A SPECIAL TYPE OF KÄHLER MANIFOLD

By W. V. D. HODGE

[Received 24 October 1950.—Read 16 November 1950]

1. In my book on harmonic integrals (**4**) I applied the theory of harmonic integrals to derive certain results concerning algebraic varieties. Subsequently (**5**) I pointed out that many of the results obtained depended only on local properties of the metric used and that some of them depended only on the fact that an algebraic variety is capable of carrying a Kähler metric, and not on the actual metric selected. Weil (**7**) pointed out that the results obtained in this way for algebraic varieties could be extended to apply to any Kähler manifold, and in a series of notes in the *Comptes Rendus* (**3**) Eckmann and Guggenheimer have shown how the arguments can be carried through in detail for a general Kähler manifold.

The topological properties of Kähler manifolds deduced by Eckmann and Guggenheimer are similar to those obtained by Lefschetz (**6**) for algebraic varieties, but are weaker, since in the case of Kähler manifolds the cycles are considered relative to the field of complex numbers, whereas in the Lefschetz theory they are combined with integral coefficients. The object of the present paper is to show that if a certain restriction is imposed on the Kähler manifold, the methods of Eckmann and Guggenheimer lead to results on the integral topology of the manifold similar to those obtained by Lefschetz for algebraic varieties, though they are still incapable of taking account of torsion. The restriction imposed has other consequences, which will also be described.

In order to describe the methods and to introduce the notation, it is necessary to describe in some detail the main results of Eckmann and Guggenheimer. This will be done as briefly as possible, and proofs will not be given at this stage. The opportunity will be taken to add to the results of Eckmann and Guggenheimer another result (Theorem I) of considerable interest. The proof of this result for algebraic varieties was indicated in (**5**), and the proof for general Kähler manifolds given below is, like the proofs of most of the theorems for Kähler manifolds which are derived by the use of harmonic integrals, similar to that for algebraic varieties.

2. A complex differentiable manifold C_m of m (complex) dimensions is a real manifold of $2m$ real dimensions which has the following property: the points of any neighbourhood N can be parametrized by means of m complex parameters $z_1, \ldots, z_m$ in such a way that if N and N' are any two intersecting neighbourhoods, and $z_1,\dots, z_m$ and $z'_1,\dots, z'_m$ are the parameters in N and N' respectively, then in $N \cap N'$ we can write

$$z'_i = f_i(z_1,\dots, z_m) \quad (i = 1,\dots, m),$$

where the functions $f_1,\dots, f_m$ are analytic functions of $z_1,\dots, z_m$ and the Jacobian

$$\left|\frac{\partial f_i}{\partial z_j}\right|$$

is never zero in $N \cap N'$. In this paper we shall further assume that the manifold C_m is compact.

If, in addition, there is a metric given on C_m, defined, in the neighbourhood N, by the positive definite Hermitian form

$$ds^2 = a_{\alpha\beta}(dz^\alpha d\bar{z}^\beta), \tag{1}$$

where the coefficients $a_{\alpha\beta}$ are functions of $z_1,\dots, z_m, \bar{z}_1,\dots, \bar{z}_m$, of class C_∞, the manifold is called a Hermitian manifold, and is denoted by H_m.

Of particular interest are the Hermitian manifolds for which the metric satisfies the Kähler property

$$\frac{\partial a_{\alpha\gamma}}{\partial z_\beta} = \frac{\partial a_{\beta\gamma}}{\partial z_\alpha}, \qquad \frac{\partial a_{\alpha\beta}}{\partial \bar{z}_\gamma} = \frac{\partial a_{\alpha\gamma}}{\partial \bar{z}_\beta},$$

for all α, β, γ; this is equivalent to saying that the exterior 2-form*

$$\omega = a_{\alpha\beta}\, dz^\alpha d\bar{z}^\beta \tag{2}$$

is exact, that is $$d\omega = 0.$$

ω, given by (2), is called the *fundamental* 2-form associated with the metric (1), and, conversely, the metric (1) associated with the 2-form ω given by (2) is called the ω-metric. Hermitian manifolds on which the metric is Kählerian are called Kähler manifolds, and will be denoted by K_m.

3. We now summarize some of the known properties of harmonic forms on a Kähler manifold K_m. In what follows, R_p denotes the pth Betti number of K_m, and $\omega_r = \omega\times\omega\times\dots\times\omega$ (r factors) is the exterior product of r forms equal to ω.

A p-form P ($p \leqslant m$) is said to be *effective* if it is harmonic, and satisfies the condition

$$P\times\omega_{m-p+1} = 0. \tag{3}$$

When P is effective (and different from zero) $P\times\omega_r$ ($r \leqslant m-p$) is harmonic (and different from zero). The effective p-forms form a vector space over the field of complex numbers of dimension $R_p - R_{p-2}$, and an

* The form in (1) is a quadratic form in $dz^1,\dots, dz^m, d\bar{z}^1,\dots, d\bar{z}^m$, that in (2) is the integrand of a double integral. To distinguish the two types of form we write the products of differentials in the former case in brackets.

independent basis for the harmonic p-forms on K_m consists of

(0) an independent basis for the effective p-forms;

(1) the p-forms obtained by constructing the exterior products of an independent basis for the effective $(p-2)$-forms, by ω_1; these p-forms are an independent basis for the *ineffective p-forms of class one*;

. . . .

(q) (where $q = [\frac{1}{2}p]$), the p-forms obtained by constructing the exterior products of an independent basis for the effective $(p-2q)$-forms by ω_q; these p-forms are an independent basis for the *ineffective p-forms of class q*.

A p-form P which can be written as

$$\sum_{\alpha_1<\ldots<\alpha_h} \sum_{\beta_1<\ldots<\beta_k} P_{\alpha_1\ldots\alpha_h\beta_1\ldots\beta_k}\, dz^{\alpha_1}\ldots dz^{\alpha_h} d\bar{z}^{\beta_1}\ldots d\bar{z}^{\beta_k} \quad (h+k = p), \tag{4}$$

is said to be a form of type (h, k). Usually we shall write $P^{h,k}$ to denote a form of type (h, k). Any p-form P can be written, in one and only one way, as a sum

$$P = P^{p,0}+P^{p-1,1}+\ldots+P^{0,p}. \tag{5}$$

For a Kähler manifold we have the property that if, in (5), P is harmonic (effective), then each $P^{h,p-h}$ on the right-hand side is harmonic (effective). Thus we obtain a further classification of the basis for harmonic p-forms obtained above. If $\rho^{h,k}$ denotes the number of linearly independent effective $(h+k)$-forms of type (h, k), we have

$$\rho^{0,p}+\ldots+\rho^{p,0} = R_p - R_{p-2};$$

and since it is clear that if $P^{h,k}$ is an effective form of type (h, k) its complex conjugate $\bar{P}^{h,k}$ is an effective p-form of type (k, h), we have

$$\rho^{h,k} = \rho^{k,h}.$$

For proofs of the above results see (**3**).

4. If P is any exact p-form on a complex manifold, there is associated with P a uniquely determined homology class of dimension $2m-p$, with coefficients in the complex field, with the property that if γ_{2m-p} is any cycle of the homology class corresponding to P

$$\int\limits_{\gamma_p} P = I(\gamma_{2m-p}, \gamma_p),$$

where, generally, $I(\Gamma_1, \Gamma_2,\ldots, \Gamma_r)$ is the intersection number of the cycles $\Gamma_1,\ldots, \Gamma_r$; this relation holding for every p-cycle γ_p of the manifold. We then write

$$P \sim \gamma_{2m-p}.$$

It is known (**2**) that if P and Q are, respectively, exact p- and q-forms, and if

$$P \sim \gamma_{2m-p}, \qquad Q \sim \gamma'_{2m-q},$$

then
$$P \times Q \sim \gamma_{2m-p}\gamma'_{2m-q}. \tag{6}$$
In particular, if $2m-q = p$, we have
$$\int\limits_{\gamma'_p} P = I(\gamma_{2m-p}, \gamma'_p) = \int\limits_{M} P \times Q. \tag{7}$$
We apply these formulae to the harmonic p-form on a Kähler manifold K_m.

Let
$$P_i^{h,k} \quad (i = 1,..., \rho^{h,k})$$
be an independent basis for the effective $(h+k)$-form of type (h,k) $(h+k \leqslant m)$, and let
$$P_i^{h,k} \sim \Gamma_i^{h,k} \quad \text{(a } (2m-h-k)\text{-cycle).}$$
We may assume that
$$P_i^{k,h} = \bar{P}_i^{h,k},$$
and we may write
$$\Gamma_i^{k,h} = \bar{\Gamma}_i^{h,k}.$$
Further, we denote by Δ a $(2m-2)$-cycle satisfying the relation
$$\omega \sim \Delta$$
and let Δ^r represent a cycle of the $(2m-2r)$-dimensional homology class determined by the intersection of r cycles each homologous to Δ.

If $h+k = p$, $\Gamma_i^{h-r,k-r}\Delta^{m-p+r}$ is a p-cycle homologous to $P_i^{h-r,k-r} \times \omega_{m-p+r}$, and it follows from what was said in § 3 that the cycles
$$\Gamma_i^{h-r,k-r}\Delta^{m-p+r}$$
$$(i = 1,..., \rho^{h-r,k-r};\ r = 0,..., \min[h,k];\ h = 0,...,p;\ k = p-h)$$
form a basis for the p-cycles of K_m. For a given value of r, the cycles of this basis form a basis for the *ineffective p-cycles of class r*; ineffective cycles of class zero are also called *effective cycles.*

The ineffective cycles of class r are the intersections of $(2m-p+2r)$-cycles with Δ^{m-p+r}. In the case of an algebraic variety, Δ^{m-p+r} is, save for a scalar factor, the cycle of dimension $2(p-r)$ corresponding to an algebraic sub-variety V_{p-r} of $p-r$ complex dimensions, and it is a consequence of Lefschetz's theory that a basis for ineffective p-cycles of class r can be found among the p-cycles of the sub-variety V_{p-r}. Hence, without confusion, we may say that a basis for the ineffective p-cycles of rank r of any Kähler manifold K_m 'lies in' Δ^{m-p+r}.

Let $h+k = p = h'+k'$. Then, by (6),
$$\begin{aligned}\Gamma_i^{h-r,k-r}\bar{\Gamma}_j^{h'-r',k'-r'}\Delta^{m-p+r+r'} &\sim P_i^{h-r,k-r} \times \bar{P}_j^{h'-r',k'-r'} \times \omega_{m-p+r+r'} \\ &= P_i^{h-r,k-r} \times P_j^{k'-r',h'-r'} \times \omega_{m-p+r+r'}.\end{aligned}$$
The right-hand side of this homology is a $2m$-form involving
$$h-r+k'-r'+m-p+r+r' = h+k'+m-p$$
differentials dz^α, and
$$k-r+h'-r'+m-p+r+r' = h'+k+m-p$$

differentials $d\bar{z}^\alpha$. It is zero unless there are m differentials dz^α and m differentials $d\bar{z}^\alpha$ present; hence it is zero unless

$$h+k' = p, \qquad h'+k = p,$$

that is, unless $h = h'$, $k = k'$.

Again, since the forms $P_i^{h-r,k-r}$ are effective $(p-2r)$-forms, it follows from (3) that $P_j^{h-r,k-r}\times\omega_{m-p+r+r'}$ is zero if $r' > r$; and similarly

$$P_j^{k'-r',h'-r'}\times\omega_{m-p+r+r'}$$

is zero if $r > r'$. Hence we conclude that

$$I(\Gamma_i^{h-r,k-r}, \bar{\Gamma}_j^{h'-r',k'-r'}, \Delta^{m-p+r+r'}) = 0$$

unless $h = h'$, $k = k'$, $r = r'$. Since the cycles

$$\Gamma_i^{h-r,k-r}\Delta^{m-p+r}$$

$$(i = 1,..., \rho^{h-r,k-r};\ r = 0,..., \min[h,k];\ h = 0,..., p;\ k = p-h)$$

form an independent basis for the p-cycles of K_m, and the cycles

$$\bar{\Gamma}_j^{h'-r',k'-r'} \quad (j = 1,..., \rho^{h'-r',k'-r'};\ r' = 0,..., \min[h',k'];\ h' = 0,..., p;\ k' = p-h')$$

form an independent basis for the $(2m-p)$-cycles of K_m, the intersection matrix of these two bases is non-singular. It follows from the results obtained above that this is only possible if each of the $\rho^{h-r,k-r}\times\rho^{h-r,k-r}$ matrices whose elements are

$$I(\Gamma_i^{h-r,k-r}, \bar{\Gamma}_j^{h-r,k-r}, \Delta^{m-p+2r}) \quad (i,j = 1,..., \rho^{h-r,k-r})$$

is non-singular.

Since

$$\int\limits_{\Gamma_j^{h'-r',k'-r'}\Delta^{m-p+r'}} P_i^{h-r,k-r}\times\omega_r = I(\Gamma_i^{h-r,k-r}, \bar{\Gamma}_j^{h'-r',k'-r'}, \Delta^{m-p+r+r'}),$$

it follows that the period of any integral of type (h,k) and ineffective of class r is zero on any p-cycle of type (k',h'), ineffective of class r', unless $h = h'$, $k = k'$, $r = r'$.

5. It is known that not every complex manifold can be made into a Kähler manifold by the construction of a Kähler metric on it. But if it is possible to construct one such on a complex manifold, many such metrics can be constructed on it. The harmonic integrals, and hence the corresponding classification of the cycles on the manifold, depend on the choice of the metric. However, some of the characters of the manifold determined by the consideration of the harmonic integrals can be shown not to depend on the choice of metric but only on the fact that it is possible to construct a Kähler metric on the manifold; these may be called *invariants of the complex structure of the Kähler manifold.* We now prove

THEOREM I. *The numbers* $\rho^{h,k}$ *are invariants of the complex structure of the manifold* K_m.

Consider two different Kähler metrics on K_m, and let ω, ν be their fundamental forms. We shall use the notation of the previous paragraphs when dealing with the ω-metric; when we deal with the ν-metric we shall replace $P_i^{h,k}$, $\rho^{h,k}$ by $Q_i^{h,k}$, $\sigma^{h,k}$. We have to prove that $\rho^{h,k} = \sigma^{h,k}$. We suppose that $h+k = p = h'+k'$.

The ν-harmonic forms of type (h,k) are all those of the form

$$\sum_{r=0}^{\min[h,k]} \sum_{i=1}^{\sigma^{h-r,k-r}} \lambda_{ri}\, Q_i^{h-r,k-r} \times \nu_r$$

and we shall call the p-cycles homologous to

$$\sum_{r=0}^{\min[h,k]} \sum_{i=1}^{\rho^{h-r,k-r}} \mu_{rj}\, \Gamma_j^{h-r,k-r} \Delta^{m-p+r}$$

the p-cycles of ω-type (h,k). Now, by (7),

$$\int\limits_{\Gamma_j^{h-r,k-r}\Delta^{m-p+r}} Q_i^{h'-r',k'-r'} \times \nu_{r'} = \int\limits_{M} Q_i^{h'-r',k'-r'} \times \nu_{r'} \times \bar{P}_j^{h-r,k-r} \times \omega_{m-p+r}.$$

The form under the integral sign on the right is of type

$$(h'+k+m-p,\ k'+h+m-p)$$

and hence is zero unless $h = h'$, $k = k'$. Now the period matrix of the R_p integrals $Q_i^{h-r,k-r} \times \nu_r$ (for all admissible values of i, r, h, k) on the R_p p-cycles $\Gamma_i^{h-r,k-r}\Delta^{m-p+r}$ (for all admissible values of i, r, h, k) is non-singular. It follows from the above that this is only possible if the number of ν-harmonic forms of type (h,k) is equal to the number of p-cycles of ω-type (h,k), for all admissible (h,k). Hence

$$\sigma^{h,k} + \sigma^{h-1,k-1} + \sigma^{h-2,k-2} + \dots = \rho^{h,k} + \rho^{h-1,k-1} + \rho^{h-2,k-2} + \dots.$$

By considering ν-harmonic $(p-2)$-forms and the $(p-2)$-cycles, we obtain the equation

$$\sigma^{h-1,k-1} + \sigma^{h-2,k-2} + \dots = \rho^{h-1,k-1} + \rho^{h-2,k-2} + \dots.$$

Subtracting, we obtain the equation

$$\sigma^{h,k} = \rho^{h,k}.$$

6. The classification of the p-cycles of K_m into cycles of various classes of ineffectiveness and different types which we have outlined above requires us to consider the cycles over the field of complex numbers. We now introduce a new condition to be satisfied by the metric on K_m, and show that when this condition is satisfied the classification of the cycles according to their class of ineffectiveness can be carried out *rationally*, that is, that a base for the ineffective p-cycles of class h can be found consisting of integral cycles.

For a general Kähler manifold, the $(2m-2)$-cycle Δ is a complex cycle. But when the Kähler manifold is constructed from an algebraic variety as in (**4**), Δ is a scalar multiple of an integral cycle. We shall say that the Kähler manifold is *of the restricted type* if the fundamental 2-form ω is homologous to $\Delta = k\Gamma$, where Γ is an integral cycle. We shall denote a Kähler manifold of the restricted type by U_m.

It is easy to see that k must be a pure imaginary number. For, since the metric is Hermitian ω is $\sqrt{(-1)}$ times a real form. This real form is homologous to a real $(2m-2)$-cycle. Hence $(k/\sqrt{(-1)})\Gamma$ is real, and therefore $k/\sqrt{(-1)}$ is real. It will be convenient to write $\Delta = -ia\Gamma$, where a is a positive real number. This amounts to assigning a particular orientation to Γ.

THEOREM II. *If U_m is a Kähler manifold of restricted type, the ineffective p-cycles which are ineffective of class r have an integral basis.*

Let $\Gamma = \Gamma^{h-r,k-r}\Delta^{m-p+r}$ be any ineffective p-cycle of class r. Then if $q \leqslant r$, $\Gamma\Delta^q = \Gamma^{h-r,k-r}\Delta^{m-p+q+r}$ is a $(p-2q)$-cycle, and it is easily seen from § 4 that it is ineffective of class $r-q$. On the other hand, if $q > r$,

$$\Gamma\Delta^q \sim P^{h-r,k-r} \times \omega_{m-p+q+r},$$

and, since $m-p+q+r > m-h-k+2r$, it follows from (3) that

$$\Gamma\Delta^q \sim 0.$$

Now let Γ_i $(i = 1,..., R_{2m-p} = R_p)$ be an integral basis for the $(2m-p)$-cycles of U_m. Then the cycles $\Gamma_i \Delta^{m-p}$ form a basis for the p-cycles of U_m. If $\sum_i \lambda_i \Gamma_i \Delta^{m-p}$ is an effective p-cycle,

$$\sum_i \lambda_i \Gamma_i \Delta^{m-p+1} \sim 0$$

and R_{p-2} of the $(p-2)$-cycles $\Gamma_i \Delta^{m-p+1}$ are independent and form a basis for the $(p-2)$-cycles of U_m. This, and the statements which follow, are immediate consequences of the properties of the cycles of a Kähler manifold given in § 4.

More generally, $$\sum_i \lambda_i \Gamma_i \Delta^{m-p+r} \sim 0$$

is a necessary and sufficient condition that $\sum_i \lambda_i \Gamma_i \Delta^{m-p}$ should be homologous to a combination of p-cycles whose class of ineffectiveness is less than r. Hence we can find a basis for the p-cycles of U_m whose class of ineffectiveness does not exceed $r-1$ by finding a basis for the solutions of the homology

$$\sum_i \lambda_i \Gamma_i \Delta^{m-p+r} \sim 0. \tag{8}$$

Similarly, to find the p-cycles whose class of ineffectiveness does not exceed r we have to solve the homology

$$\sum_i \lambda_i \Gamma_i \Delta^{m-p+r+1} \sim 0. \tag{9}$$

Now if $\Delta = -ia\Gamma$, where Γ is an integral cycle, the homologies (8) and (9) are equivalent to

$$\sum \lambda_i \Gamma_i \Gamma^{m-p+r} \sim 0$$

and

$$\sum \lambda_i \Gamma_i \Gamma^{m-p+r+1} \sim 0,$$

respectively, and hence they can be solved in integers. Let us suppose the basis $\{\Gamma_i\}$ has been rearranged so that the solutions of (8) have a basis $\lambda_i = \delta_{ij}$ $(j = 1,..., R_p - R_{p-2r})$, and that the solutions of (9), which include those of (8), have a basis $\lambda_i = \delta_{ij}$ $(j = 1,..., R_p - R_{p-2r-2})$. A necessary and sufficient condition that

$$\sum_{i=1}^{R_p - R_{p-2r-2}} \lambda_i \Gamma_i \Delta^{m-p}$$

be ineffective of rank r exactly is that its intersection with each cycle Γ_i such that the class of ineffectiveness of $\Gamma_i \Delta^{m-p}$ is less than r should be zero. Thus we find a basis for the ineffective p-cycles of class r by finding a basis for the solutions of

$$\sum_{i=1}^{R_p - R_{p-2r-2}} \lambda_i I(\Gamma_i, \Gamma_j, \Gamma^{m-p}) = 0 \quad (j = 1,..., R_p - R_{p-2r}).$$

This can be done rationally, and Theorem II is therefore proved.

7. The fact that U_m is a Kähler manifold of restricted type does not, however, permit us to obtain an integral basis for the cycles

$$\sum \lambda_i \Gamma_i^{h-r,k-r} \Delta^{m-p+r},$$

for given h, k, r. Nevertheless, we are able to obtain a theorem about the integral topology of U_m by considering these cycles. Since the theorem which we propose to prove for the ineffective p-cycles of class r is the same as that for the effective $(p-2r)$-cycles, it will be sufficient to consider the effective p-cycles of U_m.

From § 4 we know that

$$I(\Gamma_i^{h,k}, \bar{\Gamma}_j^{h',k'}, \Gamma^{m-p}) = 0 \quad (h+k = p = h'+k'),$$

unless $h = h'$, $k = k'$. Write

$$M_{ij}^{h,k} = I(\Gamma_i^{h,k}, \bar{\Gamma}_j^{h,k}, \Gamma^{m-p}). \tag{10}$$

Let $\alpha_1,..., \alpha_{\rho^{h,k}}$ be $\rho^{h,k}$ complex numbers. Then, by (6),

$$(-ia)^{m-p} \sum M_{ij}^{h,k} \alpha_i \bar{\alpha}_j \sim P \times \bar{P} \times \omega_{m-p}, \tag{11}$$

where $\quad P = \sum \alpha_i P_i^{h,k} = P_{\alpha_1,...,\alpha_h,\beta_1,...,\beta_k}\, dz^{\alpha_1}...dz^{\alpha_h}\, d\bar{z}^{\beta_1}...d\bar{z}^{\beta_k}.$

To evaluate the form on the right of (11) at a point O, we choose the local coordinates $z_1,\dots, z_m$ so that at O

$$\omega = \sum_{\alpha=1}^{m} dz^\alpha d\bar{z}^\alpha.$$

A simple calculation then shows that

$$P\times \bar{P}\times \omega_{m-p}$$
$$= (-1)^{k+\frac{1}{2}p(p-1)} \sum P_{\alpha_1\dots\alpha_h\alpha_{h+1}\dots\alpha_p} \bar{P}_{\alpha_1\dots\alpha_k\alpha_{k+1}\dots\alpha_p}\, dz^1d\bar{z}^1\dots dz^m d\bar{z}^m,$$

and if we express this in real form by writing

$$z_\alpha = x_{2\alpha-1}+ix_{2\alpha}$$

we see that

$$P^{h,k}\times \bar{P}^{h,k}\times \omega_{m-p} = (-1)^k\, i^{p(p-1)-m} A\; dx^1\dots dx^{2m},$$

where A is a positive function of $x_1,\dots, x_{2m}$, except when $P = 0$. Since

$$(-ia)^{m-p} \sum_{i,j} M_{ij}^{h,k}\alpha_i\,\bar{\alpha}_j = \int_M P\times \bar{P}\times \omega_{m-p},$$

and a is positive, we conclude that

$$\sum_{i,j} M_{ij}^{h,k}\alpha_i\,\bar{\alpha}_j = (-1)^h i^{p^2}\times c, \tag{12}$$

where c is a non-negative number, zero only when $\alpha_1 = \alpha_2 = \dots = 0$.

Let
$$\mathbf{M}^{h,k} = ||M_{ij}^{h,k}||.$$

Then
$$\bar{\mathbf{M}}^{h,k} = (-1)^p(\mathbf{M}^{h,k})'.$$

We consider first the case in which p is even. Then $\mathbf{M}^{h,k}$ is a Hermitian matrix, and by (12) it is $(-1)^h$ times a positive definite Hermitian matrix.

Let Γ_i ($i = 1,\dots, R_p-R_{p-2}$) be an integral basis for the $(2m-p)$-cycles of U_m whose intersections with Γ^{m-p} are effective p-cycles. Then

$$\Gamma_i^{h,k} \sim \sum_j \lambda_{ij}^{hk}\Gamma_j.$$

We denote by $\mathbf{\Lambda}$ the square matrix whose elements are λ_{ij}^{hk}, where the suffix j indicates the column, and the other indices refer to the rows. If $\mathbf{N}$ is the integral matrix whose elements are the intersection numbers $I(\Gamma_i, \Gamma_j, \Gamma^{m-p})$, we see that

$$\mathbf{\Lambda}\mathbf{N}\bar{\mathbf{\Lambda}}' = \begin{pmatrix} \mathbf{M}^{p,0} & 0 & 0 & . & . & . \\ 0 & \mathbf{M}^{p-1,1} & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & \mathbf{M}^{0,p} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{H}^{p,0} & 0 & 0 & . & . & . \\ 0 & -\mathbf{H}^{p-1,1} & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & & & (-1)^p\mathbf{H}^{0,p} \end{pmatrix},$$

where $\mathbf{H}^{h,k}$ is a positive definite Hermitian matrix of $\rho^{h,k}$ rows and columns. Hence we have

THEOREM III. *Let Γ_i $(i = 1,..., R_p-R_{p-2})$ be an integral basis for the $(2m-p)$-cycles of U_m whose intersections with Γ^{m-p} are effective p-cycles where p is even. Then the signature of the intersection matrix $||I(\Gamma_i, \Gamma_j, \Gamma^{m-p})||$ is*

$$(\rho^{p,0}+\rho^{p-2,2}+...+\rho^{0,p},\ \rho^{p-1,1}+\rho^{p-3,3}+...+\rho^{1,p-1}).$$

8. We now consider the period matrix of a base for the effective p-fold integrals on a base for the effective p-cycles, for any value of p not exceeding m. We consider the integrals of the forms $P_i^{h,k}$ $(h+k = p)$, arranging them so that $P_i^{h,k}$ precedes $P_j^{h',k'}$ if $h > h'$, and the cycles $\overline{\Gamma}_i^{hk}\Gamma^{m-p}$, arranging them in the same way. Then, as we have seen,

$$\int\limits_{\Gamma_j^{h',k'}\Gamma^{m-p}} P_i^{h,k} = I(\Gamma_i^{h,k}, \overline{\Gamma}_j^{h,k}, \Gamma^{m-p}) = M_{ij}^{h,k}$$

if $h = h'$, $k = k'$, and is zero otherwise. Hence the period matrix $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega} = \begin{pmatrix} \mathbf{M}^{p,0} & 0 & . & . & . \\ 0 & \mathbf{M}^{p-1,1} & . & . & . \\ . & . & . & . & . \\ . & . & . & . & \mathbf{M}^{0,p} \end{pmatrix}.$$

Let $\mathbf{R}$ be the intersection matrix of the cycles $\overline{\Gamma}_i^{h,k}$ of the base with their complex conjugates and Γ^{m-p}. Then we have

$$\mathbf{R}' = (-1)^p \begin{pmatrix} \mathbf{M}^{p,0} & 0 & . & . & . \\ 0 & \mathbf{M}^{p-1,1} & . & . & . \\ . & . & . & . & . \\ . & . & . & . & \mathbf{M}^{0,p} \end{pmatrix}.$$

Hence, if $\tilde{\mathbf{R}}$ denotes the transpose of the inverse of $\mathbf{R}$, we have

$$\mathbf{\Omega}\tilde{\mathbf{R}}\overline{\mathbf{\Omega}}' = \begin{pmatrix} \mathbf{M}^{p,0} & 0 & . & . & . \\ 0 & \mathbf{M}^{p-1,1} & . & . & . \\ . & . & . & . & . \\ . & . & . & . & \mathbf{M}^{0,p} \end{pmatrix}$$

$$= i^{p^2}(-1)^p \begin{pmatrix} \mathbf{H}^{p,0} & 0 & . & . & . \\ 0 & -\mathbf{H}^{p-1,1} & . & . & . \\ . & . & . & . & . \\ . & . & . & . & (-1)^p\mathbf{H}^{0,p} \end{pmatrix}.$$

Now let Γ_i $(i = 1,..., R_p-R_{p-2})$ be an integral base for the cycles $\overline{\Gamma}_i^{h,k}$ (Theorem II), and let $\mathbf{A}$ be the matrix of coefficients in the representation of the new base in terms of the old. If $\mathbf{S}$ is the (integral) matrix of intersection of the cycles Γ_i, Γ_j, Γ^{m-p}

$$\mathbf{S} = \mathbf{A}\mathbf{R}\bar{\mathbf{A}}',$$

and if $\mathbf{\Lambda}$ is the period matrix of the integrals of the forms $P_i^{h,k}$ on the cycles $\Gamma_i \Gamma^{m-p}$,

$$\mathbf{\Lambda} = \mathbf{\Omega}\mathbf{A}'.$$

Hence

$$\mathbf{\Lambda}\tilde{\mathbf{S}}\bar{\mathbf{\Lambda}}' = \mathbf{\Omega}\tilde{\mathbf{R}}\bar{\mathbf{\Omega}}' = i^{p^2}(-1)^p \begin{pmatrix} \mathbf{H}^{p,0} & 0 & . & . & . \\ 0 & -\mathbf{H}^{p-1,1} & . & . & . \\ . & . & . & . & . \\ . & . & . & . & (-1)^p\mathbf{H}^{0,p} \end{pmatrix}, \tag{13}$$

where $\mathbf{H}^{h,k}$ is a positive definite Hermitian matrix, and $\tilde{\mathbf{S}}$ is a matrix of rational numbers.

The most interesting case of this result arises when $p = 1$. In this case the effective integrals of type $(1, 0)$ are the exact integrals of the form $\int P_\alpha \, dz^\alpha$, which are everywhere finite on the manifold, and the effective integrals of type $(0, 1)$ are their complex conjugates. Let $\boldsymbol{\omega}$ denote the period matrix of the effective integrals of type $(1, 0)$ on U_m, with respect to the cycles $\Gamma_i \Gamma^{m-1}$. Then, in the notation used above,

$$\mathbf{\Lambda} = \begin{pmatrix} \boldsymbol{\omega} \\ \bar{\boldsymbol{\omega}} \end{pmatrix},$$

and equation (13) gives the equation

$$\begin{pmatrix} \boldsymbol{\omega} \\ \bar{\boldsymbol{\omega}} \end{pmatrix} \tilde{\mathbf{S}}(\bar{\boldsymbol{\omega}}', \boldsymbol{\omega}') = -i\begin{pmatrix} \mathbf{H} & 0 \\ 0 & -\mathbf{H} \end{pmatrix},$$

where $\mathbf{H}$ is a positive definite Hermitian matrix. Since $\tilde{\mathbf{S}}$ is a skew-symmetric matrix of rational numbers, we deduce

THEOREM IV. *On a Kähler manifold of restricted type, the period matrix of the effective integrals of type* $(1, 0)$ *is a Riemannian matrix.*

9. If C_m is any compact manifold, an exact integral of the form $\int P_\alpha \, dz^\alpha$ which is regular at all points of C_m is called a Picard integral, by analogy with the case of algebraic varieties. If C_m is not a Kähler manifold, there may be no Picard integrals on it, even if $R_1 > 0$. If C_m is a Kähler manifold, the Picard integrals are the same as the effective integrals of type $(1, 0)$.

The complete converse of Theorem IV would state that if on a compact complex manifold C_m there exist $\frac{1}{2}R_1$ linearly independent Picard integrals whose period matrix is Riemannian, then, with a suitable choice of metric,

C_m is a Kähler manifold of restricted type. We cannot prove this result completely, but we can prove a partial converse of Theorem IV as follows:

THEOREM V. *If a compact complex manifold C_m satisfies the following conditions:*

(*a*) *it possesses $\frac{1}{2}R_1$ linearly independent Picard integrals*

$$\int P_{i\alpha}\,dz^\alpha \quad (\alpha = 1,..., \tfrac{1}{2}R_1);$$

(*b*) *the matrix $||P_{i\alpha}||$ is of rank m at all points of C_m;*

(*c*) *the period matrix of $\int P_{i\alpha}\,dz^\alpha$ is Riemannian;*

then we can choose a metric on C_m so that it becomes a Kähler manifold of restricted type.

On account of condition (*b*), the theorem tells us nothing if $R_1 < 2m$. If $R_1 = 2m$ the m covariant vectors $P_{i\alpha}$ can be used to find the mth characteristic cohomology class of C_m (**1**). It follows from the properties of characteristic classes that in this case (*b*) cannot be satisfied if the mth characteristic class of C_m is not the zero class.

Assuming the conditions of the theorem satisfied, we proceed as follows. Let γ_i ($i = 1,..., R_p$) be an integral basis for the 1-cycles of C_m, and let $\boldsymbol{\omega}$ be the period matrix of the integrals $\int P_i$ with respect to these cycles. Then there exists a skew symmetric matrix of integers $\mathbf{T}$ such that

$$\mathbf{\Omega T \bar{\Omega}}'^{-1} = -i\begin{pmatrix} \mathbf{H} & 0 \\ 0 & -\mathbf{\bar{H}} \end{pmatrix}, \qquad \mathbf{\Omega} = \begin{pmatrix} \boldsymbol{\omega} \\ \bar{\boldsymbol{\omega}} \end{pmatrix},$$

where $\mathbf{H}$ is a positive definite Hermitian matrix. Let Γ_i ($i = 1,..., R_1$) be an integral basis for the $(2m-1)$-cycles of C_m. Then there is a matrix $\mathbf{b} = ||b_{ij}||$ such that

$$P_i \sim \sum_k b_{ik}\Gamma_k.$$

We have

$$\omega_{ij} = \sum_k b_{ik}(\Gamma_k \gamma_j) = \sum_k b_{ik} c_{kj},$$

where $\mathbf{c} = ||c_{ij}||$ is a matrix of integers. Thus

$$\mathbf{b} = \boldsymbol{\omega}\mathbf{c}^{-1}.$$

Let $\boldsymbol{\alpha} = ||\alpha_{ij}||$ be a $\frac{1}{2}R_1 \times \frac{1}{2}R_1$ matrix. Then

$$\sum_{i,j} \alpha_{ij} P_i \times \bar{P}_j = \tfrac{1}{2}[\sum \alpha_{ij} P_i \times \bar{P}_j - \sum \alpha_{ji} \bar{P}_i \times P_j] \sim \tfrac{1}{2}\sum d_{ij}\Gamma_i\Gamma_j,$$

where

$$d_{ij} = \sum_{h,k} \alpha_{hk} b_{hi} \bar{b}_{kj} - \sum_{h,k} \alpha_{kh} \bar{b}_{hi} b_{kj},$$

that is,

$$\mathbf{d} = ||d_{ij}|| = \mathbf{b}'\boldsymbol{\alpha}\bar{\mathbf{b}} - \bar{\mathbf{b}}'\boldsymbol{\alpha}'\mathbf{b} = \tilde{\mathbf{c}}\boldsymbol{\omega}'\boldsymbol{\alpha}\bar{\boldsymbol{\omega}}\mathbf{c}^{-1} - \tilde{\mathbf{c}}\bar{\boldsymbol{\omega}}'\boldsymbol{\alpha}'\boldsymbol{\omega}\mathbf{c}^{-1}$$

$$= \tilde{\mathbf{c}}\mathbf{\Omega}'\begin{pmatrix} \boldsymbol{\alpha} & 0 \\ 0 & -\boldsymbol{\alpha}' \end{pmatrix}\bar{\mathbf{\Omega}}\mathbf{c}^{-1}.$$

Take $\boldsymbol{\alpha} = \mathbf{H}^{-1}$. Then

$$\begin{pmatrix} \boldsymbol{\alpha} & 0 \\ 0 & -\boldsymbol{\alpha}' \end{pmatrix} = i\tilde{\boldsymbol{\Omega}}\mathbf{T}^{-1}\overline{\boldsymbol{\Omega}}^{-1},$$

and so

$$\mathbf{d} = i\tilde{\mathbf{c}}\mathbf{T}^{-1}\mathbf{c}^{-1}.$$

Since $\mathbf{c}$ and $\mathbf{T}$ are both matrices of integers, it follows that

$$\sum_{i,j} \alpha_{ij} P_i \times \overline{P}_j$$

is homologous to a scalar multiple of an integral cycle.

Now consider the quadratic form

$$\sum_{i,j} \alpha_{ij}(P_i \times \overline{P}_j).$$

Since $\boldsymbol{\alpha} = \mathbf{H}^{-1}$ is a positive definite Hermitian matrix, and condition (*b*) of the theorem is assumed to be satisfied, we conclude that this quadratic form is positive definite. Hence C_m, with the metric

$$ds^2 = \sum_{i,j} \alpha_{ij}(P_i \times \overline{P}_j),$$

is a Kähler manifold of restricted type.

10. Theorem VI. *A necessary condition that a complex manifold be a Kähler manifold of restricted type is that there should exist on it an integral $(2m-2)$-cycle Γ such that every exact integral on the manifold, of type $(m, m-2)$, should have zero period on Γ, and such that $I(\Gamma^m) > 0$.*

If C_m is of restricted type

$$\omega \sim -ia\Gamma,$$

where a is a positive real number. If P is any exact form of type $(m, m-2)$, $P \times \omega$ is a form of type $(m+1, m-1)$ and is hence zero. Now we have

$$-ia \int_\Gamma P = \int_M P \times \omega = 0.$$

Again, $\quad (-ia)^m \Gamma^m \sim \omega_m = (-i)^m \times \text{real positive } 2m\text{-form}.$

Hence $\quad I(\Gamma^m) = a^{-m} \times \text{positive number} > 0.$

By considering conjugate imaginaries, Theorem VI remains true if for 'integrals of type $(m, m-2)$' we read 'integrals of type $(m-2, m)$'.

It is not known whether these conditions are sufficient for the manifold to be of restricted type.

If U_m is a Kähler manifold for which $R_1 > 0$, and whose Picard integrals have a Riemannian period matrix, we can construct, in the usual way, Θ-functions of the Picard integrals. From these Θ-functions we can construct one-valued functions $f(z_1, \ldots, z_m)$ on U_m whose singularities are all of polar type. The locus of poles of such a function defines on U_m an integral $(2m-2)$-cycle Γ. Γ is analogous to the $(2m-2)$-cycle defined on an algebraic manifold by a sub-variety of dimension $m-1$ (complex), and it can be verified at once that the exact integrals of type $(m, m-2)$ on U_m have zero periods on Γ. But it is not always true that $I(\Gamma^m) > 0$, as can be verified by considering the special case of an algebraic variety.

REFERENCES

1. S. S. CHERN, *Annals of Mathematics*, 47 (1946), 85.
2. G. DE RHAM, *Journal de Mathématique* (9), 10 (1931), 115.
3. B. ECKMANN and H. GUGGENHEIMER, *Comptes Rendus*, 229 (1949), 464, 489, 503, 577.
4. W. V. D. HODGE, *Theory and applications of harmonic integrals* (Cambridge, 1941).
5. —— *Proc. Cambridge Phil. Soc.* 44 (1947), 37.
6. S. LEFSCHETZ, *L'Analyse situs et la géométrie algébrique* (Paris, 1924).
7. A. WEIL, *Comm. Math. Helvetici*, 20 (1947), 110.

Pembroke College,
Cambridge.