (3, 6j - 1, 18j - 1)$ we get for α the value $8j$ (see (3)).

Observe that

$$\text{Fix } T \circ C \equiv C \circ C \bmod 2 \tag{15}$$

for all curves in the graph $\mathfrak{g}_a$, a fact which is almost equivalent to our above recipe for Fix T. The quadratic form of $\mathfrak{g}_a$ has determinant ± 1 because $\Sigma^3_{(a_0,a_1,a_2)}$ is for pairwise prime a_j an integral homology sphere ([1], [2], [7]). The divisibility of α by 8 is then a consequence of a well known theorem on quadratic forms.

The manifold Σ^{2n-1}_a (see the Introduction) is diffeomorphic to the manifold $\Sigma^{2n-1}_a(\epsilon)$ given by

$$z_0^{a_0} + \ldots + z_n^{a_n} = \epsilon \tag{16}$$
$$\Sigma z_i \bar{z}_i = 1,$$

where ϵ is sufficiently small and not zero. $\Sigma^{2n-1}_a(\epsilon)$ bounds the manifold $N_a(\epsilon)$ given by

$$z_0^{a_0} + \ldots + z_n^{a_n} = \epsilon \tag{17}$$
$$\Sigma z_i z_i \leqslant 1.$$

This fact apparently cannot be used to investigate the involution T_a in the case of odd exponents because then (17) is not invariant under T_a. If, however, the exponents are all even, then (17) is invariant under T_a and for $n = 2k$ the number $\alpha(\Sigma_a^{4k-1}, T_a)$ can be calculated using like Brieskorn [1] the results of Pham on $N_a(\epsilon)$. We get in this way

Theorem. *Let* $a = (a_0, a_1, \dots, a_{2k})$ *with* $a_i \equiv 0 \bmod 2$. *Then*

$$\alpha(\Sigma_a^{4k-1}, T_a) = \sum_j \epsilon(j)(-1)^{j_0+\cdots+j_{2k}}. \tag{18}$$

The sum is over all $j = (j_0, j_1, \dots, j_{2k}) \in Z^{2k+1}$ with $0 < j_r < a_r$ and $\epsilon(j)$ is 1, -1 or 0 depending upon whether the sum $\frac{j_0}{a_0} + \dots + \frac{j_{2k}}{a_{2k}}$ lies strictly between 0 and 1 mod 2, or strictly between 1 and 2 mod 2, or is integral.

Remark. For simplicity the resolution was only constructed for the exponents a_0, a_1, a_2 being pairwise relatively prime. The resolution of the singularity

$$z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0$$

can also be done in a similar way for arbitrary exponents and gives the following information.

Theorem. *If* $a_0 \equiv a_1 \equiv a_2 \bmod 2$ *and d is any integer* $\geqslant 1$, *then*

$$\alpha(\Sigma^3_{(da_0, da_1, da_2)}, T_{da}) = d\alpha(\Sigma^3_{(a_0,a_1,a_2)}, T_a) + d - 1.$$

For a_0, a_1, a_2 all odd and $d = 2$ we get

$$\alpha(\Sigma^3_{(a_0,a_1,a_2)}, T_a) = \tfrac{1}{2}(\alpha(\Sigma^3_{(2a_0,2a_1,2a_2)}, T_{2a}) - 1)$$

and therefore a method to calculate α also for odd exponents by formula (18).

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Universität Bonn.