equations. It is also applicable to various classes of multistage continuous games, such as games of survival.

If—as is necessary in more realistic mathematical models dealing with the production of capital goods—time lags are taken into account, the complexity of the problem increases.

A complete and detailed treatment of the above problem will be presented subsequently, together with a discussion of extensions in the directions just cited.

[1] Bellman, R., "On the Theory of Dynamic Programming," PROC. NATL. ACAD. SCI., **38**, pp. 716–719 (1952).

[2] Bellman, R., Glicksberg, I., and Gross, O., "On Some Variational Problems Occurring in the Theory of Dynamic Programming," *Ibid.*, **39**, 298–301 (1953).

[3] Note added in proof: In the meantime, we have developed a method based upon simultaneous consideration of the problem and its dual, which makes the verification quite simple.

ON STEENROD'S REDUCED POWERS, THE INDEX OF INERTIA, AND THE TODD GENUS

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Introduction.—This note is a preliminary report on some results concerning the Steenrod reduced powers[1] in *oriented* manifolds M^m and the index τ of such a manifold.

Definition of τ: If the dimension m of M is not divisible by 4, then $\tau(M) = 0$. If $m = 4k$, then $\tau(M)$ is equal to the number of positive squares minus the number of negative squares of the normalized quadratic form defined by the cup-product $x \smile x$ where $x \in H^{2k}(M^{4k}, R)$.

We apply the results to almost complex manifolds, in particular we give a definition of the Todd[2] genus of an almost complex manifold and state properties of this genus. Since a special algebraic formalism will be used throughout the note, we outline this formalism in the first section. All manifolds occurring in this note are compact, differentiable, and *oriented* unless stated to the contrary. Full details with further applications will appear elsewhere.

1. Multiplicative Γ*-Sequences.*—Let $\sum_{i=0}^{\infty} a_i x^i$, $a_0 = 1$, be the power series with the indeterminates a_i as coefficients, and let Γ be a field. Let $\{K_j\}$ be a sequence of polynomials, K_j being of weight j in the a_i and having coefficients in Γ ($K_0 = 1$). If $C = \sum_{i=0}^{\infty} c_i x^i$, $c_0 \doteq 1$, is an arbitrary power series, we denote by $K(C)$ the series $\sum_{j=0}^{\infty} K_j(c_1, c_2, \ldots, c_j)x^j$. We call $\{K_j\}$ a multiplicative Γ-sequence provided that K is a homomorphism, i.e., $K(AB) = K(A)K(B)$, where A and B are power series with indeterminates a_i, b_i as coefficients ($a_0 = b_0 = 1$). We construct for every given power series $Q(x) = \sum_{i=0}^{\infty} \gamma_i x^i$, ($\gamma_i \in \Gamma$, $\gamma_0 = 1$), a multiplicative Γ-sequence. Writing formally

$$1 + a_1x + \ldots + a_mx^m = (1 + \alpha_1 x)(1 + \alpha_2 x) \ldots (1 + \alpha_m x),$$

we express $Q(\alpha_1 x)Q(\alpha_2 x) \ldots Q(\alpha_m x)$ as a power series with coefficients which are polynomials in the a_i:

$$Q(\alpha_1 x)Q(\alpha_2 x) \ldots Q(\alpha_m x) = \sum_{j=0}^{\infty} K_{j,m}(a_1, \ldots, a_j)x^j.$$

One verifies easily that $K_{j,m}$ does not depend on m for $j \leq m$. We write $K_{j,j} = K_j$ and obtain the unique multiplicative Γ-sequence with $K(1 + x) = Q(x)$.

Now let Γ be the rationals. We denote by $\{T_j\}$ the multiplicative sequence belonging to $Q(x) = -x(e^{-x} - 1)^{-1}$ and call it the Todd sequence.[2]

$$2T_1 = a_1,\ 12T_2 = a_1^2 + a_2,\ 24T_3 = a_1a_2,\ 720T_4 = -a_4 + a_3a_1 + 3a_2^2 + 4a_2a_1^2 - a_1^4.$$

For a prime $q \geq 2$ the coefficients of $q^r T_{(q-1)r}$ are integers mod q (i.e., do not contain q in the denominators).

We also consider the multiplicative sequence belonging to $Q(x) = \sqrt{x}(\text{tgh}\sqrt{x})^{-1}$ which we denote by $\{L_j\}$. We have

$$3L_1 = a_1,\ 45L_2 = 7a_2 - a_1^2,\ 945L_3 = 62a_3 - 13a_2a_1 + 2a_1^3, \ldots$$

For a prime $q \geq 3$ the coefficients of $q^r L_{1/2(q-1)r}$ are integers mod q.

2. *Reduced Powers.*[1]—Let M^m be a compact *oriented* manifold. The reduced powers are defined for every odd prime q:

$$\mathcal{P}_q^r: H^k(M^m, Z_q) \to H^{k+2(q-1)r}(M^m, Z_q).$$

In case $k + 2(q - 1)r = m$, there exists by Poincaré duality an element $s_q^r \in H^{2(q-1)r}(M^m, Z_q)$ such that

$$\mathcal{P}_q^r u = s_q^r u \qquad \text{for all} \qquad u \in H^k(M^m, Z_q).$$

Let $p_1, p_2, \ldots$ be the Pontrjagin classes[3] of M^m where $p_i \in H^{4i}(M^m, Z)$. In the notation of Wu[3] $p_i = P_0^{4i}$.

THEOREM 2.1. *The class s_q^r can be expressed as a polynomial in the Pontrjagin classes:*

$$s_q^r = q^r L_{1/2(q-1)r}(p_1, p_2, \ldots, p_{1/2(q-1)r}) \bmod q.$$

THEOREM 2.2. *If M^m is an almost complex manifold of n complex dimensions ($m = 2n$), then we can express s_q^r as a polynomial in the Chern classes c_i where $c_i \in H^{2i}(M^m, Z)$:*

$$s_q^r = q^r T_{(q-1)r}(c_1, c_2, \ldots, c_{(q-1)r}) \bmod q.$$

We have assumed that the prime q is odd, but in the case $q = 2$ we may consider the Steenrod squares Sq^i. For a manifold M^m (not necessarily oriented) Wu[4] defined the class $U^i \in H^i(M^m, Z_2)$ by $\mathrm{Sq}^i v = U^i v$ for all $v \in H^{m-i}(M^m, Z_2)$.

THEOREM 2.3. *Let $w_i \in H^i(M^m, Z_2)$ be the Stiefel-Whitney classes of M^m. We have*

$$U^i = 2^i T_i(w_1, w_2, \ldots, w_i) \bmod 2.$$

Remark:[4] T_{2i+1} is divisible by a_1. Hence $w_1 = 0$ implies $U_{2i+1} = 0$.

THEOREM 2.4. *If M^m is almost complex, we have*

$$U^{2i+1} = 0, \qquad U^{2i} = 2^i T_i(c_1, c_2, \ldots, c_i) \bmod 2.$$

The proofs of Theorems 2.1–2.4 are based on the "diagonal" method of Thom[5] and Wu,[4,6] and on a topological interpretation of the multiplicative sequences which uses the Borel-Serre[7] method of regarding the classes of Stiefel-Whitney, Chern, Pontrjagin as elementary symmetric functions, and which also uses the Whitney duality theorem. The new point in the Theorems 2.1–2.4 lies in the explicit construction of the polynomials and in the fact that for all primes q these polynomials are obtainable from one and the same (rational) multiplicative sequence by reduction mod q.

3. *The Index.*—For an oriented M^{4k} we can regard the class $L_k(p_1, \ldots, p^k) \in H^{4k}(M^{4k}, R)$ as a rational number. By using a strong theorem of Thom[8] (p. 1735, Theorem 7) we obtain:

THEOREM 3.1. *We have $\tau(M^{4k}) = L_k(p_1, \ldots, p_k)$. Hence L_k for an M^{4k} is always an integer. For example,*

$$3\,\tau(M^4) = p_1,\ 45\,\tau(M^8) = 7p_2 - p_1^2,\ 945\,\tau(M^{12}) = 62p_3 - 13p_2p_1 + 2p_1^3.$$

Remarks: It was known to Thom that τ can be expressed as a polynomial in the Pontrjagin classes.[9] Theorem 3.1 implies for an M^4: $p_1 \equiv 0(3)$ (see Wu[6]) and for an M^8: $7p_2 - p_1^2 \equiv 0(45)$. From Theorem 2.1 we obtain $p_1 \equiv 0(3)$ and $7p_2 - p_1^2 \equiv 0(15)$. Analogously for all dimensions. In case M^{4k} is almost complex, we can express the Pontrjagin classes by the Chern classes[3]

$$\sum_{i=0}^{\infty} (-1)^i p_i x^{2i} = \left(\sum_{i=0}^{\infty} c_i x^i \right) \left(\sum_{i=0}^{\infty} (-1)^i c_i x^i \right)$$

and obtain polynomials for τ in the Chern classes.

We now state two theorems about the index which follow from formal properties of the polynomials L_j.

THEOREM 3.2. *If the manifold M^m is fibred in complex projective spaces P_n of n complex dimensions with the group of all projective transformations as structure group and the manifold B^{m-2n} as base ($m \geq 2n$), then*

$$\tau(M^m) = \tau(B^{m-2n})\tau(P_n).$$

Remarks: $\tau(P_n) = 1$, if n is even; $\tau(P_n) = 0$, if n is odd (see the Introduction). For the direct product of two manifolds M, M' we have $\tau(M \times M') = \tau(M)\cdot\tau(M')$.

Consider a manifold M^{4k+2}. Every element $x \in H^2(M^{4k+2}, Z)$ can be represented by a subvariety V^{4k} of M^{4k+2} (Thom[8], p. 573, Theorem 2). The index $\tau(V^{4k})$ only depends on x and may be denoted by $\tau(x)$. If $x_1, \ldots, x_r \in H^2(M^{4k+2}, Z)$, then x_1 can be represented by a subvariety V^{4k} of M^{4k+2}, the restriction of x_2 to V^{4k} can be represented by a subvariety V^{4k-2} of V^{4k}, etc. Finally, the restriction of x_r to $V^{4k-2r+4}$ can be represented by a subvariety $V^{4k-2r+2}$ of $V^{4k-2r+4}$. The index of $V^{4k-2r+2}$ only depends on the non-ordered r-tpl $(x_1, \ldots, x_r)$ and may be denoted by $\tau(x_1, \ldots, x_r)$.

THEOREM 3.3. *We have for $x_1, x_2 \in H^2(M^{4k+2}, Z)$*

$$\tau(x_1 + x_2) = \tau(x_1) + \tau(x_2) - \tau(x_1, x_2, x_1 + x_2).$$

4. *The Todd Genus.*—Let M_n be an almost complex manifold of n complex dimensions and $c_i \in H^{2i}(M_n, Z)$ its Chern classes. We can regard $T_n(c_1, \ldots, c_n) \in H^{2n}(M_n, R)$ as a rational number which we denote by $T(M_n)$. We call $T(M_n)$ the Todd genus of M_n. Kodaira[10] proved for all algebraic varieties M_n which are a complete non-singular intersection of hypersurfaces in some projective space that

$$T(M_n) = 1 - g_1 + g_2 - \ldots + (-1)^n g_n$$

where g_i is the number of linearly independent i-pl differentials of the first kind attached to M_n. From the results of Todd,[2] Hodge,[11] and Kodaira-Spencer[12] it seems very likely that the last formula is true for all non-singular algebraic varieties. But in the present moment it is not known, even for algebraic varieties, whether $T(M_n)$ is always an integer. Therefore it seems to be interesting that one can prove by the Theorems 2.4 and 3.1:

THEOREM 4.1. *The number* $2^{n-1}T(M_n)$ *is an integer for every almost complex manifold. This means for* $n = 1, 2, 3, 4$:

$$M_1:\ c_1 \equiv 0(2),\quad M_2:\ c_1^2 + c_2 \equiv 0(6),\quad M_3:\ c_1 \cdot c_2 \equiv 0(6),$$

$$M_4:\ -c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4 \equiv 0\ (90).$$

We state two theorems about the Todd genus which follow from formal properties of the polynomials T_j and which are analogous to the Theorems 3.2 and 3.3.

THEOREM 4.2. *If the almost complex manifold* M_n *is fibred in complex projective spaces* P_k *of* k *complex dimensions with the group of all projective transformations of* P_k *as structure group and the almost complex manifold* B_{n-k} *as base, the fibering being compatible with the almost complex structures of* M_n, P_k *and* B_{n-k} *then*

$$T(M_n) = T(B_{n-k})T(P_k) = T(B_{n-k}).$$

Remarks: $T(P_k) = 1$ for all k. Since $\{T_j\}$ is a multiplicative sequence we have for the direct product of two almost complex manifolds M, M'

$$T(M \times M') = T(M) \cdot T(M').$$

This (in the algebraic case) was checked by Todd[2] for dimensions not exceeding 6.

For a class $x \in H^2(M_n, Z)$ we can define a virtual Todd genus $T(x)$, which is a polynomial of weight n in x and the Chern classes of M_n, such that $T(x)$ is the Todd genus of every admissible almost complex subvariety V_{n-1} of M_n representing x. Moreover, if $x_1, x_2 \in H^2(M_n, Z)$ we can define a virtual Todd genus $T(x_1, x_2)$ with $T(x_1, x_2) = T(x_2, x_1)$ such that for every admissible almost complex subvariety V_{n-1} representing x_1, the number $T(x_1, x_2)$ is the virtual genus with respect to V_{n-1} of the restriction of x_2 to V_{n-1}. The virtual genus $T(x_1, x_2)$ is a polynomial of weight n in x_1, x_2 and the Chern classes of M_n. The following theorem is well known in algebraic geometry:

THEOREM 4.3. *For* $x_1, x_2 \in H^2(M_n, Z)$, *we have*

$$T(x_1 + x_2) = T(x_1) + T(x_2) - T(x_1, x_2).$$

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THE IDEAS OF VARIABLE AND FUNCTION

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Investigation of the semantics of pure and applied analysis reveals that there exist three different concepts of a variable, which may conveniently be described as a logical, a scientific, and a mathematical concept. Followers of Weierstrass have treated the logical notion; French analysts, the mathematical and scientific ideas. The confusion of the three concepts accounts for the obscurity which marks the introductions to many treatises on analysis. The lack of a distinction is also one of the reasons that prompted Russell to call the notion of variable "one of the most difficult with which logic has to deal." Severally, the concepts and their mutual relations (which we propose to study within the realm of real numbers) seem clear and simple.

The Logical Concepts of Arithmetized Analysis.—A *function* is a nonempty set of ordered pairs of numbers such that no two pairs of the set contain equal first and unequal second elements. The set of all first (second) elements of the pairs, referred to as arguments (values), is called the domain (the range) of the function. In arithmetized analysis (in conformity with the usage of modern logic), a *real variable* is a symbol which stands for any element of a certain set of real numbers, called the range of the variable. E.g., the set of pairs $(x, \tan x)$, for all numbers x which are not odd multiples of $\pi/2$, is a function which we shall denote by tan. The letter x in the definition of tan and in the formula $D \tan x = \sec^2 x$ is a variable having the domain of tan as range. The symbols log and $\sqrt[3]{\ }$ denote other functions.

For the set of all pairs (x, x), in spite of the paramount importance of this function, no traditional symbol exists. We shall denote this identity function by ι, and write ι^n, $16 \cdot \iota^2$, and 16 for the functions (x, x^n), $(x, 16x^2)$, and $(x, 16)$, respectively, which traditionally are referred to as the functions x, x^n, $16x^2$, and 16, that is, by their values for x. (Even references