DIFFERENTIABLE MANIFOLDS AND QUADRATIC FORMS

F. Hirzebruch and W. D. Neumann

Mathematisches Institut der Universität Bonn
Bonn, Germany

and

S. S. Koh

Department of Mathematics
West Chester State College
West Chester, Pennsylvania

Appendix II by W. SCHARLAU

MARCEL DEKKER, INC. New York 1971

These Lecture Notes in Mathematics *are produced directly from the author's typewritten notes. They are intended to make available to a wide audience new developments in mathematical research and teaching that would normally be restricted to the author's classes and associates.*

The publishers feel that this series will provide rapid, wide distribution of important material at a low price.

MARCEL DEKKER, INC.
95 Madison Avenue, New York, New York 10016

LIBRARY OF CONGRESS CATALOG CARD NUMBER 70-176304
ISBN NO. 0-8247-1318-4

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

The original notes on "Differentiable manifolds and quadratic forms" where written by Sebastian S. Koh on the basis of lectures and seminars which I held in 1962 at Brandeis University and at the University of California in Berkeley. They were a fairly faithful version of the actual lectures, which explains their informal character.

The Marcel Dekker publishing company suggested to me that these notes appear in their Lecture Notes series. Although the material is in some respects outdated, a demand for the notes still seems to exist, so I decided to follow this suggestion.

W.D. Neumann accepted the job of updating the notes and giving them a form which does justice to the rather more permanent character of such a Lecture Notes series. This was done by reordering and rewriting a lot of the material and adding two appendices to indicate developements in the field of differentiable manifolds (Appendix I) and quadratic forms (Appendix II). Apart from one or two further minor additions, the notes have remained otherwise essentially unchanged from the original mimeographed Berkeley version.

W.D. Neumann has, with his own initiative and ideas, brought the task of editing these notes to a successful conclusion. I would like to thank him for this effort, and also W. Scharlau for the composition of Appendix II.

F. HIRZEBRUCH

Bonn, 30th June 1971

CONTENTS

§1. QUADRATIC FORMS.

Let A be an integral domain. A lattice over A (or A-lattice) is a finitely generated free unitary A-module. The terms base, rank, etc., will be used in the usual way. In particular A itself can be considered as an A-lattice of rank 1. The dual $V' = \mathrm{Hom}_A(V,A)$ of an A-lattice V is again an A-lattice, and there is a bilinear pairing

$$V' \times V \longrightarrow A$$

defined by $\langle x',x\rangle := x'(x)$, called the Kronecker product.

We define a quadratic form $f = (f,V)$ over A to be a symmetric bilinear map

$$f: V \times V \longrightarrow A,$$

where V is an A-lattice. To such a form f there corresponds a linear map

$$\varphi : V \longrightarrow V'$$

called the correlation associated with f, given by

(1.1) $$\langle \varphi(x),y\rangle = f(x,y)$$

for all $x,y \in V$. The form f is called non-degenerate if its correlation φ is injective. This is equivalent to the condition

(1.2) $$f(x,y) = 0 \text{ for all } y \in V \implies x = 0 .$$

Let $(e_i)_{i=1,\dots,r}$ be a base in V, and $(e_i')_{i=1,\dots,r}$ the dual base of V', characterised by $\langle e_i', e_j \rangle = \delta_{ij}$ for each i and j. Then the coefficients of the representation

$$\varphi(e_i) = \sum_j \alpha_{ij} e_j'$$

are given by $\alpha_{ij} = f(e_i, e_j)$. The symmetric matrix $M = M_f = (\alpha_{ij})$ is called the <u>matrix of f</u> with respect to the base (e_i), and its rank, which is independant of the base chosen, is called the <u>rank</u> of f.

<u>We shall only consider non-degenerate quadratic forms</u>, so the rank of a form (f,V) is the same as the rank of V. A form of rank r will be called an <u>r-ary</u> form (unary, binary, and so on).

If we use the base (e_i) to express each element of V as a column matrix, e.g. $x^t = (x_1, \dots, x_r)$ and $y^t = (y_1, \dots, y_r)$, where t denotes transposition, then f is given by

$$f(x,y) = x^t My = \sum \alpha_{ij} x_i x_j \quad .$$

If $(\tilde{e}_i)_{i=1,\dots,r}$ is another base for V, obtained from (e_i) by applying the invertible matrix P, then the matrix $\tilde{M}$ of f relative to the new base is clearly $\tilde{M} = P^t MP$. Hence if we define the <u>determinant</u> $\det f$ of f to be $\det M_f$, then $\det f$ is well defined up to multiplication by the square of a unit in A.

Two quadratic forms $f = (f,V)$ and $g = (g,W)$ over the same domain A are said to be <u>equivalent</u>, written $f \sim g$, if there exists an isomorphism $u: V \longrightarrow W$ with $f(x,y) = g(u(x),u(y))$ for all $x,y \in V$. Such an isomorphism u is called an <u>isometry</u>. An isometry of a form (f,V) to itself is called an <u>automorph</u> of f.

Let M_f and M_g be the matrices of f and g with respect to some bases. Then clearly

<u>LEMMA (1.3)</u>: The quadratic forms f and g are equivalent if and only if there exists an invertible matrix P with $M_f = P^t M_g P$, i.e. M_f and M_g are congruent matrices. ||

Let f_1 and f_2 be quadratic forms defined on the A-lattices V_1 and V_2. Their <u>sum</u> $f = f_1 \oplus f_2$ is the quadratic form on $V_1 \oplus V_2$ defined by

$$f(x_1 \oplus x_2, y_1 \oplus y_2) := f_1(x_1,y_1) + f_2(x_2,y_2) .$$

If a base $(e_1, \dots, e_k; e_{k+1}, \dots, e_s)$ is given for $V_1 \oplus V_2$, then the matrices M, M_1, M_2, of f, f_1, f_2, are related by

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} .$$

If $f \sim f_1 \oplus f_2$, we shall say that f_1 <u>splits off</u> from f or that f <u>decomposes into</u> f_1 and f_2. Thus a form decomposes into unary forms if and only if its matrix is equivalent to a diagonal matrix.

A quadratic form is called <u>non-singular</u> if its correlation $\varphi : V \longrightarrow V'$ is an isomorphism. This means that the matrix M of f is invertible, or equivalently, $\det f$ is a unit in A. If A is a field, the properties of being non-singular and of being non-degenerate coincide. If the restriction of f on a sublattice V_1 of V is non-singular, we say that f is <u>non-singular on V_1</u>. Notice that the sum of non-singular forms is again non-singular.

<u>LEMMA (1.4)</u>: If f is a quadratic form defined on $V_1 \oplus V_2$ which is non-singular on V_1, then the restriction f_1 of f to

V_1 splits off from f.

<u>Proof</u>: With respect to a base $(e_1, \dots, e_k; e_{k+1}, \dots, e_s)$ of $V_1 \oplus V_2$, the matrix M of f is of the form

$$M = \begin{pmatrix} M_1 & L^t \\ L & N \end{pmatrix}, \quad M_1 = M_1^t, \quad N = N^t,$$

where M_1 is, by assumption, invertible. If P is the invertible matrix

$$P = \begin{pmatrix} I & -M_1^{-1}L^t \\ 0 & I \end{pmatrix}$$

then

$$P^tMP = \begin{pmatrix} M_1 & 0 \\ 0 & -LM_1^{-1}L^t + N \end{pmatrix},$$

which, in view of (1.3), proves the lemma. ||

From now on we shall assume that the integral domain A is a <u>local domain</u>, that is, A has a unique maximal ideal $\mathfrak{m}$. Then A has the following trivial but useful properties.

(i). The set A^* of all units of A coincides with $A - \mathfrak{m}$.

(ii). If α, β are units and ξ, η are non-units in A then $\alpha\beta$ and $\alpha + \xi$ are units and $\alpha\xi$ and $\xi\eta$ are non-units.

The following theorem generalizes the classical diagonalization theorem for symmetric matrices.

<u>THEOREM (1.5)</u>: Every non-singular quadratic form f over a local domain A decomposes into unary and binary forms. If, in addition, $2 \in A^*$ then f decomposes into unary forms.

<u>Proof</u>: Let $M = (\alpha_{ij})$ be the matrix of f with respect to some base. If a diagonal entry α_{ii} is a unit, the corresponding unary form splits off by (1.4). We can thus assume that no diagonal entry is in A^*. Since $\det f$ is a unit, $\alpha_{11} \notin A^*$ implies that $\alpha_{1i} \in A^*$ for some i, so we lose nothing by assuming $\alpha_{12} \in A^*$. But then the matrix

$$(1.6) \qquad \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \quad \alpha_{ii} \in \mathfrak{m}, \quad \alpha_{12} = \alpha_{21} \in A^*,$$

has determinant $\alpha_{11}\alpha_{22} - \alpha_{12}^2 \in A^*$ and is hence invertible. The first assertion now follows from (1.4) by a trivial induction. To prove the second statement we need only show that the matrix (1.6) is congruent to one whose diagonal entries are units, for we can then decompose the corresponding binary form by (1.4). Indeed the assumption $2 \in A^*$ shows that

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

is invertible, and

$$P^t \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} P = \begin{pmatrix} \alpha_{11}+2\alpha_{12}+\alpha_{22} & * \\ * & \alpha_{11}-2\alpha_{12}+\alpha_{22} \end{pmatrix}$$

then gives the required congruence. ||

Consider now a quadratic form $f = (f,V)$. If $a \in V$ is such that $f(a,a) = \alpha \in A^*$, then the linear map $u_a : V \longrightarrow V$ given by

$$u_a(x) := \frac{2f(x,a)}{\alpha} a - x$$

is an involution, that is $u_a u_a = \mathrm{id}$. It is easy to check that it is an automorph of f and leaves the element a fixed. Such automorphs are also known as reflections.

LEMMA (1.7) : Let $f = (f,V)$ be a quadratic form over A and suppose $2 \in A^*$. If $x,y \in V$ satisfy $f(x,x) = f(y,y) = \alpha \in A^*$, then there is an automorph u of f which interchanges x and y.

Proof : Since $f(x-y,x-y) + f(x+y,x+y) = 2(f(x,x) + f(y,y)) \in A^*$, the relation $f(a,a) \in A^*$ must be satisfied by either $a = x-y$ or $a = x+y$. In the first case $u = -u_a$ and in the second case $u = u_a$ is the desired automorph. ||

THEOREM (1.8) (Witt) : Suppose $2 \in A^*$, and let f_1, f_2 and h be quadratic forms over A with h non-singular. If $f_1 \oplus h \sim f_2 \oplus h$ then $f_1 \sim f_2$.
This may be regarded as the "cancellation law" for forms.

Proof : Since h decomposes into unary forms by (1.5), we lose nothing by assuming that h itself is unary. Let the lattices on which f_1, f_2 and h are defined be V_1, V_2 and W respectively. Then W has rank 1 ; let w be a base. By assumption there is an isometry $u : V_1 \oplus W \longrightarrow V_2 \oplus W$. Then $(f_2 \oplus h)(w,w) = h(w,w) = (f_1 \oplus h)(w,w) = (f_2 \oplus h)(u(w),u(w))$, and this is a unit since h is non-singular. (1.7) gives an automorph v of $f_2 \oplus h$ with $v(u(w)) = w$. The composite $vu : V_1 \oplus W \longrightarrow V_2 \oplus W$ is an isometry which carries w into w. It follows that $vu(V_1) = V_2$, so the restriction of vu to V_1 is the required isometry $f_1 \sim f_2$. ||

We shall now investigate the classification (up to equivalence) of non-singular quadratic forms in some simple cases. The simplest case is obviously that any non-singular form decomposes into unary forms. In view of theorem (1.5) we therefore state : <u>in the rest of this section A is a local ring with $2 \in A^*$</u> .

For notational convenience we denote the multiplicatively written cyclic group of order n by C_n . Consider the subgroup $A^{*2} := \{ x^2 \mid x \in A^* \}$ of the multiplicative group of units A^* of A . Clearly, an equivalence class of non-singular unary forms over A can be identified with an element of A^*/A^{*2} . In particular, if $A^* = A^{*2}$ (e.g. if $A = \mathbb{C}$), then the "diagonalization theorem" (1.5) shows that non-singular quadratic forms over A are classified by their rank alone.

Assume now

$$A^*/A^{*2} \cong C_2 \quad , \tag{1.10}$$

and let the two cosets in A^*/A^{*2} be represented by elements 1 and ε of A^* . Then by (1.5), any non-singular form f over A is equivalent to one of the form

$$x_1^2 + x_2^2 + \dots + x_k^2 + \varepsilon x_{k+1}^2 + \dots + \varepsilon x_{k+m}^2 \quad .$$

(We follow the standard convention of denoting a form f by the expression for $f(x,x)$). Let g be another non-singular form over A and

$$g \sim y_1^2 + \dots + y_r^2 + \varepsilon y_{r+1}^2 + \dots + \varepsilon y_{r+s}^2 \quad .$$

We may assume that $k \leq r$. By the cancellation theorem (1.8) we see that $f \sim g$ if and only if

$$\varepsilon(\tilde{x}_1^2 + \dots + \tilde{x}_{m-s}^2) \sim (\tilde{y}_1^2 + \dots + \tilde{y}_{r-k}^2) \tag{1.11}$$

In particular if $A = \mathbb{R}$, we may take $\varepsilon = -1$, and the relation (1.11) holds if and only if $m-s = r-k = 0$. This gives us

Sylvester's law of inertia: the number m (called the index of f) does not depend on the diagonalization of f. Clearly rank and index classify quadratic forms over $\mathbb{R}$.

Recall that the determinant of a form f is determined up to the square of a unit. Hence for a non-singular form f over A, $\det f$ determines a well defined element $\mathrm{DET}\, f \in A^*/A^{*2}$. Now assume A satisfies in addition to (1.10) the condition:

(1.12) the matrices $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are congruent over A.

Then, with the notation as above, $f \sim g$ if and only if $m - s = r - k$ is even; in other words $\mathrm{DET}\, f = \mathrm{DET}\, g$. We have shown:

THEOREM (1.13): If A satisfies the additional conditions (1.10) and (1.12), then two non-singular quadratic forms f and g are equivalent if and only if they have the same rank and same DET. ||

Condition (1.12) can be put in the more convenient form:

(1.12)′ $\qquad \varepsilon x^2 + \varepsilon y^2 = 1$ has a solution in A.

Indeed, (1.12) $\Rightarrow$ (1.12)′ is trivial, and conversely if $(x,y) = (a,b)$ is a solution of (1.12)′, then

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^t \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

We have been working under the general assumption that A is a local domain in which 2 is a unit. Perhaps a few examples of such domains are now in order. Any field of characteristic different from 2 is such a domain (with $\mathfrak{m} = \{0\}$). For instance a). $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, the fields of rational, real and complex numbers.

b). $\mathbb{Q}_p$, the field of p-adic numbers, where p is a (rational) prime.

c). $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, the finite field of rational integers modulo p, where p is an odd prime.

As examples of local domains which are not fields we have

d). $\mathbb{Z}_p$, $Q(p)$, the ring of p-adic integers and the ring of rational p-adic integers (p an odd prime).

We may regard $Q(p)$ as $\mathbb{Z}_p \cap \mathbb{Q}$, which consists of all fractions $a/b \in Q$ with $b \not\equiv 0 \pmod p$. Notice that the maximal ideal $\mathfrak{m}$ in $\mathbb{Z}_p$ is generated by the single element $p \in \mathbb{Z}_p$. In fact $\mathbb{Z}_p$ is a principal ideal domain in which each ideal is of the form $p^t\mathbb{Z}_p$. Thus one can talk of congruences modulo p^t in $\mathbb{Z}_p$. We make the convention that $\mathbb{Z}_\infty = \mathbb{Q}_\infty = \mathbb{R}$.

The examples $A = \mathbb{R}$ and $A = \mathbb{C}$ have already been dealt with. We now consider the case $A = \mathbb{F}_p$ (p an odd prime). Then A^* is the cyclic group of even order $p-1$, so $A^*/A^{*2} \cong C_2$. We define the <u>Legendre symbol</u> $(q|p)$ for $q \in A^*$ by

$$(1.14) \qquad \begin{aligned} (q|p) &= 1 \quad \text{if} \quad q \in A^{*2}, \\ (q|p) &= -1 \quad \text{if} \quad q \notin A^{*2}. \end{aligned}$$

From (1.13) we have :

<u>COROLLARY (1.15)</u> : If $A = \mathbb{F}_p$ (p an odd prime), then two quadratic forms f and g of the same rank over A are equivalent if and only if $(\det f|p) = (\det g|p)$.

<u>Proof</u> : We need only show that (1.12)′ holds in $\mathbb{Z}_p$. This

follows from the following lemma.

<u>LEMMA (1.16)</u>: If $\alpha, \beta \in \mathbb{F}_p^*$ then $\alpha x^2 + \beta y^2 = 1$ is soluble in $\mathbb{F}_p$.

<u>Proof</u>: Let H be the subset $\{0, 1, \dots, (p-1)/2\}$ of $\mathbb{F}_p$. The maps $i, j : H \longrightarrow \mathbb{F}_p$ given by

$$i(x) = \alpha x^2 \qquad j(y) = 1 - \beta y^2$$

are both injective. Hence their image sets have at least one element in common, proving the lemma. ||

To carry the above results over to the case $A = \mathbb{Z}_p$, the ring of p-adic integers, we need the following lemma.

<u>LEMMA (1.17)</u>: Let $f = (f, V)$ be a quadratic form over $\mathbb{Z}_p$, not necessarily non-degenerate. If there exists an $x \in V$ with $f(x,x) \equiv c \pmod{p^w}$, where $c \in \mathbb{Z}_p^*$ and $w = 1$ or 3 according as p is odd or $p = 2$, then there is an $\tilde{x} \in V$ with $f(\tilde{x}, \tilde{x}) = c$.

<u>Proof</u>: Recall that a sequence $\{\alpha_i\}$ of p-adic numbers converges to a limit in $\mathbb{Z}_p$ if $(\alpha_{n+1} - \alpha_n) \longrightarrow 0$ as $n \longrightarrow \infty$ (see e.g. Van Der Waerden: Algebra I, 5te Auflage, p255). Put $f(x,x) - c = p^t u$, where $t \geq 1$ if p is odd and $t \geq 3$ if $p = 2$. Define

$$x_1 := x - \frac{1}{2} \frac{p^t u}{f(x,x)} x \quad .$$

This is meaningful, since $f(x,x)$ is invertible and the factor $\frac{1}{2}$ is admissible even when $p = 2$, since $t \geq 1$. Then

$$f(x_1, x_1) - c = \frac{1}{4} p^{2t} u^2 f(x,x) \equiv 0 \pmod{p^{t+1}} \; .$$

By this means we construct a sequence $x = x_0, x_1, x_2, \ldots$ of elements in a sublattice of rank 1 in V such that $\{x_i\}$ converges to an element $\tilde{x}$ in V and $f(\tilde{x},\tilde{x}) = c$. This proves the lemma. ||

Let $\mathfrak{m}$ be the maximal ideal in $\mathbb{Z}_p$. Then $\mathbb{Z}_p/\mathfrak{m} = \mathbb{F}_p$, the prime field of characteristic p. If

$$\pi : \mathbb{Z}_p \longrightarrow \mathbb{F}_p$$

is the projection, then for each $\alpha \in \mathbb{Z}_p^*$, $\pi\alpha$ is in $\mathbb{F}_p^*$. We have

<u>COROLLARY (1.18)</u>: If p is an odd prime then $\mathbb{Z}_p^*/\mathbb{Z}_p^{*2} \cong C_2$ and if $p = 2$, $\mathbb{Z}_2^*/\mathbb{Z}_2^{*2} \cong C_2 \times C_2$.

<u>Proof</u>: (1.17), with f taken as the unary form x^2, implies that the map $\mathbb{Z}_p^*/\mathbb{Z}_p^{*2} \longrightarrow \mathbb{F}_p^*/\mathbb{F}_p^{*2} \cong C_2$ induced by π is an isomorphism. For $p = 2$ the same argument gives $\mathbb{Z}_2^*/\mathbb{Z}_2^{*2} \cong$ $(\mathbb{Z}/8\mathbb{Z})^* / (\mathbb{Z}/8\mathbb{Z})^{*2} \cong C_2 \times C_2$, where $\mathbb{Z}/8\mathbb{Z}$ is the ring of integers modulo 8, and $(\mathbb{Z}/8\mathbb{Z})^*$ its group of units. ||

For $\alpha \in \mathbb{Z}_p^*$ (p odd) we can define the Legendre symbol by

$$(\alpha|p) := (\pi\alpha|p).$$

Then the above argument shows that $(\alpha|p) = 1$ if α is a square and $(\alpha|p) = -1$ otherwise.

A further corollary of (1.17) is:

<u>COROLLARY (1.19)</u>: If $\alpha, \beta \in \mathbb{Z}_p^*$ (p odd) then $\alpha x^2 + \beta y^2 = 1$ is soluble in $\mathbb{Z}_p$.

<u>Proof</u>: $\alpha x^2 + \beta y^2 \equiv 1 \pmod p$ is soluble in $\mathbb{Z}_p$ by (1.16), so the corollary follows by (1.17). ||

As in the case of $\mathbb{F}_p$, (1.13) now yields :

<u>COROLLARY (1.20)</u> : Two non-singular quadratic forms f and g of the same rank over $\mathbb{Z}_p$ (p odd) are equivalent if and only if $\mathrm{DET}\, f = \mathrm{DET}\, g$, that is, $(\det f \mid p) = (\det g \mid p)$. ||

It is of interest to study when $\alpha x_1^2 + \beta x_2^2 = 1$ has a solution in other local rings. For the fields $\mathbb{Q}_p$ this leads to the <u>Hilbert symbol</u> defined as follows : for $\alpha, \beta \in \mathbb{Q}_p^*$, $p = 2, 3, 5, \ldots$, or $p = \infty$

$$(\alpha,\beta)_p := \begin{cases} 1 & \text{if } \alpha x_1^2 + \beta x_2^2 = 1 \text{ is soluble in } \mathbb{Q}_p , \\ -1 & \text{otherwise.} \end{cases}$$

We list some properties of Hilbert symbols that we will need later. For the proofs of these properties and further properties see for instance B.W. Jones [11], p27ff.

1. $(\alpha\rho^2, \beta\sigma^2)_p = (\alpha,\beta)_p$
2. $(\alpha,\beta)_p = (\beta,\alpha)_p$
3. $(\alpha,\beta)_p(\alpha,\gamma)_p = (\alpha,\beta\gamma)_p$
4. $(\alpha,-\alpha)_p = 1$

<u>Remark</u> : Property 1 states essentially that the Hilbert symbol is a map

$$\mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \times \mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \longrightarrow \{ 1, -1 \} ,$$

and properties 2 and 3 state that this map is symmetric and "bilinear". In fact if one writes $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ as an additive group, then it is not hard to see that it is an $\mathbb{F}_2$-lattice and the above map becomes a non-degenerate quadratic form over $\mathbb{F}_2$ - the non-degeneracy is given by property 9 in Jones (loc. cit.).

The calculation of the Hilbert symbol is given by the following properties :

5. $(\alpha,\beta)_\infty = 1$ unless α and β are both negative.

6. If $\alpha,\beta \in \mathbb{Q}_p^*$, write α and β in the form $p^a\alpha_1$, $p^b\beta_1$, with α_1, $\beta_1 \in \mathbb{Z}_p^*$, then
$$(\alpha,\beta)_p = (-1|p)^{ab}(\alpha_1|p)^b(\beta_1|p)^a \qquad \text{if } p \neq 2 ,$$
$$(\alpha,\beta)_2 = (2|\alpha_1)^b(2|\beta_1)^a(-1)^{(\alpha_1-1)(\beta_1-1)/4} \qquad \text{if } p = 2 .$$

Here $(2|\alpha)$ is defined to be $(-1)^{(\alpha^2-1)/8}$; for $x \in \mathbb{Z}_2$, $(-1)^x$ is of course defined as 1 or -1 according as $x \equiv 0$ or $1 \pmod 2$. Observe that for $p \neq 2$, the case $a = b = 0$ of property 6 is just corollary (1.19). A final important property of the Hilbert symbol is the Hilbert product formula :

7. If $a,b \in \mathbb{Q}^*$, then $(a,b)_p = 1$ for almost all primes p and $\prod_p (a,b)_p = 1$, where the product is over all $p = 2,3,5,\ldots,\infty$.

Our next goal is to define the <u>Hasse-Minkowski symbol</u> for a quadratic form over $\mathbb{Q}_p$ through a diagonalization of its matrix. An invariant definition will be given in §2.

Let $\mathrm{diag}(\alpha_1,\alpha_2, \ldots ,\alpha_r)$ denote the diagonal matrix with diagonal entries $\alpha_1, \ldots ,\alpha_r$. A quadratic form is said to be diagonal if its matrix has been diagonalized, i.e. f is of the form $\alpha_1x_1^2 + \ldots + \alpha_rx_r^2$.

<u>LEMMA (1.21)</u> : Let f,g be equivalent diagonal forms over a field K . Then f may be carried into g by successive application of binary transformations such that at each stage the form remains diagonal.

<u>Proof</u>: Let f and g be represented by the matrices $M = \operatorname{diag}(\alpha_1, \dots, \alpha_r)$ and $N = \operatorname{diag}(\beta_1, \dots, \beta_r)$ respectively. Our proof is by induction on r. The case $r = 2$ is trivial. Using the cancellation theorem (1.8) and the induction hypothesis we see that for $r > 2$ it suffices to show that $\operatorname{diag}(\alpha_1, \dots, \alpha_r)$ can be transformed into a form $\operatorname{diag}(\beta_1, \gamma_2, \dots, \gamma_r)$ by binary transformations such that at each stage the resulting form is diagonal.

First note that any permutation of the α_i's can be realized by such a sequence of binary transformations. Now by assumption there is a non-singular matrix $R = (\rho_{ij})$ with $R^t M R = N$. Since $\beta_1 = \sum_i \alpha_i \rho_{i1}^2 \neq 0$, we may, by permuting the α_i's (and hence the ρ_{ij}'s) if necessary, assume that $\alpha_1\rho_{11}^2$, $\alpha_1\rho_{11}^2 + \alpha_2\rho_{21}^2$, ..., $\sum \alpha_i \rho_{i1}^2$ are all non-zero. Now the binary transformation with matrix

$$U = \begin{pmatrix} \rho_{11} & -\rho_{21}\alpha_2 & \\ \rho_{21} & \rho_{11}\alpha_1 & 0 \\ & 0 & I_{r-2} \end{pmatrix}$$

carries M into $U^t M U = \operatorname{diag}((\alpha_1\rho_{11}^2 + \alpha_2\rho_{21}^2), \gamma_2, \alpha_3, \dots, \alpha_r)$ which goes into $M' = \operatorname{diag}((\alpha_1\rho_{11}^2 + \alpha_2\rho_{21}^2), \alpha_3, \dots, \alpha_r, \gamma_2)$ by permuting the terms. U is non-singular since $\det U = \alpha_1\rho_{11}^2 + \alpha_2\rho_{21}^2 \neq 0$. We can now apply the same process to M', using $\rho'_{11} = 1$, $\rho'_{21} = \rho_{31}$, $\alpha'_1 = \alpha_1\rho_{11}^2 + \alpha_2\rho_{21}^2$, $\alpha'_2 = \alpha_3$, in place of ρ_{11}, ρ_{21}, α_1, α_2. Continued iteration finally leads to the desired matrix $\operatorname{diag}(\beta_1, \gamma_2, \dots, \gamma_r)$. ||

Now let f be a quadratic form over $\mathbb{Q}_p$. We may diagonalize f by (1.5). Let the matrix of f be $M_f = \operatorname{diag}(\alpha_1, \dots, \alpha_r)$. We define

(1.22) $$c_p(f) := c_p(\alpha_1, \dots ,\alpha_r) := \prod_{i<j}(\alpha_i,\alpha_j)_p \quad ,$$

which is either 1 or -1 . We must show that $c_p(f)$ depends only on f , and not on the particular diagonalization M_f . That is, if $\mathrm{diag}(\beta_1, \dots ,\beta_r)$ is another diagonalization of f then we claim

$$c_p(\alpha_1, \dots ,\alpha_r) = c_p(\beta_1, \dots ,\beta_r)$$

Indeed, for $r=1$ we define $c_p(f) := 1$ and there is nothing to prove Clearly, for $r=2$ $c_p(f) = 1$ or -1 according as $f(x,x)=1$ has or has not a solution ; this property is obviously independant of the diagonalization. For the general case observe that by properties 2 and 3 of the Hilbert symbols

$$c_p(\alpha_1, \dots ,\alpha_r) = c_p(\alpha_1,\alpha_2)c_p(\alpha_3, \dots ,\alpha_r)(\alpha_1\alpha_2,\alpha_3\alpha_4\dots\alpha_r)_p \; .$$

By property 1 of Hilbert symbols and the fact that $c_p(\alpha_1, \dots ,\alpha_r)$ is unchanged if we permute the α_i's , $c_p(\alpha_1, \dots ,\alpha_r)$ is invariant under binary transformations provided that the resulting form is again diagonal. Our claim thus follows from (1.21).

The symbol $c_p(f)$ is called the <u>Hasse-Minkowski symbol</u>.

<u>COROLLARY (1.23)</u> : Let f_1 and f_2 be quadratic forms over $\mathbb{Q}_p$. Then

$$c_p(f_1 \oplus f_2) = c_p(f_1)c_p(f_2)(\det f_1 ,\det f_2)_p \quad . \quad \|$$

A quadratic form f over $\mathbb{Z}_p$ can always be regarded as a form over $\mathbb{Q}_p$, since $\mathbb{Z}_p \subset \mathbb{Q}_p$; thus the symbol $c_p(f)$ is also defined. By (1.19) and the definition of the Hilbert symbol

<u>LEMMA (1.24)</u> : If f is a non-singular quadratic form over $\mathbb{Z}_p$, $p \neq 2$, , then $c_p(f) = 1$. $\|$

By the same token we may consider $c_p(f)$ for $p = 2,3,5, \ldots, \infty$ if f is a quadratic form over $\mathbb{Q}$. The Hilbert product formula (property 7) of the Hilbert symbol gives

<u>LEMMA (1.25)</u>: If f is a non-degenerate quadratic form over $\mathbb{Q}$ then $c_p(f) = 1$ for almost all p and

$$\prod_p c_p(f) = 1 \quad ,$$

where p ranges over $2,3,5, \ldots, \infty$. ||

§2. THE GROTHENDIECK RING.

Let B be a commutative semigroup with operation $\oplus$. Let F be the free abelian group with base B and operation denoted by $+$, and A the subgroup of F generated by all elements of the form $b_1 \oplus b_2 - b_1 - b_2$ $(b_1, b_2 \in B)$. The group $G(B) := F/A$ is called the <u>Grothendieck group</u> of B.

The obvious canonical map

$$j: B \longrightarrow G(B)$$

is a semigroup homomorphism and has the universal property: any semigroup homomorphism $h: B \longrightarrow G$ of B into an abelian group G factors uniquely over j; that is, there is a unique group homomorphism $g: G(B) \longrightarrow G$ such that the diagram

$$\begin{tikzcd} G(B) \arrow[dr, dashed, "g"] & \\ B \arrow[u, "j"] \arrow[r, "h"'] & G \end{tikzcd}$$

commutes.

<u>EXERCISE (2.1)</u>: The natural map $j: B \longrightarrow G(B)$ is injective if and only if the cancellation law holds in B.

The name "Grothendieck group" is in honour of Grothendieck, who used the above construction to define a group $K_\omega(X)$ for any algebraic manifold X, starting from a semigroup of analytic sheaves over X (see e.g. Borel, Serre: Le théorème de Riemann-Roch (d'après Grothendieck), Bull. Math. Soc. Franc., 86(1958), 97 - 136). It was this example which first made the importance of the above construction so apparant. A closely related example is the following.

EXAMPLE (2.2): Let X be a topological space and B the semigroup of equivalence classes of real (respectively complex) vector bundles over X, with Whitney sum as $\oplus$. Then the group $G(B)$ is usually denoted by $KO(X)$ (resp. $K(X)$). We shall not discuss these examples further here (see for instance Atiyah, Hirzebruch: Riemann--Roch theorems for differentiable manifolds, Bull. A.M.S., 65 (1956), 276 - 281).

As the reader will have guessed, the example which interests us here is the following.

EXAMPLE (2.3): Let A be an integral domain and let $\mathcal{F}(A)$ and $\mathcal{F}_o(A)$ be respectively the sets of equivalence classes of non-degenerate and non-singular quadratic forms over A. These are commutative semigroups with respect to the sum $\oplus$ defined in §1. The groups $G(\mathcal{F}(A))$ and $G(\mathcal{F}_o(A))$ will be denoted by $G(A)$ and $G_o(A)$.

Recall that the cancellation law holds in $\mathcal{F}_o(A)$ if A is suitably restricted (see (1.8)). Thus by (2.1), one can expect the calculation of $G_o(A)$ to yield a complete classification of non-singular quadratic forms over A in such cases. Observe also that $G(A) = G_o(A)$ if A is a field.

The rank of quadratic forms induces a homomorphism

$$\mathrm{rk}: G(A) \longrightarrow \mathbb{Z}$$

called the <u>augmentation</u>. If f_1 and f_2 are forms defined on the A-lattices V_1 and V_2 we define their <u>product</u> $f_1 \otimes f_2$ to be the form over $V_1 \otimes_A V_2$ characterised by

$$f_1 \otimes f_2(a_1 \otimes a_2, b_1 \otimes b_2) := f_1(a_1,b_1)f_2(a_2,b_2) .$$

This operation induces a ring structure in $G(A)$ and $rk: G(A) \longrightarrow \mathbb{Z}$ is then a ring homomorphism. This ring will be referred to as the <u>Grothendieck ring</u> of quadratic forms over A. The above remarks apply of course equally well to $G_o(A)$.

<u>Remark</u>: The groups of example (2.2) are also in a natural way augmented rings, see for instance Atiyah - Hirzebruch, loc. cit. .

The ring $G(A)$ has a unit element 1 represented by the unary form $f = x^2$. There is a ring homomorphism $\varepsilon: \mathbb{Z} \longrightarrow G(A)$ defined by $\varepsilon(1) = 1$. Clearly $rk \circ \varepsilon = id$, so if $\hat{G}(A)$ denotes the kernel of rk then the short exact sequence of rings

$$0 \longrightarrow \hat{G}(A) \longrightarrow G(A) \underset{\varepsilon}{\overset{rk}{\rightleftarrows}} \mathbb{Z} \longrightarrow 0$$

splits, and $G(A) = \hat{G}(A) \oplus \mathbb{Z}$. Similarly $G_o(A) = \hat{G}_o(A) \oplus \mathbb{Z}$, where $\hat{G}_o(A)$ is the kernel of $rk: G_o(A) \longrightarrow \mathbb{Z}$.

The determinant of forms induces a group homomorphism $DET: G_o(A) \longrightarrow A^*/A^{*2}$, since $DET(f_1 \oplus f_2) = DET(f_1)DET(f_2)$. Since DET restricted to the subgroup $\mathbb{Z} = \mathrm{Im}\,\varepsilon$ in the representation $G_o(A) = \hat{G}_o(A) \oplus \mathbb{Z}$ is trivial, it follows that the restriction of DET to $\hat{G}_o(A)$ is already an epimorphism. Hence we have a short exact sequence of groups

$$(2.4) \qquad 0 \longrightarrow L(A) \longrightarrow \hat{G}_o(A) \xrightarrow{DET} A^*/A^{*2} \longrightarrow 0 \quad ,$$

where $L(A)$ is the kernel of DET restricted to $\hat{G}_o(A)$. This sequence does not always split, as is shown by the examples discussed below.

Suppose now that A is a local domain and $2 \in A^*$. Then the cancellation law (1.8) holds in $\mathcal{F}_o(A)$, so by (2.1) $\mathcal{F}_o(A) \subset G_o(A)$. Since we can identify an equivalence class of non-singular unary forms with an element of A^*/A^{*2}, we have $A^*/A^{*2} \subset G_o(A)$. Furthermore, A^*/A^{*2} generates $G_o(A)$ additively by the "diagonalization theorem" (1.5). It follows that the elements of the form $a-1 \in \hat{G}_o(A)$ with $a \in A^*/A^{*2}$ generate $\hat{G}_o(A)$ additively (here and in the following $+$ and $-$ always denote operations in $G_o(A)$).

<u>EXAMPLES (2.5)</u>: $G(\mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z}$ as a group. $G(\mathbb{C}) \cong \mathbb{Z}$.

<u>Proof</u>: If $A = \mathbb{R}$ then $A^*/A^{*2} = \{1,-1\} \cong C_2$. Every quadratic form may be written uniquely as $\alpha^+ \cdot 1 \oplus \alpha^- \cdot (-1)$, so $G(\mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z}$ as a group. Note that it follows that $\hat{G}(\mathbb{R}) \cong \mathbb{Z}$ as a group, generated by $(-1) - 1$, however the ring structure is not the usual structure in $\mathbb{Z}$. For $A = \mathbb{C}$ the assertion is trivial. ||

Our aim now is to use the above comments to calculate $G(\mathbb{Q}_p)$, p a rational prime.

If p is an odd prime, then $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2} \cong C_2 \times C_2$. Indeed, by (1.18), $\mathbb{Z}_p^*/\mathbb{Z}_p^{*2} \cong C_2$; let its elements be represented by $\{1, \varepsilon\}$. Now for any element of $\mathbb{Q}_p$, the product of this element with a suitable power of p is in $\mathbb{Z}_p$, so $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ is represented by $\{1, \varepsilon, p, p\varepsilon\}$.

We hence know that $\hat{G}(\mathbb{Q}_p)$ is generated additively by the elements $\varepsilon-1$, $p-1$ and $p\varepsilon-1$, and hence also by the elements $\varepsilon-1$, $p-1$ and

$(p\varepsilon-1)-(p-1)-(\varepsilon-1) = (\varepsilon-1)(p-1)$. We must investigate the relations between these elements.

<u>LEMMA (2.6)</u>: For $\alpha,\beta \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$ we have :

$$(\alpha,\beta)_p = 1 \iff \alpha+\beta = 1+\alpha\beta \text{ in } G(\mathbb{Q}_p) .$$

<u>Proof</u>: $(\alpha,\beta)_p = 1$ means that $\alpha x_1^2+\beta x_2^2 = 1$ has a solution in $\mathbb{Q}_p$. A simple calculation shows that this is equivalent to saying that $\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 0 & \alpha\beta\end{pmatrix}$ are congruent, which proves the lemma. ||

Now by property 6 of the Hilbert symbol, $(\varepsilon,\varepsilon)_p = 1$, $(\varepsilon,p)_p = (\varepsilon|p) = -1$ and $(p,p)_p = (-1|p) = 1$ or -1 according as $p \equiv 1$ or 3 modulo 4 . Hence in $G(\mathbb{Q}_p)$ we have

$p\equiv 1 \pmod 4$	$p\equiv 3 \pmod 4$
$2\varepsilon = 2$	$2\varepsilon = 2$
$\varepsilon+p \neq 1+\varepsilon p$	$\varepsilon+p \neq 1+\varepsilon p$
$2p = 2$	$2p \neq 2$

,

where $2 = 1+1$ in $G(\mathbb{Q}_p)$.

For <u>$p\equiv 1 \pmod 4$</u> this gives $2(\varepsilon-1)=0$, $2(p-1)=0$, $2(\varepsilon-1)(p-1)=0$, $(\varepsilon-1)(p-1)\neq 0$. Also $\mathrm{DET}(\varepsilon-1)(p-1) = \mathrm{DET}((\varepsilon p-1)-(\varepsilon-1)-(p-1)) = \varepsilon p\varepsilon^{-1}p^{-1} = 1$. Hence by the exact sequence (2.4)

$$\begin{aligned} G(\mathbb{Q}_p) &= \mathbb{Z} \oplus \hat{G}(\mathbb{Q}_p) \\ &\cong \mathbb{Z} \oplus \underset{(\varepsilon-1)}{\mathbb{Z}/2\mathbb{Z}} \oplus \underset{(p-1)}{\mathbb{Z}/2\mathbb{Z}} \oplus \underset{(\varepsilon-1)(p-1)}{\mathbb{Z}/2\mathbb{Z}} , \end{aligned}$$

where the element under a group shows a generator of that group, and the last summand $\mathbb{Z}/2\mathbb{Z}$ is $L(\mathbb{Q}_p)$.

For $\underline{p \equiv 3 \pmod 4}$ we have

$$\begin{aligned} G(\mathbb{Q}_p) &= \mathbb{Z} \oplus \hat{G}(\mathbb{Q}_p) \\ &\cong \mathbb{Z} \oplus \underset{(\varepsilon-1)}{\mathbb{Z}/2\mathbb{Z}} \oplus \underset{(p-1)}{\mathbb{Z}/4\mathbb{Z}} \ . \end{aligned}$$

Indeed, property 3 of the Hilbert symbol shows that $(\varepsilon p,p)_p = (\varepsilon,p)_p(p,p)_p = 1$. Therefore $\varepsilon p + p = 1+\varepsilon$, whence $(\varepsilon-1)(p-1) = -2(p-1)$. Thus $\hat{G}(\mathbb{Q}_p)$ is generated by $(\varepsilon-1)$ and $(p-1)$ alone. Also $4(p-1) = -2(\varepsilon-1)(p-1) = 0$, so $(p-1)$ has order 4 in $G(\mathbb{Q}_p)$. The element $2(p-1)$ generates $L(\mathbb{Q}_p)$, so $L(\mathbb{Q}_p) \cong \mathbb{Z}/2\mathbb{Z}$.

Finally, we come to the case $\underline{p=2}$. As we know, $\mathbb{Z}_2^*/\mathbb{Z}_2^{*2} \cong C_2 \times C_2$; let it be represented by $\{1,\varepsilon,\varsigma,\varepsilon\varsigma\}$. Then $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2} \cong C_2 \times C_2 \times C_2$, represented by $\{1,\varepsilon,\varsigma,\varepsilon\varsigma,2,2\varepsilon,2\varsigma,2\varepsilon\varsigma\}$. We can represent ε and ς by 3 and 5 respectively modulo 8, and then property 6 of the Hilbert symbol gives $(\varepsilon,\varepsilon)_2 = -1$, $(\varepsilon,\varsigma)_2 = 1$, $(\varepsilon,2)_2 = -1$, $(\varsigma,\varsigma)_2 = 1$, $(\varsigma,2)_2 = -1$, $(2,2)_2 = 1$. Writing γ for 2 we obtain (where 2 now denotes the element $1+1 \in G(\mathbb{Q}_2)$)

$$\begin{aligned} \varepsilon + \varsigma &= 1 + \varepsilon\varsigma \ , \\ 2\varsigma &= 2 \ , \\ 2\gamma &= 2 \ , \\ \varepsilon + \varepsilon\gamma &= 1 + \gamma \ , \\ \varepsilon\varsigma + \gamma &= 1 + \varepsilon\varsigma\gamma \ . \end{aligned}$$

Using an argument similar to that used before it follows that

$$G(\mathbb{Q}_2) \cong \underset{1}{\mathbb{Z}} \oplus \underset{(\varsigma-1)}{\mathbb{Z}/2\mathbb{Z}} \oplus \underset{(\gamma-1)}{\mathbb{Z}/2\mathbb{Z}} \oplus \underset{(\varepsilon-1)}{\mathbb{Z}/4\mathbb{Z}} \ ,$$

and $L(\mathbb{Q}_2) \cong \mathbb{Z}/2\mathbb{Z}$, generated by $2(\varepsilon-1)$. We leave the details to the reader.

We conclude this section by giving an invariant definition of the Hasse-Minkowski symbol. For each finite prime p let

$$c'_p : G(\mathbb{Q}_p) \longrightarrow L(\mathbb{Q}_p)$$

be the map defined by

$$c'_p(a) = a - \mathrm{DET}a - (\mathrm{rk}\, a)\cdot 1 + 1 .$$

Identify $L(\mathbb{Q}_p)$ with $\mathbb{Z}/2\mathbb{Z}$, so $c'_p(a) = 0$ or 1, and define

$$c_p(a) = (-1)^{c'_p(a)} .$$

For $p = \infty$ (i.e. $\mathbb{Q}_p = \mathbb{R}$), write $a \in G(\mathbb{R})$ as $a = \alpha^+\cdot 1 + \alpha^-\cdot(-1)$, and define

$$c_\infty(a) = (-1)^{\alpha^-(\alpha^- - 1)/2} .$$

We shall show that if $a = f$ is a quadratic form, then this definition of $c_p(f)$ coincides with the previous one.

If p is a finite prime then trivially

$$c'_p(f \oplus g) = f + g - \mathrm{DET}f\cdot\mathrm{DET}g - \mathrm{rk}\, f - \mathrm{rk}\, g + 1$$
$$c'_p f + c'_p g = f - \mathrm{DET}f - \mathrm{rk}\, f + 1 + g - \mathrm{DET}g - \mathrm{rk}\, g + 1 ,$$

where we write $\mathrm{rk}\, a$ for $(\mathrm{rk}\, a)\cdot 1$. Hence

(2.7) $$c'_p(f \oplus g) - c'_p f - c'_p g = \mathrm{DET}f + \mathrm{DET}g - \mathrm{DET}f\cdot\mathrm{DET}g - 1 .$$

By lemma (2.6)

$$(\mathrm{DET}f, \mathrm{DET}g)_p = (-1)^{\mathrm{RHS}} ,$$

where RHS is the right hand side of (2.7). Thus

$$(-1)^{c'_p(f \oplus g) - c'_p f - c'_p g} = (\mathrm{DET}f, \mathrm{DET}g)_p ,$$

that is,

$$c_p(f \oplus g) = c_p(f)c_p(g)(\mathrm{DET}f, \mathrm{DET}g)_p \quad . \tag{2.8}$$

Since this newly defined $c_p(f)$ coincides with the one defined in §1 for unary forms, and since every form over $\mathbb{Q}_p$ decomposes into unary forms, the desired result follows by (2.8) and (1.23). For $p = \infty$ the proof is similar; alternatively by direct computation from the definition of §1, using property 5 of the Hilbert symbol.

An immediate consequence of the above definition and the exact sequences of this section is

<u>COROLLARY (2.9)</u>: An element a of $G(\mathbb{Q}_p)$, p finite, is completely determined by $\mathrm{rk}\, a$, $\mathrm{DET}\, a$ and $c_p(a)$. In particular, the equivalence class of a quadratic form over $\mathbb{Q}_p$, p finite, is determined by its rank, DET, and Hasse-Minkowski symbol. ||

§3. CERTAIN ARITHMETICAL PROPERTIES OF QUADRATIC FORMS.

Let $A \subset B$ be an inclusion of integral domains. Then clearly any quadratic form over A can be considered as a quadratic form over B. Formaly, this goes as follows. Consider B as an A-module. If $f = (f,V)$ is a form over A, denote by f^B the form over $V \otimes_A B$ (which is a B-lattice whose rank over B is the same as the rank of V over A) characterized by $f^B(x \otimes b, y \otimes c) := bcf(x,y)$. We shall simply write f for f^B when there is no possibility of confusion.

A quadratic form over A will be called <u>integral</u>, <u>rational</u> or <u>real</u> according as $A = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. Let f be a real quadratic form whose matrix has been diagonalized according to (1.5), and let α^+ and α^- denote the number of positive and negative diagonal entries respectively. The integers α^+ and α^- depend only on the form f by Sylvester's law of inertia, which we proved in §1.

Clearly $\alpha^+ + \alpha^- = \operatorname{rk} f$, the rank of f. Define the <u>signature</u> $\tau(f)$ of f by

$$\tau(f) := \alpha^+ - \alpha^- .$$

By what we said above, α^+ and α^-, and hence also $\tau(f)$ are defined also for integral and rational forms and forms over $Q(p) = \mathbb{Q} \cap \mathbb{Z}_p$, the ring of rational p-adic integers.

A quadratic form $f = (f,V)$ over $\mathbb{Z}$ or $\mathbb{Z}_2$ is called <u>even</u> if $f(x,x) \equiv 0 \pmod 2$ for all $x \in V$, otherwise f is called <u>odd</u>.

Since

$$f(x+y,x+y) = f(x,x) + f(y,y) + 2f(x,y) \,, \tag{3.1}$$

f is even if and only if the diagonal entries in its matrix are all even.

Consider a quadratic form $f = (f,V)$ over $\mathbb{Z}$ or $\mathbb{Z}_2$ with odd determinant. Then there exists an element $w \in V$, in general not unique, with

$$f(x,x) \equiv f(x,w) \pmod 2, \text{ for all } x \in V \,. \tag{3.2}$$

For if $e_1, \dots, e_r$ is a base of V and we write $x = \sum x_i e_i$, $w = \sum w_i e_i$, then

$$f(x,x) \equiv \sum_i f(e_i,e_i)x_i^2 \equiv \sum_i f(e_i,e_i)x_i \pmod 2$$

and

$$f(x,w) = \sum_{i,j} f(e_i,e_j)x_i w_j \,.$$

Hence relation (3.2) is equivalent to

$$f(e_i,e_i) \equiv \sum_j f(e_i,e_j)w_j \pmod 2 \,,$$

and since $\det f \not\equiv 0 \pmod 2$, we can solve for the coefficients w_i of w modulo 2.

Such an element w will be called a <u>characteristic element</u> of f. Clearly $\widetilde{w} \in V$ is another characteristic element of f if and only if

$$\widetilde{w} = w + 2z \tag{3.3}$$

for some $z \in V$. One then has

$$f(\widetilde{w},\widetilde{w}) = f(w,w) + 4f(w,z) + 4f(z,z) \,,$$

so by (3.2)

$$(3.4) \qquad f(\tilde{w},\tilde{w}) \equiv f(w,w) \pmod 8 .$$

In $\mathbb{Z}_2$, mod 8 of course means modulo the ideal $2^3\mathbb{Z}_2$.

THEOREM (3.5): Let f be a quadratic form over $\mathbb{Z}$ or $\mathbb{Z}_2$ with odd determinant. Then

$$(3.6) \qquad f(w,w) - r - \det f + 1 \equiv 0 \pmod 4 ,$$

where $r = \operatorname{rk} f$ is the rank of f.

Proof: Observe that (3.6) is meaningful, since $\det f$ is defined up to the square of a unit, so $\det f$ modulo 4 is well defined Since $\mathbb{Z} \subset \mathbb{Z}_2$, a form over $\mathbb{Z}$ can be considered as a form over $\mathbb{Z}_2$, so it suffices to prove the theorem for $\mathbb{Z}_2$.

Let $f = (f,V)$ be the given form. If $r = 1$ then f is the form ax^2, where $a = \det f$ is odd. Thus $w = 1$ is a characteristic element, and (3.6) is clearly satisfied.

Suppose now f is an even form of rank $r = 2$. Then f has matrix

$$\begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix} ,$$

with c odd since $\det f$ is odd. Thus $c^2 \equiv 1 \pmod 4$, and since $w = 0$ is a characteristic element of f, $f(w,w) - r - \det f + 1 = 0 - 2 - 4ab + c^2 + 1 \equiv 0 \pmod 4$.

Thus (3.5) is proved for unary and for even binary forms.

Now assume $f = f_1 \oplus f_2$ is a decomposition of f into forms of ranks r_1 and r_2 respectively. If w_i is a characteristic element of f_i for $i = 1,2$, then $w = w_1 \oplus w_2$ is a characteristic element of f. Hence

$$(3.7)\quad f(w,w)-r-\det f+1 = \begin{aligned}[t] & f_1(w_1,w_1)-r_1-\det f_1+1+ \\ & +f_2(w_2,w_2)-r_2-\det f_2+1+ \\ & +\det f_1+\det f_2-\det f_1\cdot\det f_2-1 .\end{aligned}$$

But $\det f_1$ and $\det f_2$ are odd, so

$$\begin{aligned}\det f_1+\det f_2-\det f_1\cdot\det f_2-1 &= -(\det f_1-1)(\det f_2-1) \\ &\equiv 0 \pmod 4 .\end{aligned}$$

Hence, if we assume that the theorem holds for forms of rank $<r$, then it holds for f. To complete the proof we need only show that every form of odd determinant may be written as a sum of unary forms and binary even forms with odd determinants. This needs a slight modification of the proof of (1.5) and is left to the reader. ||

For $a \in \mathbb{Z}_2$, we defined $(-1)^a$ as 1 or -1 according as a is even or odd. For a form f over $\mathbb{Z}_2$ with odd determinant we can thus define

$$\tau_2(f) := (-1)^{(f(w,w)-r-\det f+1)/4} .$$

Recall that the Hasse-Minkowski symbol $c_2(f)$ is also defined for a form over $\mathbb{Z}_2$.

<u>THEOREM (3.8)</u>: $\tau_2(f) = c_2(f)$.

<u>Proof</u>: If f_1 and f_2 and hence also $f = f_1 \oplus f_2$ are forms over $\mathbb{Z}_2$ with odd determinants, then by (3.7) and property 6 of the Hilbert symbol

$$\tau_2(f) = \tau_2(f_1)\tau_2(f_2)(\det f_1, \det f_2)_2 .$$

Thus by (1.23) τ and c behave in the same way on sums of forms, so

(3.8) is proved once we prove it for unary and even binary forms.

We use the notation of the proof of (3.5). If f is unary then $\det f = a \in \mathbb{Z}_2^*$ and we choose $w = 1$, so $\tilde{c}_2(f) = 1 = c_2(f)$.

If f is an even binary form we choose $w = 0$, so

$$\tilde{c}_2(f) = (-1)^{(-1-4ab+c^2)/4} = (-1)^{ab}$$

since $c^2 - 1 \equiv 0 \pmod 8$. If $a = b = 0$ then the matrix of f is congruent over $\mathbb{Q}_2$ to one of the form $\operatorname{diag}(\alpha, -\alpha)$, so by property 4 of the Hilbert symbol, $c_2(f) = 1 = \tilde{c}_2(f)$. We may thus assume that one of a and b, say a, is non-zero. Then the matrix of f is congruent over $\mathbb{Q}_2$ to

$$\begin{pmatrix} 2a & 0 \\ 0 & \frac{4ab-c^2}{2a} \end{pmatrix} ,$$

so

$$\begin{aligned} c_2(f) &= (2a, (4ab-c^2)/2a)_2 \\ &= (2a, (4ab-c^2)/2a)_2 (2a, -2a)_2 \\ &= (2a, c^2-4ab)_2 . \end{aligned}$$

It is now a routine matter to check that $c_2(f) = \tilde{c}_2(f)$ using property 6 of the Hilbert symbol and the fact that $(2|\alpha) = 1$ or -1 according as $\alpha = \pm 1$ or $\pm 3 \pmod 8$. ||

The lecturer is indepted to J.W.S. Cassels for a helpful letter containing the above theorem and proof. See also Cassels [7].

Now let f be a quadratic form over $Q(2) = \mathbb{Z}_2 \cap \mathbb{Q}$. Then the signature $\tau = \tau(f)$ is defined.

<u>THEOREM (3.9)</u>: For a non-singular quadratic form f over $Q(2)$,

$$c_2(f) c_\infty(f) = (-1)^{(f(w,w) - \tau - \det f + \operatorname{sign} \det f)/4} .$$

<u>Proof</u>: Let α^+ and α^- be defined as at the beginning of this section. Then $\tau = \alpha^+ - \alpha^-$, $r = \alpha^+ + \alpha^-$ and clearly $\operatorname{sign}\det f = (-1)^{\alpha^-}$. In view of theorem (3.8) we must consider

$$\begin{aligned}
\frac{f(w,w) - \tau - \det f + \operatorname{sign}\det f}{4} - \frac{f(w,w) - r - \det f + 1}{4} &= \frac{-\tau + r + \operatorname{sign}\det f - 1}{4} \\
&= \frac{2\alpha^- + (-1)^{\alpha^-} - 1}{4} \\
&\equiv \frac{\alpha^-(\alpha^- - 1)}{2} \quad (\bmod 2).
\end{aligned}$$

Since $c_\infty(f) = (-1)^{\alpha^-(\alpha^- - 1)/2}$, the theorem is proved. ||

Now suppose f is a <u>unimodular</u> quadratic form, that is a non--singular integral form. Then $\det f = \pm 1$ so $\det f - \operatorname{sign}\det f = 0$. Furthermore, by lemma (1.24), $c_p(f) = 1$ if p is an odd prime. Hence by lemma (1.25) $c_2(f)c_\infty(f) = 1$. With (3.9) this proves

<u>THEOREM (3.10)</u>: If f is a unimodular form then

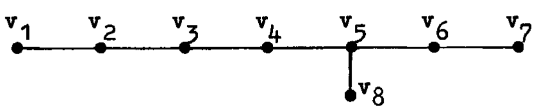
$$f(w,w) - \tau \equiv 0 \quad (\bmod 8). \quad ||$$

If f is in addition an even form, then $w = 0$ is a characteristic element, so

<u>COROLLARY (3.11)</u>: If f is an even unimodular form then

$$\tau(f) \equiv 0 \quad (\bmod 8). \quad ||$$

We remark that this last result is the best possible in the sense that there actually exists a quadratic form satisfying the hypothesis with $\tau = 8$. Indeed let E_8 be the following graph:



.

Construct an integral matrix $M = (\mu_{ij})$ by the formulae

$$\mu_{ii} = 2 \qquad 1 \le i \le 8$$

$$\mu_{ij} = 1 \qquad \text{if the vertices } v_i \text{ and } v_j \text{ are joined by an edge in } E_8 \,,$$

$$\mu_{ij} = 0 \qquad \text{otherwise.}$$

Thus

$$M = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

The corresponding even quadratic form is easily seen to be unimodular with $\tau = 8$. Until we give a more general definition of "quadratic forms of graphs" in §8, we shall refer to this quadratic form as "the" quadratic form associated with the graph E_8.

§4. INTEGRAL UNIMODULAR QUADRATIC FORMS.

This section is based both on the original lectures and on the article by J.P. Serre [24]. The proofs of the statements on the first two pages can be found both in Serre's article and in Jones [11].

Let $f = (f,V)$ be a non-degenerate integral quadratic form of rank n. The set of non-negative integers $\{|f(x,x)| \mid x \in V\}$ has a minimum which we denote by $\min f$. If $\min f = 0$ we say f is a <u>zero form</u> or f <u>represents zero</u>.

With respect to a base $e_1, \ldots, e_n$ of the lattice V the form f is given by

$$f = \sum_{i,j} a_{ij} x_i x_j \quad . \tag{4.1}$$

If f is not a zero form one can always choose the base (e_i) of V such that the expression (4.1) is <u>Hermite reduced</u>. This term is defined inductively as follows:

1). If f has rank 1, then $f = a_{11}x_1^2$ is Hermite reduced.

2). If f has rank n, the expression (4.1) is Hermite reduced if

 a). $|a_{11}| \geq 2|a_{1j}|$ for $j > 1$,

 b). $|a_{11}| = \min f$,

 c). $a_{11}f = (\sum a_{1i}x_i)^2 + f_1(x_2, \ldots, x_n)$, where f_1 is Hermite reduced of rank $n-1$.

The following theorem gives a bound for $|a_{11}|$ in an Hermite reduced expression for f.

THEOREM (4.2): If f is a non-degenerate integral form of rank n then

$$\min f \leq (4/3)^{(n-1)/2} |\det f|^{1/n} .$$

COROLLARY (4.3): If f is an integral unimodular form of rank ≤ 5 and is not a zero form, then it is equivalent to either $\sum x_i^2$ or $-\sum x_i^2$. ||

In particular an indefinite unimodular form of rank ≤ 5 represents zero. For forms of rank ≥ 5 the condition unimodular may be dropped:

THEOREM (4.4) (Meyer): Every non-degenerate indefinite integral form of rank ≥ 5 represents zero.

We now turn to study the Grothendieck ring $G_o(\mathbb{Z})$ defined in §2. An integral unimodular form f represents an element (also denoted by f) in the semigroup $\mathcal{F}_o(\mathbb{Z})$ (c.f.(2.3)). The elements in $\mathcal{F}_o(\mathbb{Z})$ represented by the forms x^2 and $-x^2$ will be denoted by 1 and -1 respectively. The element in $G_o(\mathbb{Z})$ represented by $f \in \mathcal{F}_o(\mathbb{Z})$ will be denoted by $\bar{f}$. This differs very slightly from our conventions in §2.

LEMMA (4.5): Every indefinite odd unimodular form can be decomposed into the form $1 \oplus (-1) \oplus g$, with g non-singular.

Proof: Let $f = (f,V)$ be the given form. By (4.3) and (4.4) f

is a zero form. Let $x \in V$ be an indivisible element satisfying $f(x,x) = 0$. Since f is non-singular, the correlation φ of f maps x into an indivisible element $\varphi(x)$ in V'. Hence there exists a $y \in V$ with $\langle \varphi(x), y \rangle = f(x,y) = 1$. If $f(y,y)$ is even, we choose a $z \in V$ with $f(z,z) \equiv 1 \pmod 2$ (this is possible since f is odd) and replace y by $y' = z + (1-f(x,z))y$. Then $f(y',y')$ is odd and $f(x,y') = 1$. We may hence assume that $f(y,y)$ is odd, say $f(y,y) = 2m+1$. The elements $e_1 = y-mx$ and $e_2 = y-(m+1)x$ are indivisible in V, as $f(e_1,e_1) = 1 = -f(e_2,e_2)$. The lemma hence follows from (1.4). ||

<u>THEOREM (4.6)</u>: If f is an indefinite odd unimodular form then f is equivalent to $\alpha^{+}\cdot 1 \oplus \alpha^{-}\cdot(-1)$.

<u>Proof</u>: By lemma (4.5) f is equivalent to $1 \oplus (-1) \oplus g$. Since one of $1 \oplus g$ and $(-1) \oplus g$ has to be indefinite, the theorem follows by a trivial induction. ||

<u>COROLLARY (4.7)</u>: Two indefinite odd unimodular forms are equivalent if they have the same rank and same index. ||

We are now ready to calculate $G_o(\mathbb{Z})$.

<u>THEOREM (4.8)</u>: $G_o(\mathbb{Z}) = \mathbb{Z}\cdot\bar{1} \oplus \mathbb{Z}\cdot(\overline{-1})$.

<u>Proof</u>: For $f \in \mathcal{F}_o(\mathbb{Z})$, either $1 \oplus f$ or $(-1) \oplus f$ is odd and indefinite. By (4.6) it follows that $\bar{f} = \alpha^{+}\cdot\bar{1} + \alpha^{-}\cdot(\overline{-1})$ in $G_o(\mathbb{Z})$, so $\bar{1}$ and $\overline{-1}$ generate $G_o(\mathbb{Z})$. Since the homomorphism $G_o(\mathbb{Z}) \longrightarrow G_o(\mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z}$, induced by the inclusion $\mathbb{Z} \subset \mathbb{R}$, maps these generators into free generators of $\mathbb{Z} \oplus \mathbb{Z}$, the theorem follows ||

Remark (4.9) : Corollary (3.10), which states that $f(w,w)-\tau(f) \equiv 0 \pmod 8$ for any unimodular form f now follows easily from (4.8). The map $h: \mathcal{F}_0(\mathbb{Z}) \longrightarrow \mathbb{Z}/8\mathbb{Z}$ defined by $h(f) = f(w,w) - \tau(f) \pmod 8$ is a semigroup homomorphism, so it induces a homomorphism $\bar{h}: G_0(\mathbb{Z}) \longrightarrow \mathbb{Z}/8\mathbb{Z}$. (3.10) is equivalent to saying that $\bar{h}$ vanishes identically on $G_0(\mathbb{Z})$; but this is clear since $\bar{h}$ vanishes on the generators $\bar{1}$ and $\overline{-1}$ of $G_0(\mathbb{Z})$.

The structure of integral unimodular indefinite odd forms is completely determined by theorem (4.6). The structure of unimodular indefinite even forms can also be determined (see Serre [24]) : two such forms are equivalent if and only if they have the same rank and same index. Combining this with (4.7) we have

THEOREM (4.10) : Two unimodular indefinite quadratic forms are equivalent if and only if they have the same rank, index and type (even or odd). ||

As for definite forms, our knowledge is meagre. We have for example :

THEOREM (4.11) : Two unimodular quadratic forms of the same rank ≤ 8 are equivalent if they have the same signature and type.

The last theorem is no longer true for rank > 8. For example, let f be the form associated with E_8 as defined in §3, and let $f' = f \oplus 1$. Then f' and $g = 1 \oplus 1 \oplus \ldots \oplus 1$ (9 times) are of the same rank, signature and type. But they are not equivalent as $f'(x,x) = 1$ has only 2 solutions but $g(x,x) = 1$ has 18.

§5. QUADRATIC FORMS OVER $\mathbb{Z}_p$; THE GENUS OF INTEGRAL FORMS.

So far we have considered exclusively quadratic forms which are non-degenerate. This is not an essential restriction; for if $f = (f,V)$ is any quadratic form, the kernel $\mathrm{Rad}(V)$ of the correlation $\varphi : V \longrightarrow V'$ of f is a direct summand of V, and f induces a non-degenerate form $\tilde{f}$ on $V/\mathrm{Rad}(V)$. We shall denote $\det \tilde{f}$ and $\mathrm{DET}\,\tilde{f}$ by $\widetilde{\det}\, f$ and $\widetilde{\mathrm{DET}}\, f$ respectively, and agree that if f is totally isotropic (that is $f(x,y) = 0$ for all $x,y \in V$) then $\widetilde{\mathrm{DET}}\, f = 1$.

<u>THEOREM (5.1)</u>: Every non-degenerate quadratic form f over $\mathbb{Z}_p$ may be decomposed into

$$f = f_o \oplus pf_1 \oplus p^2 f_2 \oplus \cdots \oplus p^k f_k \quad , \tag{5.2}$$

where f_i $(i = 0, \dots, k)$ is a non-singular form of rank $r_i \geq 0$.

<u>COROLLARY (5.3)</u>: Every non-degenerate quadratic form over $\mathbb{Z}_p$ $(p \neq 2)$ decomposes into unary forms.

<u>Proofs</u>: The corollary follows immediately from the theorem and (1.5). To prove (5.1) let

$$M = (\alpha_{ij}) \qquad (i,j = 1,2, \dots ,r)$$

be the matrix of f with respect to some base. If every α_{ij} is divisible by p in $\mathbb{Z}_p$ we say that M is divisible by p. We can write $M = p^t M'$, where $M' = (\alpha'_{ij})$ is not divisible by p. Notice that M' need not be invertible.

a). If not all the diagonal entries of M' are divisible by p, then some of them are units. Applying (1.4), we may split off these entries.

b). If all the diagonal entries of M' are divisible by p, then at least one entry off the diagonal, say α'_{ij}, is a unit. But then the minor

$$\begin{pmatrix} \alpha'_{ii} & \alpha'_{ij} \\ \alpha'_{ji} & \alpha'_{jj} \end{pmatrix}$$

is non-singular and hence splits off.

By repeated application of a) and b) we get $f = p^t(f_t \oplus g)$, where f_t is non-singular and the matrix of g is divisible by p. An obvious induction completes the proof. ||

The ranks r_i of the forms f_i in (5.2) are uniquely determined by the equivalence class of f. Indeed, by the structure theorem for finitely generated modules over a principal ideal domain (c.f. Bourbaki [6] Ch VII, §4, th 2), the cokernel of the correlation $\varphi : V \longrightarrow V'$ is isomorphic to $\bigoplus_j \mathbb{Z}_p/\mathfrak{a}_j$, where $\{0\} \neq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots \subseteq \mathfrak{a}_m \neq \mathbb{Z}_p$ are ideals uniquely determined by φ, and hence by f. In $\mathbb{Z}_p$ the only non-zero ideals are of the form $p^t\mathbb{Z}_p$. The number of times $\mathbb{Z}_p/p^t\mathbb{Z}_p$ appears in $\bigoplus_j \mathbb{Z}_p/\mathfrak{a}_j$ is clearly r_t, giving an invariant definition of the r_i for $i \geq 1$. Since $\sum r_i = \operatorname{rank} f$, r_0 is then also determined.

Each f_i in (5.2) is non-singular, so $\det f_i$ is a unit. For $p \neq 2$ we define $\gamma_i := (\det f_i | p)$ for each i, so

$$\gamma_i = \begin{cases} 1 & \text{if } \det f_i \in \mathbb{Z}_p^{*2} \\ -1 & \text{if } \det f_i \notin \mathbb{Z}_p^{*2} \end{cases}$$

In other words, if we identify $\mathbb{Z}_p^*/\mathbb{Z}_p^{*2}$ with $\{1,-1\}$, then $\gamma_i = \mathrm{DET}\, f_i$. We shall show that the γ_i only depend on the equivalence class of f; they are called the Minkowski invariants of f.

To prove the invariance of the γ_i we give an invariant definition. Let $f = (f,V)$. We can consider f as a quadratic form over $V \otimes \mathbb{Q}_p$ (see beginning of §3). Consider $V \subset V \otimes \mathbb{Q}_p$, and let V^+ be the "dual" of V in $V \otimes \mathbb{Q}_p$, i.e.

$$V^+ = \{x \in V \otimes \mathbb{Q}_p \mid f(x,y) \in \mathbb{Z}_p \text{ for all } y \in V\} .$$

Then $U = V^+/V$ is a finite abelian p-group, and f induces a bilinear pairing

$$t : U \times U \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p .$$

Filter the group U by

$$\{0\} = U_0 \subseteq U_1 \subseteq \dots \subseteq U ,$$

where $U_i = \{x \in U \mid p^i x = 0\}$. It is clear that every element of the quotient $W_i = U_i/U_{i-1}$ is of order p, so we may consider W_i as a vector space over the finite field $\mathbb{F}_p$. The above pairing t induces a bilinear pairing

$$f_i' : W_i \times W_i \longrightarrow \frac{1}{p^i}\mathbb{Z}_p/\frac{1}{p^{i-1}}\mathbb{Z}_p \cong \mathbb{F}_p .$$

Therefore f_i' is a quadratic form over $\mathbb{F}_p$ and $\widetilde{\mathrm{DET}}\, f_i'$ is defined. Identify $\mathbb{F}_p^*/\mathbb{F}_p^{*2}$ with $\{1,-1\}$, so $\widetilde{\mathrm{DET}}\, f_i' = \pm 1$. We claim that $\gamma_i = \widetilde{\mathrm{DET}}\, f_i'$ $(i \geq 1)$. Indeed let $f_i = (f_i, V_i)$ be as in (5.2). Then

$V = \bigoplus_i V_i$ and $V^+ = \bigoplus_i \frac{1}{p^i} V_i \subset V \otimes \mathbb{Q}_p$. It is now easy to see that $\tilde{f}'_i$ is just f_i reduced modulo p , so $\gamma_i = \mathrm{DET}\,\tilde{f}'_i$. The invariance of the γ_i is hence proved for $i \geq 1$, and for γ_o it follows since $\gamma_o(f) = \gamma_1(pf)$. Alternatively, it is easily seen that γ_o is determined by $\gamma_1, \gamma_2, \ldots$ and $\det f$. Observe that $r_i = \mathrm{rk}\,\tilde{f}'_i$, giving a second invariant definition of the r_i .

An immediate consequence of (1.20) is :

<u>THEOREM (5.4)</u> : The integers r_i and γ_i $(i \geq 0)$ constitute a complete set of invariants for the equivalence classes of non-degenerate quadratic forms over $\mathbb{Z}_p$ $(p \neq 2)$. ||

The following lemma is not hard to prove (c.f. Jones [11] p91). For $p \neq 2$ it is an immediate consequence of (1.20) and (2.9).

<u>LEMMA (5.5)</u> : Let f and g be non-singular quadratic forms over $\mathbb{Z}_p$. For $p \neq 2$ f and g are equivalent if and only if they are equivalent over $\mathbb{Q}_p$. For $p = 2$ they are equivalent if and only if they have the same type (even or odd) and are equivalent over $\mathbb{Q}_2$. ||

We now return to integral quadratic forms. Two such forms f and g are called <u>semi-equivalent</u> or said to have the same <u>genus</u> if f is equivalent to g over $\mathbb{Z}_p$ for $p = 2, 3, \ldots, \infty$.

<u>THEOREM (5.6)</u> : Two integral even quadratic forms f and g are semi-equivalent if they are equivalent over $\mathbb{Z}_p$ for $p = 3, 5, \ldots, \infty$.

Proof: In virtue of lemma (5.5) we need only show that f and g are equivalent over $\mathbb{Q}_2$. $f \sim g$ over $\mathbb{Z}_\infty$ implies $\operatorname{rk} f = \operatorname{rk} g$. Furthermore (1.25) implies $c_2(f) = c_2(g)$ and hence $\tilde{c}_2(f) = \tilde{c}_2(g)$. Since f and g are even forms, the last equality implies $\det f \equiv \det g \pmod 8$, and hence $\operatorname{DET} f = \operatorname{DET} g$ (c.f. proof of (1.18)) Thus by (2.9) $f \sim g$ over $\mathbb{Q}_2$. ||

THEOREM (5.7): The rank, index and type (even or odd) are a complete system of invariants for the genus of unimodular forms.

Proof: If f and g are unimodular forms with the same rank, index and type, then by (1.24) and (1.25) $c_2(f) = c_2(g)$. Since clearly $\det f = \det g$, it follows by (2.9) that $f \sim g$ over $\mathbb{Q}_2$. The theorem thus follows by (1.20) and (5.5). ||

Remark (5.8): Let $f = (f,V)$ be an integral quadratic form. We denote the invariants r_i and γ_i of $f^{\mathbb{Z}_p}$ by $r_i^{(p)}$ and $\gamma_i^{(p)}$. These are invariants of f, and can also be defined directly, without reference to $\mathbb{Z}_p$. Namely, consider f as a form defined on $V \otimes \mathbb{Q}$ and let V^+ be the "dual" of $V \subset V \otimes \mathbb{Q}$ given by

$$V^+ := \{x \in V \otimes \mathbb{Q} \mid f(x,y) \in \mathbb{Z} \quad \text{for all} \quad y \in V\} .$$

Then $U := V^+/V$ is a finite abelian group of order $|\det f|$, and f induces a bilinear pairing

$$L: U \times U \longrightarrow \mathbb{Q}/\mathbb{Z} .$$

For any odd prime p, this pairing restricted to the p-component of U is essentially the pairing t considered earlier in this section. Thus if one puts $U_i^{(p)} = \{u \in U \mid p^i u = 0\}$ and $W_i^{(p)} = U_i^{(p)}/U_{i-1}^{(p)}$ then

L induces a bilinear pairing

$$W_i^{(p)} \times W_i^{(p)} \longrightarrow \mathbb{F}_p$$

which can be used to define $\gamma_i^{(p)}$ and $r_i^{(p)}$

§6. THE QUADRATIC FORM OF A 4k-DIMENSIONAL MANIFOLD.

Unless otherwise specified, by a <u>manifold</u> we mean a connected compact orientable differentiable manifold with or without boundary together with a given orientation. Thus all manifolds under consideration are <u>oriented</u>.

Let M be a $4k$-dimensional manifold (briefly: $4k$-manifold). The homology is finitely generated, so $V := H_{2k}(M,\mathbb{Z})/\text{Torsion}$ is a $\mathbb{Z}$-lattice. The intersection numbers of cycles induces a quadratic form

$$S_M\colon V \times V \longrightarrow \mathbb{Z} \ .$$

S_M is called the <u>quadratic form of M</u>, and its signature $\tau(S_M)$ is called the <u>signature of M</u> and will also be written $\tau(M)$.

S_M can also be defined by means of the cup product in cohomology as follows. The Poincaré duality for an n-manifold may be expressed by an isomorphism

$$H_i(M;\mathbb{Z}) \cong H^{n-i}(M,\partial M;\mathbb{Z})$$

for each $i = 0,\dots,n$. In particular for $n = 4k$, $i = 2k$, we have

$$H_{2k}(M;\mathbb{Z}) \cong H^{2k}(M,\partial M;\mathbb{Z}) \ .$$

Denote the elements in $H^{2k}(M,\partial M;\mathbb{Z})$ which correspond to $a,b,\dots \in H_{2k}(M,\mathbb{Z})$ under this isomorphism by $\alpha,\beta,\dots$ respectively.

Since

$$\alpha \cup \beta \in H^{4k}(M, \partial M; \mathbb{Z}) = \mathbb{Z} \quad ,$$

we may consider $\alpha \cup \beta$ as an integer. This integer is precisely the intersection number of a and b (see for instance Hilton and Wylie [9] p156). We may thus regard S_M as the pairing defined on $H^{2k}(M, \partial M; \mathbb{Z})/\text{Torsion}$ into $\mathbb{Z}$ by cup product.

Exercise (6.1) : Show that if M is a 4k-manifold with empty boundary then S_M is unimodular (see for instance Milnor [14]).

Many of the results stated below have higher dimensional analogues but our main concern for a while will be 4-manifolds.

Let M be a 4-manifold with empty boundary and let $H^*(M; \mathbb{Z}/2\mathbb{Z})$ be its cohomology ring with coefficients in $\mathbb{Z}/2\mathbb{Z}$. The quadratic form of M over $\mathbb{Z}/2\mathbb{Z}$ is again given by cup product: $S(x,y) = x \cup y$. In this case there exists a unique characteristic element $w_2 = w_2(M)$ in $H^2(M; \mathbb{Z}/2\mathbb{Z})$ with $x \cup x = x \cup w_2$ for each $x \in H^2(M; \mathbb{Z}/2\mathbb{Z})$. We remark that (for instance by the Wu formula) w_2 is actually the middle Stiefel Whitney class of M.

The following theorems are known (Rohlin [21], Borel Hirzebruch [4], see also Kervaire Milnor [12] and [13]).

THEOREM (6.2) (Rohlin) : If M is a 4-manifold with $\partial M = \emptyset$ and $w_2 = 0$ then

$$\tau(M) \equiv 0 \pmod{16} \quad .$$

We remind the reader that all manifolds under consideration are differentiable. We do not know whether (6.2) holds in general for oriented topological 4-manifolds.

Let $\pi\colon H^2(M;\mathbb{Z}) \longrightarrow H^2(M;\mathbb{Z}/2\mathbb{Z})$ be the reduction modulo 2, that is the homomorphism induced by the coefficient epimorphism $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$.

<u>THEOREM (6.3)</u> (Borel Hirzebruch): If $d \in H^2(M;\mathbb{Z})$ is such that $\pi d = w_2$, then $S_M(d,d) \equiv \tau(M) \pmod 8$, where S_M is the integral quadratic form of M.

<u>Proof</u>: The assumption of (2.3) implies that d is a characteristic element of S_M, as cup product commutes with the homomorphism π. Hence (6.3) follows from (6.1) and (3.10). ||

A weaker version of (6.2), namely $\tau(M) \equiv 0 \pmod 8$ is of course a special case of (6.3).

The short exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ of coefficients induces an exact sequence

(6.4) $$\cdots \longrightarrow H^2(M;\mathbb{Z}) \xrightarrow{2} H^2(M;\mathbb{Z}) \xrightarrow{\pi} H^2(M;\mathbb{Z}/2\mathbb{Z}) \longrightarrow \cdots,$$

where the number 2 over an arrow means multiplication by 2. If M has no 2-torsion (that is, the groups $H^i(M;\mathbb{Z})$ have no elements of even order), then (6.4) leads to a short exact sequence

$$0 \longrightarrow H^2(M;\mathbb{Z}) \xrightarrow{2} H^2(M;\mathbb{Z}) \xrightarrow{\pi} H^2(M;\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0,$$

so there exists a $d \in H^2(M;\mathbb{Z})$ with $\pi d = w_2$. The set of all such d, considered as elements of $H^2(M;\mathbb{Z})/\text{Torsion}$, is just the set of characteristic elements for S_M. We conclude that if M has no 2-torsion, then $w_2 = 0$ if and only if the form S_M is even. Thus we may state

THEOREM (6.2a): If M is a 4-manifold with no 2-torsion and S_M is even, then $\tau(M) \equiv 0 \pmod{16}$. ||

Thus the quadratic form associated with E_8 given in §3 cannot occur as the quadratic form of a 4-manifold with no 2-torsion.

The following theorem of Kervaire and Milnor sharpens the results of (6.2) and (6.3).

THEOREM (6.5): In the statement of (6.3), if the dual class of d can be represented by a differentiably imbedded 2-sphere in M then $S_M(d,d) \equiv \tau(M) \pmod{16}$.

The proof of this theorem and some examples will be given in §10.

§7. AN APPLICATION OF ROHLIN'S THEOREM, μ-INVARIANTS.

If G is an abelian group, then a <u>G-homology k-sphere</u> is a manifold X with $\partial X = \emptyset$ which has the same G-homology as the k-sphere; that is $H_i(X;G) \cong G$ for $i = 0, k$, and $H_i(X;G) = 0$ otherwise. If $G \neq 0$, this implies $\dim X = k$.

We intend to define an invariant μ for $\mathbb{Z}/2\mathbb{Z}$-homology 3-spheres, but first we need a lemma.

<u>LEMMA (7.1)</u>: Any $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere bounds a 4-manifold Y with

a). $H_1(Y;\mathbb{Z})$ has no 2-torsion;

b). s_Y is an even quadratic form.

The proof will be postponed to the end of this section.

Now let X be a $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere and Y be as in lemma (7.1). Define

$$\mu(X) := \frac{-\tau(Y)}{16} \in \mathbb{Q}/\mathbb{Z} ,$$

that is $\frac{-\tau(Y)}{16}$ reduced modulo 1.

<u>THEOREM (7.2)</u>: $\mu(X)$ is an invariant of the oriented diffeomorphism type of X.

For this reason we call $\mu(X)$ the <u>μ-invariant</u> of X. This is a special case of the μ-invariants studied by Eells and Kuiper in [8]. Before we prove (7.2) we need

LEMMA (7.3): If X is a $\mathbb{Z}/2\mathbb{Z}$-homology k-sphere then

a). $H_i(X;\mathbb{Z})$ is a torsion group of odd order for $i \neq 0,k$;

b). X is a $\mathbb{Q}$-homology sphere;

c). X is a $\mathbb{Z}_2$-homology sphere (we remind topologists that $\mathbb{Z}_2$ denotes the ring of 2-adic integers).

Proof: Consider the exact sequence

$$(7.4) \qquad \cdots \longrightarrow H_{i+1}(X;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H_i(X;\mathbb{Z}) \xrightarrow{2} H_i(X;\mathbb{Z}) \longrightarrow \cdots$$

induced by the exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \quad .$$

If $i \neq 0,k$ then $H_i(X;\mathbb{Z}) \xrightarrow{2} H_i(X;\mathbb{Z})$ is an epimorphism, so since the groups involved are finitely generated, a) follows. b) is an obvious consequence of a), so it remains to prove c).

Recall that the ring $\mathbb{Z}_2$ of 2-adic integers is a principal ideal domain and $H_i(X;\mathbb{Z}_2)$ is finitely generated. Hence

$$(7.5) \qquad H_i(X;\mathbb{Z}_2) \cong \bigoplus_{k=1}^{m} \mathbb{Z}_2/\mathfrak{a}_k \quad ,$$

where $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_m \subset \mathbb{Z}_2$ are ideals in $\mathbb{Z}_2$ (see for instance Bourbaki [6] ch7, §4, th2). But the only ideals in $\mathbb{Z}_2$ are the zero ideal and the ideals $2^j\mathbb{Z}_2$, so each summand in (7.5) is of the form $\mathbb{Z}_2$ or $\mathbb{Z}_2/2^j\mathbb{Z}_2 \cong \mathbb{Z}/2^j\mathbb{Z}$. However, using the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_2 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

one can replace $\mathbb{Z}$ by $\mathbb{Z}_2$ in (7.4), showing that

$H_i(X;\mathbb{Z}_2) \xrightarrow{2} H_i(X;\mathbb{Z}_2)$ is epic for $i \neq 0,k$. It follows that $H_i(X;\mathbb{Z}_2) = 0$ for $i \neq 0,k$. ||

We now return to the proof of (7.2).

<u>Proof</u> (7.2) : It clearly suffices to show that $\mu(X)$ is independant of the choice of Y. Let Y_1 and Y_2 be two 4-manifolds as given by lemma (7.1). Let $M = Y_1 \cup -Y_2$ pasted together along the common boundary X. Here $-Y_2$ of course means Y_2 with orientation reversed. M has, after smoothing if necessary, a differentiable structure compatible with those on Y_1 and $-Y_2$. We refer the reader to Milnor [17], [18] for details about pasting and smoothing.

We claim that M has the following properties:

1). M has no 2-torsion;

2). S_M is even;

3). $S^{\mathbb{Q}}_{Y_1} \oplus -S^{\mathbb{Q}}_{Y_2} = S^{\mathbb{Q}}_M$, where $S^{\mathbb{Q}}$ is defined as in §3.

Before we prove these properties, observe that they suffice to prove (7.2). For by property 3), $\tau(M) = \tau(Y_1) - \tau(Y_2)$, and properties 1) and 2) together with Rohlin's theorem (6.2a) imply that $\tau(M) \equiv 0 \pmod{16}$. Theorem (7.2) then follows.

To prove 1), 2) and 3) above, apply the Mayer-Vietoris sequence to $M = Y_1 \cup -Y_2$ to obtain

$$(7.6) \qquad \cdots \longrightarrow H_i(X) \longrightarrow H_i(Y_1) \oplus H_i(Y_2) \longrightarrow H_i(M) \longrightarrow H_{i-1}(X) \longrightarrow \cdots ,$$

where the coefficient group is not specified.

By (7.3) we know that X is a $\mathbb{Q}$-homology sphere, so (7.6) gives

$$(7.7) \qquad 0 \longrightarrow H_2(Y_1;\mathbb{Q}) \oplus H_2(Y_2;\mathbb{Q}) \xrightarrow{\cong} H_2(M;\mathbb{Q}) \longrightarrow 0 .$$

Taking orientations into account, it is not hard to see that $S^{\mathbb{Q}}_{Y_1} \oplus -S^{\mathbb{Q}}_{Y_2} = S^{\mathbb{Q}}_M$, proving property 3).

Next we take integral coefficients in (7.6) to obtain

$$H_1(X;\mathbb{Z}) \longrightarrow H_1(Y_1;\mathbb{Z}) \oplus H_1(Y_2;\mathbb{Z}) \longrightarrow H_1(M,\mathbb{Z}) \longrightarrow \widetilde{H}_0(X;\mathbb{Z}) ,$$

where $\widetilde{H}_0(X,\mathbb{Z}) = 0$ is the reduced homology group. Since Y_1 and Y_2 have no 2-torsion and $H_1(X;\mathbb{Z})$ is a torsion group (by (7.3a)), it follows that $H_1(M;\mathbb{Z})$ has no 2-component. Poincaré duality of homology groups (see for instance [23] p245, Satz III) now proves property 1).

Finally, we take coefficients $\mathbb{Z}_2$ in (7.6) and apply (7.3c) to obtain

$$0 \longrightarrow H_2(Y_1;\mathbb{Z}_2) \oplus H_2(Y_2;\mathbb{Z}_2) \xrightarrow{\cong} H_2(M;\mathbb{Z}_2) \longrightarrow 0 ,$$

just as in (7.7). Since $\mathrm{Tor}(\text{odd torsion},\mathbb{Z}_2) = 0$, the universal coefficient theorem gives that $H_2(Y_i;\mathbb{Z}_2) = H_2(Y_i;\mathbb{Z}) \otimes \mathbb{Z}_2$ and $H_2(M;\mathbb{Z}_2) = H_2(M;\mathbb{Z}) \otimes \mathbb{Z}_2$. Recall that $S^{\mathbb{Z}_2}_{Y_i}$ is defined on $[H_2(Y_i;\mathbb{Z})/\text{Torsion}] \otimes \mathbb{Z}_2 = [H_2(Y_i;\mathbb{Z}) \otimes \mathbb{Z}_2]/\text{Torsion}$. Thus

$$S^{\mathbb{Z}_2}_M = S^{\mathbb{Z}_2}_{Y_1} \oplus -S^{\mathbb{Z}_2}_{Y_2} .$$

The terms on the right hand side are even by assumption, so $S^{\mathbb{Z}_2}_M$ is even. This implies that S_M itself is even, proving property 2) and completing the proof of (7.2). ||

As an example we shall apply the μ-invariant to lens spaces. Let

$$S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \mid z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}$$

be the standard 3-sphere, with canonical orientation inherited from $\mathbb{C}^2$. Let n and q be integers with greatest common divisor

$(n,q)=1$. Define an operation of the group $\mathbb{Z}/n\mathbb{Z}$ on S^3 by

$$(7.8) \qquad \gamma(z_1,z_2) = (e^{2\pi i\gamma/n}z_1, e^{2\pi iq\gamma/n}z_2)$$

for each $\gamma \in \mathbb{Z}/n\mathbb{Z}$. Then $\mathbb{Z}/n\mathbb{Z}$ acts freely and differentiably on S^3 . The quotient space (space of orbits) $S^3/(\mathbb{Z}/n\mathbb{Z}) =: L(n,q)$ is, by definition, the lens space of type (n,q), with orientation and differentiable structure inherited from S^3 . Since $L(n,q)$ has S^3 as universal covering space and $\mathbb{Z}/n\mathbb{Z}$ as group of covering transformations,

$$\pi_1(L(n,q)) = \mathbb{Z}/n\mathbb{Z} = H_1(L(n,q);\mathbb{Z}) \quad .$$

Thus the Betti numbers of $L(n,q)$ are $1,0,0,1$ and the only torsion coefficient is n at dimension 1 . If n is odd it follows that $L(n,q)$ is a $\mathbb{Z}/2\mathbb{Z}$-homology sphere.

Clearly $L(n,q)$ only depends on the residue of q modulo n , so one may assume $0<q<n$. Furthermore

<u>LEMMA (7.9)</u> : With respect to the canonical orientations

$$L(n,q) = -L(n,n-q) \quad .$$

<u>Proof</u> : The differentiable orientation reversing involution $\alpha\colon (z_1,z_2) \mapsto (z_1,\bar{z}_2)$ on S^3 carries $L(n,q)$ onto $-L(n,n-q)$, since

$$\alpha(e^{2\pi i\gamma/n}z_1, e^{2\pi iq\gamma/n}z_2) = (e^{2\pi i\gamma/n}z_1, e^{2\pi i(n-q)\gamma/n}\bar{z}_2) \quad . \qquad \|$$

Since clearly $\mu(-X) = -\mu(X)$ for any $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere X , we need only consider lens spaces $L(n,q)$ where q takes even values. Recall that a fraction $n/q>1$ can be expanded into a finite continued fraction :

$$n/q = b_1 - \cfrac{1}{b_2 - \ddots - \cfrac{1}{b_s}}$$

$$= [b_1, b_2, \ldots, b_s] \qquad \text{(notation)},$$

where each b_i is an integer with $|b_i| \geq 2$. It is easy to prove that if n is odd and q is even, there is a unique expansion of this type with each b_i even. Denote by $p^+ = p^+(n,q)$ and $p^- = p^-(n,q)$ the number of positive and negative b_i's in this unique expansion.

RECIPE (7.10): $\mu(L(n,q)) = \dfrac{p^+ - p^-}{16} \in \mathbb{Q}/\mathbb{Z}$.

A proof will be given in §8. As examples we have:

$$7/6 = [2,2,2,2,2,2], \quad p^+ = 6, \quad p^- = 0;$$
$$7/2 = [4,2] \qquad\qquad, \quad p^+ = 2, \quad p^- = 0.$$

Therefore,

$$\mu(L(7,6)) = 3/8 \quad,$$
$$\mu(L(7,2)) = 1/8 \quad,$$
$$\mu(L(7,1)) = -\mu(L(7,6)) = -3/8 = 5/8 \quad (\mathrm{mod}\ 1).$$

Recall (J.H.C. Whitehead [28]):

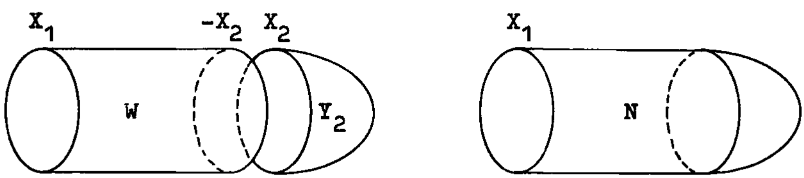
THEOREM (7.11): Two lens spaces $L(n,q)$ and $L(n,q')$ have the same homotopy type if and only if either qq' or $-qq'$ is a quadratic residue modulo n. ||

Thus $L(7,1)$ and $L(7,2)$ have the same homotopy type; but they are not diffeomorphic since $\mu(L(7,1))$ differs from $\mu(L(7,2))$ and $-\mu(L(7,2))$.

Now let X_1 and X_2 be two n-manifolds without boundaries. Recall that X_1 and X_2 are called <u>h-cobordant</u> if and only if there exists an n+1-manifold W such that $\partial W = X_1 \cup -X_2$ (disjoint union) and X_1 and X_2 are both deformation retracts of W.

<u>THEOREM (7.12)</u>: Let X_1 and X_2 be h-cobordant $\mathbb{Z}/2\mathbb{Z}$-homology 3-spheres. Then $\mu(X_1) = \mu(X_2)$.

<u>Proof</u>: Let $X_i = \partial Y_i$ $(i = 1,2)$ where the Y_i are as in lemma (7.1), and let W be the manifold of the h-cobordism. Let N be $W \cup Y_2$ pasted along X_2.



Since X_2 is a deformation retract of W, N is homotopy equivalent to Y_2, so $S_N = S_{Y_2}$. On the other hand, N is a manifold as in (7.1) with $\partial N = X_1$. Hence

$$\mu(X_1) = -\frac{\tau(N)}{16} \pmod 1 = -\frac{\tau(Y_2)}{16} \pmod 1 = \mu(X_2) . \quad \|$$

It follows that $L(7,1)$ and $L(7,2)$ are not even h-cobordant.

Observe that the set γ_3 of all $\mathbb{Z}/2\mathbb{Z}$-homology 3-spheres is closed under connected sum (the connected sum $X_1 \# X_2$ of two n-manifolds X_1 and X_2 is obtained by cutting a small open disc out of each and pasting together along the resulting boundaries S^{n-1}). With respect to the operation $\#$, γ_3 is a commutative semigroup with identity. One easily proves the following

<u>THEOREM (7.13)</u>: As a map $\mathcal{V}_3 \longrightarrow \mathbb{Q}/\mathbb{Z}$, μ is a semigroup homomorphism which maps the identity onto the identity. ||

The operation # is compatible with the h-cobordism relation in $\mathcal{V}_3$ (see Milnor [17] lemma 2.3). Hence μ is even a homomorphism of the semigroup of h-cobordism classes in $\mathcal{V}_3$ into $\mathbb{Q}/\mathbb{Z}$.

An easy consequence of the definition of μ and the Alexander duality theorem is

<u>Exercise (7.14)</u>: Let X be a $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere. If X is embeddable in $\mathbb{R}^4$ then $\mu(X) = 0$.

The spherical dodecahedral space is a classical example of a $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere (see for example Seifert and Threlfall [23] p218). The μ-invariant of this space, as we shall show in §8, is 1/2, so this space is not embeddable in $\mathbb{R}^4$.

We now give the promised proof of lemma (7.1). Recall the well-known fact that every 3-manifold is parallelizable (Stiefel [26]). Therefore, by Milnor [17] (see also M.W. Hirsch [10]), every 3-manifold with empty boundary bounds a simply connected π-manifold. Let X be a $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere and Y a simply connected π-manifold with $\partial Y = X$. Then (7.1a) is trivially satisfied. We verify b).

Let M be the double of Y, that is $M = Y \cup -Y$ pasted along the common boundary X. We claim $w_2(M) = 0$. Indeed $w_2(Y) = 0$ since Y is a π-manifold, so the claim follows immediately from the

Mayer-Vietoris sequence with coefficients in $\mathbb{Z}/2\mathbb{Z}$

$$0 = H^1(X) \longrightarrow H^2(M) \longrightarrow H^2(Y) \oplus H^2(Y) \longrightarrow H^2(X) = 0$$

and the naturality of the Stiefel-Whitney classes. It follows, as in §6, that S_M is even. On the other hand

$$S_M^{\mathbb{Z}2} = S_Y^{\mathbb{Z}2} \oplus -S_Y^{\mathbb{Z}2}$$

(see proof of (7.2)). It follows that S_Y is even, as was to be shown. ||

We end this section with a digression. Let Y be an oriented $4k$-manifold with boundary $\partial Y = X$ and assume

1). Y only has homology in dimensions 0 and $2k$ and this is torsion-free (integer homology);

2). $\partial Y = X$ is a rational homology sphere.

(These assumptions may be weakened). Consider

$$\begin{tikzcd}
H_{2k}(Y) \arrow[r] \arrow[dr, "\varphi"'] & H_{2k}(Y,X) \arrow[r] & H_{2k-1}(X) \arrow[r] & 0 \\
& \mathrm{Hom}(H_{2k}(Y),\mathbb{Z}) \arrow[ur] &
\end{tikzcd}$$

$$H_{2k}(Y,X) \cong H^{2k}(Y) \cong \mathrm{Hom}(H_{2k}(Y),\mathbb{Z})$$

(coefficients in $\mathbb{Z}$). Let $V = H_{2k}(Y)$, so the quadratic form S_Y of Y is defined on V . Define, as in remark (5.8), $U = V^+/V$, where $V^+ = \{x \in V \otimes \mathbb{Q} \mid S_Y(x,y) \in \mathbb{Z} \text{ for all } y \in V\}$. Now φ is the correlation for S_Y , and $\mathrm{cokern}(\varphi) \cong H_{2k-1}(X)$ is a finite torsion group, so S_Y is non-degenerate. Furthermore $\mathrm{Hom}(H_{2k}(Y),\mathbb{Z}) = H_{2k}(Y,X)$ is essentially V^+ and $H_{2k-1}(X)$ can be identified with

U. As in (5.8), the quadratic form S_Y induces a bilinear pairing

$$L: U \times U \longrightarrow \mathbb{Q}/\mathbb{Z}$$

which can be used to define the invariants $\gamma_i^{(p)}$ and $r_i^{(p)}$ of S_Y, p an odd prime. However, as the informed reader may already have observed, L gives the linking numbers of homology classes in X (see Seifert and Threlfall [23] §77). Thus L, and consequently also $\gamma_i^{(p)}$ and $r_i^{(p)}$ are invariants of the oriented homotopy type of X.

Theorem (5.6) yields the following consequence: If Y_1 and Y_2 are $4k$-manifolds satisfying 1) and 2) above and the quadratic forms S_1 and S_2 of Y_1 and Y_2 are even with odd determinants and equivalent over $\mathbb{Z}_\infty = \mathbb{R}$, then S_1 and S_2 have the same genus if ∂Y_1 and ∂Y_2 have the same oriented homotopy type.

We shall first describe plumbing in arbitrary dimensions before going into more detail in the case which interests us here, namely plumbing of 2-disc bundles over S^2.

Recall (Steenrod [25] p 96) that principal $SO(n)$ bundles over S^n, and hence also n-disc bundles over S^n with structure group $SO(n)$, are classified up to equivalence by the elements of $\pi_{n-1} SO(n)$. This classification is as follows.

Consider S^n as the unit sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ in $\mathbb{R}^{n+1}$. S^n is the union of the two discs $D^n_+ = \{x \in S^n \mid x_{n+1} \geq 0\}$ and $D^n_- = \{x \in S^n \mid x_{n+1} \leq 0\}$, which intersect in their common boundary $S^{n-1} = \{x \in S^n \mid x_{n+1} = 0\}$. If E is a principal $SO(n)$ bundle over S^n, let

$$f_+ : D^n_+ \times SO(n) \longrightarrow E|D^n_+ ,$$

$$f_- : D^n_- \times SO(n) \longrightarrow E|D^n_- ,$$

be trivialisations of E restricted to each of the discs. Then the map

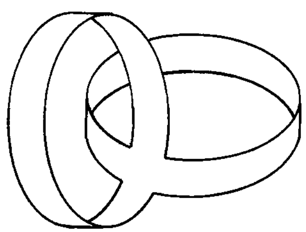
$$f_+^{-1} \cdot f_- : S^{n-1} \times SO(n) \longrightarrow S^{n-1} \times SO(n)$$

is of the form $(t,x) \longmapsto (t, \rho(t)x)$, where ρ is a map $\rho : S^{n-1} \longrightarrow SO(n)$. The homotopy class $[\rho] \in \pi_{n-1} SO(n)$ is the classifying element of the bundle.

Let $\xi_1 = (E_1, p_1, S_1^n)$ and $\xi_2 = (E_2, p_2, S_2^n)$ be two oriented n-disc bundles over S^n. Let $D_i^n \subset S_i^n$ be embedded n-discs in the base spaces and let

$$f_i : D_i^n \times D^n \longrightarrow E_i | D_i^n$$

be trivialisations of the restricted bundles $E_i | D_i^n$ for $i = 1,2$. To <u>plumb</u> ξ_1 and ξ_2 we take the disjoint union of E_1 and E_2 and identify the points $f_1(x,y)$ and $f_2(y,x)$ for each $(x,y) \in D^n \times D^n$.



We obtain an orientable manifold with boundary, and this manifold is differentiable except along $f_1(S^{n-1} \times S^{n-1})$, where it has a "corner". By Milnor [17] this corner can be smoothed, and the smoothing is essentially unique. We give a brief idea of how this can be done: A point $x \in f_1(S^{n-1} \times S^{n-1})$ has a neighbourhood which looks like $(S^{n-1} \times S^{n-1}) \times (\mathbb{R}_+ \times \mathbb{R} \cup \mathbb{R} \times \mathbb{R}_+)$, where $\mathbb{R}_+$ denotes the non-negative half line in $\mathbb{R}$. The second factor may be "straightened" by the transformation

$$(r\cos(\theta), r\sin(\theta)) \longmapsto (r\cos\frac{2\theta+\pi}{3}, r\sin\frac{2\theta+\pi}{3})$$

which is differentiable except at $r = 0$.

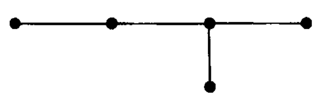
If n is even, the plumbed manifold inherits the same orientation from E_1 and E_2 and is thus canonically oriented. If n is odd one must make a choice.

We now want to describe how to plumb several bundles together according to a "tree". A <u>tree</u> T is a finite contractible 1-dimensional simplicial complex.

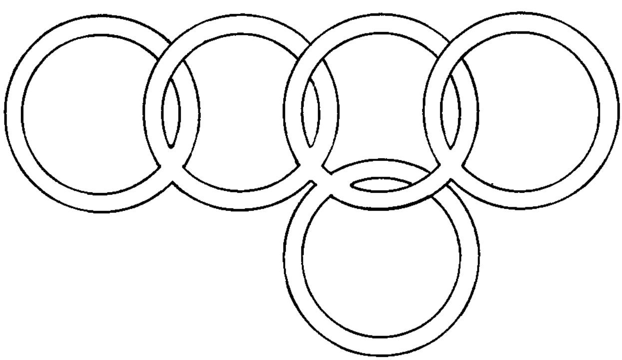
Suppose we have a tree (T,m_i) <u>weighted in</u> $\pi_{n-1}SO(n)$. That is, to each vertex v_i of T is assigned an element $m_i \in \pi_{n-1}SO(n)$ called the <u>weight</u> of v_i . To each vertex v_i of T we choose an n-disc bundle $\xi_i = (E_i, p_i, S_i^n)$ classified by the element m_i of $\pi_{n-1}SO(n)$. We then plumb ξ_i to ξ_j whenever v_i and v_j are joined by an edge of T . If several edges of T meet in one vertex v_i , we choose the necessary embeddings of the disc D^n in S_i^n to be disjoint. A theorem of Thom (Milnor [16]) assures that the result of plumbing is independant of the choice of these embedded discs.

We denote the resulting manifold after smoothing by $P(T,m_i)$.

<u>Example</u> :

T = 

Plumbing trivial 1-disc bundles according to T gives :



From now on we restrict n to be even, say $n = 2k$. Let $Y = P(T, m_i)$ be the $4k$-dimensional oriented manifold obtained by plumbing according to the weighted tree (T, m_i), and let $X = \partial Y$. We wish to calculate the homology of X and Y.

Denote the bundles which have been plumbed by $\xi_i = (E_i, p_i, S_i^{2k})$ $i = 1, \dots, s$. Y has as deformation retract the one-point union $S^{2k} \vee S^{2k} \vee \dots \vee S^{2k}$ of s copies of S^{2k}, so the only non-zero homology groups are $H_0(Y;\mathbb{Z}) \cong \mathbb{Z}$ and $H_{2k}(Y;\mathbb{Z}) \cong \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$ (s times). $H_{2k}(Y;\mathbb{Z})$ has a basis $a_1, \dots, a_s$, where a_i is the homology class represented by the zero section $S^n \subset E_i$.

Denote the <u>euler number</u> of the bundle ξ_i by $e(m_i)$, since it only depends on the classifying element $m_i \in \pi_{2k-1} SO(2k)$. There are many equivalent definitions of the euler number, for instance, considered as a map $e: \pi_{2k-1} SO(2k) \longrightarrow \mathbb{Z}$, e is just $-p_*$, where $p_*: \pi_{2k-1} SO(2k) \longrightarrow \pi_{2k-1} S^{2k-1} = \mathbb{Z}$ is the map induced by the fibre map $SO(2k) \longrightarrow SO(2k)/SO(2k-1) = S^{2k-1}$ (see proof of theorem 2 in Milnor [15] or theorem 35.12 in Steenrod [25]). The "classical" definition is essentially that $e(m_i)$ is equal to the self-intersection number of the zero section $S^{2k} \subset E_i$. Using this definition the following theorem is obvious.

<u>THEOREM (8.1)</u>: The matrix of the quadratic form S_Y of Y with respect to the basis $a_1, \dots, a_s$ of $H_{2k}(Y)$ is given by

$$M = (\alpha_{ij})_{1 \le i,j \le s} ,$$

with

$$\alpha_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ and } v_i \text{ and } v_j \text{ are} \\ & \text{connected by an edge in } T, \\ 0 & \text{if } i \neq j \text{ otherwise,} \\ e(m_i) & \text{if } i = j . \end{cases}$$ ||

Since the above quadratic form depends only on the tree $(T, e(m_i))$ weighted in $\mathbb{Z}$, we call it the <u>quadratic form of</u> $(T, e(m_i))$.

With theorem (8.1) we are in a position to calculate the homolog of $X = \partial Y$. Namely let

$$\varphi : \mathbb{Z}^s \longrightarrow \mathbb{Z}^s$$

be the linear map given by the matrix M above.

<u>THEOREM (8.2)</u>: $H_i(X;\mathbb{Z}) = 0$ for $i \neq 0, 2k-1, 2k, 4k-1$, and

$$H_{2k-1}(X;\mathbb{Z}) \cong \operatorname{Coker} \varphi$$
$$H_{2k}(X;\mathbb{Z}) \cong \operatorname{Ker} \varphi \quad .$$

<u>Proof</u>: Consider the exact homology sequence with integral coefficients

$$\cdots \longrightarrow H_i(Y,X) \longrightarrow H_{i-1}(X) \longrightarrow H_{i-1}(Y) \longrightarrow \cdots \quad .$$

By Poincaré-Lefschetz duality, $H_i(Y,X) \cong H^{4k-i}(Y)$, and since Y has no torsion, $H^{4k-i}(Y) = \operatorname{Hom}(H_{4k-i}(Y),\mathbb{Z})$, which vanishes for $i \neq 2k, 4k$. Hence $H_i(X) = 0$ for $i \neq 0, 2k-1, 2k, 4k-1$. In the middle dimensions we obtain the commutative diagram with exact row

$$\begin{tikzcd}
0 \arrow[r] & H_{2k}(X) \arrow[r] & H_{2k}(Y) \arrow[r] \arrow[dr, "\varphi"'] & H_{2k}(Y,X) \arrow[r] \arrow[d, phantom, "\wr\|"] & H_{2k-1}(X) \arrow[r] & 0 \\
 & & & H^{2k}(Y) \arrow[d, phantom, "\wr\|"] & & \\
 & & & \operatorname{Hom}(H_{2k}(Y),\mathbb{Z}) \arrow[uur] & &
\end{tikzcd} \quad .$$

φ is the correlation associated with S_Y, and hence has matrix M, so the theorem follows. ||

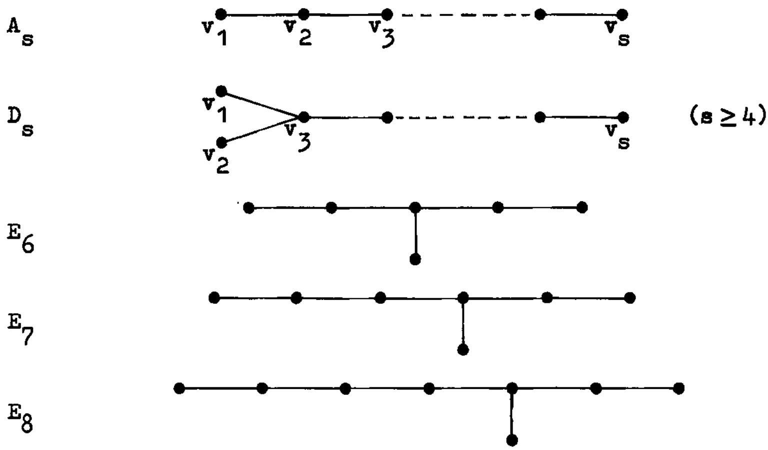
In particular if $\det M \neq 0$ then $H_{2k}(X) = 0$ and $H_{2k-1}(X)$ is finite of order $|\det M|$. Thus if $\det M$ is odd, X is a $\mathbb{Z}/2\mathbb{Z}$-homology sphere and if $\det M = \pm 1$ then X is a $\mathbb{Z}$-homology sphere. In fact for $k > 1$ it is not hard to see that X is simply connected, so in this case $\det M = \pm 1$ implies that X is even a homotopy sphere. However for $k = 1$, that is $\dim Y = 4$, this is not true in general. We shall now discuss this 4-dimensional case in more detail.

Plumbing Bundles over S^2.

The principal $SO(2)$ bundles over S^2 are classified by $\pi_1 SO(2) \cong \mathbb{Z}$, with the Hopf fibration $S^3 \longrightarrow S^2$ corresponding to a generator in $\mathbb{Z}$ (see for instance Steenrod [25] p99 and p105; here and in the following we identify $SO(2)$ with S^1). By the remarks preceding theorem (8.1), the euler number of bundles gives an isomorphism $e: \pi_1 SO(2) \longrightarrow \mathbb{Z}$, so we can identify the classifying element of a principal $SO(2)$ bundle over S^2 with its euler number (this identification of $\pi_1 SO(2)$ with $\mathbb{Z}$ is the negative of the usual identification given by degrees of maps).

The trees we need for plumbing are hence trees (T, m_i) weighted in $\mathbb{Z}$, and the quadratic form of the manifold $Y = P(T, m_i)$ is equal to the quadratic form of the tree (T, m_i).

As examples we mention the following trees which arise in the classification theory of simple Lie algebras (see for instance Séminaire Sophus Lie, 1e année, Exp.13). These are the only trees, when weighted by -2, whose quadratic forms are negative definite.



In the following table each vertex of the tree T is weighted by -2.

TABLE

T	$H_1(\partial P(T);\mathbb{Z})$	$\pi_1(\partial P(T)) =: F_T$	$\mu(\partial P(T))$
A_s	$\mathbb{Z}/(s+1)\mathbb{Z}$	C_{s+1}	$s/16$ (s even)
D_s	$\begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & s \text{ even} \\ \mathbb{Z}/4\mathbb{Z}, & s \text{ odd} \end{cases}$	D'_{s-2}	
E_6	$\mathbb{Z}/3\mathbb{Z}$	T'	$6/16$
E_7	$\mathbb{Z}/2\mathbb{Z}$	W'	
E_8	0	I'	$8/16$

The groups F_T are the only finite subgroups of S^3 (= unit quaternions), as can be seen as follows: the finite subgroups of $SO(3)$ are known to be the cyclic groups C_n, the dihedral groups

D_n (not to be confused with the tree D_n), and the tetrahedral, octahedral and icosahedral groups T, W and I . Regarding S^3 as a double covering of SO(3) , one deduces easily that the only finite subgroups of S^3 are the cyclic groups C_n and the binary groups D'_n , T′, W′ and I′ obtained by lifting the subgroups D_n , T , W and I of SO(3) .

We sketch a proof of the statements in the table. The homology groups $H_1(\partial P(T))$ can be checked by theorem (8.2). For $T = A_s$ (s even ~~odd~~), $T = E_6$ and $T = E_8$ it follows that the manifold $\partial P(T)$ is a $\mathbb{Z}/2\mathbb{Z}$ - homology sphere, so the μ-invariant is defined. The values of the μ-invariant are obvious, since the quadratic form of T is negative definite and even for each of the trees listed and is equal to the quadratic form of P(T) . Finally the fundamental groups $\pi_1\partial P(T)$ can be computed directly by finding explicit generators and relations, however these groups can be obtained much more easily from the general results of von Randow [20] which we shall now describe briefly.

An oriented 3-manifold will be called a <u>Seifert manifold</u> if it can be fibred, with exceptional fibres, over S^2 with fibre S^1 (Seifert [22]). To such a manifold X , Seifert associated a system of integers

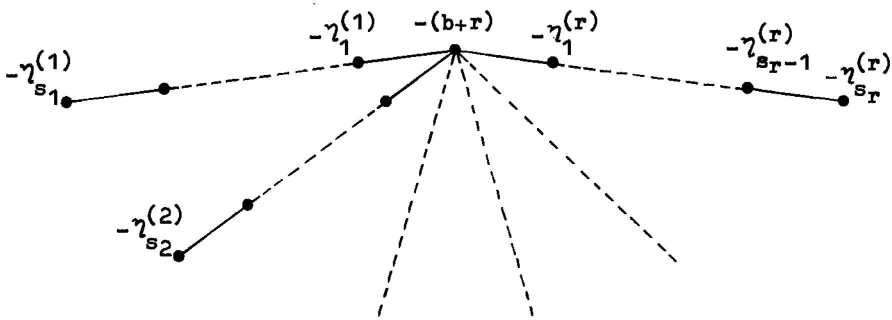
$$(b\ ;\ (\alpha_1, \beta_1),\ \dots\ , (\alpha_r, \beta_r))$$

known as the <u>Seifert invariants</u>, where r is the number of exceptional fibres and the integers α_i , β_i are coprime and satisfy $0 < \beta_i < \alpha_i$ for each $i = 1, \dots, r$.

Expand $\alpha_i/(\alpha_i - \beta_i)$ into a continued fraction

$$\alpha_i/(\alpha_i - \beta_i) = [\eta_1^{(i)}, \dots , \eta_{s_i}^{(i)}] \quad , \quad |\eta_j^{(i)}| \geq 2 ,$$

and let T be the star-shaped tree weighted by the $-\eta_j^{(i)}$:



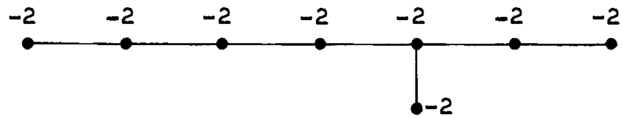
Then $X = \partial P(T)$. Von Randow's proof involves a special construction of lens spaces via plumbing, which we shall study later in this section.

<u>Remark</u> : Von Randow's orientation conventions for Seifert manifolds and lens spaces are opposite to ours, so his weighted trees are the negatives of those here. We discuss various orientation conventions occurring in the literature in the appendix.

Let us return to the fundamental groups of the table. For any finite subgroup F of S^3 , the coset space S^3/F is a Seifert manifold (the fibration is given by the action of a subgroup S^1 of S^3 by left multiplication). We claim that each manifold $\partial P(T)$ listed in the table is in fact diffeomorphic to S^3/F_T . For instance, the spherical dodecahedral space S^3/I' has Seifert invariants $(-1\,;\,(5,1),(3,1),(2,1))$ (see Seifert [22]). Here $r = 3$ and

$$5/(5-1) = 5/4 = [2,2,2,2] \ ,$$
$$3/(3-1) = 3/2 = [2,2] \ ,$$
$$2/(2-1) = 2/1 = 2 \ .$$

By the above, we get the star-shaped tree :



.

But this is E_8 ; in other words $\partial P(E_8)$ is the spherical dodecahedral space if E_8 is weighted by -2 . In particular $\pi_1 \partial P(E_8) = I'$. The other cases in the table may be checked in the same way.

To investigate plumbing in more detail, we need a precise description of the bundles involved. We shall identify S^1 with $\mathbb{R}/\mathbb{Z}$, so that the operation in S^1 is written additively.

Denote the principal S^1 bundle over S^2 with euler number m by $\xi_{(m)} = (X_m, p_m, S^2)$. Then $\xi_{(m)}$ is classified by the element in $\pi_1 S^1$ represented by a map $S^1 \longrightarrow S^1$ of degree $-m$. Hence X_m is obtained as the union of two trivial S^1 bundles over the disc D^2 as follows :

$$X_m = D^2 \times S^1 \underset{f}{\cup} D^2 \times S^1$$

where $f\colon \partial(D^2 \times S^1) = S^1 \times S^1 \longrightarrow S^1 \times S^1 = \partial(D^2 \times S^1)$ is the map

$$f\colon (x,y) \longmapsto (-x, y-mx)$$

(the first entry $-x$ on the right assures that the orientation of each $D^2 \times S^1$ is compatible with that of X_m). This map f may also be represented by the matrix

$$\begin{pmatrix} -1 & 0 \\ -m & 1 \end{pmatrix} .$$

Note that the above is a well known description of the lens space $L(-m,1)$, so $X_m = L(-m,1)$. If $-m$ is negative, $L(-m,1)$ denotes $-L(m,1) = L(m,m-1)$ in standard notation.

Another way of seeing that $X_m = L(-m,1)$ is by observing that the Hopf map $S^3 \longrightarrow S^2$, represented as the map $(z_1,z_2) \longmapsto [z_1,z_2]$ from the unit sphere in $\mathbb{C}^2$ to the complex projective line, is compatible with the $\mathbb{Z}/m\mathbb{Z}$-action (7.8) on S^3 when $q = 1$. Using the fact that the Hopf map is $\xi_{(-1)}$, it is not hard to see that the induced map $S^3/(\mathbb{Z}/m\mathbb{Z}) = L(m,1) \longrightarrow S^2$ is just $\xi_{(-m)}$.

The principal S^1 bundle $\xi_{(m)}$ can be identified with the boundary of the associated 2-disc bundle over S^2, which is the bundle we need for plumbing. However as we have seen, in many cases we are interested in the manifold $\partial P(T)$ rather than $P(T)$ itself, where T is a weighted tree. The assignment $\partial P\colon T \longmapsto \partial P(T)$ will also be called <u>plumbing</u> ; it may be defined directly without help of the operator P . We describe this in three steps :

1). If T has just one vertex and this is weighted by m , define $\partial P(T) = X_m$.

2). Let T be the tree $\overset{m_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{m_2}{\bullet}$. Let D_1 and D_2 be 2-discs embedded in the base spaces of the bundles $\xi_{(m_1)} = (X_{m_1},p,S^2)$ and $\xi_{(m_2)} = (X_{m_2},p,S^2)$ respectively and let

$$f_i\colon D_i \times S^1 \longrightarrow X_{m_i} | D_i$$

be trivialisations of the restricted bundles $X_{m_i} | D_i$. Denote by X'_{m_i} the restricted bundle $X_{m_i} | (S^2 - \operatorname{Int} D_i)$ and consider the composite map

$$f\colon \partial X'_{m_1} = \partial(X_{m_1} | D_1) \xrightarrow{f_1^{-1}} \partial(D_1 \times S^1) \xrightarrow{t} \partial(D_2 \times S^1) \xrightarrow{f_2} \partial X'_{m_2}$$

where $t\colon \partial(D_1 \times S^1) = S^1 \times S^1 \longrightarrow S^1 \times S^1 = \partial(D_2 \times S^1)$ is defined by

$t(x,y) = (y,x)$. Then $P(T) = X'_{m_1} \cup_f X'_{m_2}$ after smoothing.

3). The procedure for general weighted trees is now clear.

The map t above is given by the matrix

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Denote by Δ the annulus obtained by cutting a small open disc from the center of the disc D^2 . Δ can be identified with $[0,1] \times S^1$. In the notation of 2) above, it is clear by our previous comments that we can write

$$X'_{m_1} = D^2 \times S^1 \underset{f_1}{\cup} \Delta \times S^1 ,$$

$$X'_{m_2} = \Delta \times S^1 \underset{f_2}{\cup} D^2 \times S^1 ,$$

where $f_i : S^1 \times S^1 \longrightarrow S^1 \times S^1$ is given by the matrix

$$\begin{pmatrix} -1 & 0 \\ -m_i & 1 \end{pmatrix}$$

for $i = 1,2$. Thus

$$\begin{aligned} \partial P(T) &= D^2 \times S^1 \underset{f_1}{\cup} \Delta \times S^1 \underset{t}{\cup} \Delta \times S^1 \underset{f_2}{\cup} D^2 \times S^1 \\ &= D^2 \times S^1 \underset{f}{\cup} D^2 \times S^1 , \end{aligned}$$

where $f = f_2 \circ t \circ f_1$ is given by the matrix

$$\begin{pmatrix} -1 & 0 \\ -m_2 & 1 \end{pmatrix} \cdot J \cdot \begin{pmatrix} -1 & 0 \\ -m_1 & 1 \end{pmatrix} .$$

More generally let us consider the weighted tree $(A_s, m_i)_{1 \leq i \leq s}$

where the m_i are integers. If we denote by f_i the map $S^1 \times S^1 \longrightarrow S^1 \times S^1$ with matrix

$$\begin{pmatrix} -1 & 0 \\ -m_i & 1 \end{pmatrix} ,$$

then it is clear that

$$\partial P(A_s, m_i) = D^2 \times S^1 \underset{f_1}{\cup} \Delta \times S^1 \underset{t}{\cup} \Delta \times S^1 \underset{f_2}{\cup} \Delta \times S^1 \underset{t}{\cup} \cdots$$
$$\cdots \underset{t}{\cup} \Delta \times S^1 \underset{f_s}{\cup} D^2 \times S^1$$

$$(8.3) \qquad = D^2 \times S^1 \underset{f}{\cup} D^2 \times S^1 ,$$

where $f = f_s \circ t \circ f_{s-1} \circ t \circ \cdots \circ t \circ f_1$. Observing that $t \circ f_i$ has matrix

$$J \cdot \begin{pmatrix} -1 & 0 \\ -m_i & 1 \end{pmatrix} = \begin{pmatrix} -m_i & 1 \\ -1 & 0 \end{pmatrix} ,$$

we see that f has matrix

$$(8.4) \qquad \begin{pmatrix} -q & p \\ n & q' \end{pmatrix} := \begin{pmatrix} -1 & 0 \\ -m_s & 1 \end{pmatrix} \cdot \begin{pmatrix} -m_{s-1} & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -m_1 & 1 \\ -1 & 0 \end{pmatrix} .$$

(8.3) is thus the well-known description of the lens space $L(n,q)$ as the union of two solid tori. By multiplying each side of (8.4) from the left by J we get the handier equation

$$(8.5) \qquad \begin{pmatrix} n & q' \\ -q & p \end{pmatrix} = \begin{pmatrix} -m_s & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -m_{s-1} & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -m_1 & 1 \\ -1 & 0 \end{pmatrix} ,$$

so we have proved

<u>THEOREM (8.6)</u> : Let (A_s, m_i) be the weighted tree as above. Then $\partial P(A_s, m_i) = L(n,q)$, where n and q are defined by equation (8.5). ||

The matrix of (8.5) has determinant 1 since each factor on the right hand side has determinant 1. This shows

$$np + qq' = 1 \quad .$$

In particular n and q are coprime, as they should be, and also $qq' \equiv 1 \pmod{n}$. If we reverse the order of the tree and plumb according to the reversed tree we get $L(n,q')$, so $L(n,q) = L(n,q')$. This is a classical result on lens spaces (see Seifert and Threlfall [23] p215 Satz II).

For given coprimes n and q, $0 < q < n$, we can find a tree (A_s, m_i) such that $\partial P(A_s) = L(n,q)$. To see this we let $\lambda_o = n$, $\lambda_1 = q$, and we use the euclidean algorithm to obtain

$$\begin{array}{lll}
\lambda_o = a_1\lambda_1 - \lambda_2 & 0 \le \lambda_2 < \lambda_1 \ , & a_1 > 1 \\
\lambda_1 = a_2\lambda_2 - \lambda_3 & 0 \le \lambda_3 < \lambda_2 \ , & a_2 > 1 \\
\quad \cdots & \quad \cdots & \quad \cdots \\
\quad \cdots & \quad \cdots & \quad \cdots \\
\end{array} \tag{8.7}$$
$$\lambda_{s-1} = a_s\lambda_s - \lambda_{s+1}, \lambda_{s+1} = 0 \ , \ \lambda_s = 1 \ , \ a_s > 1 \quad .$$

This is equivalent to saying that n/q can be expanded into the continued fraction $n/q = [a_1, a_2, \cdots, a_s]$, that is

$$n/q = a_1 - \cfrac{1}{a_2 - \ddots - \cfrac{1}{a_s}} \quad .$$

Using (8.7) we get by induction

$$(8.8) \qquad \begin{pmatrix} a_s & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_i & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_{i-1} & \lambda_i \\ * & * \end{pmatrix}$$

so

$$\begin{pmatrix} a_s & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} n & q \\ -q' & * \end{pmatrix}$$

Comparing this with theorem (8.6) and the remarks following this theorem we obtain

<u>THEOREM (8.9)</u>: If $A_s = (A_s, m_i)$ is the tree with weights $m_i = -a_i$, where the a_i are given by (8.7), then $L(n,q) = \partial P(A_s)$. ||

The lens space $L(n,q)$ obtained in this theorem bounds a 4-manifold $P(A_s)$ whose quadratic form is the same as that of the tree (A_s, m_i) with $m_i = -a_i$. As the integers a_i may not be even, this quadratic form cannot be used to compute $\mu(L(n,q))$ when the latter is defined. This situation can be remedied. By §7, the μ-invariant $\mu(L(n,q))$ is only defined for odd n, and in this case we may assume without loss of generality that q is even. For such n and q (8.7) may be modified to yield

$$(8.10) \qquad \begin{array}{lll} \lambda_0 = n \quad \lambda_1 = q & & \\ \lambda_0 = b_1\lambda_1 - \lambda_2 & |\lambda_1| \geq |\lambda_2| & |b_1| > 0 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \lambda_{s-1} = b_s\lambda_s - \lambda_{s+1} & \lambda_{s+1} = 0\ ,\ |\lambda_s| = 1 & |b_s| > 0\ , \end{array}$$

where each b_i is even. This, by the way, proves the assertion preceding recipe (7.10). Now let $A_s = (A_s, n_i)$ with weights $n_i = -b_i$.

As above, one can easily show that $\partial P(A_s) = L(n,q)$, even though (8.8) must be modified slightly if $\lambda_s = -1$. In this construction $L(n,q)$ bounds a 4-manifold $Y = P(A_s)$ whose quadratic form S_Y is even, and $\mu(L(n,q))$ can be calculated as $-\tau(Y)/16 \pmod 1$.

In (7.10) we stated that $\mu(L(n,q)) = (p^+ - p^-)/16 \pmod 1$, where p^+ and p^- are the number of positive and negative b_i respectively To prove this, we must show that $\tau(Y) = p^- - p^+$. Hence, since the quadratic form S_Y is represented by the matrix

$$(8.11) \qquad M = \begin{pmatrix} -b_1 & 1 & & & \\ 1 & -b_2 & & 0 & \\ & & \ddots & & \\ & 0 & & & 1 \\ & & & 1 & -b_s \end{pmatrix} ,$$

we need only prove the following lemma.

<u>LEMMA (8.12)</u> : Let M be a matrix of the form (8.11) with $|b_i| > 1$. Then $\tau(M) = p^- - p^+$.

<u>Proof</u> : An easy induction shows that M is congruent over $\mathbb{R}$ to the diagonal matrix $\mathrm{diag}(-c_1, \dots, -c_s)$ where

$$(8.13) \qquad c_i = b_i - \cfrac{1}{b_{i+1} - \ddots - \cfrac{1}{b_s}} = [b_i, \dots, b_s] .$$

If $|b_i| > 1$ for all i , then clearly $\operatorname{sign} c_i = \operatorname{sign} b_i$, which obviously proves the lemma. ||

We end this section with a digression. Let n be an odd integer and q an integer prime to n . Suppose that $n = p_1^{\beta_1} p_2^{\beta_2} \dots p_t^{\beta_t}$ is the prime decomposition of n. The <u>Jacobi symbol</u> $(q|n)$ is defined via the Legendre symbol $(q|p_i)$ by the formula

$$(q|n) = (q|p_1)^{\beta_1} \dots (q|p_t)^{\beta_t} .$$

It is clear that if q and q' are integers such that qq' is a quadratic residue modulo n (that is, $qq' = x^2$ has a solution in $\mathbb{Z}/n\mathbb{Z}$), then

$$(q|n) = (q'|n) .$$

By theorem (7.11) it follows that $(q|n)$ is an orientation preserving homotopy type invariant for lens spaces $L(n,q)$ with odd n. Notice that the h-cobordism invariant $\mu(L(n,q))$ is also defined for such spaces.

<u>THEOREM (8.14)</u>: If n is odd and q coprime to n then $16\mu(L(n,q)) + (n-1) \equiv 0 \pmod 4$ and

$$(q|n) = (-1)^{4\mu L(n,q) + (n-1)/4} .$$

<u>Proof</u>: Suppose we have proved the theorem for q even. Then using the identities

(8.15) $$(-1|n) = (-1)^{(n-1)/2}$$

and

$$L(n,q) = - L(n,n-q)$$

the theorem also follows for q odd. Hence we can restrict ourselves to the case that q is even.

Expand n/q into the continued fraction $n/q = [b_1, \dots, b_s]$ according to (8.10), that is $|b_i| > 1$ and b_i even for $i = 1, \dots, s$. We have shown that $L(n,q)$ bounds a 4-manifold $Y = P(A_s)$ whose quadratic form S_Y may be described as follows. Let $S = -S_Y$; then

$$\begin{aligned} S &= \sum b_i x_i^2 - \sum 2x_i x_{i+1} \\ &\sim \sum_{i=1}^{s} c_i y_i^2 \end{aligned}$$

where the rational numbers c_i are given by (8.13). We have

$$\mu L(n,q) = -\tau(S_Y)/16 = \tau(S)/16 \pmod 1 .$$

We now compute the Hasse-Minkowski symbol $c_p(S)$ for all odd primes p. Note first that by theorem (8.2) and the fact that $H_1(L(n,q)) = \mathbb{Z}/n\mathbb{Z}$,

$$\det S = \pm n .$$

Hence if p does not divide n then $c_p(S) = 1$ by (1.24). We hence assume p is a prime factor of n, say $p = p_i$, where $n = p_1^{\beta_1} \dots p_t^{\beta_t}$ is the prime decomposition of n. Since $(q,n) = 1$ we have that $p \nmid q$.

Write S in the form

$$S = S_1 \oplus S_2 ,$$

where $S_1 = c_1 y_1^2 = \frac{n}{q} y_1^2$ and $S_2 = \sum_{i=2}^{s} c_i y_i^2$. It follows that $\det S_2 = \pm q$, so since $p \nmid q$, $c_p(S_2) = 1$. Also, since S_1 is a unary form, $c_p(S_1) = 1$. Hence by (1.23)

$$\begin{aligned} c_p(S) &= c_p(S_1) c_p(S_2) (\det S_1, \det S_2)_p \\ &= (n/q, q \cdot \operatorname{sign} \det S_2)_p . \end{aligned}$$

Since $p = p_i$ and $(n,q) = 1$ we may write $n/q = p^{\beta_i} \alpha_1$, and by property 6 (§1) of the Hilbert symbol

$$(n/q, q\cdot \mathrm{sign}\,\det S_2)_p = (q\cdot \mathrm{sign}\,\det S_2|p)^{\beta_i}$$
$$= (q|p)^{\beta_i}(\mathrm{sign}\,\det S_2|p)^{\beta_i}$$

Hence

$$\prod_{p \text{ odd prime}} c_p(S) = \prod_{i=1}^{t} (q|p_i)^{\beta_i}(\mathrm{sign}\,\det S_2|p_i)^{\beta_i}$$
$$= (q|n)(\mathrm{sign}\,\det S_2|n) \quad .$$

We may assume $n/q > 0$, so $\mathrm{sign}\,\det S_2 = \mathrm{sign}\,\det S$. Applying lemma (1.25), we get

(8.16) $$c_2(S)c_\infty(S) = (q|n)(\mathrm{sign}\,\det S|n) \; .$$

On the other hand, since S is even, theorem (3.9) gives

(8.17) $$c_2(S)c_\infty(S) = (-1)^{(\tau(S) + \det S - \mathrm{sign}\,\det S)/4} \; .$$

Combining (8.15), (8.16) and (8.17) now easily proves the theorem. ||

The lecturer is again indepted to the letter of J.W.S. Cassels, mentioned in conection with theorem (3.8), for essentially the above proof.

In this section we indicate the connection between plumbing and the resolution of singularities of 2-dimensional complex varieties.

We shall express complex dimension by superscripts in parentheses. Let $M = M^{(n)}$ be a complex manifold, not necessarily compact. A complex analytic subset is said to have <u>maximal dimension</u> if it has complex codimension 1. Recall that such a subset N is given locally as the set of zeros of a holomorphic function; that is, to each point $p \in N$ there is an open neighbourhood U of p in M and a holomorphic function f defined on U and not identically zero on any component of U such that $N \cap U = \{q \in U \mid f(q) = 0\}$. The point $p \in N$ is called <u>regular</u> if the above f can be chosen as a coordinate function about p, otherwise p is called <u>singular</u>.

The complex analytic subset N is called <u>irreducible</u> if it is not the union of two smaller non-empty closed analytic subsets. A <u>divisor</u> D on M is a formal linear combination

$$D = \sum n_i N_i$$

of irreducible closed analytic subsets of maximal dimension in M with integer coefficients.

A divisor D can also be defined by an indexed family $\{f_i\}_{i \in I}$ of meromorphic functions, where each f_i is defined an an open subset U_i of M, and

(a) $\{U_i\}_{i\varepsilon I}$ is an open covering of M,

(b) f_i is not identically zero on any component of U_i, and

(c) f_i/f_j is holomorphic and has no zeros on $U_i \cap U_j$.

We express this by $D \sim \{f_i\}$. Note that condition (c) implies that the locus of zeros and poles of the family $\{f_i\}$ is well defined. This locus, together with the multiplicities of its irreducible components (positive multiplicities for zeros, negative for poles) describes the divisor D according to the previous definition.

If, in particular, each function in the family $\{f_i\}$ is holomorphic, the corresponding divisor $D \sim \{f_i\}$ is called <u>non-negative</u>, and is called <u>positive</u> if in addition the locus of zeros N_D is not empty. Perhaps the simplest non-negative divisor is one given by a single globally defined holomorphic function, i.e. $D \sim \{f\}$. For such a divisor we simply write $D = (f)$ and $N_D = |f = 0|$.

To each divisor $D \sim \{f_i\}_{i\varepsilon I}$ there is an associated complex line bundle $[D]$ over M given by the transition functions

$$f_{ij} = f_i/f_j \quad \text{in} \quad U_i \cap U_j \quad .$$

The <u>characteristic class</u> $c_1(D)$ of D is defined to be the Chern class $c_1([D])$ of the bundle $[D]$. If $D = (f)$, then clearly $[D]$ is trivial and hence $c_1(D) = 0$.

Recall that the usual singular homology H_* is a theory with compact supports. Let $\mathcal{H}_*$ denote a homology theory with closed supports (see for instance Borel Moore [5]. A singular theory of this type, defined on the category of locally compact spaces and proper maps, is given by allowing infinite, but locally finite chains). The main property of $\mathcal{H}_*$ that we need here is that there is a Poincaré duality isomorphism

$$\Delta : \mathcal{H}_{m-i}(M;\mathbb{Z}) \longrightarrow H^i(M;\mathbb{Z})$$

for any (not necessarily compact) oriented m-manifold M without boundary.

Using this, one can generalize the intersection numbers of cycles to the case where one cycle may have non-compact support. Namely if $x \in \mathcal{H}_{m-i}(M;\mathbb{Z})$ and $y \in H_i(M;\mathbb{Z})$, define

$$x \cdot y = \Delta(x)(y) \;;$$

that is, the result of evaluating the cohomology class $\Delta(x)$ on the homology class y. Intersection of cycles is then defined via the homology classes they represent. One can check that this gives the usual definition in the compact case.

Returning to the complex manifold $M^{(n)}$, if D is a divisor written as $D = \sum n_i N_i$, then each irreducible component N_i has a fundamental homology class in the sense of Borel and Haeflinger [1], so D represents a homology class $h(D) \in \mathcal{H}_{2n-2}(M;\mathbb{Z})$. Borel and Haeflinger proved :

<u>LEMMA (9.1)</u> : If D is a positive divisor, then the class $h(D) \in \mathcal{H}_{2n-2}(M;\mathbb{Z})$ is dual to the characteristic class $c_1(D) \in H^2(M;\mathbb{Z})$. ||

This clearly follows also for arbitrary divisors, but we shall not need this.

Now suppose $D = (f)$. Then by the above, $\Delta h(D) = c_1(D) = 0$, so $h(D) = 0$, so certainly $h(D) \cdot x = 0$ for any $x \in H_2(M;\mathbb{Z})$. Hence considering D as a cycle on M we have shown :

<u>COROLLARY (9.2)</u>: If $D = (f)$, then the intersection number $D\cdot x$ vanishes for any cycle x with compact support in M. ||

Now let p be a point in the complex manifold $M^{(n)}$. The complex line elements at p in M form a complex projective space $\mathbb{C}P^{(n-1)}$. One can "replace" p in M by $\mathbb{C}P^{(n-1)}$ to obtain a new complex manifold $\sigma_p M$; this process is known as a <u>σ-process</u>, <u>blowing-up</u> operation or <u>quadratic transformation</u>.

To make this process precise we take a coordinate neighbourhood U of p in M with local coefficients $z = (z_1, \dots, z_n): U \longrightarrow \mathbb{C}^n$ centered at p (i.e. $z(p) = 0$). There is an obvious map

$$\varphi : U - \{p\} \longrightarrow \mathbb{C}P^{(n-1)}$$

given by $\varphi(u) = \langle z_1(u), \dots, z_n(u)\rangle$ (homogeneous coordinates). Let

$$\Gamma \subset U \times \mathbb{C}P^{(n-1)}$$

be the graph of φ, and

$$K_p := \{p\} \times \mathbb{C}P^{(n-1)} \subset U \times \mathbb{C}P^{(n-1)} .$$

Then $N = \Gamma \cup K_p$ is a non-singular analytic subset of $U \times \mathbb{C}P^{(n-1)}$. Indeed, if $w_1, \dots, w_n$ are homogeneous coordinates in $\mathbb{C}P^{(n-1)}$, let V_i be the open subset of $U \times \mathbb{C}P^{(n-1)}$ given by $w_i \neq 0$. The V_i cover $U \times \mathbb{C}P^{(n-1)}$. By normalizing so that $w_i = 1$ in V_i we can consider

$$z_1, \dots, z_n, w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n$$

to be local coordinates in V_i. With respect to these coordinates N is given by

$$z_j = z_i w_j \qquad (1 \le j \le n\ ,\quad j \ne i\ )\ , \tag{9.3}$$

so $w_1, \dots, w_{i-1}, z_i, w_{i+1}, \dots, w_n$ give a system of local coordinates in $N \cap V_i$.

Using the projection $\Gamma \longrightarrow U - \{p\}$, we identify $U - \{p\}$ with $\Gamma = N - K_p$ in the disjoint union of $M - \{p\}$ and N to obtain $\sigma_p M$. It is not hard to see that $\sigma_p M$ depends only on M and p and not on the various choices used in the construction. There is a projection

$$\pi_p : \sigma_p M \longrightarrow M$$

which maps K_p onto p and maps $\sigma_p M - K_p$ biholomorphically onto $M - \{p\}$.

<u>Complex Dimension 2</u> .

In the following M will always have complex dimension 2 . Let $p \in M$ with local coordinates z_1, z_2 (centred at p) defined on a neighbourhood U of p in M . In the complex manifold $\sigma_p M$ these coordinates are replaced by two systems u, v and $\tilde{u}, \tilde{v}$ such that

$$\begin{cases} z_1 = u \\ z_2 = uv \end{cases} \qquad \begin{cases} z_1 = \tilde{u}\tilde{v} \\ z_2 = \tilde{v} \end{cases}\ , \tag{9.4}$$

as one can see by (9.3). The subspace K_p of $\sigma_p M$ is clearly given locally by the equations $u = 0$ and $\tilde{v} = 0$. It is also clear that K_p is a 2-sphere embedded in $\sigma_p M$ and hence represents a cycle with compact support in $\sigma_p M$.

LEMMA (9.5) : The self-intersection number $K_p \circ K_p$ is -1 .

Proof : Consider the function z_1 on $\pi_p^{-1}(U)$. (Strictly speaking, z_1 is a function defined on U only. When we consider z_1 as a function on $\pi_p^{-1}(U)$ we actually mean the function $z_1 \circ \pi_p$ which is given in the (u,v)-chart by u and in the $(\tilde{u},\tilde{v})$-chart by $\tilde{u}\tilde{v}$. This is a very convenient abuse of notation which we have already used in (9.4) and shall use again in (9.7).)

The divisor $|z_1 = 0|$ in $\pi_p^{-1}(U)$ consists of K_p and $|\tilde{u} = 0|$; in other words $|z_1 = 0| = K_p + |\tilde{u} = 0|$. Now the intersection number $|\tilde{u} = 0| \circ K_p$ is clearly $+1$, so applying (9.2) to the open manifold $\pi_p^{-1}(U)$ gives

$$0 = |z_1 = 0| \circ K_p = K_p \circ K_p + |\tilde{u} = 0| \circ K_p = K_p \circ K_p + 1 \quad .$$

This clearly proves the lemma. ||

Remark (9.6) : If we ignore the complex analytic structure on M and consider $M = M^4$ as a C^∞-manifold, then $\sigma_p M$ is diffeomorphic to the connected sum $M \# (-\mathbb{C}P^{(2)})$.

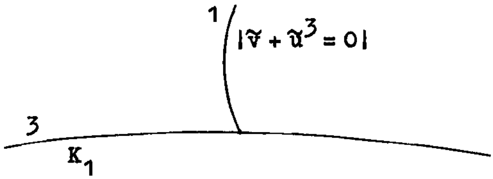
The blowing-up process may be iterated. Let $p = p_1$ be a point in M and consider $\sigma_{p_1} M$. Pick a point p_2 in $K_{p_1} = \pi_{p_1}^{-1}(p)$ and blow up at this point to give $\sigma_{p_1 p_2} M := \sigma_{p_2} \sigma_{p_1} M$. Denote K_{p_i} by K_i ; then K_1 and K_2 intersect at exactly one point and the projection $\pi_{p_1 p_2} := \pi_{p_2} \circ \pi_{p_1} : \sigma_{p_1 p_2} M \longrightarrow M$ maps $K_1 \cup K_2$ onto p . We now blow up at a point $p_3 \in K_1 \cup K_2$, and so on. We finally reach a complex manifold $\tilde{M} = \sigma_{p_1 \dots p_s} M$ and projection $\tilde{\pi} : \tilde{M} \longrightarrow M$ with $\tilde{\pi}^{-1}(p) = K_1 \cup \dots \cup K_s$. Notice that K_i and K_j $(i \neq j)$ are either disjoint or intersect at one point regularly.

We can construct a weighted tree T with s vertices $v_1, \dots, v_s$ by joining v_i and v_j by an edge in T if and only if K_i and K_j intersect, and weighting each vertex v_i with the self-intersection number of K_i in $\tilde{M}$. This tree is called the <u>dual weighted tree</u> of the "spherical space" $\tilde{\pi}^{-1}(p)$. The self-intersection numbers of the K_i can easily be computed by applying (9.2) to the open manifold $\tilde{\pi}^{-1}(U)$, where U is a coordinate neighbourhood of p in M.

We are now ready to investigate how the resolution of singularities in the "Riemann surface" of an algebroid function by means of the blowing-up process is related to the plumbing operation studied in §8. An example should suffice to clarify the situation.

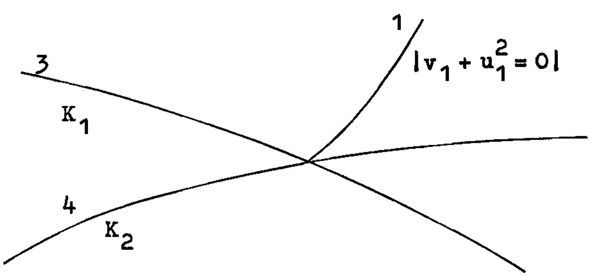
<u>Example (9.7)</u>: Let $M = \mathbb{C}^2$ and let f be the algebroid function $f = (z_1^3 + z_2^4)^{1/2}$ defined on M. The origin $p_1 = (0,0)$ is the only non-uniformizable singularity of f; that is, the complex 2-dimensional "Riemann surface" R_f of f has a non-uniformizable singularity at $q = \psi^{-1}(p_1)$, where $\psi\colon R_f \longrightarrow M$ is the projection of the Riemann surface onto M. We want to resolve this singularity by blowing M up at p_1.

Let $w = z_1^3 + z_2^4$. Then the locus $|w=0|$ is the complex analytic subset along which the branching of f occurs. Blow up M at p_1 and in $\sigma_{p_1}M$ consider the local coordinates u, v and $\mathfrak{u}, \mathfrak{v}$ given by (9.4). We consider w as a function on $\sigma_{p_1}M$ (see remark in the proof of (9.5)). The divisor $|w=0|$ in $\sigma_{p_1}M$ is clearly expressed by $|u^3(1+uv^4)=0|$ in the (u,v)-chart and by $|\mathfrak{v}^3(\mathfrak{v}+\mathfrak{u}^3)=0|$ in the $(\mathfrak{u},\mathfrak{v})$-chart. It thus has two irreducible components, namely $|u=0|$ and $|\tilde{v}=0|$ give the 2-sphere $K_1 = K_{p_1}$, which occurs with multiplicity 3, and $|(1+uv^4)=0| = |(\tilde{v}+\tilde{u}^3)=0|$ is a curve which meets K_1 in the point $(\tilde{u},\tilde{v})=(0,0)$. We represent this divisor by the diagram



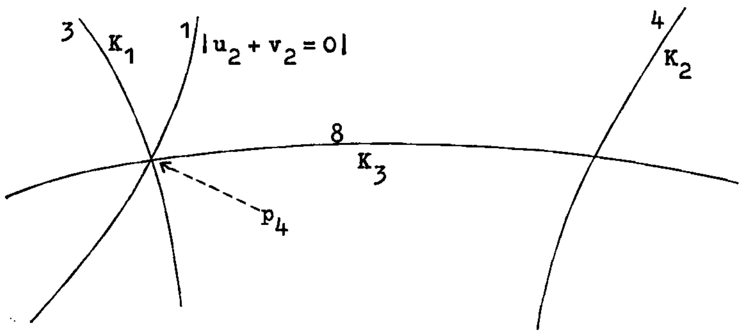
where the integers 1 and 3 represent the multiplicities of the irreducible components indicated. Now the non-uniformizable singularity of $f = w^{1/2}$ (considered as a function on $\sigma_{p_1}M$) is at the point p_2 where $(\tilde{u},\tilde{v})=(0,0)$. Since this point does not appear in the (u,v)--chart, we need only consider the restriction of w to the $(\tilde{u},\tilde{v})$--chart, that is we only consider $\tilde{v}^3(\tilde{v}+\tilde{u}^3)$.

Blow up $\sigma_{p_1}M$ at p_2 and in $\sigma_{p_1p_2}M$ take local coordinates u_1,v_1 satisfying $\tilde{u}=u_1$, $\tilde{v}=u_1v_1$ (for reasons similar to those given above, the other chart may be discarded). In this chart $|w=0|$ is given by $|u_1^4v_1^3(v_1+u_1^2)=0|$, or in a diagram:

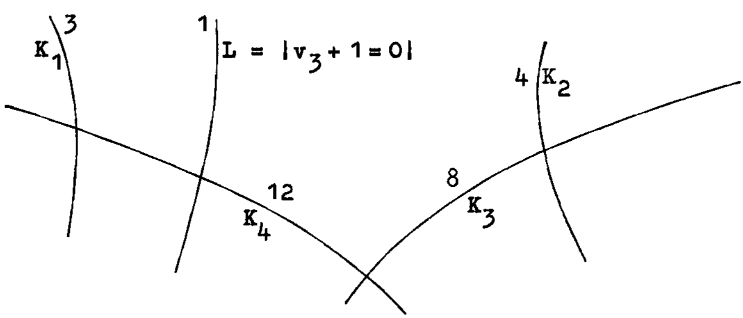


Next let p_3 be a point where $(u_1,v_1)=(0,0)$. Blow up at p_3 and consider the charts (u_2,v_2) and $(\tilde{u}_2,\tilde{v}_2)$ in $\sigma_{p_1p_2p_3}M$ satisfying $u_1=u_2$, $v_1=u_2v_2$; $u_1=\tilde{u}_2\tilde{v}_2$, $v_1=\tilde{v}_2$. In these charts

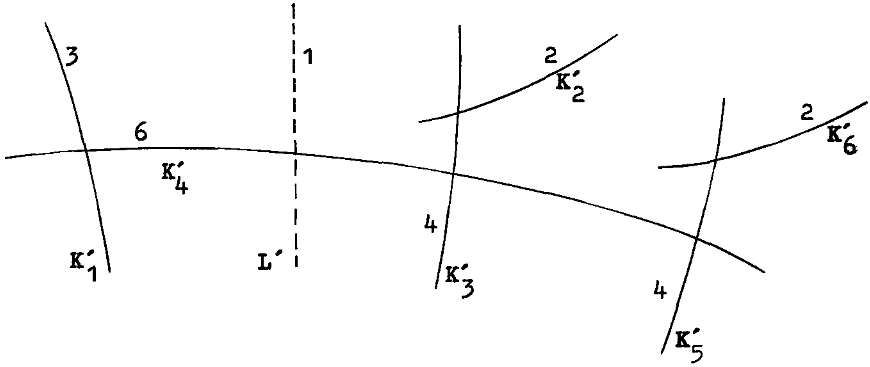
$|w=0|$ is given by $|u_2^8 v_2^3(u_2+v_2)=0|$ and $|\tilde{u}_2^4 \tilde{v}_2^8(1+\tilde{u}_2^2\tilde{v}_2)=0|$ respectively.



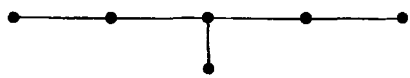
Finally we blow up at p_4 where $(u_2,v_2)=(0,0)$ to obtain a manifold $\widetilde{M}=\sigma_{p_1p_2p_3p_4}M$ in which the divisor $|w=0|$ is expressed by $|u_3^{12}v_3^3(v_3+1)=0|$ and $|\tilde{u}_3^8\tilde{v}_3^{12}(1+\tilde{u}_3)=0|$. In a diagram this is



The function $f=w^{1/2}$ in $\widetilde{M}$ is now uniformizable. Branching of f occurs only along K_1 and $L=|v_3+1=0|$ where the multiplicities are odd. In the Riemann surface $\widetilde{R}_f$ of the function f on $\widetilde{M}$, K_4 is lifted to a curve K_4' with two branches K_1' and L' sticking out. K_4' is a two-fold branched cover of K_4 with just two branch points (the intersection points with K_1 and L) and is hence again a 2-sphere ; K_1' covers K_1 once and L' covers L once. Thus the divisor $D = |f=0|$ in $\widetilde{R}_f$ can be expressed by the diagram



where K_3' and K_5' cover K_3 , and K_2' and K_6' cover K_2 . All the loci except L' are 2-spheres. The dual graph of the the spherical space $K_1' \cup \dots \cup K_6'$ is just E_6 (with vertices renamed)



We now use (9.2) to compute the self-intersection numbers $K_i' \cdot K_i'$ $(i = 1, \dots, 6)$. Note that if K_i' and K_j' $(i \neq j)$ are not disjoint, then they intersect in one point regularly, so $K_i' \circ K_j' = +1$. Also $K_4' \circ L' = +1$. We have for example

$$0 = K_1' \cdot D = 6 + 3K_1' \circ K_1' \quad ,$$
$$0 = K_4' \circ D = 3 + 1 + 4 + 4 + 6K_4' \circ K_4' \ .$$

Therefore $K_1' \circ K_1' = K_4' \circ K_4' = -2$. The reader will have no trouble in showing that $K_i' \circ K_i' = -2$ also for the remaining curves. Thus the dual weighted tree of the union of 2-spheres $K = K_1' \cup \dots \cup K_6'$ is E_6 weighted by -2 ; let us denote this tree by $(E_6;-2)$.

It is now clear that if V is a tubular neighbourhood of K in the Riemann surface $\widetilde{R}_f$ of (the modified) f , then V is diffeomorphic to $P(E_6;-2)$. We have investigated this space in §8. In particular $\partial V \cong S^3/T'$, where T' is the binary tetrahedral group.

We now return to the unmodified function $f = (z_1^3 + z_2^4)^{1/2}$ on $\mathbb{C}^2$. Let ψ be the projection of the Riemann surface R_f of f onto $\mathbb{C}^2$, and let B be the unit ball in C^2. Then $\psi^{-1}(B)$ is a neighbourhood of the singular point $\psi^{-1}(0) = q$ of R_f. The boundary of this neighbourhood is clearly diffeomorphic to $\partial V \cong S^3/T'$. Note that for this f, R_f is isomorphic to the graph of f, that is the set of points $(z_1, z_2, z_3) \varepsilon \mathbb{C}^2 \times \mathbb{C} = \mathbb{C}^3$ satisfying $z_1^3 + z_2^4 = z_3^2$. We have thus calculated the diffeomorphism type of a "neighbourhood boundary" of the singular point $q = 0$ of this surface in $\mathbb{C}^3$.

<u>Exercise (5.8)</u>: Let $M = \mathbb{C}^2$. Show that for $f = \sqrt{z_1(z_1^2 + z_2^3)}$, $\sqrt{z_1^3 + z_2^5}$, $\sqrt{z_1(z_2^2 + z_1^n)}$ $(n \geq 2)$, $\sqrt{z_1^2 - z_2^n}$ $(n \geq 2)$, the dual trees obtained by the preceding process are respectively E_7, E_8, D_{n+1}, A_{n-1}, every tree weighted by -2.

In general, resolving a singularity of an algebroid function f defined on a complex manifold $M^{(2)}$ amounts to replacing the singular point p in the "Riemann surface" of f by a finite number of curves $K_1, K_2, \ldots, K_n$ $(\dim_{\mathbb{R}} K_i = 2)$ of various genera, such that for any pair i,j $(i \neq j)$ the curves K_i and K_j are either disjoint or intersect in one point regularly. Thus the intersection relations give rise to a dual weighted graph, where each vertex corresponds to a curve; furthermore this graph is connected. The quadratic form associated with this dual graph (in the same way as that associated with a tree) may be represented by the intersection matrix $M = (\alpha_{ij})$, where $\alpha_{ij} = K_i \circ K_j$. We have:

<u>THEOREM (9.9)</u>: The quadratic form S defined above is negative definite.

This is essentially a classical theorem (see Du Val [27]). The proof we are about to give is due to D. Mumford [19].

<u>Proof</u>: Let $\bar{f} = f - f(p)$, where p is the singular point of the Riemann surface. Then the multiplicity m_i of the curve K_i in the divisor $|\bar{f} = 0|$ is a positive integer for each $i = 1, \dots, n$. Define a quadratic form

$$S' \sim \sum_{i,j} \alpha'_{ij} x_i x_j ,$$

where $\alpha'_{ij} = m_i m_j \alpha_{ij} = m_i K_i \circ m_j K_j$. Clearly S is negative definite if and only if S' is so. Notice that

$$\alpha'_{ij} \geq 0 \quad \text{if } i \neq j . \tag{9.10}$$

In virtue of (9.2) we have

$$\sum_i \alpha'_{ij} = (\sum_i m_i K_i) \circ m_j K_j \leq 0 \quad (j = 1, \dots, n) , \tag{9.11}$$

and

$$\sum_i \alpha'_{ij} < 0 \quad \text{for some } j . \tag{9.12}$$

Write S' in the form

$$\sum_{i,j} \alpha'_{ij} x_i x_j = \sum_j (\sum_i \alpha'_{ij}) x_j^2 - \sum_{i<j} \alpha'_{ij} (x_i - x_j)^2 .$$

It is then clear that (9.10) and (9.11) imply that S' is negative semidefinite. To prove definiteness equate the above expression to zero. Then considering the first term on the right hand side, (9.12) shows that $x_j = 0$ for some j. Considering the second term, the fact that the dual graph is connected proves that $x_1 = x_2 = \dots = x_n$. ∥

In this section we prove theorem (6.5) which was stated without proof in §6. This theorem and the proof given here are due to Kervaire and Milnor [13].

Let M be a compact oriented differentiable 4-manifold without boundary. Let $\pi: H^2(M;\mathbb{Z}) \longrightarrow H^2(M;\mathbb{Z}/2\mathbb{Z})$ be the reduction modulo 2 and let $d \in H^2(M;\mathbb{Z})$ be such that $\pi d = w_2(M)$. Under Poincaré duality d corresponds to an element $b \in H_2(M;\mathbb{Z})$. Theorem (6.5) states that

<u>THEOREM</u>: If b can be represented by a differentiable embedding $f: S^2 \longrightarrow M$, then the self-intersection number $b \circ b = (d \smile d)[M]$ is congruent to $\tau(M)$ modulo 16.

<u>Proof</u>: First let us consider the case where $b \cdot b = -1$. Then $f(S^2)$ in M has normal bundle associated to the principal $SO(2)$-bundle $\xi_{(-1)}$ defined in §8. In other words $f(S^2)$ has in M a tubular neighbourhood A whose boundary ∂A, considered as the normal S^1-bundle of $f(S^2)$, is the Hopf fibration $S^3 \longrightarrow S^2$. Put $M_1 = (M - \operatorname{Int} A) \cup e^4$, where e^4 is a 4-cell whose boundary $\partial e^4 = S^3$ is attached to $\partial A = S^3$ by the identity map. Then

$$M = M_1 \# (-\mathbb{C}P^{(2)}) \quad .$$

Clearly $w_2(M_1) = 0$ and hence $\tau(M_1) \equiv 0 \pmod{16}$ by (6.2). Since $\tau(M) = \tau(M_1) + \tau(-\mathbb{C}P^{(2)}) = \tau(M_1) - 1$, the theorem is verified for this particular case.

In the general case we may assume that $b \circ b = s \geq 0$, by reversing the orientation of M if necessary. Let $P_1, \dots, P_{s+1}$ be $s+1$ copies of $-\mathbb{C}P^{(2)}$ and let

$$M' = M \# P_1 \# P_2 \# \dots \# P_{s+1} .$$

Using the natural isomorphism

$$j:\ H_2(M;\mathbb{Z}) \oplus H_2(P_1;\mathbb{Z}) \oplus \dots \oplus H_2(P_{s+1};\mathbb{Z}) \longrightarrow H_2(M';\mathbb{Z}) ,$$

Let

$$c = j(b \oplus g_1 \oplus \dots \oplus g_{s+1}) ,$$

where g_i denotes a generator of $H_2(P_i;\mathbb{Z})$. Then

$$c \cdot c = b \circ b + \sum_{i=1}^{s+1} g_i \circ g_i = -1 .$$

Using the hypothesis that b can be represented by a differentiable embedding of S^2 in M, it follows easily that c can be represented by a differentiable embedding of S^2 in M'. Since c is clearly dual to a cohomology class whose reduction modulo 2 is $w_2(M')$, the special case of the theorem that we have just proved shows that

$$\tau(M') \equiv -1 \pmod{16} .$$

Since $\tau(M') = \tau(M) - (s+1)$, we deduce that

$$\tau(M) \equiv s \pmod{16} ,$$

as was to be proved. ||

<u>EXAMPLE (10.1)</u> : Let $M = S^2 \times S^2$ and let $\alpha, \beta \in H^2(M;\mathbb{Z})$ be the standard generators. Then α and β together form a basis of $H^2(M;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, and with respect to this basis the quadratic form S_M of M is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The reduction modulo 2 of an element $m\alpha + n\beta \in H^2(M;\mathbb{Z})$ is equal to $w_2(M)$ if and only if m and n are both even. Assume this is the case and let $m = 2m'$, $n = 2n'$. Using the relation

$$(2m'\alpha + 2n'\beta)^2 = 8m'n'\alpha \cup \beta \quad ,$$

it follows that the dual homology class of $m\alpha + n\beta$ can be represented by a differentiably embedded 2-sphere only if $8m'n' \equiv 0 \pmod{16}$; that is $m'n'$ is even. In particular $2\alpha + 2\beta$ is not representable by a differentiably embedded 2-sphere in M .

<u>EXAMPLE (10.2)</u> : Let $M = \mathbb{C}P^{(2)}$ and let $g \in H^2(M;\mathbb{Z})$ be the standard generator. The dual homology class of g is obviously represented by a differentiably embedded 2-sphere ; however the dual class of $3g$, for example, is not. Indeed, an element $ng \in H^2(M;\mathbb{Z})$ satisfies $\pi(ng) = w_2(M)$ if and only if n is odd. Consider the element $(2k+1)g \in H^2(M;\mathbb{Z})$. If its dual homology class is representable by a differentiably embedded 2-sphere then $(2k+1)^2 \equiv 1 \pmod{16}$; that is $k \equiv 0$ or $3 \pmod 4$.

On the other hand, the dual homology class of every element ng can be represented by a combinatorially embedded 2-sphere in $M = \mathbb{C}P^{(2)}$. To see this let z_0, z_1, z_2 be homogeneous coordinates in $\mathbb{C}P^{(2)}$, and let $P(z_0, z_1, z_2)$ be a homogeneous polynomial of (total) degree n in z_o, z_1 and z_2. This polynomial defines a divisor of $\mathbb{C}P^{(2)}$ in the obvious fashion (e.g. in the chart where $z_o \neq 0$ consider the function $P(1, z_1/z_o, z_2/z_o)$), and the homology class of this divisor is represented by the algebraic curve $P=0$. If $Q = Q(z_o, z_1, z_2)$ is another homogeneous polynomial of the same degree, then the meromorphic function P/Q is globally defined on $\mathbb{C}P^{(2)}$, so as a divisor it gives the zero homology class. In other words P and Q represent the same homology class.

Since the dual class of g is represented by the projective line $z_o = 0$, for P of degree n the class represented by $P=0$ is dual to ng. In particular let

$$P = z_o z_1^{n-1} - z_2^n \quad .$$

The curve $P=0$ has just one singularity, a cusp at $(1,0,0)$. Using the classical Plücker formula

$$\text{genus} = (n-1)(n-2)/2 - \text{local terms} \quad ,$$

one sees that this curve has genus zero (the multiplicity m_p of the curve at the cusp is $n-1$, so the local term is $m_p(m_p-1)/2 = (n-1)(n-2)/2$). Thus the curve $P=0$, considered as a real surface in the 4-manifold $\mathbb{C}P^{(2)}$, is a 2-sphere and is clearly combinatorially embedded. Another way of seeing this is by checking that the map of $\mathbb{C}P^{(1)}$ into $\mathbb{C}P^{(2)}$ given in homogeneous coordinates by

$$(w_o, w_1) \longrightarrow (w_o^n, w_1^n, w_1^{n-1} w_o)$$

maps $\mathbb{C}P^{(1)} = S^2$ bijectively onto the curve $P=0$.

We have thus represented the dual class of ng, $n \geq 1$, by a combinatorially embedded 2-sphere. This is trivially also possible for

$n = 0$, and for $n \leq -1$ one can clearly do it by reversing the orientation of the embedded sphere.

Remark (10.3) : Kervaire and Milnor proved also for the manifold $S^2 \times S^2$ that any 2-dimensional homology class can be represented by a combinatorial embedding of S^2 .

REFERENCES

[1] A. Borel, A. Haeflinger, La classe d'homologie fundamentale d'un espace analytique, Bull. Soc. Math. de France 89 (1961) 461-513.

[2] A. Borel, F. Hirzebruch, Characteristic classes and homogeneous spaces I, Am. J. Math. 80 (1958) 459-538.

[3] ------ ditto II, Am. J. Math. 81 (1959) 315-382.

[4] ------ ditto III, Am. J. Math. 82 (1960) 491-504.

[5] A. Borel, J. Moore, Homology theory for locally compact spaces, Mich. Math. J. 7 (1960) 137-159

[6] N. Bourbaki, Algèbre, Hermann, Paris, 1952.

[7] J.W.S. Cassels, Über die Äquivalenz 2-adischer quadratischer Formen, Comm. Math. Helv. 37 (1962/63) 61-64.

[8] J. Eells, N.H. Kuiper, An invariant for certain smooth manifolds, Annali di Mat. pura et appl. 60 (1963) 93-110.

[9] P.J. Hilton, S. Wylie, Homology theory, Cambridge, 1960.

[10] M.W. Hirsch, The imbedding of bounding manifolds in euclidean space, Ann. of Math. 74 (1961) 494-497.

[11] B.W. Jones, The arithmetic theory of quadratic forms, Carus Math. Monograph No. 10, New York, 1950.

[12] M.A. Kervaire, J. Milnor, Bernoulli numbers, homotopy groups, and a theorem of Rohlin, Proc. Internat. Congr. of Math. (Edinburgh 1958) 454-458.

[13] ------ On 2-spheres in a 4-manifold, Proc. Nat. Acad. Sci. U.S.A. 49 (1961) 1651-1657.

[14] J. Milnor, On simply connected 4-manifolds, Symp. Int. de Topol. Algebraica, Mexico 1956, (Mexico 1958) 122-128.

[15] ------ Some consequences of a theorem of Bott, Ann. Math. 68 (1958) 444-449.

[16] ------ Differentiable structures (mimeographed), Princeton 1960.

[17] ------ Differentiable manifolds which are homotopy spheres (mimeographed), Princeton, 1959.

[18] ------ A procedure for killing homotopy groups of differentiable manifolds, Proc. Symposia in Pure Math. 3, A.M.S. (1961) 39-55.

[19] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. Math. No. 9 I.H.E.S., Paris, 1961.

[20] R. von Randow, Zur Topologie von dreidimensionalen Baummannigfaltigkeiten, Bonner Math. Schriften 14, Bonn, 1962.

[21] V.A. Rohlin, A new result in the theory of 4-dimensional manifolds (russian), Dokl. Akad. Nauk. SSSR 84 (1952) 221-224.

[22] H. Seifert, Topologie dreidimensionaler gefaserter Raüme, Acta Math. 60 (1933) 147-238.

[23] H. Seifert, W. Threlfall, Lehrbuch der Topologie, Leipzig, 1934.

[24] J.P. Serre, Formes bilinéares symétriques entières à discriminant ±1, Sem. H. Cartan, 14e année (1961/62), Exp. 14.

[25] N. Steenrod, The topology of fibre bundles, Princeton, 1951.

[26] E. Stiefel, Richtungsfelder und Fernparallelismus in Mannigfaltigkeiten, Comm. Math. Helv. 8 (1936) 3-51.

[27] P. Du Val, On absolute and non-absolute singularities of algebraic surfaces, Revue de la Faculté de Science de l'Université d'Istanbul (A) 91 (1944) 159-215.

[28] J.H.C. Whitehead, On incidence matrices, nuclei and homotopy type, Ann. of Math. 42 (1941) 1197-1237.

APPENDIX I.

In this appendix we comment on the subject matter of these notes from a 1971 viewpoint, thus giving an indication of newer developements in the field.

1. Definition of Quadratic Forms.

Nowadays the space V on which a bilinear form is defined is usually allowed to be a finitely generated projective (rather than free) A-module, in order to assure that a direct summand of V is again of the same type. If A is a principal ideal domain, as is the case for all rings considered in these notes, then finitely generated projective A-modules are free, so this is no change.

If V is a finitely generated projective A-module then a <u>quadratic form</u> on V is defined as a map

$$q : V \longrightarrow A$$

such that $q(\alpha x) = \alpha^2 q(x)$ $(\alpha \in A)$ and $b_q(x,y) := q(x+y) - q(x) - q(y)$ is bilinear ; q is called non-singular or non-degenerate according as the associated bilinear form b_q is non-singular or non-degenerate. A form as defined in §1 is simply called a <u>symmetric bilinear</u>

form. If 2 is a unit in A, then the two definitions are essentially the same, since a quadratic form q determines a symmetric bilinear form b and vice versa by

$$(1) \qquad b(x,y) = b_q(x,y) = q(x+y) - q(x) - q(y) ,$$

$$(2) \qquad q(x) = \tfrac{1}{2} b(x,x) .$$

However if $2 \notin A^*$ then the theory of quadratic forms and the theory of symmetric bilinear forms diverge.

Example: If $A = \mathbb{Z}$ or $\mathbb{Z}_2$ (2-adic integers) then (1) and (2) define a one-one correspondence between quadratic forms and even symmetric bilinear forms, so over $\mathbb{Z}$ and $\mathbb{Z}_2$ the theory of quadratic forms is essentially the theory of even bilinear symmetric forms.

In view of the above comments, when we remarked in §1 that the Hilbert symbol is a non-degenerate quadratic form over $\mathbb{F}_2$, we "meant" symmetric bilinear form, and in §4 it was the Grothendieck ring of non-singular integral symmetric bilinear forms (not non-singular quadratic forms) which was calculated.

2. The Grothendieck and Witt Groups.

By the above comments, the ring $G_o(A)$ defined in §2 would nowadays be called the Grothendieck group of non-singular symmetric bilinear forms, which we shall now denote by $KU_o(A)$. The analogously defined Grothendieck group of non-singular quadratic forms is generally denoted by $GU_o(A)$. Assigning to a quadratic form q the associated symmetric bilinear form b_q leads to a map

$$GU_o(A) \longrightarrow KU_o(A) ,$$

which is an isomorphism if $2 \in A^*$.

The quotient groups of $KU_o(A)$ and $GU_o(A)$ obtained by setting hyperbolic forms (see Appendix II) equal to zero are called the Witt groups and denoted by $B(A)$ and $W(A)$ respectively. The theory of Grothendieck and Witt groups has taken enormous strides since these notes were first written. In particular many connections with other fields such as algebraic number theory and algebraic K-theory have developed. We refer the reader to Appendix II (by W. Scharlau) for more information.

3. Integral Forms.

In §4 we stated that little is known about the classification of definite unimodular integral forms. This is still true, though a little more is known now. We refer the reader to the charming book "Cours d'arithmétique" by J.P. Serre [29] for a brief discussion, see also Niemeyer [22].

4. Symbols.

In §2 we used the Hilbert symbol to calculate the Grothendieck ring $KU_o(\mathbb{Q}_p) = GU_o(\mathbb{Q}_p)$, there denoted by $G(\mathbb{Q}_p)$.

More generally if F is a field, a symbol on F is a bimultiplicative symmetric map

$$F^* \times F^* \longrightarrow C$$

into an abelian group C, satisfying $(a^2,b) = 1$, and $(a,1-a) = 1$ for $a \neq 1$. Given such a symbol one defines a <u>Hasse - Minkowski</u> or <u>Hasse - Witt</u> invariant $c(f)$ via a diagonalization of the form f by

$$c(\mathrm{diag}(a_1, \ldots, a_n)) := \prod_{i<j} (a_i, a_j) \quad .$$

The methods of §2 can then be generalized. For instance it is not hard to see that every Hasse-Minkowski invariant factors over the map $c' \colon G(F) \longrightarrow L(F)$ defined in the same way as c'_p in §2, so if this map is itself a Hasse-Minkowski invariant then the result of corollary (2.9) applies.

There is a universal symbol which turns out to be given by the algebraic K - group $K_2(F)$, which leads to a universal Hasse-Minkowski invariant (see Milnor [20]). However K_2 is generally difficult to calculate, so one often works with a concrete symbol with values in the Brauer group of F, given by setting (a,b) equal to the class of the quaternionic algebra of (a,b), see for instance W. Scharlau [27].

<u>5. Signature and the μ-Invariant</u>.

In the proof of the invariance of the μ- invariant we proved a special case of the following "additivity property" of signature. If M is a compact oriented manifold obtained by pasting two compact oriented manifolds M_1 and M_2 together along boundary components,

$$M = M_1 \cup -M_2 \quad ,$$

then

$$\tau(M) = \tau(M_1) - \tau(M_2) \quad .$$

This property was first observed by S.P. Novikov; a short proof (also in the equivariant case) can be found in Atiyah Singer [3], see also Jänich [14].

This property was fundamental in the definition of the μ-invariant. In a similar way, any additive invariant defined on some set of manifolds and zero on closed manifolds defines an invariant for boundaries of manifolds in the set, so long as the set is closed under the operation of pasting manifolds along their boundaries. In conjunction with the Atiyah-Singer index theorem [3] this has proved useful in obtaining interesting invariants for manifolds with group actions or other structure. See for instance [2], [10], [11], [19], [25].

The minor application of the μ-invariant to detecting h-cobordism type of lens spaces in special cases (§7) has been superceded by a complete h-cobordism classification (Atiyah Bott [1] p479). It turns out to be the same as the diffeomorphism classification. A general reference for this sort of classification problem is C.T.C. Wall's book [30] and the literature quoted there.

6. Plumbing and Spheres.

As remarked in §9, the boundary of a manifold $Y = P(T,m_i)$ obtained by plumbing n-disc bundles over the n-sphere ($n > 2$) is a homotopy sphere if and only if the intersection form S_Y, which is equal to the form of the weighted tree $(T, e(m_i))$, has determinant ± 1. A lot is known about smooth manifolds which are homotopy k-spheres for $k \geq 5$ ([16], [7], [18]). They are all homeomorphic to the standard sphere S^k, but not necessarily diffeomorphic. All the possible differentiable structures on the k-sphere form a finite group Θ_k with respect to the connected sum operation $\#$. Those spheres which bound a stably parallelizable manifold represent a cyclic subgroup $bP_{k+1} \subset \Theta_k$ which has order 1 for $k \equiv 0 \pmod 2$, 1 or 2 for $k \equiv -1 \pmod 4$, and rapidly increasing order for increasing $k \equiv 1 \pmod 4$.

Clearly, if one only plumbs stably trivial bundles, then $Y = P(T, m_i)$ is stably parallelizable, so ∂Y, if a sphere, represents an element of bP_{2n}. In this case the element is easily determined from the intersection form S_Y; for instance if one plumbs copies of the tangent disc bundle of S^n $(n \geq 3)$ according to the tree E_8 (n even) or A_2 (n odd), the resulting sphere ∂Y represents a generator of bP_{2n}. A convenient reference for plumbing of spheres is Hirzebruch Mayer [12].

7. Plumbing Seifert Manifolds.

In §9 we described von Randow's result on obtaining Seifert manifolds by plumbing. To apply this to calculating the μ-invariant one must alter it slightly. The Seifert manifold X with invariants

$$(b; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$$

is a $\mathbb{Z}/2\mathbb{Z}$ - homology sphere if and only if

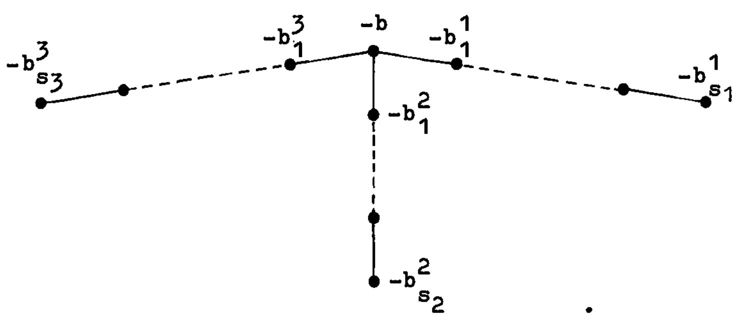
$$(1) \qquad b\alpha_1\alpha_2 \cdots \alpha_r + \beta_1\alpha_2 \cdots \alpha_r + \alpha_1\beta_2\alpha_3 \cdots \alpha_r + \alpha_1 \cdots \alpha_{r-1}\beta_r$$

is odd, as follows immediately from Seifert's results [28]. For any Seifert fibration one can drop the condition $0 < \beta_i < \alpha_i$ on the Seifert invariants and alter each β_i modulo α_i ; if one simultaniously alters b to keep the above expression (1) constant, the resulting "generalized" Seifert invariant set still determines the Seifert fibration in a well defined way [21]. If one now makes the restriction $0 < \beta_i < 2\alpha_i$, von Randow's result can be applied to the generalized invariants with no formal change to obtain alternative methods of obtaining X by plumbing. In particular if X is a $\mathbb{Z}/2\mathbb{Z}$ - homology sphere and one α_i is even one can use this to represent X as the boundary $X = \partial Y$ of a manifold $Y = P(T, m_i)$ with even intersection form (i.e. the m_i are even), which can hence be used to calculate $\mu(X)$.

8. Resolution of Singularities and Plumbing.

In §9 we illustrated the uniformization of singularities of algebroid functions in two variables by means of the example $\sqrt{z_1^3 + z_2^4}$. The relatively naive method of this example - blowing up "nasty" points (points where the "Riemann surface" has a non-uniformizable singularity) until none are left - will work for instance for functions of the form $\sqrt{f(z_1, z_2)}$ with f a polynomial, but already breaks down for the function $(z_1 z_2^{n-q})^{1/n}$ $(0 < q < n,\ n > 2,\ (n,q) = 1)$. However this function can be uniformized by replacing the singular point of its Riemann surface by a "spherical space" (c.f. p.81) having dual graph equal to the tree (A_s, m_i) of theorem (8.9). Furthermore, it turns out that essentially the above "naive method" can be used to reduce the general problem to this latter case, giving a general method of uniformizing such singularities. For details see Hirzebruch [9].

As we indicated in §9, uniformizing the singularity of the algebroid function $\sqrt{z_1^3 + z_2^4}$ is equivalent to resolving the isolated singularity of the variety $z_1^3 + z_2^4 = z_3^2$ in $\mathbb{C}^3$. This variety is clearly isomorphic to $z_1^3 + z_2^4 + z_3^2 = 0$. More generally one can consider the variety $X(a_1, a_2, a_3)$ given by $z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0$ in $\mathbb{C}^3$ $(a_i > 1)$, which has an isolated singularity at zero. This singularity has been resolved when the a_i are pairwise coprime by Hirzebruch Jänich [11] and in the general case by Orlik and Wagreich [24]. In the pairwise coprime case the result is that the singularity can be resolved by inserting a spherical space $S = \bigcup_\alpha S_\alpha$ whose intersection behaviour is given by the following dual tree $(T, -b_i^j)$:



Here the integers b and b_i^j are determined by the relations

$$a_j/b_j = [b_1^j, \dots, b_{s_j}^j] \quad , \quad b_i^j \geq 2 \; ,$$

$$ba_1a_2a_3 = 1 + \beta_1 a_2 a_3 + a_1 \beta_2 a_3 + a_1 a_2 \beta_3 \quad ,$$

where the β_k are given by $0<\beta_k<a_k$ and

$$a_i a_j \beta_k = -1 \pmod{a_k} \quad \text{for} \quad \{i,j,k\} = \{1,2,3\} .$$

Let $\Sigma(a_1,a_2,a_3) = X(a_1,a_2,a_3) \cap S^5$, where S^5 is the unit sphere in $\mathbb{C}^3$. The result quoted above shows that if the a_i are pairwise coprime, $\Sigma(a_1,a_2,a_3)$ is given by plumbing as $\Sigma(a_1,a_2,a_3) = \partial P(T,-b_i^j)$. This can also be seen as follows: $\Sigma(a_1,a_2,a_3)$ has a natural S^1-action

$$t(z_1,z_2,z_3) = (t^{a_2a_3}z_1, t^{a_1a_3}z_2, t^{a_1a_2}z_3) \qquad (t \in S^1) ,$$

which gives $\Sigma(a_1,a_2,a_3)$ the structure of a Seifert space. The Seifert invariants can be calculated directly [21], and one can then apply von Randow's results quoted in these notes. By part 7 of this appendix one can in this way also obtain alternative representations $\Sigma(a_1,a_2,a_3) = \partial P(T',m_i)$ of $\Sigma(a_1,a_2,a_3)$ by plumbing, and in some cases (namely if one a_i is even) one can do this with even m_i.

We mention two applications: firstly one can calculate the μ-invariant $\mu\Sigma(a_1,a_2,a_3)$ in many cases where it is defined. Secondly and more interestingly, $\Sigma(a_1,a_2,a_3)$ is a $\mathbb{Z}$-homology sphere if the a_i are pairwise coprime. Hence by theorem (8.2) the quadratic form of the tree (T',m_i) mentioned above has determinant ± 1 and can be used to plumb in higher dimensions to give homotopy spheres (see Hirzebruch [10] and Mayer [19] for an application to involutions on spheres). Of course if some m_i are odd one needs bundles with odd euler characteristic for this, which only exist in dimensions 2, 4 and 8.

To resolve the singularity of $X(a_1,a_2,a_3)$ in general (a_i not necessarily pairwise coprime) one can "replace" the singular point by $\Sigma(a_1,a_2,a_3)/S^1$, which is a complex curve by Brieskorn and Van de Ven [6]. One is then only left with singular points equivalent to singularities of the form $z_3^n = z_1 z_2^{n-q}$ which can be resolved as described at the beginning of this section. One thus also gets a star-shaped dual tree, but the central curve usually has higher genus and the tree more branches than in the coprime case. The details of this method are a bit messy and Orlik and Wagreich (loc. cit.) use a similar but more economical method to resolve in fact all 2-dimensional normal singularities which have an effective $\mathbb{C}^*$-action.

In higher dimensions one has the analogous variety $X(a_1, \dots, a_{n+1})$ in $\mathbb{C}^{n+1}$ given by $z_1^{a_1} + \dots + z_{n+1}^{a_{n+1}} = 0$ $(a_i > 1)$, which has an isolated singularity at zero. These singularities have been studied extensively by Brieskorn and others; it turns out that the neighbourhood boundary $\Sigma(a_1, \dots, a_{n+1}) = X(a_1, \dots, a_{n+1}) \cap S^{2n+1}$ of such a singularity is often a homotopy sphere, and that all elements of bP_{2n} can be represented by such spheres [4], see also Hirzebruch and Mayer [12].

We mention that theorem (9.9) has a converse due to Grauert ([8] p.367). Suppose that $K = \cup K_i$ is a union of compact irreducible curves in a non-singular complex surface $M^{(2)}$ such that K is connected and the bilinear form S defined by the intersection numbers of the K_i is negative definite; then the space M' obtained by collapsing K to a point is a complex surface with at most an isolated normal singularity at this point. Thus the classification of 2-dimensional normal singularities is essentially reduced to the determination of the dual graphs which can occur, together with the possible complex structures in a neighbourhood of such a union of curves K. This is a very difficult problem and is not solved in general. A connected account will appear in the notes by Henry Laufer [17] ; see also Brieskorn [5] for a discussion of the case of rational singularities.

9. Orientation Conventions.

As mentioned in §8, von Randow's orientation conventions for Seifert spaces and lens spaces are opposite to those adopted here, so his weighted trees are actually the negatives of the trees we gave when quoting his results on Seifert spaces and lens spaces.

In these notes we have tried to follow Raymond's orientation conventions of [26] and [23]. This is also the convention adopted in [21]. In the original mimeographed version of these notes (apart from a sign error at one point) and in Orlik Wagreich [24] the opposite convention has been adopted for Seifert spaces (but not for lens spaces). This leads to compatible results on obtaining the manifolds $\Sigma(a_1,a_2,a_3)$ by plumbing, but negative values to the above for Seifert invariants (except for lens spaces).

10. Representing Homology Classes by Embedded Spheres.

Using the Atiyah Singer equivariant signature theorem, W.C. Hsiang and R.H. Szczarba [13] have obtained results on representing integral homology classes by differentiably embedded surfaces in 4-manifolds, among other things greatly improving the results of examples (10.1) and (10.2). For $S^2 \times S^2$ they show a class in $H_2(S^2 \times S^2)$ can be represented by a differentiably embedded 2-sphere only if it is dual to $m\alpha + n\beta$ (notation of (10.1)) with m and n coprime. For $\mathbb{C}P^{(2)}$ they have complete results, namely only the dual classes of $0, \pm g, \pm 2g$ can be so represented.

These two examples are both simply connected, so $H_2(M) = \pi_2(M)$. In dimensions $n > 2$ Kervaire [15] has complete results on representing classes in $\pi_n(M^{2n})$ by differentiably embedded n-spheres.

REFERENCES

[1] M.F. Atiyah, R. Bott, A Lefschetz fixed point formula for elliptic complexes, II Applications, Ann. of Math. 88 (1968) 451-491.

[2] M.F. Atiyah, F. Hirzebruch, Spin-manifolds and group actions, Essays on Topology and Related Topics, 17-28, Springer Verlag, 1970.

[3] M.F. Atiyah, I.M. Singer, The index of elliptic operators III, Ann. of Math. 87 (1968) 546-604.

[4] E. Brieskorn, Beispiele zur Differentialtopologie von Singularitäten, Inv. Math. 2 (1966) 1-14.

[5] ------ Rationale Singularitäten komplexer Flächen, Inv. Math. 4 (1968) 336-358.

[6] E. Brieskorn, A. Van de Ven, Some complex structures on products of homotopy spheres, Topology 7 (1968) 389-393.

[7] W. Browder, The Kervaire invariant of framed manifolds and its generalization, Ann. of Math. 90 (1969) 157-186.

[8] H. Grauert, Über Modifikation und exzeptionelle analytische Mengen, Math. Ann. 146 (1962) 331-368.

[9] F. Hirzebruch, Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen, Math. Ann. 126 (1953) 1-22.

[10] ------ Involutionen auf Mannigfaltigkeiten, Proc. Conf. on Transf. Groups, New Orleans 1967, Springer 1968, 148-166.

[11] F. Hirzebruch, K. Jänich, Involutions and singularities, Algebraic Geometry, Papers presented at the Bombay Coll. 1968 Oxford University Press, 1969, 219-240.

[12] F. Hirzebruch, K.H. Mayer, O(n)-Mannigfaltigkeiten, exotische Sphären und Singularitäten, Springer Lecture Notes 57, 1968.

[13] W.C. Hsiang, R.H. Szczarba, To appear in Proc. Symposia in Pure Math., A.M.S.

[14] K. Jänich, Charakterisierung der Signatur von Mannigfaltigkeiten durch eine Additivitätseigenschaft, Inv. Math. 6 (1968) 35-40.

[15] M.A. Kervaire, Geometric and algebraic intersection numbers, Comm. Math. Helv. 39 (1968) 271-280.

[16] M.A. Kervaire, J. Milnor, Groups of homotopy spheres I, Ann. of Math. 77 (1963) 504-537.

[17] H. Laufer, Lecture notes, Princeton University Press, to appear.

[18] M. Mahowald, The order of the image of the J-homomorphism, Bull. A.M.S. 76 (1970) 1310-1313.

[19] K.H. Mayer, Fixpunktfreie Involutionen auf 7-Sphären, Math. Ann. 185 (1970) 250-258.

[20] J. Milnor, Algebraic K-theory and quadratic forms, Inv. Math. 9 (1970) 318-344.

[21] W.D. Neumann, S^1-actions and the α-invariant of their involutions, Dissertation Bonn, 1969, to appear as Bonner Math. Schriften 44.

[22] H. V. Niemeyer, Definite quadratische Formen der Dimension 24 und Discriminante 1, Dissertation, Göttingen, 1968.

[23] P. Orlik, F. Raymond, Actions of SU(2) on 3-manifolds, Proc. Conf. on Transf. Groups, New Orleans 1967, Springer, 1968, 297-318.

[24] P. Orlik, P. Wagreich, Isolated singularities of algebraic surfaces with $\mathbb{C}^*$ action, Ann. of Math. 93 (1971) 205-228.

[25] E. Ossa, Cobordismustheorie von fixpunktfreien und semifreien S^1- Mannigfaltigkeiten, Dissertation, Bonn, 1969.

[26] F. Raymond, Classification of actions of the circle on 3-manifolds, Trans. A.M.S. 131 (1968) 51-78.

[27] W. Scharlau, Quadratic forms, Queens Paper on Pure and Appl. Math. No. 22, Kingston 1969.

[28] H. Seifert, Topologie dreidimensionaler gefaserter Räume, Acta Math. 60 (1933) 147-238.

[29] J.P. Serre, Cours d'arithmétique, Presses Universitaires de France, Paris 1970.

[30] C.T.C. Wall, Surgery on compact manifolds, London Math. Soc. Monograph No. 1, Academic Press, London, 1970.

APPENDIX II : GROTHENDIECK AND WITT RINGS

W. Scharlau

Mathematisches Institut,
Universität Münster, Münster, Germany.

1. Generalities.

Let R be a commutative ring. A non-singular symmetric bilinear space over R is a pair (M,b) consisting of a finitely generated projective R-module M and a non-singular symmetric bilinear form $b : M \times M \longrightarrow R$. The Grothendieck group of non-singular symmetric bilinear spaces will be denoted by $KU_o(R)$.

A non-singular quadratic space over R is a pair (M,q) consisting of a finitely generated projective R-module M and a quadratic form $q : M \longrightarrow R$ whose associated bilinear form

$$b_q(x,y) := q(x+y) - q(x) - q(y)$$

is non-singular. The Grothendieck group will be denoted by $GU_o(R)$. There is an obvious natural homomorphism $GU_o(R) \longrightarrow KU_o(R)$ which is an isomorphism if 2 is a unit in R.

The theory of non-singular quadratic spaces can be developed analogously to the developement of algebraic K-theory. This has been done by H. Bass [2], A. Roy [16], and above all A. Bak [1]. The main results are the analogons of the stability and cancellation theorems of Bass and Serre. The theory can be developed in a more general setting by considering a ring R with involution instead of just a ring; one must then consider hermitian forms.

An important rôle is played by the <u>hyperbolic forms</u>, defined as follows.

Let M be a finitely generated projective R-module and $b : M \times M \longrightarrow R$ any symmetric bilinear form, respectively $q : M \longrightarrow R$ any quadratic form, not necessarily non-singular. The space $\mathbb{H}(M,b)$ or $\mathbb{H}(M,q)$ is defined as follows : the underlying module is $M \oplus M^*$ and the form is respectively

$$(x,f) \times (y,g) \longmapsto b(x,y) + g(x) + f(y)$$

or

$$(x,f) \longmapsto q(x) + f(x) \ .$$

Both forms are non-singular. The hyperbolic forms are the special case $\mathbb{H}(M) := \mathbb{H}(M,o)$.

One checks that

$$\mathbb{H}(M,b) \oplus \mathbb{H}(M,-b) \cong \mathbb{H}(M,o) \oplus \mathbb{H}(M,-b)$$

and

$$\mathbb{H}(M,q) \oplus \mathbb{H}(M,-q) \cong \mathbb{H}(M,o) \oplus \mathbb{H}(M,-q) \quad ,$$

so the subgroup of $KU_o(R)$ or $GU_o(R)$ generated respectively by the $\mathbb{H}(M,b)$ or $\mathbb{H}(M,q)$ is already generated by the hyperbolic forms $\mathbb{H}(M,o)$.

The quotient group by this subgroup is called the <u>Witt group</u> and denoted by $B(R)$ and $W(R)$ respectively.

<u>Remark</u> : $KU_o(R)$ and $B(R)$ are rings via tensor product of forms

and $GU_o(R)$ is in a natural way a $KU_o(R)$ - module and $W(R)$ a $B(R)$ - module.

2. Dedekind Domains.

From now on let R be a Dedekind ring. We first consider the local case, that is, R is a discrete valuation ring. Let π be a uniformizing element, that is, $(\pi) = \mathfrak{p}$ is the maximal ideal. Let F be the quotient field and assume $\operatorname{char}(F) \neq 2$. Any non-singular symmetric bilinear form b over F can be diagonalized, and by multiplying the diagonal coefficients by suitable squares one can write b in the form

$$b = \operatorname{diag}(a_1, \ldots, a_m, b_1\pi, \ldots, b_n\pi)$$

with $a_i, b_j \in R - \mathfrak{p}$. The form $\partial^2(b) := \operatorname{diag}(\bar{b}_1, \ldots, \bar{b}_n)$ gives a well defined element of $B(R/\mathfrak{p})$ and one obtains in this way exact sequences

$$0 \longrightarrow B(R) \longrightarrow B(F) \xrightarrow{\partial^2} B(R/\mathfrak{p}) \longrightarrow 0$$

and

$$0 \longrightarrow KU_o(R) \longrightarrow KU_o(F) \xrightarrow{\partial^2} B(R/\mathfrak{p}) \longrightarrow 0 \ .$$

∂^2 (which depends on the choice of π) is called the <u>second residue class form</u>. The element $\partial^1(b) := \operatorname{diag}(\bar{a}_1, \ldots, \bar{a}_n)$ is also well defined in the Witt group $B(R/\mathfrak{p})$ and is called the <u>first residue class form</u>.

<u>THEOREM</u>: If R is complete and $\operatorname{char}(R/\mathfrak{p}) \neq 2$ then there exist canonical isomorphisms

$$B(R) \cong B(R/\mathfrak{p}) \quad , \quad KU_o(R) \cong KU_o(R/\mathfrak{p}) \ .$$

Furthermore the above exact sequences split by means of the first residue class form, so one has direct sum representations

$$B(F) \cong B(R/\mathfrak{p}) \oplus B(R/\mathfrak{p}) \quad , \quad KU_o(F) \cong KU_o(R/\mathfrak{p}) \oplus B(R/\mathfrak{p}) \ .$$

Literature : T.A. Springer [22]. This theorem can be generalized to complete semilocal rings : C.T.C. Wall [24] . See also Knebusch [5], Scharlau [21].

Now let R be an arbitrary Dedekind ring with quotient field F and $\mathrm{char}(F) \neq 2$ (not a very essential restriction).

THEOREM : There is an exact sequence

$$0 \longrightarrow B(R) \longrightarrow B(F) \xrightarrow{\partial} \coprod_{\mathfrak{p} \in \mathrm{Max}(R)} B(R/\mathfrak{p}) \ ,$$

where the map ∂ is given by the second residue class forms.

Proof : Knebusch [5], Knebusch Scharlau [8], Milnor [12], Fröhlich [3].

THEOREM : There is a canonical exact sequence

$$0 \longrightarrow C/C^2 \longrightarrow GU_o(R) \longrightarrow GU_o(F) \quad ,$$

where C denotes the ideal class group of R and the map $C/C^2 \longrightarrow GU_o(R)$ assigns to an ideal $\mathfrak{a}$ of R the element $\mathbb{H}(\mathfrak{a}) - \mathbb{H}(R)$ of $GU_o(R)$.

Proof : Kneser [9], Knebusch [5], Fröhlich [3].

For symmetric bilinear forms the situation is similar but there are difficulties at the dyadic prime places (c.f. Knebusch loc. cit.)

3. Algebraic Number Fields.

The calculation of the Witt group has been done so far only for a few fields. One of the main results is :

THEOREM (Hasse - Minkowski) : Let F be an algebraic number field. Then the canonical map

$$W(F) \longrightarrow \prod_{\wp} W(F_{\wp})$$

is injective, where $\wp$ ranges over all finite and infinite prime places.

The image of this map can also be determined.

Literature : O.T. O'Meara [13], Witt [26].

Let R be the ring of integers in the algebraic number field F. Then the first exact sequence of the preceding page can be extended to an exact sequence

$$0 \longrightarrow B(R) \longrightarrow B(F) \xrightarrow{\partial} \coprod_{\wp \in \mathrm{Max}(R)} B(R/\wp) \longrightarrow C/C^2 \longrightarrow 0 \quad .$$

A detailed investigation of this situation, taking into account also the infinite primes and the quadratic reciprocity law, can be found in [8].

Example : The second residue class forms give an exact sequence

$$0 \longrightarrow B(\mathbb{Z}) \longrightarrow B(\mathbb{Q}) \longrightarrow \coprod_{p=2,3,\dots} B(\mathbb{Z}/p\mathbb{Z}) \longrightarrow 0$$

On the other hand $B(\mathbb{Q})$ can be calculated directly as

$$B(\mathbb{Q}) \cong \mathbb{Z} \oplus \coprod B(\mathbb{Z}/p\mathbb{Z}) \quad ,$$

so $B(\mathbb{Z}) \cong \mathbb{Z}$ (application to the invariant $b(w,w)$! - see remark (5.9) of the notes).

<u>Further Literature</u>: Fröhlich [3], Milnor [12], Scharlau [20], C.T.C. Wall [25].

<u>4. Function fields.</u>

A similar situation to the case $\mathbb{Q}$ exists also for rational function fields $F = k(t)$.

<u>THEOREM</u> (Milnor [11]) : The second residue class forms yield an exact sequence

$$0 \longrightarrow B(k) \longrightarrow B(F) \longrightarrow \coprod_{p \text{ irred.}} B(k[t]/p) \longrightarrow 0 \ .$$

On the other hand, $B(k[t])$ also lies in the kernel of $B(F) \longrightarrow \coprod B(k[t]/p)$. Since $B(k) \subset B(k[t])$, it follows that

$$B(k) = B(k[t]) \ .$$

This has also been proved by Harder (unpublished).

Little is known for arbitrary function fields in one variable. The known results mainly concern the quadratic reciprocity law, see for instance Geyer Harder Knebusch and Scharlau [4], Scharlau [21]. However if the field of constants is finite or equal to $\mathbb{R}$ then the theory is about as complete as in the algebraic number field case

<u>5. Characteristic Classes.</u>

One can define invariants for quadratic forms which are in a certain way analogous to characteristic classes for vector bundles. This can for instance be done by means of Galois cohomology in a very similar way to how it is done for vector bundles, see Scharlau [17], Springer [23]. The definitions vary a little according to whether one works with the Witt group or the Grothendieck group.

Literature: Witt [26], O'Meara [13], Scharlau [17],[18], Milnor [11].

6. Pfister Theory.

Let F be a field of characteristic $\neq 2$. The theorems of Pfister and subsequent investigations tell us a certain amount about the structure of $B(F) = W(F)$.

THEOREM: The torsion subgroup $W_t(F)$ of $W(F)$ is a 2-group and is precisely the kernel of the homomorphism

$$W(F) \longrightarrow \prod_{\alpha} (W(F_\alpha)) \quad ,$$

where F_α runs through the real closures of F (observe that $W(F_\alpha) \cong \mathbb{Z}$ by Sylvester's law of inertia). In particular if F is not formally real then $W(F)$ is a 2-group.

Literature: Pfister [14], [15], Lorenz [10], Scharlau [18],[19], Knebusch Scharlau [7], Knebusch Rosenberg Ware [6].

REFERENCES

[1] A. Bak, The stable structure of quadratic modules, Preprint 1970.

[2] H. Bass, Lectures on topics in algebraic K-theory, Tata Inst. Fund. Res. Bombay, 1967.

[3] A. Fröhlich, On the K-theory of unimodular forms over rings of algebraic integers, Preprint 1970.

[4] W.D. Geyer, G. Harder, M. Knebusch, W. Scharlau, Ein Residuensatz für symmetrische Bilinearformen, Inv. Math. 11 (1970) 319-328.

[5] M. Knebusch, Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen, Sitzungsber. Heidelberger Akad. Wiss. 1969/70, 3. Abh., Springer-Verlag 1970.

[6] M. Knebusch, A. Rosenberg, R. Ware, Structure of Witt rings, quotients of abelian group rings, and orderings of fields, Bull. A.M.S. 77 (1971) 205-210.

[7] M. Knebusch, W. Scharlau, Über das Verhalten der Wittgruppe bei Körpererweiterungen, Math. Ann., to appear.

[8] ------ Quadratische Formen und quadratische Reziprozitätsgesetze über algebraischen Zahlkörpern, Math. Zeitschr., to appear 1971.

[9] M. Kneser, Unpublished manuscript, 1962.

[10] F. Lorenz, Quadratische Formen über Körpern, Springer Lecture Notes, 1970.

[11] J. Milnor, Algebraic K-theory and quadratic forms, Inv. Math. 9 (1970) 318-344.

[12] ------ Symmetric inner product spaces over a Dedekind domain, Preprint, 1970.

[13] O.T. O'Meara, Introduction to quadratic forms, 2nd. Ed., Springer 1971.

[14] A. Pfister, Quadratische Formen in beliebigen Körpern, Inv. Math. 1 (1966) 116-132.

[15] ------ Quadratic forms, Cambridge lecture notes 1967.

[16] A. Roy, Cancellation of quadratic forms over commutative rings, J. Algebra 10 (1968) 286-298.

[17] W. Scharlau, Quadratische Formen und Galois-Cohomologie, Inv. Math. 4 (1967) 238-264.

[18] ------ Quadratic forms, Queen's paper on pure and applied math. No. 22, Kingston 1969.

[19] ------ Induction theorems and the structure of the Witt group, Inv. Math. 11 (1970) 37-44.

[20] ------ Quadratic reciprocity laws, J. Number Theory, to appear.

[21] W. Scharlau, Klassifikation hermitescher Formen über lokalen Körpern, Math. Ann. 186 (1970) 201-208.

[22] T. A. Springer, Quadratic forms over a field with a discrete valuation, Indagationes Math. 17 (1955) 352-362.

[23] ------ On the equivalence of quadratic forms, Proc. Acad. Amsterdam 62 (1959) 241-253.

[24] C.T.C. Wall, On the classification of hermitian forms III, complete semilocal rings, Preprint May 1971.

[25] ------ ditto IV , Global rings, Preprint June 1971.

[26] E. Witt, Theorie der quadratischen Formen in beliebigen Körpern, J. reine u. angew. Math. 176, (1937) 31-44.

AUTHOR INDEX

Numbers in square brackets are reference numbers and indicate that an author's work is referred to although his name is not cited in the text. Numbers that are underlined show the page on which a complete reference is listed.