been made by P. Du Val†. He shows that the prime sections of the surfaces for which m is even can be represented by all quadric sections of normal rational ruled surfaces in space $[p+2]$. From this representation it follows at once that these surfaces have the property of the surfaces F'. For if to the quadric sections of a normal rational ruled surface in space $[p+2]$ there are assigned $p+1$ general base-nodes, the resulting system consists of pairs of curves in a pencil of prime sections of the ruled surface; and this system is of grade zero.

Finally, an example may be given, the case $p = 1$. The corresponding surfaces are the three octavic Del Pezzo surfaces in eight dimensions, F^8, F^{8*}, and F^{8*} with a node. Then through a general point of the space passes a unique trisecant plane of F^8, but no trisecant plane of F^{8*} or of F^{8*} with a node.

THE GEOMETRIC GENUS OF A SURFACE AS A TOPOLOGICAL INVARIANT

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In a paper now in the press§ I have established the following theorem:

Let M be an analytic construct of n dimensions which has the topological properties of an orientable absolute manifold, with Betti numbers $R_1, \ldots, R_n$, and which has attached to it a Riemannian (positive definite) metric, given by the quadratic differential form

$$g_{ij}\, dx^i\, dx^j$$

in the region in which the parameters $(x^1, \ldots, x^n)$ are valid. Then there are exactly R_p linearly independent skew-symmetric tensors $B_{i_1 \ldots i_p}$ satisfying the tensor equations

$$\sum_{r=1}^{p+1} (-1)^{r-1} B_{i_1 \ldots i_{r-1} i_{r+1} \ldots i_{p+1}, i_r} = 0, \tag{1}$$

$$g^{rs} B_{i_1 \ldots i_{p-1} r, s} = 0 \tag{2}$$

† *Journal London Math. Soc.*, 8 (1933), 306-307.

‡ Received 20 May, 1933; read 15 June, 1933.

§ *Proc. London Math. Soc.* (in the press).

which are finite (and, indeed, analytic) everywhere on M. ",i" as a suffix denotes covariant differentiation with respect to x^i, the fundamental metrical tensor being g_{ij}.

This result has important applications to the theory of Abelian integrals attached to an algebraic variety, and the present note shows how we can deduce from it a topological definition of the geometric genus of a surface.

1. Consider a complex projective linear space S_r of r dimensions $(z_1, \ldots, z_{r+1})$. Mannoury* has given the following simple representation of the Riemannian of this space, which has many interesting metric properties. We subject the coordinates to the condition that

$$|z_1|^2+\ldots+|z_{r+1}|^2 = 1.$$

Let $\bar{x}$ denote the conjugate imaginary of x, and write

$$X_i = \sqrt{2}\, z_i \bar{z}_i \qquad (i = 1, \ldots, r+1),$$

$$X_{ij} = z_i \bar{z}_j + z_j \bar{z}_i = X_{ji} \qquad (i \neq j),$$

$$Y_{ij} = i(z_i \bar{z}_j - z_j \bar{z}_i) = -Y_{ji} \quad (i \neq j).$$

This defines a locus in the Euclidean space of $(r+1)^2$ dimensions in which (X_i, X_{ij}, Y_{ij}) are rectangular Cartesian coordinates, which is the Riemannian R of S_r. The distance between the images of $(z_1, \ldots, z_{r+1})$ and $(z_1', \ldots, z_{r+1}')$ is given by

$$4\,[1+|z_1 \bar{z}_1' + \ldots + z_{r+1} \bar{z}_{r+1}'|^2].$$

To discuss the neighbourhood of a point on R, we shall find it more convenient to use non-homogeneous coordinates. Except on $z_{r+1} = 0$, we take these to be

$$u^j + i u^{r+j} = \frac{z_j}{z_{r+1}} \quad (u^j,\, u^{r+j} \text{ real},\ \ j = 1, \ldots, r).$$

Then a simple calculation shows that the metric on R (defined by the Euclidean metric of the space in which it lies) is given by

$$g_{ij}\, du^i du^j,$$

where
$$g_{ij} = \frac{\partial^2 \psi}{\partial u^i \partial u^j} + \frac{\partial^2 \psi}{\partial u^{r+i} \partial u^{r+j}},$$

* G. Mannoury, *Nieuw Archief voor Wiskunde* (2), 4 (1898), 112.

and $$\psi = \log\{1+(u^1)^2+\ldots+(u^{2r})^2\},$$

and where we write $$u^{2r+j} = -u^j.$$

Now let us put u^j+iu^{r+j} equal to an analytic function of m complex variables $(m \leqslant r)$ x^k+ix^{m+k}. Remembering that

$$\frac{\partial u^j}{\partial x^k} = \frac{\partial u^{r+j}}{\partial x^{m+k}} \quad (\text{all } i, j, \quad x^{2m+k} = -x^k),$$

we find that, on the locus of $2m$ dimensions so defined on R, the metric is given by

$$g_{ij}\,dx^i\,dx^j, \tag{3}$$

where $$g_{ij} = \frac{\partial^2\psi}{\partial x^i \partial x^j} + \frac{\partial^2\psi}{\partial x^{m+i}\partial x^{m+j}} = \psi_{ij}+\psi_{m+i,\,m+j}. \tag{4}$$

This fails on $z_{r+1} = 0$, but we readily verify that all that is necessary for the discussion of an analytic locus in this case is that we should replace ψ by $\psi' = \psi - \chi$, where ψ' remains finite on $z_{r+1} = 0$, and the metric defined by formulae (3) and (4) is the same wherever ψ and ψ' are both defined.

2. We now consider an algebraic surface without singularities lying in S_r, and denote by M the corresponding four-dimensional manifold in R. We consider two complex parameters valid in the portion of the surface in which we are interested. We have now the case $m = 2$ of the last paragraph. The matrix (g_{ij}) will now be written

$$\begin{pmatrix} a & b & 0 & c \\ b & d & -c & 0 \\ 0 & -c & a & b \\ c & 0 & b & d \end{pmatrix}.$$

Let B_{ij} be a skew-symmetric tensor satisfying (1) and (2). These imply that

$$\iint B_{ij}\,dx^i\,dx^j \quad \text{and} \quad \iint \Sigma \sqrt{g}\, g^{i_1 i_2,\, jk}\, B_{jk}\, dx^{i_3}\, dx^{i_4}$$

are integrals of total differentials, where, in the second integral, the summation is over even permutations (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$, and

$$g^{ij,\,kl} = \begin{vmatrix} g^{ik} & g^{il} \\ g^{jk} & g^{jl} \end{vmatrix}, \quad \sqrt{g} = ad - b^2 - c^2.$$

Write

$$\phi = B_{12}\,dx^1\,dx^2 + B_{13}\,dx^1\,dx^3 + B_{14}\,dx^1\,dx^4 + B_{23}\,dx^2\,dx^3 + B_{24}\,dx^2\,dx^4 + B_{34}\,dx^3\,dx^4,$$

$$\phi' = \sqrt{g}\,(g^{12,12}\,B_{12} + g^{12,13}\,B_{13} + g^{12,14}\,B_{14} + g^{12,23}\,B_{23} + g^{12,24}\,B_{24} + g^{12,34}\,B_{34})\,dx^3\,dx^4 + \ldots$$

(the point being that, if we were to use the ordinary conventions of tensor calculus, numerical factors would enter which we wish to avoid). We call ϕ' the conjugate of ϕ. Then we find that

$$\phi' = B_{12}\,dx^3\,dx^4 - B_{13}\,dx^2\,dx^4 + B_{14}\,dx^2\,dx^3 + B_{23}\,dx^1\,dx^4 - B_{24}\,dx^1\,dx^3 + B_{34}\,dx^1\,dx^2 + (B_{13} + B_{24})\,(dx^1\,dx^3 + dx^2\,dx^4) - \chi\omega,$$

where

$$(ad - b^2 - c^2)\,\chi = cB_{12} + dB_{13} - bB_{14} - bB_{23} + aB_{24} + cB_{34},$$

$$\omega = a\,dx^1\,dx^3 + d\,dx^2\,dx^4 - c\,(dx^1\,dx^2 + dx^3\,dx^4) + b\,(dx^1\,dx^4 + dx^2\,dx^3).$$

We consider the R_2 independent integrals

$$\iint B_{ij}\,dx^i\,dx^j$$

satisfying (1) and (2), which are everywhere finite on M (harmonic integrals of the first kind). From the fact that $(\phi')' = \phi$, we see that we can take them in two sets: (*a*) those for which $\phi = \phi'$, and (*b*) those for which $\phi = -\phi'$. The more interesting integrals are those of the second set. In this case, we find that ϕ is of the form

$$A\,(dx^1\,dx^2 - dx^3\,dx^4) - B\,(dx^1\,dx^4 - dx^2\,dx^3) + C\omega. \tag{5}$$

The only conditions to which the coefficients are now subject are that (5) should be a total differential, and that it should be finite everywhere.

Consider the form ω. If we remember that

$$a = \psi_{11} + \psi_{33}, \quad b = \psi_{12} + \psi_{34}, \quad c = \psi_{14} - \psi_{23}, \quad d = \psi_{22} + \psi_{44},$$

we verify at once that ω is a total differential, and we have indeed

$$\omega = \psi_{ij}\,dx^i\,dx^{2+j} = \tfrac{1}{2}g_{ij}\,dx^i\,dx^{m+j}.$$

If we make an "analytic" change of variable, *i.e.* one satisfying the conditions

$$\frac{\partial x^i}{\partial x'^j} = \frac{\partial x^{2+i}}{\partial x'^{2+j}} \quad (\text{all } i, j),$$

ω becomes $\frac{1}{2}g'_{ij}\,dx'^{i}\,dx'^{2+j}$ and hence

$$\iint \omega$$

is a harmonic integral of the first kind attached to M.

Consider now the general form (5). If this is a total differential, we have

$$\begin{aligned}
-A_2+B_4 \qquad\quad -cC_2-dC_3+bC_4 &= 0,\\
-A_1+B_3-cC_1 \qquad\quad -bC_3+aC_4 &= 0,\\
A_4+B_2+dC_1-bC_2 \qquad\quad -cC_4 &= 0,\\
A_3+B_1+bC_1-aC_2-cC_3 \qquad\quad &= 0.
\end{aligned}$$

Now these equations can be written

$$\begin{aligned}
-A_2+B_4 &= \sqrt{g}\,g^{3i}\,C_i,\\
A_1-B_3 &= \sqrt{g}\,g^{4i}\,C_i,\\
-A_4-B_2 &= \sqrt{g}\,g^{1i}\,C_i,\\
A_3+B_1 &= \sqrt{g}\,g^{2i}\,C_i,
\end{aligned}$$

and hence
$$\frac{\partial}{\partial x^2}\left(\sqrt{g}\,g^{ij}\,C_j\right) = 0. \tag{6}$$

If we make any analytic change of variable, C is unaltered. Hence C is a one-valued finite function defined everywhere on the manifold. But (6) shows that it is harmonic. Hence it must be a constant*.

In the integrability conditions,

$$C_1 = C_2 = C_3 = C_4 = 0,$$

and the conditions then show that $A+iB$ is a function of

$$X = x^1+ix^3, \quad Y = x^2+ix^4,$$

and that
$$\iint (A+iB)\,dX\,dY$$

is everywhere finite on M, and is therefore an Abelian integral of the first kind. Thus the harmonic integrals of the first kind on M which satisfy the condition $\phi = -\phi'$ are the real and imaginary parts of the double

* Proved in my paper previously mentioned.

integrals of the first kind, together with the integral

$$\iint \omega;$$

and there are hence $2p_g+1$ integrals of this kind.

3. Let $\Gamma_1, \ldots, \Gamma_{R_2}$ be a fundamental base for the 2-cycles of the manifold M. Let $(\Gamma_i \Gamma_j) = a_{ij}$, and let A be the inverse of the symmetric matrix a. Let $\phi_1, \ldots, \phi_{R_2}$ be differentials of R_2 harmonic integrals of the first kind on M such that

$$\iint_{\Gamma_j} \phi_i = \delta_i^j,$$

and let ϕ_i' be the conjugate of ϕ_i. Then, since ϕ_i' is the differential of a harmonic integral of the first kind,

$$\phi_i' = \sum_j c_{ij} \phi_j \quad (c_{ij} = \text{constant}).$$

It is known* that, if ϕ, ψ are the total differentials of everywhere finite integrals on M, having periods $\omega_1, \ldots, \omega_{R_2}, \nu_1, \ldots, \nu_{R_2}$, then

$$\int_M \phi\psi = \sum_{ij} A_{ij}\, \omega_i \nu_j.$$

Hence
$$\sum_k c_{jk} A_{ki} = \int_M \phi_i \phi_j' = \int_M \phi_j \phi_i' = \sum_k c_{ik} A_{kj},$$

therefore
$$cA = Ac',$$

where c' is the transpose of c.

Also
$$c^2 = 1.$$

Further,
$$\int_M \sum \lambda_i \phi_i \sum \lambda_i \phi_i'$$

is positive for all values of the constants λ; hence cA is a symmetric matrix defining an essentially positive form. Let

$$c = \lambda^{-1} \begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix} \lambda,$$

where λ is real, and $\begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix}$ is a diagonal matrix with $\alpha = R_2 - 2p_g - 1$

* *Cf.* W. V. D. Hodge, *Journal London Math. Soc.*, 5 (1930), 283.

positive elements and $\beta = 2p_g+1$ negative elements. Then

$$\lambda^{-1}\begin{pmatrix}1 & \cdot \\ \cdot & -1\end{pmatrix}\lambda A = A\lambda'\begin{pmatrix}1 & \cdot \\ \cdot & -1\end{pmatrix}\lambda'^{-1},$$

i.e.
$$\begin{pmatrix}1 & \cdot \\ \cdot & -1\end{pmatrix}\lambda A\lambda' = \lambda A\lambda'\begin{pmatrix}1 & \cdot \\ \cdot & -1\end{pmatrix}.$$

Hence $\lambda A \lambda'$ is of the form
$$\begin{pmatrix}p & \cdot \\ \cdot & q\end{pmatrix},$$

where p and q are symmetrical matrices of orders a and β respectively; and further

$$\begin{pmatrix}p & \cdot \\ \cdot & -q\end{pmatrix}$$

defines an essentially positive form. Hence A is such that, when it is transformed to a diagonal matrix $\mu A \mu'$, it has β negative terms. This number is an invariant of the matrix A, and hence p_g is expressed as a topological invariant of the manifold.

4. This result can easily be verified in the few cases in which the matrix A can be determined. In the case of a rational surface we know* that every 2-cycle is homologous to an algebraic cycle. A base is most readily constructed from a plane representation and is given by the curves represented by straight lines in the plane, and by certain curves represented by points in the plane. In this case A is a diagonal matrix of which the first element is -1, and the other elements are $+1$, the signs being the reverse of those usually given in Severi's theory of the base, in consequence of our unusual arrangement of the parameters. Hence

$$\beta = 1 = 2p_g + 1.$$

A more interesting verification of our theorem is provided by the surface which represents the product of two algebraic curves. Let the 0-, 1-, and 2-cycles of the first curve be denoted by c; $\gamma_1, \ldots, \gamma_{2p}$; C, and those of the second by d; $\delta_1, \ldots, \delta_{2q}$; D, where

$$(\gamma_i \gamma_j) = 0 \qquad (i \neq j \pm p),$$
$$(\gamma_i \gamma_{i+p}) = -(\gamma_{i+p}\gamma_i) = 1,$$
$$(\delta_i \delta_j) = 0 \qquad (i \neq j \pm q),$$
$$(\delta_i \delta_{i+q}) = -(\delta_{i+q}\delta_i) = 1.$$

* S. Lefschetz, "L'analysis situs et la géométrie algébrique" (*Borel Tract*), 72.

Then a base for the 2-cycles is given by

$$c \times D, \quad \gamma_i \times \delta_j, \quad C \times d.$$

A is a symmetrical matrix such that in each row or column there is only one element different from zero, and this is ± 1. By rearranging the order of the cycles, and possibly changing certain orientations, we can arrange that the quadratic form in $4pq+2$ variables defined by A is

$$2x_1 x_2 + 2x_3 x_4 + \ldots + 2x_{4pq+1} x_{4pq+2}.$$

If we put

$$x_{2r+1} = y_{2r+1} - y_{2r+2}, \quad x_{2r+2} = y_{2r+1} - y_{2r+2} \quad (r = 0, 1, \ldots, 2pq),$$

this becomes
$$2(y_1^2 - y_2^2 + y_3^2 - \ldots - y_{4pq+2}^2).$$

Hence our theorem gives

$$2pq + 1 = 2p_g + 1, \quad i.e. \quad p_g = pq,$$

as is well known to be the case. The theorem can also be verified for hyperelliptic surfaces.

LIBRARY

Presents.

Between 30 June, 1932, and 30 June, 1933, the following presents were made to the library (by their respective authors or publishers, when not otherwise stated) :—

Fisher, R. A. : *Bibliography (of statistical publications by R. A. Fisher)*, 1932 ; *The concepts of inverse probability and fiduciary probability referring to unknown parameters*, 1933 ; *Inverse probability and the use of likelihood*, 1932 ; *The sampling error of estimated deviates, etc.*, 1931 ; *Forschungen und Fortschritte* : Jahrgang 1932, nos. 19–26 ; Jahrgang 1933, nos. 1–18.

Gauthier-Villars : Batault, E., *Le mystère et le paradoxe du vol animal*, 1933 ; Blance, E., *Problèmes et compléments de mécanique*, 1931 ; Chazy, J., *Cours de mécanique rationnelle*, tome 1–2, 1933 ; Galbrun, H., *Théorie mathématique de l'assurance* (Borel, E., *Traité du calcul des probabilités*, tome 3, fasc. 4–5, 1933) ; Julia, G., *Essai sur le développement de la théorie des fonctions de variables complexes*, 1933 ; *Exercices d'analyse*, tome 1–2, 1933 ; Montel, P., *Leçons sur les fonctions univalentes ou multivalentes*, 1933 ; De Montessus de Ballore, R., *La méthode de corrélation*, 1932 ; Pomey, J. B., *Application des imaginaires au calcul vectoriel*, 1933 ; Risser, R., *Application de la statistique à la démographie et à la biologie* (Borel, E., *Traité du calcul des probabilités*, tome 3, fasc. 3, 1932) ; Risser, R., and Traynard, C. E., *Les principes de la statistique mathématique* (Borel, E., *Traité du calcul des probabilités*, tome 1, fasc. 4, 1933) ; Wavre, R., *Figures planétaires et géodésie*, *Cahiers scientifiques*, 12, 1932.