34.

The topology of normal singularities of an algebraic surfac (d'après un article de D. Mumford)

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* We shall study Mumford's results in the complex-analytic case.

1. Regular graphs of curves

Let X be a complex manifold of complex dimension 2. A regular grap of curves on X is defined as follows.

i. $\Gamma = \{E_1, E_2, \ldots, E_n\}$.
ii. Each E_i is a compact connected complex submanifold of X of com dimension 1.
iii. Each point of X lies on at most two of the E_i.
iv. If $x \in E_i \cap E_j$ and $i \neq j$, then E_i, E_j intersect regularly in x $E_i \cap E_j = \{x\}$.

Γ defines a graph Γ' in the usual sense (i.e. a one-dimensional finite plicial complex) by associating to each E_i a vertex e_i and by joining e_i and an edge if and only if E_i and E_j intersect. Γ' becomes a "weighted graph attaching to each e_i the self-intersection number $E_i \cdot E_i$, i.e. the Euler nur of the normal bundle of E_i in X. We have the symmetric matrix

$$S(\Gamma) = ((E_i \cdot E_j))$$

where $E_i \cdot E_j$ $(i \neq j)$ equals 1 if $E_i \cap E_j \neq \emptyset$ and equals 0 if $E_i \cap E_j = \emptyset$. matrix is called the intersection matrix of Γ and defines a bilinear symm form S over the $\mathbb{Z}$-module $V = \mathbb{Z}\, e_1 + \mathbb{Z}\, e_2 + \ldots + \mathbb{Z}\, e_n$. The matrix $S(\Gamma$ pends (up to the ordering of the e_i) only on the weighted graph and ma denoted by $S(\Gamma')$. The subset A of X is called a tubular neighbourhood of

i. $A = \bigcup_{i=1}^{n} A_i$,
where A_i is a (compact) tubular neighbourhood of E_i,
ii. $E_i \cap E_j = \emptyset$ implies $A_i \cap A_j = \emptyset$,
iii. $E_i \cap E_j = \{x\}$ implies the existence of a local coordinate system (z with center x and a positive number ε such that the open neight hood

$$U = \{p \mid p \in X \wedge |z_1(p)| < 2\varepsilon \wedge |z_2(p)| < 2\varepsilon\}$$

is defined in this coordinate system and

$$A_i \cap U = \{p \mid p \in U \cap |z_2(p)| \leqq \varepsilon\},$$

$$A_j \cap U = \{p \mid p \in U \cap |z_1(p)| \leqq \varepsilon\},$$

$$A_i \cap A_j \subset U.$$

Such tubular neighbourhoods always exist.

A is a compact 4-dimensional manifold (differentiable except "corners") whose boundary M is a 3-dimensional manifold (without boundary). It is easy to see that A has $E = \bigcup_{i=1}^{n} E_i$ as deformation retract. Thus

$$\text{(1)} \qquad H_i(A) \cong H_i(E).$$

Suppose that the graph Γ' is connected. This is the case if M is connected. If, moreover, Γ' has no cycles, then E is homotopically equivalent to a wedge of n compact oriented topological surfaces with the genera $g_i = \text{genus}(E_i)$. If Γ' has p linearly independent cycles, then the homotopy type of E is the wedge of n surfaces as above and p one-dimensional spheres. The first Betti number of E is given by the formula

$$\text{(2)} \qquad b_1(E) = 2\sum_{i=1}^{n} g_i + p.$$

We have the exact sequence (rational cohomology)

$$\text{(3)} \qquad H^1(A, M) \to H^1(A) \to H^1(M).$$

By Poincaré duality $H^1(A, M) \cong H_3(A)$ which vanishes by (1).

Therefore $H^1(A)$ maps injectively into $H^1(M)$ which proves in virtue of (1) and (2):

Lemma. *If the regular graph of curves $\Gamma = \{E_1, \dots, E_n\}$ has a tubular neighbourhood A whose boundary M is a rational homology sphere, then the graph Γ' is a tree (i.e. Γ' is connected and has no cycles). Furthermore, the genera of the curves are all* 0, *thus all the E_i are 2-spheres.*

2. The fundamental group of the "tree manifold" M

Suppose M is obtained as in Section 1, assume that Γ' is a tree and all the E_i are 2-spheres. By the lemma of Section 1 this is true if M is a rational homology sphere. The fundamental group $\pi_1(M)$ is presented by the following theorem.

Theorem. *Put $S(\Gamma) = ((E_i \cdot E_j)) = (s_{ij})$. Then, with the above assumptions, $\pi_1(M)$ is isomorphic with the free group generated by the vertices $e_1, \dots, e_n$ of Γ' modulo the relations*

$$\text{(a)} \qquad e_i e_j^{s_{ij}} = e_j^{s_{ij}} e_i$$

$$\text{(b)} \qquad 1 = \prod_{1 \leqq j \leqq n} e_j^{s_{ij}},$$

the product in (b) being ordered from left to right by increasing j. Recall that t exponents s_{ij} are all 1 *or* 0 (for $i \neq j$).

Remark. Each weighted tree with a numbering of its vertices defines by tł recipe a group. A change of the numbering gives an isomorphic group. This not difficult to prove. Thus it makes sense to speak (up to an isomorphism) $\pi_1(\Gamma')$ where Γ' is any weighted tree.

We sketch a proof of the theorem. The boundary of A_i, denoted by ∂A_i, is circle bundle over S^2 with Euler number s_{ii}. A generator e_i of $\pi_1(\partial A_i)$ represented by a fibre. The only relation is

$$e_i^{s_{ii}} = 1.$$

Recall $M = \partial A$ and put $B_i = \partial A \cap A_i$ which is a 3-dimensional manifo obtained from ∂A_i by removing for each j with $j \neq i$ and $s_{ij} \neq 0$ a fib preserving neighbourhood of some fibre. This neighbourhood to be remov has in local coordinates (Section 1, (iii)) the description ($|z_1| < \varepsilon$, $|z_2| = \varepsilon$) aı thus is of the type $D^2 \times S^1$. The boundary of B_i consists of a certain number 2-dimensional tori (one for each j with $j \neq i$ and $s_{ij} \neq 0$). The fundament group $\pi_1(B_i)$ has generators e_j ($j = i$ or $s_{ij} \neq 0$) with the only relations

$$\text{(a)} \qquad e_i e_j = e_j e_i$$

$$\text{(b)} \qquad e_i^{-s_{ii}} = \prod e_j,$$

the product is in increasing order of j (over those e_j with $j \neq i$ and $s_{ij} \neq$ (Here e_i is representable by any fibre, thus also by a fibre on the j^{th} torus. e_j represented on the j^{th} torus by ($z_1 = \varepsilon e^{2\pi i t}$, $z_2 =$ constant of absolute value ε). becomes a fibre in B_j. Since $M = \cup B_i$, we can use van Kampen's theorem present $\pi_1(M)$ as the free product of the $\pi_1(B_i)$ modulo amalgamation certain subgroups $\pi_1(S^1 \times S^1)$. This gives the theorem. Our notation takes aut matically care of the amalgamation because for $s_{ij} \neq 0$ and $i \neq j$ the symbols e_j denote elements of $\pi_1(B_i)$ and of $\pi_1(B_j)$. Of course, *there is all the troul with the base point* which we have neglected in this sketch. The trouble is r serious, mainly because Γ' is a tree. A further remark to visualize the rel tions: B_i, as a circle bundle over $S^2 -$ (disjoint union of small disks), is trivi Thus e_i lies in the center of $\pi_1(B_i)$. There is a section of ∂A_i over the orient S^2 with one singular point. This gives an "oriented disk-like 2-chain" in ∂ with $e_i^{-s_{ii}}$ as boundary (characteristic class = negative transgression!). T small disks lift to disks in that 2-chain. They have to be removed and have t e_j ($j \neq i$, $s_{ij} \neq 0$) as boundary. Knowledge of the fundamental group of a di with small disks removed gives (b).

Corollary. *The determinant of the matrix (s_{ij}) is different from* 0 *if and only $H_1(M; \mathbb{Z})$ is finite. If this is so, then $|\det(s_{ij})|$ equals the order of $H_1(M; \mathbb{Z})$.*

Proof. Recall that $H_1(M; \mathbb{Z})$ is the abelianized $\pi_1(M)$. The corollary follo from relation (b) of the theorem. The result can also be obtained directly frc the exact homology sequence of the pair (A, M) which identifies $H_1(M;$

with the cokernel of the homomorphism $V \to V^*$ defined by the quadratic form S (for the notation see Section 1). $H_2(A;\mathbb{Z})$ may be identified with V and $H_2(A, M; \mathbb{Z})$ by Poincaré duality with $V^* = \mathrm{Hom}(V, \mathbb{Z})$.

3. Elementary trees

In this section we shall prove a purely algebraic result.

A weighted tree is a finite tree with an integer associated to each vertex.

An elementary transformation (of the first kind) of a weighted tree adds a new vertex x, joins it to an old vertex y by a new edge, gives x the weight -1 and y the old weight diminished by 1. Everything else remains unchanged.

An elementary transformation (of the second kind) adds a new vertex x, joins it to the two vertices y_1, y_2 of an edge k by edges k_1, k_2, removes k, gives x the weight -1 and y_i $(i=1,2)$ the old weight of y_i diminished by 1. The following proposition is easy to prove.

Proposition. *If Γ' is a weighted tree and Γ'' obtainable from Γ' by an elementary transformation, then $S(\Gamma'')$ is negative definite if and only if $S(\Gamma')$ is. Furthermore $\pi_1(\Gamma') \cong \pi_1(\Gamma'')$* (for the notation see Section 1 and the Remark in Section 2).

An elementary tree is a weighted tree obtainable from the one-vertex-tree with weight -1 by a finite number of elementary transformations.

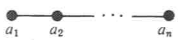
Theorem. *Let Γ' be a weighted tree. Suppose that $\pi_1(\Gamma')$ is trivial and that the matrix (integral quadratic form) $S(\Gamma')$ is negative definite. Then Γ' is an elementary tree.*

For the proof a group theoretical lemma is essential whose proof we omit.

Lemma. *Let G_1, G_2, G_3 be non-trivial groups, and $a_i \in G_i$. Then the free product $G_1 * G_2 * G_3$ modulo the relation $a_1 a_2 a_3 = 1$ is a non-trivial group.*

Inductive proof of the theorem. Suppose it is proved if the number of vertices in the weighted tree is less than n. Let Γ' have n vertices $e_1, \ldots, e_n$.

First case. There is no vertex in Γ' which is joined by edges with at least three vertices.

Then Γ' is linear 

where a_i is the associated weight. It follows that one of the a_i must be -1, if not $\det S(\Gamma')$ would be up to sign the numerator of the continued fraction

$$|a_1| - \cfrac{1}{|a_2| - \cfrac{\ddots}{\ddots - \cfrac{1}{|a_n|}}} \qquad (a_i \leqq -2)$$

which is not 1. This contradicts the corollary in Section 2. Thus Γ' is a elementary transform of a tree Γ'' with $n-1$ vertices. By the proposition an the induction assumption Γ' is elementary.

Second case. There is a vertex e_1, say, joined with $e_2, \ldots, e_m$ $(m \geqq 4)$.

We may choose this notation since the numbering plays no role for th fundamental group (see the Remark in Section 2).

Take Γ', remove e_1 and the edges joining it to $e_2, \ldots, e_m$. The remainin one-dimensional complex is a union of $m-1$ trees $T_2, \ldots, T_m$ where T_i has as vertex. The free product of the $\pi_1(T_i)$, $i=2, \ldots, m$, modulo the relatio $e_2 e_3 \ldots e_m = 1$ gives obviously (see Section 2) the group $\pi_1(\Gamma')$ modulo $e_1 =$ By assumption $\pi_1(\Gamma')$ is trivial. By the lemma at least one of the groups $\pi_1(T_i$ say $\pi_1(T_2)$, is trivial. By induction assumption T_2 is elementary and thus ca be reduced by removing a vertex x with weight -1 to give a weighted tree of which T_2 is an elementary transform of first or second kind. If $x \neq e_2$ or $x = e_2$ and joined only with one vertex in T_2, then Γ' is an elementary transfor of the tree consisting of the T_i $(i=3, \ldots, m)$, T_2', and e_1 (with the weig unchanged or increased by 1 respectively). By induction and the propositio Γ' would be elementary. In the remaining case $x = e_2$ and e_2 is joined wi exactly three vertices in Γ', namely e_1 and, say, e_{m+1}, e_{m+2} of T_2. Again, eith Γ' would be an elementary transform of a smaller tree, or the weight of or e_{m+1} or e_{m+2} would be -1. But the latter case cannot occur, since tl quadratic form takes on $e_r + e_s \in V$ (see Section 1) the value 0, if e_r, e_s ha weight -1 and are joined by an edge, and this would be true for $r=2$ ar $s=1, m+1$ or $m+2$ and contradict the negative definiteness of $S(\Gamma')$.

4. A blowing-down theorem

Theorem. *Let X be a complex manifold of complex dimension 2 a $\Gamma = \{E_1, E_2, \ldots, E_n\}$ a regular graph of curves on X. Suppose the boundary some tubular neighbourhood of Γ be simply-connected and the matrix $S(\Gamma$ negative-definite. Then the topological space X/E* (i.e. X *with $E = \bigcup_{i=1}^{n} E_i$ collaps to a point) is a complex manifold in a natural way: The projection $X \to X/E$ holomorphic and the bijection $X - E \to X/E - E/E$ is biholomorphic.*

Proof. By the lemma in Section 1 and the theorem in Section 3 all curves E_i a 2-spheres and Γ' is an elementary tree. If Γ' has only one vertex, then t above theorem is due to Grauert or, in the classical algebraic geometric ca to Castelnuovo-Enriques. By the very definition of an elementary tree a easy properties of "quadratic transformations" the result follows.

5. Resolution of singularities

Let Y be a complex space of complex dimension 2 in which all points a non-singular except possibly the point y_0 which is supposed to be normal. T theorem on desingularization states that there exist a complex manifold X, a regular (see Section 1) graph Γ of curves $E_1, \ldots, E_n$ on X, a holomorphic map $\pi: X \to Y$ with

$$\pi(E) = \{y_0\}, \quad \text{where} \quad E = \bigcup_{i=1}^{n} E_i,$$

$$\pi_{|X-E}: X - E \to Y - \{y_0\} \text{ biholomorphic.}$$

Thus the topological investigation of A and M (Section 1) which we have carried through so far contains as special case the investigation of singularities. A theorem, which we do not prove here, states *that $S(\Gamma)$ is negative-definite if Γ comes from desingularizing a singularity.*

6. The Main theorem of Mumford

Theorem. *Let Y, y_0 be as in Section 5. Suppose that y_0 has in Y a neighbourhood U homeomorphic to $\mathbb{R}^4$ by local coordinates $t_1, \ldots, t_4$. Then y_0 is non-singular.*

"Desingularize" y_0 as in Section 5. Take a tubular neighbourhood A of Γ. We can find a positive number δ such that $K = \pi^{-1}\{p \mid p \in U \wedge \sum t_i^2(p) < \delta\} \subset A$. There exists a tubular neighbourhood A' with

$$A' \subset K \subset A$$

and such that A' is obtained from A just be multiplying the "normal distances" by a fixed positive number $r < 1$. Any path in $A - E$ is homotopic to a path in $A' - E$ which is nullhomotopic in $A - E$ because $\pi_1(K - E) = \pi_1(\mathbb{R}^4 - \{0\})$ is trivial. The theorem in Section 4 together with the theorem mentioned at the end of Section 5 completes the proof.

7. Further remarks

For any weighted tree Γ' the construction in Section 1 can be topologized (assume genus $g(E_i) = 0$). In this way we may attach to each weighted tree Γ' a 3-dimensional manifold $M(\Gamma')$ (see von Randow [5]) which, as can be shown, depends only on Γ' (up to a homeomorphism).

We have $\pi_1(M(\Gamma')) = \pi_1(\Gamma')$ (see Section 2). Von Randow [5] has investigated the tree manifold $M(\Gamma')$ and shown in analogy to Mumford's theorem (Section 6) that $M(\Gamma')$ is homeomorphic to S^3 if $\pi_1(\Gamma')$ is trivial. Thus there is no counter-example to Poincaré's conjecture in the class of tree manifolds $M(\Gamma')$. Von Randow's investigations and also the topological part of Mumford's paper are in close connection to the classical paper of Seifert [6]. The oriented Seifert manifolds (fibred in circles over S^2 with a finite number of exceptional fibres) are special tree manifolds [5].

Interesting trees (always with genus $g(E_i) = 0$) occur when desingular the singularities

$$(z_1^2 + z_2^n)^{1/2} \ (n \geqq 2), \quad (z_1(z_2^2 + z_1^n))^{1/2} \ (n \geqq 2),$$

$$(z_1^3 + z_2^4)^{1/2}, \quad (z_1(z_1^2 + z_2^3))^{1/2}, \quad (z_1^3 + z_2^5)^{1/2}.$$

Each of these algebroid function elements generates a complex space w singular point at the origin.

These singularities give rise to the well known trees A_{n-1}, D_{n+2}, E_6, E of Lie group theory (all vertices weighted by -2). The corresponding n folds M are homeomorphic to S^3/G where G is a finite subgroup of S^3 (c binary dihedral, binary tetrahedral, binary octahedral, binary pentagc decahedral). Up to inner automorphisms these are the only finite subgrou S^3. The manifold $M(E_8)$ is specially interesting. Since $\det S(E_8) = 1$, it the corollary in Section 2 a Poincaré manifold, i.e. a 3-dimensional man with non-trivial fundamental group and trivial abelianized fundamental gı $M(E_8)$ was constructed by "plumbing" 8-copies of the circle bundle ove with Euler number -2. By replacing this basic constituent by the taı bundle of S^{2k} one obtains a manifold $M^{4k-1}(E_8)$ of dimension $4k-1$. carries a natural differentiable structure. For $k \geqq 2$ it is homeomorph S^{4k-1}, but not diffeomorphic (Milnor sphere).

The above mentioned singularities are classical (e.g. Du Val [1]). Fo preceding remarks see also [3].

For quadratic transformations, desingularization, etc. see the pape Zariski and also [2]. We have only been able to sketch some aspec Mumford's paper, leaving others aside, e.g. the local Picard variety, etc.

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