HOMOLOGY BOUNDARY LINKS AND BLANCHFIELD FORMS: CONCORDANCE CLASSIFICATION AND NEW TANGLE-THEORETIC CONSTRUCTIONS

TIM D. COCHRAN* AND KENT E. ORR*

ABSTRACT. Seifert forms and Blanchfield forms are defined for homology boundary links. New tangle constructions are used to show that any pair (pattern, Blanchfield form) can be realized by a homology boundary link. A classification theorem is proved for homology boundary links of fixed pattern modulo homology boundary link concordance. This is done from the points of view of Seifert matrices, Blanchfield forms and Γ -groups. The analogous notions for links in \mathbf{Z}_p -homology spheres are discussed.

§0. Introduction.

This work forms part of our on-going effort to classify the set of concordance classes of links. Recall that a link $L = \{K_1, \ldots, K_m\}$ in S^{n+2} is a locally flat piecewise-linear, oriented submanifold of S^{n+2} of which each component K_i is homeomorphic to S^n . The exterior E(L) of a link L is the closure of the complement of a small regular neighborhood N(L) of L. A longitude of a component K_i is a parallel of K_i lying on the boundary of the tubular neighborhood (untwisted if n = 1). A meridian μ_i is a path from a basepoint to $\partial N(L)$ which traverses a fiber of $\partial N(L)$ and returns. A Seifert Surface for K_i is a connected, compact, oriented, (n + 1)-manifold $V_i \subseteq E(L)$ such that ∂V_i is a longitude of K_i . Links L_0 , L_1 are concordant (or cobordant) if there is a smooth, oriented submanifold $C = \{C_1, \ldots, C_m\}$ of $S^{n+2} \times [0, 1]$ which meets the boundary transversely in ∂C , is piecewise-linearly homeomorphic to $L_0 \times [0, 1]$, and meets $S^{n+2} \times \{i\}$ in L_i for i = 0, 1. In the mid-60's M. Kervaire and J. Levine gave an algebraic classification of knot concordance groups (m = 1) in high dimensions (n > 1) [L3]. For even n these are trivial and for odd n they are infinitely generated, being isomorphic to certain Witt groups obtained from information garnered from the Seifert surface.

Extending Levine's knot cobordism classification to links is difficult for several reasons. Firstly, if m > 1, the natural operation of connected-sum is not well-defined on concordance classes so there is no obvious group structure. Secondly, the Seifert surfaces for different components of a link may intersect, obstructing at least the naive generalization of the Seifert form information.

Typeset by $\mathcal{A}_{\!\mathcal{M}}\!\mathcal{S}\text{-}T_{\!E}\!X$

^{*}partially supported by the National Science Foundation

However the techniques do extend well to the class of *boundary links*. A boundary link is one which admits a collection of *m disjoint* Seifert surfaces, one for each component. In fact, S. Cappell and J. Shaneson classified boundary links modulo *boundary link cobordism* in 1980 using their homology surgery groups, followed later by Ki Ko and W. Mio who accomplished this via Seifert surfaces [CS1, Ko, Mi]. A *boundary link cobordism* is a cobordism *C* between L_0 and L_1 for which there exist disjointly embedded 2*n*-manifolds $IV = \{IV_1, \ldots, IV_m\}$ in E(C) such that $IV \cap (S^{n+2} \times \{i\})$ is a system of Seifert surfaces for the boundary link L_i , i = 0,1, and such that

$$(\partial N(C), IV \cap \partial N(C)) \cong (\partial N(L_0) \times [0, 1], (IV \cap \partial N(L_0)) \times [0, 1])$$

These successes focussed intense scrutiny on the question of whether or not every link were concordant to a boundary link (if n = 1, Milnor's $\overline{\mu}$ -invariants were known obstructions). If this had been the case, the concordance classification of links (at least if n > 1) would have been almost complete.

Unfortunately the situation was not so simple. In 1989 the present authors defined new invariants which showed that many odd-dimensional homology boundary links are not concordant to any boundary link [CO1, CO2]. This development focussed attention on the previously obscure class of homology boundary links, first defined by N. Smythe in 1965 [S]. To define an homology boundary link, let us first define a more general notion of Seifert surface which we use for the remainder of this paper. Let F be the free group on m letters $\{x_i\}$. Consider the subset of F^m consisting of those (w_1, \ldots, w_m) for which $w_i \equiv x_i$ in the abelianization and for which $\{w_1, \ldots, w_m\}$ normally generates the free group. Consider the equivalence relation on this subset where $(w_1, \ldots, w_m) \sim (w'_1, \ldots, w'_m)$ if and only if there are elements $\eta_i \in F$ and an automorphism ϕ of F such that $w'_i = \phi(\eta_i w_i \eta_i^{-1})$ for all i. An element (w_1, \ldots, w_m) of this set of equivalence classes is called a pattern P. A system of Seifert surfaces of pattern P for L is a collection $\mathcal{V} = \{V_1, \ldots, V_m\}$ of pairwisedisjoint, connected, compact, oriented, (n + 1)-dimensional submanifolds of E(L) such that ∂V_i consists of a union of longitudes (up to orientation) of various K_j in such a way that if one traverses μ_i and "reads out" x_j (or x_j^{-1}) as one transversely encounters V_j (or $-V_i$), then one spells out the word w_i and such that the homomorphism $\phi: \pi_1(E(L)) \to F$ associated to the system (by the Thom-Pontryagin construction mapping E(L) to a wedge of m circles [Ko; 2.1]) is surjective. An homology boundary link of m components with pattern P may then be defined as one which possesses such a system of Seifert surfaces. In [CL] it is shown that the pattern is an invariant of the isotopy class of L. Boundary links are, of course, those with pattern (x_1, \ldots, x_m) . Therefore a homology boundary link is seen to be a sort of "algebraic" boundary link since ∂V_i is homologous to a single longitude of K_i . The class of homology boundary links first received attention (from the point of view of link concordance) when the first author observed in [C1, C2] that fusions of boundary links gave examples of non-boundary, non-ribbon links with vanishing Milnor's $\overline{\mu}$ -invariants and that these were in fact *sublinks* of homology boundary links. Confirmation that sublinks of homology boundary links was the *correct* class upon which to focus study was provided by [CL], [L4] and [LMO] where it was shown that, the classes of sublinks of homology boundary links and fusions of boundary links are identical up to concordance,

and that the vanishing of Jean Le Dimet's homotopy invariant of (disk link) concordance was essentially equivalent to being concordant to a sublink of a homology boundary link. This culminated in the above-mentioned result of the authors that, in fact, many homology boundary links are *not* concordant to boundary links. It is unknown whether or not every link (with vanishing $\overline{\mu}$ -invariants if n = 1) is concordant to a homology boundary link. Therefore we turn to the project of classifying concordance classes of *homology* boundary links.

Recall that there were two ingredients to the invariants of [CO1, CO2]. The first was "complexity" which was there explained to be purely a function of the *pattern* P. The second was a function of the universal Blanchfield form B of the homology boundary link, which may also be viewed in terms of "cobordism classes of Seifert matrices." The primary aim of this paper is to show that any such pair (P, B) may be realized by an explicit geometric construction. An important secondary goal is to classify homology boundary links modulo a suitable cobordism relation.

Recall the group $G(m, \epsilon)$ of cobordism classes of Seifert matrices of type (m, ϵ) defined as in [Ko; §3]. If (L, \mathcal{V}) is an *m*-component link in S^{2q+1} , q > 1, with system \mathcal{V} with pattern P then one can associate to (L, \mathcal{V}) an element of $G(m, (-1)^q)$ by taking the "Seifert form" $\frac{H_q(\mathcal{V})}{\langle \text{torsion} \rangle} \times \frac{H_q(\mathcal{V})}{\langle \text{torsion} \rangle} \xrightarrow{\theta} \mathbf{Z}$ given by $\theta(x, y) = lk(x, y^+)$. If q = 1 we must consider $H_q(\mathcal{V})/H_q(\partial \mathcal{V})$ instead of $H_q(\mathcal{V})$, and the restriction on the pattern (w_1, \ldots, w_m) guarantees that θ descend to a form on the quotient. Note that θ can be defined from any set of disjoint codimension-2 oriented submanifolds of S^{2q+1} each component of which is labeled by an element of $\{1, \ldots, m\}$, as long as, when q = 1, the boundaries of the surfaces have zero linking numbers with all elements of $H_1(\mathcal{V})$.

Specifically, our main theorem will be a stronger form of the following.

Theorem 3.6. Given any pattern P, any $q \ge 1$ and $\alpha \in G(m, (-1)^q)$, there is an mcomponent homology boundary link in S^{2q+1} with system of Seifert surfaces of pattern Pand Seifert form equivalent to α . (If q = 2, α must lie in the index 2^m subgroup of G(m, 1)described by [Ko] to account for Rochlin's theorem).

In [CL] it was shown how to construct a link with arbitrary P and $\alpha \cong 0$ (actually a ribbon link) although it would be nice to have a more constructive algorithm. At the other extreme, it is relatively easy to construct a boundary link with $P = (x_1, \ldots, x_m)$ and arbitrary α (see Theorem 3.4 of [Ko] for a proof generalizing Seifert's proof for q = 1). The general idea of Seifert's method (for q = 1) is that, given a Seifert matrix $A = (a_{ij})$ of type (m, ϵ) , one can take m disjoint wedges of appropriate numbers of circles and modify them so that the linking number between the *i*th circle and *j*th circle is a_{ij} . Then one can "thicken" the wedges of circles to create punctured surfaces in such a way that the "self-linking" of the *i*th circle is a_{ii} . These will be the Seifert surfaces of a boundary link whose Seifert matrix is A with respect to those surfaces. This procedure always produces a boundary link (as opposed to an arbitrary homology boundary link). No such simple-minded procedure has been found for homology boundary links.

Theorem 3.6 will be a corollary of a new and interesting construction for links that is useful in creating homology boundary links with prescribed properties. This method was employed in [CO2] to generate examples. On the other hand our work will be helpful in *calculation* as well, since many examples in knot theory consist of a simple knot or link with some bands of its Seifert surface tied into knots. Therefore the Seifert forms of such links are easily computed by our techniques.

We also recover a classification theorem for homology boundary links which parallels the classification theorem for boundary links but is much more complicated. Unlike boundary links, the set of homology boundary links is not closed under connected sum. Consequently we fix the pattern P and consider only those homology boundary links with pattern P. Just as in [Ko], we must consider pairs (L, \mathcal{V}) where L is such an homology boundary link in S^{2q+1} and \mathcal{V} is a system of Seifert surfaces for L. However for homology boundary links we must narrow our focus further by only considering \mathcal{V} whose surfaces meet the link components in a fixed combinatorial scheme S. For any pattern P many schemes are possible, and only links with identical schemes may be summed in such a way that their Seifert forms also naively sum. We define the set of scheme cobordism classes, C(m, q, S), of such pairs where two are scheme-cobordant if there is a concordance between the links and an embedded cobordism between the Seifert surface systems, that preserves the scheme (is a product on its boundary). We show in §5 that C(m, q, S) is naturally a group isomorphic to $G(m, (-1)^q)$, if q > 1, where this isomorphism is given by the Seifert form. We also interpret this as a relative L-group and a Γ -group by using the Blanchfield form. This much is perfectly analogous to the previous work on boundary links.

When we analyze the effect on the Seifert form of changing \mathcal{V} for a fixed L, we begin to see some surprising complications in the case of general homology boundary links of pattern P. In §7 we define two such links L, L' to be homology-boundary-link-concordant if they are concordant in such a way that for some \mathcal{V} , \mathcal{V}' the pairs (L, \mathcal{V}) , (L, \mathcal{V}') are scheme-cobordant. The set of these equivalence classes is denoted $\mathcal{P}(m, q, P)$. We then analyze this set and find that:

Theorem 6.3. For any fixed pattern P and any representative (w_1, \ldots, w_m) of P, this is a bijection $\overline{\theta} : \mathcal{P}(m, q, P) \to G(m, (-1)^q) / \operatorname{Aut}_{w_i} F$ given by taking the Seifert form of a system of Seifert surfaces with scheme $S = (w_1, \ldots, w_m)$. Similarly the Blanchfield form induces a bijection $\overline{B} : \mathcal{P}(m, q, P) \to L^{(-1)^{q+1}}(\mathbf{Z}[F], \Sigma) / \operatorname{Aut}_{w_i} F$. Here $\operatorname{Aut}_{w_i} F$ is the subgroup of automorphisms of the free group which send w_i to a conjugate of w_i (the actions are given in 3.4 and 4.5). (If q = 2 we need to take certain index 2^m subgroups to account for Rochlin's theorem).

A very surprising aspect of 6.3 is that $\mathcal{P}(m, q, P)$ depends on P (whereas C(m, q, S) is independent of S)! A translation of 6.3 in terms of Γ -groups yields the following.

Theorem 6.4. Suppose q > 2. For any fixed pattern P and any representative (w_1, \ldots, w_m) of P, there are functions $\widetilde{\Gamma}_{2q+2}(\mathbb{Z}F \longrightarrow \mathbb{Z})/\operatorname{Aut}_{w_i}F \xrightarrow{\phi_S} \mathcal{P}(m,q,P) \xrightarrow{\pi} L_{2q+1}(F)$ such that π is surjective and ϕ_S is an injection with image $\pi^{-1}(0)$. (Here $\widetilde{\Gamma}$ is the gamma group modulo the image of L_{2q+1}).

Finally, in §7, we discuss the analogues in the case of links in \mathbb{Z}_p -homology spheres to establish some claims made in [CO1, CO2].

$\S1$. Generalized basings and tangle sums.

The method of construction we shall presently detail is perhaps best described as an "action" of the set of boundary links on the set of homology boundary links with pattern P. There are actually many actions depending on various initial data. Given an homology boundary link L, and a sort of generalized basing which effectively decomposes L into two tangles, one of which is trivial; we "act" on L by removing the trivial tangle and inserting a boundary disk link (suitably modified for this purpose). To be more specific, we need to set up some notation.

Definition. A generalized basing b of a link L is an embedding b of the 2-disk $\Delta = I \times I$ into S^{n+2} such that, with regard to the subdivision of the 2-disk shown in Figure 1

FIGURE 1

- i) b is transverse to L
- ii) (image b) $\cap L$ lies interior to $\bigcup_{i=1}^{m} \Delta_i$ along the line $b(I \times \{1/2\})$.

Now suppose $\Delta_i \cap L$ is $\{K_{i_1}, \ldots, K_{i_{n_i}}\} \cap \Delta_i$ reading left to right. Then this will be called a generalized basing of type (b_1, \ldots, b_k) where $b_i = x_{i_1}^{\pm 1} \ldots x_{i_{n_i}}^{\pm 1}$ and the plus sign is used if $K_{i_j} \cap \Delta$ is +1. Note that a basing of type (x_1, \ldots, x_m) is the usual (strong) basing that decomposes L as the closure of a disk link. Also note that it is not necessary that k = m.

A generalized basing of L may be slightly thickened to given an embedding of $\Delta \times D^n = I \times I \times D^n$ whose intersection with L is the product $(L \cap \Delta) \times D^n$. Therefore b induces a "tangle" decomposition of L along $\partial(I \times I \times D^n)$, one "summand" of which is a standard trivial disk link of type (b_1, \ldots, b_k) . Since a "strand" of one of these tangles inherits a label i if it was part of the ith component knot of L, these tangles are unusual as ordered links in that the set of strands labelled i may be disconnected.

Suppose L and L' are endowed with basings b, b' of the same type. Then we can define $L_b \oplus_{b'} L$ by deleting the induced "trivial" tangles of type (b_1, \ldots, b_k) from each and identifying along the common boundaries of the remainders by the (orientation-reversing) restriction of the homeomorphism $r : I \times I \times (I \times D^{n-1}) \bigcirc$ given by r(x, y, z, w) = (x, y, -z, w). If n = 1 this tangle-sum may not yield a true *m*-component link because the

"*i*th component knot" of the result might be disconnected as is seen in Figure 3 for m = 1, L = L', $b = (x_1 x_1^{-1})$. Even if n > 1, this tangle-sum may have components homeomorphic to the connected sum of copies of $S^1 \times S^{n-1}$.

FIGURE 3

However if b' separates L' into *pure* tangles (like pure braids) then the tangle sum will be a true *m*-component link. Specifically, a labelled oriented tangle is called a *pure* tangle if each connected component of the strands labelled *i* is homeomorphic to the *n*-disk. A "link" is called a *true* link if the union of the spheres labelled *i* is connected. In what follows, we shall require situations where L' is *not* a true link but whose components are parallel copies of the components of a true link. As long as *L* is a true link and *b'* separates L' into 2 pure tangles, the sum will be a true link.

FIGURE 4

Moreover if the above links have Seifert surface system which are "compatible" then we ought to be able to "add" these as well. Here the situation is slightly more complicated. If L has a system \mathcal{V} , we may and shall assume that Δ has been isotoped, relative to L, so it meets \mathcal{V} transversely in one of a fixed set of standard schemes as shown by example in Figure 4. This is possible because Δ may be isotoped to look like Figure 5a and hence the intersections with \mathcal{V} may be assumed to be as in 5b, for example. The set of possible intersection schemes is larger than the set of possible $((b_1, \ldots, b_k), (w_1, \ldots, w_m))$ where

FIGURE 5

FIGURE 6

the latter is the pattern. For example, the schemes in Figure 6 are different although both have $b = b_1 = x_1 x_1^{-1} x_2 x_2^{-1}$ and pattern $(x_1, \text{ arbitrary})$ with respect to the meridian μ_1 .

Instead we need the extra data of the words $\gamma_1, \ldots, \gamma_k$ in the alphabet $\{x_1, \ldots, x_m\}$ obtained by traversing $\partial \Delta_i \ i = 1, \ldots, k$, in a counter-clockwise fashion, and reading $x_i^{\pm 1}$ upon encountering $\pm V_i$. In fact, precisely what we need is the factorization of γ_i as $\xi_i r_i \xi_i^{-1}$ where the letters of r_i correspond only to those components of $\mathcal{V} \cap \Delta$ which have boundary on $\partial(E(L))$. Therefore, given any *m*-tuple of words $(\xi_i r_i \xi_i^{-1})$ we shall say that (L, \mathcal{V}, b) has scheme $S = (\xi_i r_i \xi_i^{-1})$ if the words γ_i are identical to the words $\xi_i r_i \xi_i^{-1}$ such that the letters r_i correspond precisely to those components of $\mathcal{V} \cap \Delta$ which have boundary on $\partial E(L)$. A scheme is called *reduced* if each ξ_i is empty. *S* determines *P*, or more specifically, *S* determines w_i up to conjugacy for those *i* which have strands intersecting Δ .

Therefore b induces a tangle decomposition of (L, \mathcal{V}) , one of which is a standard trivial disk link of type b, with standard trivial Seifert surface system of some scheme S.

Proposition 1.1. The tangle sum $(L, \mathcal{V}) \bigoplus_{bb'} (L', \mathcal{V}')$ of two links of basings b, b' of the same type and scheme S may be added to yield $(L \oplus L', \mathcal{V} \oplus \mathcal{V}')$. Here L is a true link but L' may have disconnected components. If b' separates L' into 2 pure tangles then the sum is a true link of as many components as L.

Proof. Since the boundaries of (L, \mathcal{V}) and (L, \mathcal{V}') , after deleting the standard trivial tangle of type S, are standard of type S, the result is clear. The orientation-reversing nature of the gluing map r ensures that the orientations extend. \Box

§2. The geometric actions of boundary links on links of pattern P.

Now, suppose (L, \mathcal{V}) is an *m*-component homology boundary link of pattern *P* and $b = (b_1, \ldots, b_k)$ is a generalized basing of (L, \mathcal{V}) . We describe how to "twist" (L, \mathcal{V}) by a *k*-component boundary link (B, \mathcal{W}) , resulting in a new *m*-component homology boundary link of pattern *P* but whose Seifert form has been altered. Here we explain the action and give examples. In the next section we describe the effect on the Seifert form.

First, given $b = (b_1, \ldots, b_k)$ we describe how to alter the k-component boundary link (B, \mathcal{W}) to get a boundary link of more components with a natural basing of type (b_1,\ldots,b_k) . This is done merely by forming parallel copies of the components of \mathcal{W} dictated by the b_i . Specifically if $b_1 = x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}$ where $\epsilon_i \in \{\pm 1\}$ then form n parallel copies of W_1 , so that the jth copy is oriented "oppositely" to W_1 if $\epsilon_j = -1$. Proceeding around a positively oriented unbased meridian of $K_1 = \partial W_1$, one encounters these copies in succession. Label the j^{th} copy with the number i_i as it appears in b_1 . Similarly do the same for $\{W_2,\ldots,W_k\}$. Let W'_i be the union of all copies appearing with the label *i*. Thus we have formed a new boundary link (B', \mathcal{W}') where we shall say $\mathcal{W}' = (b_1, \ldots, b_k)^{\#}(\mathcal{W})$. This boundary link has many components as were involved in $\Delta \cap L$. Note that B' is very likely not a *true* link since for any fixed *i*, more than one of its components may have the label i. Also note that since (B, \mathcal{W}) has a basing b' of type (x_1, \ldots, x_k) this basing becomes a generalized basing b' of type (b_1, \ldots, b_k) for (B', W'), by construction. Therefore we may form $L_b \oplus_{b'} B'$ (remember that identical basings is sufficient to enable tangle addition of links, whereas tangle sum of Seifert surfaces requires identical schemes). Since b' separates B' into 2 pure tangles, $L_b \oplus_{b'} B'$ is indeed a true link with m connected components. The result may be denoted $(b, b', B)^{\#}(L)$. The definition of this action is independent of the pattern P of L. Indeed L need not have been an homology boundary link. In the next section, we see how to endow B' with a Seifert system with scheme S and calculate the effect on θ . For now we consider examples of this action.

FIGURE 7

Example 1: Connected Sum: If $b = (x_1, \ldots, x_m)$ then $\mathcal{W}' = \mathcal{W}$ and $(b, b', B)^{\#}(L)$ is merely the usual connected-sum L # B.

Example 2: Tying a local knot in L: If $b = (b_1)$ then B is a knot K and $(b, b', K)^{\#}(L)$ is obtained by "seizing some strands" of L according to b_1 and tying the whole thing in the knot K as shown in Figure 7 for a link in S^3 and $b_1 = x_1 x_1^{-1} x_2$.

$\S3$. Effect of Action on the Seifert form.

Suppose (L, \mathcal{V}, b) is an *m*-component homology boundary link with pattern P and generalized basing $b = (b_1, \ldots, b_k)$ inducing a scheme S. Suppose that (B, \mathcal{W}) is a boundary link of k-components. In the previous section we described how to use parallel copies of \mathcal{W} to form (B', \mathcal{W}', b') where b' is of the same type as b. This allowed us to form the tangle sum. Now we endow B' with a new system \mathcal{W}'' of Seifert surfaces with scheme S so that the tangle sum can be performed on the surfaces as well. This will endow the tangle sum with a surface system of pattern P. Consider Δ_1 as in Figure 1 induced by b. The word γ_1 obtained by traversing $\partial \Delta_1$ counter-clockwise is necessarily a product of conjugates $\gamma_1 = \prod_{j=1}^n \xi_j w_{i_j}^{\pm 1} \xi_j^{-1}$ where $P = (w_1, \dots, w_m)$ and $b_1 = x_{i_1}^{\pm 1} x_{i_2}^{\pm 1} \dots x_{i_n}^{\pm 1}$ as shown by example in Figure 8a (see Figure 5 and surrounding discussion). The corresponding (trivial) scheme for the boundary link is shown in 8b. Now merely form parallel copies of V_{i_i} , changing orientations and relabeling to achieve the identical γ_1 as in 8a. This is shown in 8c. Note that since B' is a boundary link, ∂V_{i_i} is connected so these relabellings will not be inconsistent. The (reduced) scheme of 8c is not quite the same as the (perhaps unreduced) scheme of 8a so we must join the oppositely oriented copies of Seifert surfaces that correspond to the conjugating elements in the word $\gamma_1 = \prod_{j=1}^n \xi_j w_{i_j}^{\pm 1} \xi_j^{-1}$. Note that

this is done by attaching an "annulus" $S^{2q-1} \times [-1,1]$ from ∂V_{ξ_1} to $\partial (-V_{\xi_1})$. Note that this last process does not change H_q so does not alter the Seifert form. Doing similar modifications for Δ_i , $1 \leq i \leq k$, completes the description of (B', \mathcal{W}'', b') .

Definition 3.1. Given b, the result of acting on (L, \mathcal{V}) by (B, \mathcal{W}, b') is the tangle sum $(L_b \oplus_{b'} B', \mathcal{V} \oplus \mathcal{W}'')$ which is an *m*-component homology boundary link of pattern P.

To calculate the effect of this action on the cobordism class of the Seifert form, first we will investigate the additivity of the Seifert form under tangle sum of links in S^{2q+1} . We find that this additive if $q \neq 1$ but, surprisingly, that additivity fails in general for q = 1. Fortunately, since B' is a boundary link the additivity will hold for $L \oplus B'$.

Theorem 3.2. Suppose (L, \mathcal{V}, b) , (L', \mathcal{V}', b') are links in S^{2q+1} with generalized basings of identical type and scheme S. Suppose L is a true link of m components, image b' intersects every component of L', the non-trivial tangle associated to b' is pure and suppose that $\{j_1, \ldots, j_n \mid j_1 < \cdots < j_n\}$ is the subset of $\{1, 2, \ldots, m\}$ such that V_{j_i} intersects Δ . Then $(L \oplus L', \mathcal{V} \oplus \mathcal{V}')$ is a true m-component link and $\theta(\mathcal{V} \oplus \mathcal{V})$ equals $\theta(\mathcal{V}) \oplus i_* \theta(\mathcal{V}')$ where $i_* : G(n, (-1)^q) \longrightarrow G(m, (-1)^q)$ is the natural map induced by the inclusion $\{j_1, \ldots, j_n\} \longrightarrow$

FIGURE 8

 $\{1, \ldots, m\}$. If q = 1, it must also be assumed that $H_0(\mathcal{V}_L \cap \mathcal{V}_T = \mathcal{V}'_{L'} \cap \mathcal{V}'_{T'}) \longrightarrow H_0(\mathcal{V}_L) \oplus H_0(\mathcal{V}'_{L'})$ is a monomorphism where these symbols are explained below.

Proof. The basing b decomposes (L, \mathcal{V}) into two tangles, one of which is the standard trivial tangle of type b with surfaces of scheme S. Let \mathcal{V}_T be the intersection of \mathcal{V} with this tangle and \mathcal{V}_L be the intersection of \mathcal{V} with the other tangle. Note that $\mathcal{V}_T \cap \mathcal{V}_L$ is a union of (2q-1)-dimensional disks and spheres. It follows, if $q \neq 1$ that $H_q(\mathcal{V}) \cong H_q(\mathcal{V}_L) \oplus$ $H_q(\mathcal{V}_T) \cong H_q(\mathcal{V}_L)$ since the components of \mathcal{V}_T are contractible. If q = 1, the observation that $H_0(\mathcal{V}_T \cap \mathcal{V}_L) \longrightarrow H_0(\mathcal{V}_T)$ is injective yields the same result. Moreover, since the ambient space of a tangle is B^{2q+1} , all linking numbers between elements of $H_q(\mathcal{V}_L)$ may be computed "inside" that tangle and agree with the linking numbers computed in S^{2q+1} . Therefore it makes sense to speak of $\theta(\mathcal{V}_L)$ and clearly $\theta(\mathcal{V}_L) = \theta(\mathcal{V})$. Similarly $\theta(\mathcal{V}') = \theta(\mathcal{V}'_{L'})$.

By the same token, the surface system $\mathcal{V} \oplus \mathcal{V}'$ decomposes as $\mathcal{V}_L \cup \mathcal{V}_{L'}$ along a union of (2q-1)-dimensional disks and spheres. Therefore, if $q \neq 1$, $H_q(\mathcal{V} \oplus \mathcal{V}') \cong H_q(\mathcal{V}) \oplus H_q(\mathcal{V}_{L'})$. Again, since each tangle is a ball, no elements of $H_q(\mathcal{V}_{L'})$ will link any element of $H_q(\mathcal{V}_L)$. Hence $\theta(\mathcal{V} \oplus \mathcal{V}') \cong \theta(\mathcal{V}) \oplus i_*\theta(\mathcal{V}')$ as claimed.

If q = 1, then when two tangles are joined, the surfaces are joined by either boundaryconnected-sum or by identifying two circle boundary components. These circle boundary components are ones which arise when a component of \mathcal{V}'_T is a disk as in Figure 9. In particular these circles are null-homologous in \mathcal{V} and in \mathcal{V}' . If the two surfaces being joined by \natural or by identifying such circles are *disjoint*, then these operations do not affect H_1 (modulo the subgroup generated by the longitudes). Of course, int V_L and int $V'_{L'}$ are disjoint (lying in disjoint tangles) but once one connection is made, there can be problems. The final hypothesis of 3.2 ensures that there are no problems as can be seen by examining a Mayier-Vietoris sequence for $(\mathcal{V}_L \cup \mathcal{V}'_{L'}, \mathcal{V}_L, \mathcal{V}'_{L'})$. \Box

FIGURE 9

Corollary 3.3. With respect to the notation of 3.1 and preceding discussion, $\theta((L_b \oplus_{b'} B', \mathcal{V} \oplus \mathcal{W}'')) = \theta(L) \oplus i_* \theta(\mathcal{W}'').$

Proof. Recall B was a k-component link with a trivial basing. Since B' is formed from parallels, b' intersects every component of B'. If q = 1, since each surface of \mathcal{W}' had a single boundary component, $H_0(\mathcal{W}''_{B'} \cap \mathcal{W}''_T) \longrightarrow H_0(\mathcal{W}''_{B'})$ is a monomorphism.

Now we need to calculate $\theta(\mathcal{W}'')$. Since this depends only on \mathcal{W}'' (not $\partial \mathcal{W}''$) we see that $\theta(\mathcal{W}'') = \theta((\gamma_1, \ldots, \gamma_k)^{\#}(\mathcal{W}))$ where $(\gamma_1, \ldots, \gamma_k)^{\#}(\mathcal{W})$ is the surface system obtained by forming parallel copies of $\{W_1, \ldots, W_k\}$, re-orienting and relabelling to achieve the word γ_i when traversing $\partial \Delta_i$. Here Δ is a basing of type (x_1, \ldots, x_k) for B. Said another way, if we look at \mathcal{W}'' in S^{2q+1} instead of E(B'), we see that it is indistinguishable from $(\gamma_1, \ldots, \gamma_k)^{\#}(\mathcal{W})$ (except for the extra "annuli" added to alter Figure 8c, which we already remarked had no effect on θ). Therefore the problem reduces to studying the effect of $(\gamma_1, \ldots, \gamma_k)^{\#}$ on the Seifert form of a boundary link. This effect, although easily described in terms of Seifert matrices, is normally quite radical. In this section we show that it depends only on the classes of γ_i in the free group $F(x_1, \ldots, x_k)$ and that it satisfies certain "functorial" properties.

Proposition 3.4. Suppose $f : F \langle x_1, \ldots, x_k \rangle \longrightarrow F \langle x_1, \ldots, x_m \rangle$ is a homomorphism such that $f(x_i)$ is represented by the word w_i , $1 \leq i \leq k$. Then f induces a homomorphism $f_* : G(k, \epsilon) \longrightarrow G(m, \epsilon)$ which is geometrically defined by choosing a simple boundary link with surface system \mathcal{V} representing $\alpha \in G(k, \epsilon)$ then letting $f_*(\alpha) \equiv \theta((w_1, \ldots, w_k)^{\#}(\mathcal{V}))$. In addition $(\mathrm{id})_* = \mathrm{id}$ and $(g \circ f)_* = g_* \circ f_*$.

Remark. Since $G(m, (-1)^q)$ has essentially been identified with S. Cappell and J. Shaneson's *L*-theoretic group $\Gamma_{2q+2}(\mathbb{Z}F \xrightarrow{\epsilon} \mathbb{Z})$, 3.4 reflects the functoriality of the Γ -groups. In section 5 we shall discuss these connections.

Proof. Although the matrix representing $f_*(\alpha)$ may be described in algebraic terms, it is more intuitive to use Seifert surfaces. Suppose $(L, \{V_1, \ldots, V_k\})$ is a simple boundary link in S^{2q+1} with $\theta(\mathcal{V}) = \alpha$ [Ko; Thm. 3.4]. We form $(w_1, \ldots, w_k)^{\#}(\mathcal{V})$ as described earlier. Specifically, if $w_1 = x_{i_1}^{\epsilon_1} \ldots x_{i_n}^{\epsilon_n}$, consider *n* parallel copies of V_1 , the *j*th of which is oriented oppositely to V_1 if $\epsilon_j = -1$. Proceeding around a positively-oriented meridian to ∂V_1 , one encounters these copies in succession and assigns the label i_j to the j^{th} copy. Do the same for V_2 through V_k to complete the definition of $(w_1, \ldots, w_k)^{\#}(\mathcal{V})$. Finally set $f_*(\alpha)$ equal to $\theta((w_1, \ldots, w_k)^{\#}(\mathcal{V}))$. \Box

We need to show $f_*(\alpha)$ is independent of the representatives w_i and of (V_1, \ldots, V_k) . For simplicity let $(w_1, \ldots, w_k)^{\#}(\mathcal{V})$ be abbreviated $w^{\#}(\mathcal{V})$. First, suppose $(J, \{W_1, \ldots, W_k\})$ is another such representative of α . We may form a connected sum of L with the concordance inverse of J in such a way that $L^{\#} - J$ is a simple boundary link admitting the system $\mathcal{V} \natural - \mathcal{W}$, and θ of this system is $\alpha - \alpha = 0$ by 3.3. But it is easy to see that $w^{\#}(\mathcal{V} \natural - \mathcal{W}) =$ $w^{\#}(\mathcal{V}) \natural w^{\#}(-\mathcal{W})$, so that the block sum of $\theta(w^{\#}(\mathcal{V}))$ and $-\theta(w^{\#}(\mathcal{W}))$ is represented by $\theta(w^{\#}(\mathcal{V} \natural - \mathcal{W}))$. It suffices to show the latter is zero. Since $\theta(\mathcal{V} \natural - \mathcal{W}) = 0$, there is a choice of basis of H_q of each component of $\mathcal{V} \natural - \mathcal{W}$ with respect to which the Seifert matrix is composed of blocks N_{ij} each of the form

$$\begin{pmatrix} 0 & C_{ij} \\ D_{ij} & E_{ij} \end{pmatrix}$$

as described in [Ko; p.668]. Thus, with respect to the "same bases", the (i, j) block of the Seifert matrix for $w^{\#}(\mathcal{V}\natural - \mathcal{W})$ will consist of sub-blocks each of which is some $\pm N_{s,t}$. But such a block is congruent to one of the form

$$\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$$

by merely re-ordering basic elements. Thus $\theta(w^{\#}(\mathcal{V}\natural - \mathcal{W})) = 0$, so $\theta(w^{\#}(\mathcal{V})) = \theta(w^{\#}(\mathcal{W}))$ as desired.

Now suppose w_i and z_i are words which are equal in the free group. It suffices to consider the case that z_i is obtained from w_i by inserting $x_j x_j^{-1}$ somewhere in w_i . Suppose (L, \mathcal{V}) and $\mathcal{V}' = w^{\#}(V)$ are as above in the definition of f_* . Let $\mathcal{V}'' = z^{\#}(\mathcal{V})$, so \mathcal{V}'' is \mathcal{V}' together with 2 more copies of V_i (oppositely oriented) which form part of \mathcal{V}''_j . Consider the product $(S^{2q+1} \times [0,1], L \times [0,1], \mathcal{V}' \times [0,1])$. This is the product concordance from L to -L together with the product "cobordism" from \mathcal{V}' to $-\mathcal{V}'$. Now in $S^{2q+1} \times \{0\}$ insert the extra manifolds, $V_i \amalg -V_i$, to form \mathcal{V}'' and in $S^{2q+1} \times [0,1]$ insert the product $V_i \times [0,1]$ in such a way that $\partial(V_i \times [0,1]) = V_i \amalg -V_i$. Then the resulting collection is what we might call a boundary cobordism from (L, \mathcal{V}'') to $(-L, -\mathcal{V}')$. In particular, we may look at the union of \mathcal{V}'' with $-\mathcal{V}'$ together with $\partial \mathcal{V}' \times [0,1]$ as a collection \mathcal{W} of closed 2q-manifolds. The argument of [Ko; Lemma 3.3 and page 671] applies to show $\theta(\mathcal{W}) = 0 = \theta(-\mathcal{V}') \oplus \theta(\mathcal{V}'')$. Therefore $\theta(\mathcal{V}') = \theta(\mathcal{V}'')$ showing that $f_*(\alpha)$ is only dependent on the class of w_i in the free group.

The "functorial" properties of f_* are straightforward to verify. \Box

We can now evaluate the effect of our actions on θ .

Theorem 3.5. Suppose (L, \mathcal{V}, b) is an *m*-component homology boundary link of pattern P with fixed generalized basing (b_1, \ldots, b_k) for which the loops $\{\partial \Delta_1, \ldots, \partial \Delta_k\}$ intersect \mathcal{V} in the words $\{w_1, \ldots, w_k\}$ (see section 1). Suppose (B, \mathcal{W}) is a k-component boundary link. Then the result of acting on (L, \mathcal{V}, b) by (B, \mathcal{W}) has Seifert form equivalent to

$$\theta(\mathcal{V}) \oplus f_*(\theta(\mathcal{W}))$$

where $f_* : G(k, (-1)^q) \longrightarrow G(m, (-1)^q)$ is induced by $f : F \langle x_1, \ldots, x_k \rangle \longrightarrow F \langle x_1, \ldots, x_m \rangle$ given by $x_i \longrightarrow w_i$.

Proof. The result of the action is $(L_b \oplus_{b'} B', \mathcal{V} \oplus \mathcal{W}'')$ as defined previously so, by 3.3, θ is $\theta(\mathcal{V}) \oplus \theta(\mathcal{W}'')$. But $\theta(\mathcal{W}'') = \theta((w_1, \ldots, w_k)^{\#}(\mathcal{W}))$ as remarked below 3.3. Then apply 3.4. \Box

FIGURE 10

Theorem 3.6. Given any scheme $S = (\eta_1 w_1 \eta_1^{-1}, \ldots, \eta_m w_m \eta_m^{-1})$ inducing the pattern Pand any $\alpha \in G(m, (-1)^q)$ (subject to the usual signature restrictions if q = 2), there is an *m*-component ribbon link (R, \mathcal{V}) and an (ordinary) basing b such that (R, \mathcal{V}, b) has scheme S (and pattern P) and $\theta(R, \mathcal{V}) = 0$. By acting on (R, \mathcal{V}) appropriately by a boundary link with Seifert form α , one obtains an homology boundary link (L, \mathcal{V}', b') with scheme S, pattern P and $\theta(L, \mathcal{V}') = \alpha$.

FIGURE 11

Proof. According to [CL; Thm. 2.3], every pattern P is the pattern of a ribbon homology boundary link in S^{2q+1} . More precisely, for every *m*-tuple of words $(\eta_i w_i \eta_i^{-1})$ representing a pattern P, there is a ribbon homology boundary link R, a map $g_*: \pi_1(ER) \to \pi_1(ER)$ $F\langle x_1,\ldots,x_m\rangle$ and a basing $b = (u_1,\ldots,u_m)$ such that $g_*([u_i]) = [\eta_i w_i \eta_i^{-1}]$. But then there exists a map $g: ER \to \bigvee_{i=1}^m S^1$ inducing g_* such that pulling back points under g yields (via the Thom-Pontryagin construction) $\mathcal{V} = (V_i)$ a system of Seifert surfaces (perhaps disconnected) for R such that, with respect to b, (R, \mathcal{V}) has scheme $(\xi_1 r_1 \xi_1^{-1}, \ldots, \xi_m r_m \xi_m^{-1})$ where $\xi_i r_i \xi_i^{-1} = \eta_i w_i \eta_i^{-1}$ in the free group F. We shall now alter \mathcal{V} by moves called elementary reductions and enlargements, until (R, \mathcal{V}, b) has scheme S. To explain these moves, consider Figures 10–13. A general scheme $(\Delta_i, \Delta_i \cap \mathcal{V})$ is shown in 10a which can be encoded by the unreduced word $\xi_i r_i \xi_i^{-1}$. The first elementary reduction (Figure 10b) fuses adjacent copies V_j and $-V_j$ allowing for a potential cancellation of any occurrence of $x_j x_j^{-1}$ or $x_j^{-1} x_j$ in r_i . The first elementary expansion involves adding a small $S^{2q-1} \times [0,1]$ as a new component of V_i as shown in 11. This allows for the insertion of $x_i x_i^{-1}$ or $x_i^{-1} x_i$ in r_i . The second elementary reduction and its inverse are shown in Figure 12. Using this move we may alter \mathcal{V} to assume $\xi_i = n_i$ and $r_i = w_i$ as elements of the free group. Using the first moves, we can assume $r_i = w_i$ as words. Finally, the third elementary reduction

FIGURE 12

Figure 13

(respectively expansion) is shown in Figure 13a (13b). Using this move we can assume \mathcal{V} has precisely the given pattern S. These moves do not change the Seifert form of \mathcal{V} except

for the second elementary reduction, which changes V_j by an ambient 1-handle addition and thus do change the cobordism class of the Seifert form. The resulting (R, \mathcal{V}, b) may not be satisfactory since V_j may not be connected. We must alter \mathcal{V} further to remedy this. However before proceeding note that the Thom-Pontryagin construction applied to (R, \mathcal{V}) yields a map g' homotopic to g. If A and B are two components of V_j , choose a path δ in E(L) from the positive side of A to the negative side of B, which meets \mathcal{V} transversely and misses the basing disk b.

Let * denote the wedge point of $\bigvee_{i=1}^{m} S^{1}$ and let y denote the mid-point of the *j*th circle

so $(g')^{-1}(y) = V_j$. The image of δ under g' represents an element of $\pi_1 \left(\bigvee_{i=1}^m S^1, y\right)$. Since g'_* is surjective, the path δ can be altered so that its image under g' represents zero in π_1 . Thus δ hits \mathcal{V} in a pattern such that the corresponding word may be reduced to the empty word by deleting occurences of $x_i x_i^{-1}$ or $x_i^{-1} x_i$. Hence by tubing of V_i along δ , say, we may alter V_i so that it misses δ until δ is a path in complement of \mathcal{V} connecting A to B. Then A may be joined to B by tubing. The resulting (R, \mathcal{V}, b) is the desired ribbon link with scheme S. Moreover, if g'' represents the associated Thom-Pontryagin map, then $g''_* = g'_* = g_*$ by the same argument as that of [Ko; 2.2].

Since $\{\eta_i w_i \eta_i^{-1} \mid 1 \leq i \leq m\}$ normally generates the free group, there are disjointly embedded loops $\gamma_1, \ldots, \gamma_m$ in E(R) sharing the common basepoint *, disjoint from b(each of which travels to a component R_i , traverses a meridian, returns along nearly the same path and sets off again, et cetera) such that $g_*([\gamma_i]) = x_i$. These loops γ_i induce a generalized basing b where $\gamma_i = \partial \Delta_i$. Choose a boundary link (B, \mathcal{W}, b') with $\theta(\mathcal{W}) = \alpha$ (and trivial basing b'). Act on (R, \mathcal{V}, b) by (B, \mathcal{W}, b') . The result, by 3.1, is an *m*-component homology boundary link of scheme S with Seifert form equivalent to $\theta((R, \mathcal{V})) \oplus \mathrm{id}_*(\alpha) = \theta(R, \mathcal{V}) \oplus \alpha$ by 3.5.

We must now see that $\theta(R, \mathcal{V}) = 0$. Recall the system of Seifert surfaces induces, by the Pontryagin construction, a map $g: E(R) \longrightarrow \bigvee_{i=1}^{m} S^1$ such that the inverse image of $\{1\}$ on the i^{th} circle is V_i . Now, the proof of Theorem 2.3 [CL] shows (see the proof of Theorem 3.1 of [L1] for a more complete argument) that R may be chosen to possess slice disks $\{D_1, \ldots, D_m\} = \mathcal{D}$ in B^{2q+2} such that $\pi_1(E(R)) \xrightarrow{j_*} \pi_1(E(\mathcal{D}))$ is an epimorphism (isomorphism if $q \neq 1$). Extend g over the boundaries of the tubular neighborhoods of the D_i in the obvious way. Since $H_2(\pi_1(E(\mathcal{D}))) = 0$, a theorem of Stallings [St] ensures that j_* induces an isomorphism modulo the intersection of the finite terms of the lowercentral series. Since free groups are ω -nilpotent, g extends to $\hat{g}: E(\mathcal{D}) \longrightarrow \bigvee_{i=1}^m S^1$. After modifying \hat{g} by an isotopy rel g, let W_i be the inverse image of $\{1\}$ on the i^{th} circle. Then ∂W_i is V_i together with various copies of D^{2q} glued along the components of ∂V_i . This collection $\{W_i\}$ shows that $\theta(\mathcal{V}) = 0$ as in [Ko; Lemma 3.3 and page 24]. \Box

Recall that we have failed to establish addivity of Seifert form under tangle sum when q = 1. The following shows that this will hold for *ordinary* connected sum of classical

links if those links are obtained from acting on ribbon links by boundary links and the connected sum avoids the boundary link tangles. This establishes details of certain claims of additivity in Chapter 3, section B of [CO2].

Theorem 3.7. Suppose (L, \mathcal{V}) is an homology boundary link with scheme S in S^3 which is obtained from the boundary link (B_0, \mathcal{W}_0) acting on the ribbon link $(R_0, \mathcal{V}_0, b_0)$. Similarly suppose (L', \mathcal{V}') is another such obtained from (B_1, \mathcal{W}_1) acting on the ribbon link $(R_1, \mathcal{V}_1, b_1)$. Finally suppose that $b = (x_1, \ldots, x_m)$ is a (normal) basing of (L, \mathcal{V}) and b' of (L', \mathcal{V}') (with respect to which $L \oplus L'$ is the ordinary connected sum of links) each of which is disjoint from their respective boundary link tangle summand. Then $\theta(L \oplus L') =$ $\theta(L, \mathcal{V}) \oplus \theta(L', \mathcal{V}')$.

Proof. Since *b* and *b'* lie entirely within the "ribbon link tangle" summands of *L* and *L'* respectively, $(L \oplus L', \mathcal{V} \oplus \mathcal{V}')$ is merely the result of acting on $(R_0, \mathcal{V}_0, b) \oplus (R_1, \mathcal{V}_1, b')$ first by (B_0, \mathcal{W}_0) and then by (B_1, \mathcal{W}_1) (or vice-versa). By 3.5, $\theta(L, \mathcal{V}) = \theta(R_0, \mathcal{V}_0) \oplus f_*(\theta(B_0, \mathcal{W}_0))$ and $\theta(L', \mathcal{V}') = \theta(R_1, \mathcal{V}_1) \oplus f'_*(\theta(B_1, \mathcal{W}_1))$ where f_* is defined by the way $\partial \Delta_i \subset b_0$ intersects \mathcal{V}_0 and f'_* by the way $\partial \Delta_i \subset b_1$ intersects \mathcal{V}_1 . Similarly $\theta(L \oplus L')$ is, using our first remark, $\theta(R_0 \oplus R_1) \oplus f_*(\theta(B_0, \mathcal{W}_0)) \oplus f'_*(\theta(B_1, \mathcal{W}_1))$. Since $\theta(R_0, \mathcal{V}_0) = \theta(R_1, \mathcal{V}_1) = 0$ by the proof of 3.6, we need only show that $\theta((R_0, \mathcal{V}_0, b) \oplus (R_1, \mathcal{V}_1, b'))$ is zero. Since *b* and *b'* are ordinary basings we may use the well-known fact that the connected-sum of two ribbon links is a ribbon link. By 1.1, $R_0 \oplus R_1$ is an homology boundary link with surface system $\mathcal{V}_0 \oplus \mathcal{V}_1$. Then the proof of 3.6 shows that $\theta(R_0 \oplus R_1) = 0$. Hence $\theta(L \oplus L') = \theta(L) \oplus \theta(L')$ as desired. \Box

FIGURE 14

Example 3.8. We will show how to construct a 2-component homology boundary link in S^{2q+1} with arbitrary Seifert form α and with pattern (x, yw), where w is an element of the subgroup of $F \langle x, y \rangle$ generated by $\{x, x^{-1}, yxy^{-1}, yx^{-1}y^{-1}\}$ (and also lies in the commutator subgroup). First we construct a ribbon homology boundary link with the correct pattern. This link will be a *fusion* of a 3 component trivial link [C1, C2] and in fact what has been called a *strong fusion* of a 2-component trivial link by U. Kaiser [Ka]. As an aside, we note the fascinating fact that Theorem 3.15 of [Ka] proves that the patterns (x, wy) of type above are the *only* ones possible for a strong fusion of a 2-component boundary link. Express w as a word in $\{x, x^{-1}, yxy^{-1}, yx^{-1}y^{-1}\}$ so $w = w_1 \dots w_n$. Form a trivial link of n components in S^{2q+1} by nesting as in Figure 14a.

FIGURE 15

The "first" component is innermost, et cetera. Orient the i^{th} component counterclockwise if $w_i = x$ or yxy^{-1} , otherwise clockwise. Join all components corresponding to $w_i = yx^{\pm 1}y^{-1}$ to the left as in 14b, respecting orientation, and join all components corresponding to $w_i = x^{\pm 1}$ to the right as shown in 14b. The result is a trivial link of 2 components $\{J_1, J_2\}$. Form a ribbon knot K_1 by "fusing" J_1 to J_2 using a single "band" b (tube if q > 1) that originates at $*_1$, dives down through all the nested circles and terminates at $*_2$ as shown in Figure 15. Lastly add a trivial component K_2 as shown in Figure 15.

Then there is a system of Seifert surfaces $\mathcal{V} = (V_x, V_y)$ for the homology boundary link $R = \{K_1, K_2\}$ such that μ_1 spells the word x while μ_2 spells yw. The Seifert surface V_x for K_1 is a union of "disks with holes" and tubes as shown in Figure 16. The tubes are nested and run along b, each terminating as a longitude of K_2 . V_y is a union of "cocoons with holes" and tubes as shown in Figure 17. The tubes are nested (with each other and with the tubes of V_x) and run along b, terminating in longitudes of K_2 . Thus R is the desired ribbon link with surface system.

Now, as in the proof of 3.6, we must find paths $\{\gamma_1, \gamma_2\}$ which spell $\{x, y\}$. These are shown in Figure 18. Now, for example, suppose α were the Seifert form for the *split* link $\{J_1, J_2\}$. Then the desired homology boundary link with pattern (x, wy) and form equivalent to α would be as shown in Figure 19. FIGURE 16

FIGURE 17

We remark in passing that the links in [CO2; A. The Simplest Examples] are of this general type with $w = [x, y^{-1}]^m$, $\gamma_1 = [x, y^{-1}] \gamma_2$ = the empty word and α the form of a knot J. In addition the examples in Section B, Figure 3.12 of that paper are of the same family with $\gamma_1 = y^{-1}[y^{-1}, x]^{m-1}$, γ_2 being the empty word and α being the form of a knot J.

§4. Blanchfield forms of Simple Homology Boundary Links.

FIGURE 18

FIGURE 19

We follow the development of [Du] where Blanchfield forms were defined for boundary $\frac{22}{22}$

links and mimic developments of [H1, 14–15, 122–124] [H2, page 372]. Suppose L is an *m*-component homology boundary link in S^{2q+1} equipped with a homomorphism ϕ : $\pi_1(E(L)) \longrightarrow F(x_1, \ldots, x_m)$. Then (L, ϕ) induces a regular covering space \widetilde{X} of X = E(L)whose group of deck translations is identified with the free group. X is unique up to covering space isomorphism and the identification is unique up to a global conjugation in the group of deck translations. If ϕ were surjective then X would merely be the usual connected covering space associated to the kernel of ϕ . Any such cover is covered by the $(\pi_1(E(L)))/(\pi_1(E(L)))_\omega \cong F\langle x_1,\ldots,x_m\rangle$ cover. If ϕ is not surjective then \widetilde{X} is a disjoint union of copies of the connected cover associated to $\phi : \pi_1(E(L)) \twoheadrightarrow \operatorname{image} \phi$. Let $A = \mathbb{Z}[F]$ endowed with the involution $\overline{\sum n_i w_i} = \sum n_i w_i^{-1}$, let $H_*(X, A)$ denote the right A-module $H_*(\widetilde{X}; \mathbf{Z})$, and $M = H_q(X, A)$. If q = 1, some modification is necessary. There seem to be two ways to proceed. The first is to consider the quotient module $M = H_1(X)/H_1(\partial X)$ which is the same as the quotient of $H_1(X, A)$ by the A-submodule, denoted \mathcal{L} , generated by lifts of longitudes which must lie in kernel ϕ if L is an homology boundary link. Later in this section we shall explicitly investigate this situation and see that a Blanchfield form can be defined on this module. The second way is to restrict to those (L, ϕ) for which there exists a ribbon homology boundary link (R, ψ) and a degree one map relative boundary $f: E(L) \longrightarrow E(R)$ such that $\psi \circ f_* = \phi$. If $\widetilde{E(L)}$ and $\widetilde{E(R)}$ are the covering spaces associated to ϕ and ψ , let \widetilde{Z} be the mapping fiber of $\widetilde{f} : \widetilde{E(L)} \longrightarrow \widetilde{E(R)}$ [Wh, p. 43]. Let M denote the A-module $H_*(\widetilde{Z}; \mathbb{Z})$ in this case. In 4.4, we shall show that these two Blanchfield forms, while not isomorphic, are equivalent in the relevant Witt group.

Now we return to the general case. Let Λ denote the Cohn localization of A with respect to the augmentation $\epsilon : A \longrightarrow \mathbb{Z}$ (see [Du]). Recall that $A \stackrel{i}{\longrightarrow} \Lambda$ is an embedding with the property that any square matrix over A which is invertible when augmented, is invertible over Λ . Recent work of M. Farber and P. Vogel has identified Λ as the ring of "rational functions" in non-commuting variables [FV]. We wish now to restrict ourselves to "simple" homology boundary links.

Definition 4.1. (compare [Du, §6] [Ko; 2.8]) An homology boundary link (L, \mathcal{V}) in S^{2q+1} is *simple* if each Seifert surface V_i is (q-1)-connected.

Then we define a $(-1)^{q+1}$ -Hermitian "Blanchfield linking form" $B : \overline{H_q(X, A)} \longrightarrow$ Hom_A $(H_q(X, A), \Lambda/A)$ (see B' in [Du, 624]) as follows. Consider the intersection form $C_q(\widetilde{X}) \otimes_{\mathbf{Z}} C_{q+1}(\widetilde{X}) \longrightarrow A$ denoted by \cdot , inducing $I : H_{q+1}(X, \Lambda/A) \otimes_{\mathbf{Z}} H_q(X, A) \longrightarrow \Lambda/A$ given by $I(C \otimes \alpha, y \otimes \beta) = \overline{\beta} \left(\sum_{\lambda \in F} (C \cdot y\lambda)\lambda \right) \alpha$ where $\alpha, \beta \in \Lambda/A, C \in C_{q+1}(\widetilde{X}), y \in C_q(\widetilde{X})$. Consider also $\partial_* : H_{q+1}(X, \Lambda/A) \longrightarrow H_q(X, A)$. Then set $B(x, y) = I(\partial_*^{-1}x, y)$. In case A were commutative this agrees with [CO, §1] and [H1; 120] but differs slightly from [Du; 624]. The pair (M, B) shall be referred to as the Blanchfield form associated to (L, ϕ) . One key point of [Du] was to ensure that $\partial_* : H_{q+1}(X, \Lambda/A) \longrightarrow H_q(X, A)$ be an isomorphism by showing $H_{q+1}(X, \Lambda) \cong H_q(X, \Lambda) \cong 0$. Suppose q > 1 and let W be a wedge of m circles. Then ϕ induces $\phi : X \longrightarrow W$ and $\tilde{\phi} : \tilde{X} \longrightarrow \tilde{W}$. Since ϕ is an integral homology equivalence up to and including dimension 2q - 1, $\tilde{\phi}$ is a Λ -homology equivalence in the same range (see page 624 of [Du]). But \widetilde{W} is a 1-complex so $H_{q+1}(X,\Lambda) \cong H_q(X,\Lambda) \cong 0$. If q = 1, since $f : E(L) \longrightarrow E(R)$ is an isomorphism on integral homology, \tilde{f} is an isomorphism on Λ -homology and so $\overline{H}_*(X,\Lambda) = 0$.

Strictly speaking, the above extension of DuVal serves to define only the Blanchfield form associated to the "free" cover of E(L) associated to the epimorphism ϕ : $\pi_1(E(L)) \twoheadrightarrow F \langle x_1, \ldots, x_m \rangle$ induced by V. An arbitrary homomorphism $\phi'' : \pi_1(E(L)) \longrightarrow F \langle x_1, \ldots, x_m \rangle$ factors as

$$\pi_1(E(L)) \xrightarrow{\phi} \pi_1(E(L)) / (\pi_1(E(L)))_{\omega} \xrightarrow{\psi} F \langle x_1, \dots, x_k \rangle \xrightarrow{f} F \langle x_1, \dots, x_m \rangle$$

where ψ is onto and f is injective. Suppose (M, B) is the Blanchfield form associated to ϕ . The Blanchfield form associated to ϕ' is defined to be (M', B') where $M' = M \otimes_{\mathbf{Z}H} \mathbf{Z}F \langle x_1, \ldots, x_k \rangle$ where $H = \pi_1/(\pi_1)_\omega$ and if $x, y \in M, \alpha, \beta \in F \langle x_1, \ldots, x_k \rangle$, $B'(x \otimes \alpha, y \otimes \beta) = \bar{\beta}\psi_*(B(x,y))\alpha$. Alternatively it is easily seen that DuVal's work and the definitions above extend trivially to *these* "reduced free covers" and so the previous definition may be used and agrees with this one. The Blanchfield form (M'', B'') associated to ϕ'' is then given by $M'' = M' \bigotimes_{\mathbf{Z}F[x_1, \ldots, x_k]} \mathbf{Z}F[x_1, \ldots, x_m]$ and $B''(x \otimes \alpha, y \otimes \beta) = \bar{\beta}f_*(B(x,y))\alpha$. Here the covering space associated to ϕ'' is a union of disjoint copies of that associated to ϕ' .

the covering space associated to ϕ'' is a union of disjoint copies of that associated to ϕ' . The fact that this definition of B'' agrees with the obvious generalization of DuVal (given by our original formula for B), is obtained in a manner precisely like the proof immediately preceeding Theorem 1.9 of [CO2].

Another key point for DuVal was that the module on which B is defined be of type S. We shall presently see that this is the case, also implying that they are Z-torsion free [Du; Proposition 4.1].

We shall show that the Blanchfield form is determined by the Seifert matrix for a simple homology boundary link (L, \mathcal{V}) where by the Blanchfield form of (L, \mathcal{V}) we mean that associated to the map $E(L) \xrightarrow{\phi} \bigvee_{i=1}^{m} S^1$ by the Pontryagin construction applied to \mathcal{V} .

For such a simple homology boundary link let $Y = E(L) - \bigcup_{i=1}^{m} W_i$ where W_i is an open tubular neighborhood $V_i \times [-1, 1]$ of V_i . Let Z be the complex obtained by identifying all those boundary components of \mathcal{V} which are an i^{th} longitude, $i = 1, \ldots, m$. Then $H_j\left(\bigcup_{i=0}^{m} W_i\right) \cong H_j(Z)$ if $j \neq 0, 1, 2q-1$. By Alexander Duality, $H_q(Y) \cong H_q(S^{2q+1}-Z) \cong$ $H^q(Z) \cong \text{Hom}(H_q(Z), \mathbb{Z}) \oplus \text{Ext}(H_{q-1}(Z), \mathbb{Z}) \cong \text{Hom}(H_q(\mathcal{V}) ; \mathbb{Z})$ if $q \neq 1$ (note if q = 2, $H_{q-1}(Z)$ is torsion-free). Therefore $H_q(\mathcal{V}) \cong H_q(Y)$ are free abelian of the same rank. Choose a basis $\{\alpha_{ik} \mid 1 \leq k \leq r(i)\}$ for $H_q(V_i), 1 \leq i \leq m$.

Since the isomorphism above is detected by ordinary linking number in S^{2q+1} , we may choose a basis $\{\hat{\alpha}_{jn}\}$ for $H_q(Y)$ such that $lk(\alpha_{ik}, \hat{\alpha}_{jn}) = \delta_{ij}\delta_{kn}$.

Suppose now that (L, \mathcal{V}) is an *m*-component simple homology boundary link in S^{2q+1} .

Then there is a continuous map $f: E(L) \longrightarrow \bigvee_{i=1}^{m} S^{1}$ and points p_{i} on the i^{th} circle such that $f^{-1}(p_{i}) = V_{i}$. Such an f induces homomorphism $f_{*} = \phi$ as above. If f_{*} is onto then the covering space \widetilde{X} so induced may be constructed as in [H1, page 14] by splitting E(L) open along V. Then there is a Mayer-Vietoris sequence:

$$A \otimes H_q(\mathcal{V}) \xrightarrow{d} A \otimes H_q(Y) \xrightarrow{i} H_q(X; A) \xrightarrow{\partial} A \otimes H_{q-1}(\mathcal{V})$$

where $d(\gamma \otimes \alpha_j) = \gamma x_j \otimes (i_{j+})(\alpha_j) - \gamma \otimes (i_{j-})(\alpha_j)$ for $\alpha_j \in H_q(V_j)$ and $i_{j\pm}$ the two inclusions $V_j \longrightarrow Y$. Since L is simple, $H_{q-1}(\mathcal{V}) = 0$. By our remarks above, if $q \neq 1$ then with respect to the bases $\{\alpha_{ik}\}, \{\hat{\alpha}_{jn}\},$ the matrix of $(i_{+})_* : H_q(\mathcal{V}) \longrightarrow H_q(Y)$ is merely θ where θ is the Seifert matrix for \mathcal{V} relative to $\{\alpha_{ik}\}$. Moreover the map $d : A^{\sum r(i)} \longrightarrow A^{\sum r(i)}$ is given by the matrix $\Delta = \Gamma \theta + \epsilon \theta^T$ where $\epsilon = (-1)^q$, Γ is the block diagonal matrix $(x_1 I_{r(1)}, \ldots, x_m I_{r(m)})$ with $I_{r(i)}$ the identity matrix of rank r(i). Therefore Δ yields a presentation matrix for the module $H_q(X; A)$. Since $\theta + \epsilon \theta^T$ is unimodular, Δ is invertible when augmented. Therefore, by definition of the Cohn localization Λ , Δ is invertible in the larger ring Λ . In particular d and Δ are injective, establishing that $H_q(X, A)$ is of type S when q > 1.

We may now compute the Blanchfield form, mirroring [H1; 122–123]. Suppose C_{ik} denotes a fixed translate of the (q+1)-chain $[-1,1] \times \alpha_{ik}$ in \widetilde{X} . Note that $\partial C_{kn} = x_k \otimes \alpha_{kn}^+ - 1 \otimes \alpha_{kn}^- = d(\alpha_{kn})$, so for any $w \in H_q(\mathcal{V}) \otimes A$, $w = \sum w_{kn} \alpha_{kn}$ and $\partial(\sum w_{kn} C_{kn}) = dw$. Now, to compute B(z, y) where $z = ir = i(\sum r_{kn} \hat{\alpha}_{kn})$ and $y = i(\sum s_{jm} \hat{\alpha}_{jm})$, set $w = d^{-1}r$ or $w_{kn} = (\Delta^{-1} \cdot r)_{kn}$. Then one sees that $z = i \circ \partial(\sum (\Delta^{-1} \cdot r)_{kn}C_{kn})$. Thus

$$B(z,y) = I\left(i\left(\sum (\Delta^{-1} \cdot r)_{kn}C_{kn}\right), i\left(\sum s_{jm}\hat{\alpha}_{jm}\right)\right)$$
$$= \sum_{j,m} \bar{s}_{jm}\left(\left(\sum_{k,n} I(C_{kn}, i(\hat{\alpha}_{jm}))\right)(\Delta^{-1} \cdot r)_{kn}\right)$$

But $I(A_{kn}, i(\hat{\alpha}_{jm})) = \delta_{jk}\delta_{mn}(1 - x_k)$ so

$$B(z,y) = \bar{s}^T (I - \Gamma) \Delta^{-1} r \mod A$$

where r and s are here viewed as column vectors. Summarizing, we have shown the following for q > 1. The proof for q = 1, using our first definition of Blanchfield forms, is immediately below.

Theorem 4.2. If (L, \mathcal{V}) is a simple homology boundary link in S^{2q+1} , then with respect to the generators $i(\hat{\alpha}_{kn})$ as defined above, the Blanchfield form is represented by the square matrix $(I - \Gamma)(\Gamma\theta + (-1)^q\theta^T)^{-1}$ where θ is the Seifert matrix with respect to α_{kn} and Γ is the block diagonal matrix defined above, and I is the identity matrix. The module $H_q(X, A)$ is presented by the matarix $\Gamma\theta + (-1)^q\theta^T$. While the terminology is fresh in the reader's mind, we turn to the case q = 1. We shall show that there is a Blanchfield pairing on the quotient module $H_1(\widetilde{X})/H_1(\partial \widetilde{X}) \cong H_1(X, A)/\mathcal{L}$. Since $H_1(\partial \widetilde{X})$ is generated by lifts of longitudes, the inclusion-induced map $H_1(\partial X - V) \otimes A \longrightarrow H_1(\partial \widetilde{X})$ is onto. The argument of [H2; p. 373] works almost word for word even though that argument concerned the Blanchfield form on the universal *abelian* covering space. One special argument is necessary to establish that the map $i: H_1(X - V) \otimes A \longrightarrow H_1(\widetilde{X})$ of the Mayer-Vietoris sequence is onto in our case. For this consider the map $\phi: X \longrightarrow \bigvee_{i=1}^m S^1$ such that $\phi^{-1}(\{p_i\} \in S^1) = V_i$, which induces $\widetilde{\phi}: \widetilde{X} \longrightarrow \widetilde{W}$ where W is the wedge. Therefore there is a map of chain complexes as below

$$\begin{array}{cccc} H_1(\widetilde{X}) & \xrightarrow{\partial} & H_0(V) \otimes A & \xrightarrow{d_0} & H_0(Y) \otimes A & \xrightarrow{\epsilon} & \mathbf{Z} \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & &$$

Since \widetilde{W} is contractible, $(d_0)'$ is injective. Since the middle vertical maps are isomorphisms, d_0 is also injective implying that *i* above is onto.

Hillman's arguments result in the exact sequence

$$\frac{H_1(V)}{H_1(\partial V)} \otimes A \xrightarrow{d} \frac{H_1(Y)}{H_1(\partial X - V)} \otimes A \xrightarrow{i} \frac{H_1(\widetilde{X})}{\mathcal{L}} \longrightarrow 0$$

where the first two terms are shown to be free A-modules of the same rank $(\operatorname{rank} H_1(V) - \operatorname{rank} H_1(\partial V) + m)$. Moreover, if $\{\alpha_{ik}\}$ is a basis for $H_1(V)/H_1(\partial V)$ represented by loops on V and $\hat{\alpha}_{ik}$ the corresponding elements in $H_1(Y)$ such that $lk(\alpha_{ik}, \hat{\alpha}_{jn}) = \delta_{ij}\delta_{kn}$, then clearly $\{[\hat{\alpha}_{jn}]\}$ generates $H_1(Y)/H_1(\partial X - V)$ since each V_i is homotopy equivalent to a 1-complex. Furthermore this set is linearly independent because if $\sum n_{jn}\hat{\alpha}_{jn} = \gamma \in$ $H_1(\partial X - V)$ then $0 = lk(\alpha_{ij}, \gamma) = n_{ij}$ since $H_1(\partial X - V)$ is generated by longitudes and the V_i give null-homologies for the longitudes (disjoint from α_{ij}^+). Therefore the matrix of d is given by the same square matrix as in the case q > 1 and all of our conclusions for that case apply. In particular $H_1(\tilde{X})/\mathcal{L}$ is of type S and is **Z**-torsion-free. In this way we recover 4.2 for the case q = 1, at least under our first definition of the Blanchfield form. \Box

By [Du; Prop. 4.1, 4.2, 4.3] the Blanchfield forms defined herein are $(-1)^{q+1}$ -linking forms (M, B) in the sense of [V1]. A "Witt" group of such ϵ -linking forms is then defined by DuVal [Du; §8] which we shall denote by $L^{\epsilon}(A, \Sigma)$ where $\epsilon = (-1)^{q+1}$, $A = \mathbf{Z}[F \langle x_1, \ldots, x_m \rangle]$ and Σ is the group of square matrices which, when augmented, are invertible over \mathbf{Z} . Then it is not difficult to see that **Corollary 4.3.** The matrix correspondence of 4.2 induces a homomorphism $G(m, (-1)^q)$ $\xrightarrow{\psi} L^{\epsilon}(A, \Sigma), \ \epsilon = (-1)^{q+1}$, which sends a representative of the Seifert matrix of an homology boundary link (L, \mathcal{V}) to the class of its Blanchfield linking form (when q = 2we have taken an index 2^m subgroup of the usual G(m, -1) so the definition of $L^{\epsilon}(A, \Sigma)$ would need to be similarly restricted in this case).

We can now show that the two Blanchfield forms defined in case q = 1 are "cobordant" (equal in G(m, -1)). They are certainly not isomorphic, for, in the case that L were itself a ribbon homology boundary link in S^3 , our second Blanchfield form could be taken to be defined on the trivial module, whereas the first Blanchfield form would, in general, be non-trivial.

Theorem 4.4. In case q = 1, the Blanchfield form B, defined on $H_1(\widetilde{X})/H_1(\widetilde{\partial X})$, is equivalent in G(m, -1) to the Blanchfield form B', defined on the kernel $H_1(\widetilde{X}) \xrightarrow{\widetilde{f}_*} H_1(\widetilde{E(R)})$ (see the beginning of this section for terminology).

Proof of 4.4. We are given that $f: (E(L), \partial E(L)) \longrightarrow (E(R), \partial E(R))$ is a degree 1 map of simple Poincaré pairs in the sense of Wall [W; §2]. Let X = E(L) and Y = E(R). By Lemma 2.2 of [W] the horizontal short exact sequence below is split exact, and since f is a homeomorphism on ∂X , the upper map is an isomorphism

It follows that the following is exact

 $0 \longrightarrow \ker i_X \longrightarrow \ker i_Y \longrightarrow M \longrightarrow \operatorname{cok} i_X \longrightarrow \operatorname{cok} i_Y \longrightarrow 0.$

But since $H_2(X; \partial X; A) \xrightarrow{\tilde{f}_*} H_2(Y, \partial Y; A)$ is onto, it is easily seen that ker $i_X \longrightarrow$ ker i_Y is surjective. Therefore $0 \longrightarrow M \longrightarrow \operatorname{cok} i_X \longrightarrow \operatorname{cok} i_Y \longrightarrow 0$ is exact, and in fact split exact.

The latter observation necessitates showing that g_* , when restricted to the image of $H_1(\partial Y; A)$, is an inverse to f_* , that is to say, if $\alpha \in H_1(\partial Y; A)$ then $g_*i_Y(\alpha) = i_X f_*^{-1}(\alpha)$. This may be shown directly using the fact that $g_*(\beta)$ is given by the Poincaré dual of f^* of the Poincaré dual of β . Thus $g_*i_Y(\alpha) = (f^*(i_Y\alpha)^{\wedge}) \cap \Gamma_X$ where ()^ denotes Poincaré dual and Γ_X is the fundamental class. But $(i_Y\alpha)^{\wedge} = \delta_Y(\hat{\alpha})$ ([GH; 28.18]), and $f^*\delta_Y(\hat{\alpha}) = \delta_X f^*\hat{\alpha}$. By the same fact, $(\delta_X f^*\hat{\alpha}) \cap \Gamma_X = i_X(f^*\hat{\alpha} \cap \Gamma_{\partial X})$. Finally $f_*(f^*\hat{\alpha} \cap \Gamma_{\partial X}) = \hat{\alpha} \cap f_*\Gamma_{\partial X}$ by [GH; 24.14], which in turn equals α since f is a homeomorphism on ∂X . Therefore there is an isomorphism

$$H_1(X;A)/H_1(\partial X;A) \longleftarrow M \oplus H_1(Y;A)/H_1(\partial Y;A)$$

27

given by $(m, y) \longrightarrow m+g_*(y)$. Since we have already established that $H_1(X; A)/H_1(\partial X; A)$ and $H_1(Y; A)/H_1(\partial Y; A)$ are of type S, it follows that M is **Z**-torsion-free and of type L[Du; 3.1i], hence of type S [Du; 4.1]. Consider the intersection forms I_X , I_Y used to define the Blanchfield forms. It is a small exercise to show that $I_X(\alpha, g_*\beta) = I_Y(f_*\alpha, \beta)$ using the fact that f is degree 1. Thus $B_Y(f_*\alpha, \beta) = B_X(\alpha, g_*\beta)$. It follows that $B_X(m, g_*y) = 0$ for all $m \in M$ and $y \in H_1(Y; A)/H_1(\partial Y; A)$, and that $B_X(g_*y_1, g_*y_2) = B_Y(y_1, y_2)$. Hence B_X is isomorphic to B_Y (on $H_1(Y; A)/H_1(\partial Y; A)$) plus the Blanchfield form on M (which we have called B'). But B_Y is trivial in G(m, -1) as shown in the proof of Theorem 3.6 (a ribbon S-link is scheme null-cobordant). Hence $B_X \cong B'$ in G(m, -1). \Box

Proposition 4.5. Suppose $f : F \langle x_1, \ldots, x_k \rangle \longrightarrow F \langle x_1, \ldots, x_m \rangle$ is an homomorphism. Then there is a commutative diagram

$$G(k,\epsilon) \xrightarrow{\psi_k} L^{-\epsilon}(\mathbf{Z}[F\langle x_1,\ldots,x_k\rangle],\Sigma)$$

$$\downarrow f_* \qquad \qquad \qquad \downarrow f_*$$

$$G(m,\epsilon) \xrightarrow{\psi_m} L^{-\epsilon}(\mathbf{Z}[F\langle x_1,\ldots,x_m\rangle],\Sigma)$$

where the left-hand f_* is defined in 3.4 and the right-hand f_* is the usual homomorphism induced by an augmentation-preserving, involution-preserving ring homomorphism f, namely $f_*((M,B)) = \left(M \bigotimes_{\mathbf{Z}F\langle x_1,\ldots,x_k \rangle} \mathbf{Z}F\langle x_1,\ldots,x_m \rangle, B'\right)$ where $B'(x \otimes \alpha, y \otimes \beta) = \bar{\beta}f_*(B(x,y))\alpha$ for $x, y \in M$ and $\alpha, \beta \in F\langle x_1,\ldots,x_m \rangle$.

Proof of 4.5. We know that f_* is realized by taking parallel copies of Seifert surfaces for a boundary link of k-components and labelling them appropriately as in 3.4. Therefore we go from the Blanchfield form associated to the standard epimorphism $\phi : \pi_1(E(L)) \twoheadrightarrow$ $F \langle x_1, \ldots, x_k \rangle$ defining the usual free cover of the exterior of the boundary link, to one associated to $f \circ \phi : \pi_1(E(L)) \longrightarrow F \langle x_1, \ldots, x_m \rangle$. Thereby the result is reduced to showing that the one definition of the Blanchfield form, namely that given by 4.3, is the same as the other one we gave. We leave the details to the reader. \Box

This allows us to re-state our major theorems 3.5 and 3.6 in terms of Blanchfield linking forms. To do so we need the algebraic fact that ψ (see 4.2) is onto. In our exposition this is postponed until just before Theorem 5.7. We beg the reader's indulgence.

In summary, any pattern and any linking form may be realized by acting on a (simple) ribbon link with a simple boundary link. The following, in particular, justifies Theorem 3.16 of [CO1] which was there used for several computations.

Theorem 4.6. (see Theorem 3.5) Under the hypotheses of Theorem 3.5, the result of acting on (L, \mathcal{V}, b) by the boundary link (B, \mathcal{W}) has Blanchfield linking form equivalent to the sum of the linking form of (L, \mathcal{V}, b) and the image under f_* of the linking form of (B, \mathcal{W}) . Here we also assume that (L, \mathcal{V}) and (B, \mathcal{W}) are simple.

Theorem 4.7. (see Theorem 3.6) Given any pattern P, any $q \ge 1$ and any $\lambda \in L^{\epsilon}(\mathbb{Z}[F \langle x_1, \ldots, x_m \rangle], \Sigma)$, $\epsilon = (-1)^{q+1}$, (subject to the usual restriction if q = 2), there is a simple m-component homology boundary link (L, \mathcal{V}) in S^{2q+1} with pattern P and Blanchfield linking form equivalent to λ . This link is obtained by acting on a ribbon link with pattern P by a simple boundary link with linking form λ .

§5. Scheme Cobordism classes of homology boundary links.

Definition 5.1. Suppose (L, \mathcal{V}) and (L', \mathcal{V}') are *m*-component homology boundary links in S^{2q+1} which "have the same scheme" S in the sense that there exist basings b, b' of type (x_1, \ldots, x_m) (i.e., ordinary basings) inducing the scheme $S = (w_1, \ldots, w_m)$. Then we say that (L, \mathcal{V}) is scheme-cobordant to (L', \mathcal{V}') if in $S^{2q+1} \times [0, 1]$ there is a link concordance $g : \prod_{i=1}^{m} S^{2q-1} \times [0, 1] \hookrightarrow S^{2q+1} \times [0, 1]$ from L to L' and a set $I\mathcal{V} = \{IV_1, \ldots, IV_m\}$ of connected compact, oriented (2q + 1)-dimensional manifolds embedded in the exterior of the concordance such that $\partial(IV_i) = V_i \cup (-V'_i) \cup (\partial V_i \times [0, 1])$ for $i = 1, \ldots, m$ and such that the intersection of $I\mathcal{V}$ with a tubular neighborhood of the concordance is a product of its intersection with $\partial E(L)$ (or $\partial E(L')$) by [0, 1].

In the case that the scheme is (x_1, \ldots, x_m) (boundary links) this agrees with [Ko; §2]. This is clearly an equivalence relation, abbreviated $L \sim L'$. We have already sketched a proof that any ribbon homology boundary link with scheme S is scheme-cobordant to a trivial link with scheme S. It is known that any even-dimensional homology boundary link is scheme-cobordant to the trivial one [C3] [De2].

Proposition 5.2. If q > 1, the addition $(L, \mathcal{V}) \oplus (L', \mathcal{V}')$ of two m-component homology boundary links of reduced scheme S (S-links) given by the tangle sum using any basings b, b' of type (x_1, \ldots, x_m) which induce S, is a well-defined, commutative and associative operation on scheme-cobordism classes of S-links. Any ribbon homology boundary link with scheme S acts as identity.

Proof. Firstly, the tangle sum using a basing of type (x_1, \ldots, x_m) and *reduced* scheme is just the usual connected-sum along arcs which do not intersect \mathcal{V} as defined in [Ko; §2], together with the boundary-connected-sum along the same arcs to join up each sheet of the Seifert surfaces. The proof of [Ko; Prop. 2.11] works to show that \oplus is well-defined up to scheme-cobordism since the present situation is so clearly related. The commutativity and associativity are clear from the "connected-sum along arcs" definition. Any ribbon homology boundary link with scheme S will serve as identity. \Box

Theorem 5.3. (compare [De, 5.2 and 6.2]) Any homology boundary link (L, \mathcal{V}) with scheme S is scheme-cobordant to a simple homology boundary link with scheme S.

Proof of 5.3. The proof in [Ko; 2.8] generalizes to these generalized Seifert surfaces, but our Lemma 6.10 is needed to get the scheme-cobordism. \Box

Definition. The set of scheme-cobordism classes of homology boundary links (L, \mathcal{V}) in S^{2q+1} with scheme S will be denoted C(m, q, S) (or sometimes merely C(S) or C(q, S)). (We will shortly see that, if q > 1, this is an abelian group and will use the same symbol for the group).

Proposition 5.4. The cobordism class of the Seifert form $\theta : C(m, q, S) \longrightarrow G(m, (-1)^q)$ is a well-defined and, if q > 1, additive function sending the identity to the identity.

Proof. We have shown additivity in 3.2. The well-definedness is proved as in ([Ko]; see just prior to Theorem 3.4).

Theorem 5.5. If $\theta((L, \mathcal{V})) = 0$ then $(L, \mathcal{V}) \sim 0$.

After proving 5.5 we will get immediately that C is a group.

Corollary 5.6. If q > 1 and S is reduced, C(m, q, S) is a group and θ_S is an isomorphism. Thus the group of scheme-cobordism classes of homology boundary links with reduced scheme S is isomorphic to $G(m, (-1)^q)$.

Proof of 5.6. We showed θ surjective in 3.6. Define the inverse of L to be an element in the inverse image of $-\theta(L)$. Then $\theta(L \oplus -L) = \theta(L) \oplus -\theta(L) = 0$ so $L \oplus (-L) \sim 0$ by 5.5. Therefore C is a group. But θ has been shown to be additive, injective and surjective so it is an isomorphism. \Box

Proof of 5.5. It suffices to assume that (L, \mathcal{V}) is a simple *m*-component homology boundary link in S^{2q+1} where q > 1 and $\theta(\mathcal{V}) = 0$. We shall first show that (L, \mathcal{V}) is "S-slice", that is that the components of L bound disjoint 2q-dimensional disks $\Delta = \{\Delta_1, \ldots, \Delta_m\}$ in B^{2q+2} and there is a collection of (2q + 1)-manifolds \mathcal{W} embedded disjointly in the exterior of Δ such that

$$\partial W_i = V_i \cup (\partial W_i \cap N(\Delta))$$

and the intersection of $\partial \mathcal{W}$ with the boundary of a tubular neighborhood $(S^1 \times \Delta)$ of Δ is a product $(\mathcal{V} \cap (S^1 \times \{p\})) \times \Delta$ for $p \in \partial \Delta$. The desired result follows easily from this.

Suppose $\phi: E(L) \longrightarrow \bigvee_{i=1}^{m} S^1$ is induced by \mathcal{V} . Let $\mathcal{S}(L)$ be the result of stably-framed

surgery on the components of L. Thus $\mathcal{S}(L) = E(L) \bigcup_{\partial E(L)} \left(\prod_{i=1}^{m} D^{2q} \times S^{1} \right)$ and we can extend ϕ to $\mathcal{S}(L)$ by $\hat{\phi} \mid_{D^{2q} \times S^{1}} = \phi \mid_{p \times S^{1}}$ for $p \in \partial D^{2q}$. To show that (L, \mathcal{V}) is "S-slice" it suffices to show that the triple $(\mathcal{S}(L), \text{ stable framing, } \hat{\phi}_{*})$ is the boundary of $(Y^{2q+2},$ stable framing, $\psi_{*})$ where $\psi_{*} : \pi_{1}(Y) \longrightarrow F \langle x_{1}, \ldots, x_{m} \rangle$, $H_{*}(Y) \cong H_{*}(\natural S^{1} \times D^{2q+1})$ and $\pi_{1}(Y)$ is normally generated by the meridians of L (their images in $\pi_{1}(\mathcal{S}(L))$). For then $(Y, \partial Y)$ is transformed to (B^{2q+2}, S^{2q+1}) by attaching m 2-handles along the meridians for L. Since ϕ_* extends, the reverse of the Pontryagin construction applied to ψ yields the necessary W_i .

To produce Y, begin with B^{2q+2} and attach m 2q-handles h_1, \ldots, h_m along the components of L in such a way that the resulting (2q+2)-manifold Z is stably-parallelizable. Then $\partial Z \equiv S(L)$. Note $H_*(Z) \cong H_*(\natural_{i=1}^m D^2 \times S^{2q})$. It only remains to find disjointly embedded 2q-spheres representing a basis for $H_{2q}(Z)$ (which have trivial normal bundle since $q \neq 1$) and perform framed surgery on these, resulting in the desired Y. We also need to ensure that $\hat{\phi}_*$ extends to the exterior (in Z) of these 2q-spheres and $\pi_1(\partial Y) \longrightarrow \pi_1(Y)$ is a "normal surjection". Consider the Seifert surface V_i capped off along each of its boundary (2q-1)-spheres by copies of the 2q-disk which are parallels of cores of the handles $\{h_1, \ldots, h_m\}$. Then these capped-off manifolds, \hat{V}_i may be ambiently surgered along q-spheres to yield the desired 2q-spheres, exactly as in the injectivity part of the proof of Theorem 3.5 of [Ko]. This necessitates q > 1. Finally note that π_1 of the complement in B^{2q+2} of a set of Seifert surfaces pushed-in slightly is a free group on a set of "meridians" x_i to these surfaces. Since ϕ_* is onto, each of these is in the normal closure of the meridians of L. In fact we may take ψ_* to be what amounts to the identity map. Note that the ambient surgeries on q-spheres are of high codimension and irrelevant to π_1 . \Box

We may now summarize all of these relationships in Diagram 20. We assume q > 1. When q = 2, the index 2^m subgroups must be used as previously discussed.

$$C(m, q, (x_1, \dots, x_m))$$

$$T \qquad \theta \downarrow \qquad B$$

$$C(m, q, S) \xrightarrow{\theta_S} G(m, (-1)^q) \xrightarrow{\psi} L^{(-1)^{q+1}}(\mathbf{Z}[F], \Sigma)$$

 B_S

Diagram 20

Here $C(m, q, (x_1, \ldots, x_m))$ can be seen to be identical to Ko's group, $C_{2q-1}(B_m)$, of boundary cobordism classes of boundary links with chosen Seifert surface systems. Here $T(\alpha)$ is defined to be the result of a simple boundary link with Seifert form α acting on a ribbon homology boundary link with scheme S. Both θ and θ_S are isomorphisms by [Ko, Thm. 3.5] and by 5.6, so T is also an isomorphism. The map B is an isomorphism for $q \geq 3$ by Theorem 9.1 of [Du] and Theorem 2.7 of [Ko]. It follows that ψ and B_s are isomorphisms for $q \geq 3$. But since, if $q \neq 2$, the domain and range of ψ depend only on the parity of q, ψ is an isomorphism for q = 1. Since the index 2^m subgroup of G(m, +1)used when q = 2 is merely the subgroup of matrices A such that the blocks A_{ii} have signatures multiples of 16, and this is carried over naturally to the L-group, ψ is seen to be an isomorphism in all cases, with the understanding, when q = 2, that we restrict to the subgroup of L. It follows that B and B_s are isomorphisms onto this subgroup for q = 2. Thus all maps are isomorphisms if $q \neq 1$. When q = 1, C(m, q, S) and $C(m, q, (x_1, \ldots, x_m))$ are not groups but merely sets of equivalence classes, all maps are defined, ψ is a isomorphism and θ , θ_s , B, B_s are surjective.

Therefore we have

Theorem 5.7. (compare [Du; Thm. 9.1]) If q > 1, the group C(m, q, S) of schemecobordism classes of m-component homology boundary links (with surface systems of reduced scheme S) in S^{2q+1} is isomorphic to $L^{(-1)^{q+1}}(\mathbf{Z}[F], \Sigma)$ (when q = 2, replace L by the appropriate index 2^m subgroup). This isomorphism is given by the Blanchfield form associated to the free cover associated to the system of Seifert surfaces. Hence $C(m, q, S) \cong C(m, q, \{x_1, \ldots, x_m\})$ for all reduced schemes S.

Let C(m,q) stand for $C(m,q,\{x_1,\ldots,x_m\})$. Let \mathcal{F} stand for $\mathbf{Z}[F\langle x_1,\ldots,m\rangle] \xrightarrow{\epsilon} \mathbf{Z}$. Let $\Gamma_{2q+2}(\mathcal{F})$ stand for the homology-surgery group of Cappell and Shaneson [CS2] and $\widetilde{\Gamma}_{2q+2}(\mathcal{F})$ be its quotient by the image of $L_{2q+2}(F\langle x_1,\ldots,x_m\rangle) \xrightarrow{i} \Gamma_{2q+2}(\mathcal{F})$. Recall that Cappell, Shaneson and DuVal established the exact sequences below [Du; p. 633–634].

Therefore we can conclude the following using 5.6 and 5.7. The first exact sequence below was (essentially) obtained by DeMeo (unpublished) in [De; Thm. 7.2]. There he deals with a group analogous to the F_m -cobordism classes of Cappell and Shaneson but the equivalence to scheme-cobordism classes is not hard to deduce (see 6.10).

Theorem 5.8. If q > 2, there are exact sequences for any reduced scheme S

where B_S is the Witt class of the Blanchfield linking form associated to the free covering space dictated by the system of Seifert surfaces. Moreover $\phi_S = T \circ \phi$ (see 5.6) so that $\phi_S(\alpha)$ is obtained by allowing the boundary link $\phi(\alpha)$ to act on a ribbon homology boundary link of scheme S.

Proof of 5.8. Merely replace C(m,q) in the Cappell-Shaneson-DuVal sequence by C(m,q,S) using the isomorphism T of Diagram 20. \Box

§6. Classification of Homology Boundary Links Modulo Homology Boundary Link Concordance.

In this section we investigate the question of when two homology boundary links of pattern P are concordant respecting that pattern. In the proof of 6.3, we shall see that this is the same as asking that for *some* Seifert surface systems the links are scheme-cobordant, indicating that this is the proper analogue of boundary link cobordism of boundary links, and justifying the equivalent name of homology boundary link cobordism. This necessitates computing the effect on Seifert form of choosing *different* Seifert surface systems. This mirrors the analysis of Ko in the case of boundary links, but is much more complicated.

Definition 6.1. Two *P*-links (links of pattern *P*) *L* and *L'*, are *P*-cobordant, or patterncobordant or homology boundary link cobordant if there is a concordance *C* from *L* to *L'* and an epimorphism $g: \pi_1(E(G)) \to F$ such that $g \circ i: \pi_1(E(L)) \to F$ and $g \circ i': \pi_1(E(L'))$ are epimorphisms.

It follows that the "pattern" of the concordance is P. Let $\mathcal{P}(m, q, P)$ denote the set of P-cobordism classes of m-component homology boundary links in S^{2q+1} with pattern P. Suppose (w_1, \ldots, w_m) is in the equivalence class of the fixed pattern P.

Definition 6.2. Aut_{w_i} F is the subgroup of automorphisms of the free group F on m letters which send w_i to a conjugate of w_i for $1 \le i \le m$.

Theorem 6.3. For any fixed pattern P and any representative (w_1, \ldots, w_m) of P, there exists a bijection $\overline{\theta} : \mathcal{P}(m, q, P) \longrightarrow G(m, (-1)^q) / \operatorname{Aut}_{w_i} F$ where the action is defined as in 3.4 (and if q = 2 we mean the usual index 2^m subgroup of G). $\overline{\theta}(L)$ is defined by finding (for any scheme S compatible with (w_1, \ldots, w_m)) a system of Seifert surfaces \mathcal{V} for L which induces the scheme S (for some basing) and setting $\overline{\theta}(L) = \theta_S(\mathcal{V})$. Similarly the map given by the Blanchfield Form of a simple representative induces a bijection \overline{B} : $\mathcal{P}(m, q, P) \longrightarrow L^{(-1)^{q+1}}(\mathbf{Z}[F], \Sigma) / \operatorname{Aut}_{w_i} F$ where the action is as in 4.5.

A translation of 6.3 in terms of Γ -groups yields the following.

Theorem 6.4. Suppose q > 2. For any fixed pattern P and any representative (w_1, \ldots, w_m) of P, there are functions

$$\widetilde{\Gamma}_{2q+2}(\mathbf{Z}F \to \mathbf{Z}) / \operatorname{Aut}_{w_i} F \xrightarrow{\phi_S} \mathcal{P}(m,q,P) \xrightarrow{\pi} L_{2q+1}(F)$$

such that π is surjective and ϕ_S is an injection with image $\pi^{-1}(0)$. Here $\widetilde{\Gamma}$ is a gamma group modulo the image of L_{2q+2} .

Most of the rest of this chapter will be devoted to proving 6.3. We should note that this answer surprised us. We had thought that the answer would be $G/\operatorname{Aut}_0 F$ where $\operatorname{Aut}_0 F$ are those automorphisms inducing the identity on homology. This is tempting to conclude given the work of Cappell and Shaneson and the following propositions.

Definition 6.5. A splitting map for the *m*-component homology boundary link (with basepoint) (L, *) is an epimorphism $\phi : \pi_1(E(L), *) \longrightarrow F$, where *F* is free of rank *m*, such that, for some meridional map $\mu : F \longrightarrow \pi_1(E(L), *)$ ($\mu(x_i)$ is an *i*th meridian), $\phi \circ \mu$ induces the identity map on abelianizations. Clearly ϕ is a splitting map with respect to some μ if and only if ϕ is a splitting map with respect to each possible μ .

Let $\operatorname{Aut}_0(F)$ be the group of automorphisms of F which induce the identity on abelianization.

Proposition 6.6. If ϕ is a splitting map for (L, *) then for any $\psi \in \operatorname{Aut}_0(F)$, $\psi \circ \phi$ is a splitting map for (L, *). If ϕ and ϕ' are splitting maps for (L, *) then there exists $\psi \in \operatorname{Aut}_0(F)$ such that $\phi' = \psi \circ \phi$.

Proof of 6.6. The first claim is obvious. For the second claim, let $G = \pi_1(E(L), *)$ and let π be the quotient map $G \longrightarrow G/G_{\omega}$. By our remark above, ϕ and ϕ' are splitting maps with respect to some μ . By Stallings' theorem, ϕ and ϕ' induce isomorphisms ϕ_{ω} , ϕ'_{ω} from G/G_{ω} to F such that $\phi = \phi_{\omega} \circ \pi$, $\phi' = \phi'_{\omega} \circ \pi$. Setting $\psi = \phi'_{\omega} \circ \phi_{\omega}^{-1}$ we see that $\phi' = \psi \circ \phi$. Moreover, upon abelianization, $\psi(x_i) \equiv \psi(\phi\mu(x_i)) \equiv \phi'_{\omega} \circ \phi_{\omega}^{-1} \circ \phi \circ \mu(x_i) \equiv$ $\phi'_{\omega} \circ \pi \circ \mu(x_i) \equiv \phi'(\mu(x_i)) \equiv x_i$. \Box

Recall that a scheme S of basing type (x_1, \ldots, x_m) has pattern P if the circles $\{\partial \Delta_i \mid i = 1, \ldots, m\} = \{\gamma_i \mid i = 1, \ldots, m\}$ trace out words (w_1, \ldots, w_m) in $F \times \ldots \times F$ which has pattern P. Recall that if (w_1, \ldots, w_m) and (w'_1, \ldots, w'_m) have pattern P then $w'_i = \psi(g_i w_i g_i^{-1})$ for some $\psi \in \operatorname{Aut}_0(F)$ and some $g_i \in F$. If (L, \mathcal{V}) is an homology boundary link with pattern P, and (r_1, \ldots, r_m) is an m-tuple of words of F such that $f(x_i) = r_i$ defines an element of $\operatorname{Aut}_0(F)$, then we can define a new system (L, \mathcal{V}'') , denoted $f^{\#}(L, \mathcal{V})$, as follows. Assume $r_i = x_{i_1}^{\epsilon_{i_1}} \ldots x_{i_k}^{\epsilon_{i_k}}$. Merely replace each V_i by a disjoint union $\prod_{j=1}^k \epsilon_{i_j} V_i$ of parallel copies of V_i , with orientation varying according to $\epsilon_{i_j} = \pm 1$, the relabelling the n^{th} copy with the letter i_n . Set V'_i equal to the union of the components labelled with j. Since each V_i was connected, $S^{2q+1} - \mathcal{V}'$ is connected. Thus we can tube components of V'_i together to form V''_i which is connected. This (L, \mathcal{V}'') is the desired surface system. This is very similar to the $f^{\#}L$ defined in §3 except that here we are not changing the link L, merely making the Seifert surfaces more complicated. Note, however that $f^{\#}L$ depends on more than f and as such involves arbitrary choices. The following is then immediate from the definitions.

Proposition 6.7. Suppose r_i are words such that the endomorphism defined by $f(x_i) = r_i$ lies in Aut₀(F). If (L, \mathcal{V}) induces the splitting map ϕ (by the Pontryagin construction), then $f^{\#}(L, \mathcal{V})$ induces the splitting map $f \circ \phi$.

Given 6.6 and 6.7 it is very tempting to think that θ gives, somehow, a well-defined bijection from $\mathcal{P}(m,q,P)$ to $G(m,(-1)^q)/\operatorname{Aut}_0 F$. It does not. To correctly analyze the situation, it is helpful to introduce an intermediate, more algebraic, notion.

Given (w_1, \ldots, w_m) which represents a pattern P, consider pairs (L, ϕ) where L is an homology boundary link and $\phi : \pi_1(EL) \twoheadrightarrow F$ is an epimorphism such that, for some meridians μ_i , $\phi(\mu_i) = w_i$. Given two such (L_0, ϕ_0) , (L_1, ϕ_1) we say $(L, \phi_0) \sim (L_1, \phi_1)$ if there is a concordance C from L_0 to L_1 and an epimorphism $\psi : \pi_1(EC) \twoheadrightarrow F$ which restricts on $\pi_1(EL_0)$ to ϕ_0 and on $\pi_1(EL_1)$ to ϕ_1 (after an inner automorphism of $\pi_1(EL_1)$ to change the basepoint of $\pi_1(EC)$). Let $\mathcal{H}(m, q, w_i)$ or simply $\mathcal{H}(w_i)$ represent the set of equivalence classes. This set was defined by Cappell and Shaneson in the case $w_i = x_i$ and by De Meo in general [De].

Now fix a pattern P. The group $\operatorname{Aut}_0 F$ acts on the disjoint union $\amalg H(w_i)$ (taken over all (w_1, \ldots, w_m) which are in the equivalence class of the pattern P) as follows: $f^{\#}([L, \phi)] = [(L, f \circ \phi)].$

Lemma 6.8. The forgetful map $F : \frac{\amalg H(m,q,w_i)}{\operatorname{Aut}_0 F} \longrightarrow \mathcal{P}(m,q,P)$ is a bijection. Here the disjoint union is over the set of all m-tuples (w_i) in the equivalence class of P.

Proof of 6.8. First note that F is well-defined on $H(w_i)$ since the equivalence relation \sim is stronger than P-cobordism. It is independent of the action of $\operatorname{Aut}_0 F$ since the action changes only the splitting map, not the link.

Since F is obviously surjective, we need only show injectivity. Suppose $(L, \phi) \in H(w_i)$, $(L', \phi') \in H(w'_i)$ and suppose that L is homology boundary link concordant to L'. Thus there exists a concordance C from L to L' and an epimorphism $g : \pi_1(EC) \twoheadrightarrow F$ such that $\psi = g \circ i$ is a splitting map for L and $\psi' = g \circ \kappa \circ i'$ is a splitting map for L' (here κ is an automorphism of $\pi_1(EC)$ to change basepoints). Since $(L, \phi) \in H(w_i)$, there exist meridians $\mu_i \in \pi_1(EL)$ such that $\phi(\mu_i) = w_i$.

By 6.6 there exist elements f, f' of Aut₀ F such that $f^{\#}(L, \phi) = (L, f \circ \phi) = (L, \psi)$ and $(f')^{\#}(L', \phi') = (L, \psi')$. Therefore it suffices to show that $(L, \psi) \sim (L, \psi')$ in $H(f(w_i))$. Note that $\psi(\mu_i) = f(w_i)$ so indeed $(L, \psi) \in H(f(w_i))$. Now choose meridians $\mu'_i \in \pi_1(EL')$ such that $\kappa \circ i'(\mu'_i) = i(\mu_i)$. Then $\psi'(\mu'_i) = g \circ \kappa \circ i'(\mu'_i) = g \circ i(\mu_i) = \psi(\mu_i) = f(w_i)$ so $(L', \psi') \in H(w_i)$, and the concordance (C, g) shows $[(L, \psi)] = [(L', \psi')]$. \Box

Note that if $[(L, \phi)] \in H(w_i)$ and $f \in \operatorname{Aut}_{w_i} F$, $f^{\#}(L, \phi)$ is still in $H(w_i)$. For, if $f(w_i) = \eta_i w_i \eta_i^{-1}$, then choose ξ_i such that $f \circ \phi(\xi_i) = \eta_i^{-1}$ and observe that $f \circ \phi(\xi_i \mu_i \xi_i^{-1}) = w_i$.

Lemma 6.9. For any m-tuple (w_1, \ldots, w_m) inducing the pattern P, the inclusion map $\frac{H(w_i)}{\operatorname{Aut}_{w_i} F} \xrightarrow{i} \frac{\amalg H(w_i)}{\operatorname{Aut}_0(F)}$ is a bijection.

Proof of 6.9. First we show surjectivity. Suppose $(L, \phi) \in H(w'_i)$. Since (w'_i) is in the same pattern P as (w_i) , $w_i = f(\eta_i w'_i \eta_i^{-1})$ for some $f \in \operatorname{Aut}_0 F$. Choose meridians μ_i such that $\phi(\mu_i) = w'_i$. Consider $f^{\#}(L, \phi) = (L, f \circ \phi)$. Choose ξ_i such that $\phi(\xi_i) = \eta_i$. Then $f \circ \phi(\xi_i \mu_i \xi_i^{-1}) = f(\eta_i w'_i \eta_i^{-1}) = w_i$, showing that $(L, f \circ \phi) \in H(w_i)$. Thus i is onto.

Now suppose (L_0, ϕ_0) and $(L_1, \phi_1) \in H(w_i)$ and $i((L_0, \phi_0)) = i((L_1, \phi_1))$. It follows that there is a $g \in \text{Aut}_0 F$ such that $g^{\#}[(L_1, \phi_1)] = [(L_0, \phi_0)]$ in $H(w_i)$. In particular this implies $(L_1, g \circ \phi_1)$ lies in $H(w_i)$!! This places strong restrictions on g since (L_1, ϕ_1) also lies in $H(w_i)$. Suppose μ_i are meridians such that $\phi_1(\mu_i) = w_i$. Then there must be meridians $\eta_i \mu_i \eta_i^{-1}$ such that $g \circ \phi_1(\eta_i \mu_i \eta_i^{-1}) = w_i$. But this immediately implies $g(w_i)$ is conjugate to w_i . Therefore $g \in \operatorname{Aut}_{w_i}(F)$ and (L_0, ϕ_0) is equal to (L_1, ϕ_1) in the domain of our map *i*, concluding our proof that *i* is injective. \Box

Lemma 6.10. Suppose $S = (w_1, \ldots, w_m)$ is a scheme. The Pontryagin construction yields a bijection $p : C(m, q, S) \longrightarrow H(m, q, w_i)$. Therefore $H(m, q, w_i)$ is naturally a group if q > 1.

Proof of 6.10. First we show p is well-defined. Suppose (L, \mathcal{V}) and (L', \mathcal{V}') are S-links for which the Pontryagin construction using basings b, b' (see 5.1) yields splitting maps ϕ and ϕ' respectively where b and b' induce the scheme S. If (L', \mathcal{V}') is scheme-cobordant to (L, \mathcal{V}) via C and $I\mathcal{V}$, then we can show that $(L, \phi) \sim (L', \phi')$ in $H(w_i)$ by using the basepoint of b and applying the Pontryagin construction to $I\mathcal{V}$ to yield a homomorphism $\psi : \pi_1(EC, b_*) \longrightarrow F$ such that $\psi \circ i = \phi$ and $\psi \circ \kappa \circ i' = \phi'$ where κ is a change of basepoint from b_* to b'_* . Thus p is well-defined.

The map p is onto by the techniques of the proof of Theorem 3.6, which shows that given any link (L, \mathcal{V}) and splitting map g such that $g_*(\mu_i) = [w_i]$ and $S = (w_1, \ldots, w_m)$ is any scheme, \mathcal{V} can be modified, *preserving* g_* , until \mathcal{V} induces S precisely.

Now suppose $p((L, \mathcal{V})) = p((L', \mathcal{V}'))$. Then there exists a concordance C and an epimorphism $\psi : \pi_1(EC) \twoheadrightarrow F$ such that $\psi \circ i_C = \phi$ and $\psi \circ \kappa \circ i' = \phi'$ as usual. Let f, f' be the maps from EL, EL' respectively to $\bigvee_{i=1}^m S^1$ induced by \mathcal{V} , \mathcal{V}' as above where $f_* = \phi$, $(f')_* = \phi'$. Under an identification $\partial_+ EC \equiv \partial N(L) \times [0,1] = \partial N(L') \times [0,1]$, we can extend f and f' to $F : \partial EC \longrightarrow \bigvee_{i=1}^m S^1$ by letting $F = f \circ p_1$ ($p_1 =$ projection onto 1st factor) on $\partial N(L) \times [0,1]$. This is possible because f and f' induce the same scheme. Notice that F_* necessarily agrees with ψ , and F extends over E(C) since F_* is extended by ψ . After a small perturbation, the inverse of the Pontryagin construction then produces the "Seifert surfaces" $I\mathcal{V}$ which exhibit that (L, \mathcal{V}) is S-cobordant to (L', \mathcal{V}') . Hence p is injective.

To see that $H(m, q, w_i)$ is a group, note that clearly H depends only on the image of w_i in F. Thus we can choose a *reduced* scheme S compatible with w_i , and apply 5.6. \Box

Corollary 6.11. If $S = (w_1, \ldots, w_m)$ then there is an action of $\operatorname{Aut}_{w_i} F$ on the set C(m, q, S) of scheme cobordism classes of S-links, with respect to which the bijection p of 6.10 is equivariant.

Proof of 6.11. Given $f \in \operatorname{Aut}_{w_i} F$ simply define $f^{\#}[(L, \mathcal{V})]$ to be $p^{-1}(f^{\#}(p([L, \mathcal{V}])))$. It is then also clear that the geometric description of $f^{\#}([L, \mathcal{V}])$ in terms of copies of the Seifert surfaces (see above 6.7) realizes this action and hence that the geometric description of the action is independent, up to scheme cobordism, of the choices involved. \Box **Lemma 6.12.** Suppose (w_1, \ldots, w_m) represents the reduced scheme S. The isomorphism given by taking the Seifert form, $\theta_S : C(m, q, S) \longrightarrow G(m, (-1)^q)$, is equivariant with respect to the actions of $\operatorname{Aut}_{w_i} F$ defined in 6.11 and 3.4 respectively.

Proof of 6.12. By Theorem 3.6 and Corollary 5.6, we may assume that an arbitrary scheme cobordism class (L, \mathcal{V}_L) takes the form of a boundary link (B, \mathcal{V}_B) with Seifert form α acting on a ribbon homology boundary link (R, \mathcal{V}, b) for which the loops $\{\partial \Delta_1, \ldots, \partial \Delta_m\}$ intersect \mathcal{V} in words which reduce to $\{x_1, \ldots, x_m\}$ in the free group. Of course $\theta_S(L, \mathcal{V}_L) = \alpha$. Now consider acting on (L, \mathcal{V}_L) by $f \in \operatorname{Aut}_{w_i} F$ such that $f(x_i) = r_i$. By 6.11, we may use the geometric definition of $f^{\#}(L, \mathcal{V})$ as described above 6.7. But changing the Seifert surface system of L does not change the fact that it is obtained as the boundary link (B, \mathcal{V}_B) acting on ribbon link because L itself is unchanged by $f^{\#}$. However now (B, \mathcal{V}_B) is acting on (R, \mathcal{V}', b) and the loops $\{\partial \Delta_1, \ldots, \partial \Delta_m\}$ now intersect \mathcal{V}' in words which reduce to $\{f_*(x_1), \ldots, f_*(x_m)\} = \{r_1, \ldots, r_m\}$ in the free group. Thus $\theta_S(f^{\#}(L, \mathcal{V}_L)) = \theta(R, \mathcal{V}') \oplus f_*(\alpha)$ by 3.5. Since (R, \mathcal{V}') is clearly still a ribbon homology boundary link, $\theta(R, \mathcal{V}') = 0$. Hence $\theta_S(f^{\#}(L, \mathcal{V}_L)) = f_*\theta_S(L, \mathcal{V})$ as desired. \Box

We have now completed the proof of 6.3. Given any pattern P and any representative (w_1, \ldots, w_m) of P, we may combine 6.8–6.11 to show that the forgetful map from $C(m, q, S) / \operatorname{Aut}_{w_i} F$ to $\mathcal{P}(m, q, P)$ is a bijection. Lemma 6.12 then completes the argument. Theorem 6.4 then follows formally from 6.3 and the functoriality of the Γ -groups and L-groups. \Box

Corollary 6.13. Two homology boundary links L, L' are homology boundary link cobordant if and only if there exist Seifert surface systems such that (L, \mathcal{V}) and (L', \mathcal{V}') are scheme-cobordant (see 5.1).

§7. Z_p -Homology Boundary Links in Z_p -Homology Spheres.

Suppose S is a closed, oriented (2q + 1)-manifold which has the \mathbf{Z}_p -homology of S^{2q+1} (let $J = \mathbf{Z}_{(p)}$, the integers localized at p). Suppose $L = \{K_1, \ldots, K_m\}$ is an ordered, oriented, embedded collection of (2q - 1)-spheres in S (whose longitudes are torsion in $H_1(E(L))$ if q = 1). Then we call (L, S) a link in a $\mathbf{Z}_{(p)}$ -homology sphere. If L admits a system $\mathcal{V} = \{V_1, \ldots, V_m\}$ of "Seifert surfaces" where ∂V_i is homologous to the i^{th} longitude in $H_{2q-1}(\partial E(L); \mathbf{Z}_p)$ then we call L a \mathbf{Z}_p -homology boundary link (see [H1]). We restrict to such L with (q-1)-connected "Seifert surfaces" as before and continue to use the term simple. The Pontryagin construction associates to (L, \mathcal{V}) a map $E(L) \longrightarrow \bigvee_{i=1}^m S^1$ as before and hence a free covering space \tilde{X} . Then $H_q(\tilde{X}; \mathbf{Z}_{(p)})$ is an A-module $(A = \mathbf{Z}_{(p)}[F])$ and we may define on it a Blanchfield form, as in §4, taking values in Λ/A where Λ is the Cohn localization of $A \stackrel{\epsilon}{\longrightarrow} \mathbf{Z}_{(p)}$. The specific analysis using the Mayer-Vietoris sequence

also holds to show that this Blanchfield form is determined by a "Seifert matrix". Here, to

 $i^+\alpha_{ij}$, that is the matrix of i^+ with respect to the dual bases $\{\alpha_{ij}\}$, $\{\hat{\alpha}_{ij}\}$ for $H_q(\mathcal{V}; \mathbf{Z}_{(p)})$ and $H_q(\mathcal{E}(L) - \mathcal{V}; \mathbf{Z}_{(p)})$. Observing that $i^+\alpha - i^-\alpha = \pm \Sigma(\alpha \cdot \alpha_{ij})\hat{\alpha}_{ij}$ where the latter is the intersection form on $H_q(\mathcal{V}; \mathbf{Z}_{(p)})$, one sees that the matrix of i^+ is θ and the matrix of i^- is $\theta \pm \mathcal{I}$ where \mathcal{I} is the intersection matrix on $H_q(\mathcal{V}; \mathbf{Z}_{(p)})$ with respect to $\{\alpha_{ij}\}$ (we do not stop here to get the sign correct). Then the map d is represented by $\Delta = \Gamma \theta \pm \mathcal{I} - \theta$. Note that Δ is invertible when augmented since \mathcal{I} is invertible. Hence the entire proof of Theorem 4.2 goes through using the matrix $(I - \Gamma)(\Gamma \theta + \mathcal{I} - \theta)^{-1}$.

Recall that new invariants of links were introduced in [CO1], [CO2] to show that not all links are concordant to boundary links. The initial step of the definition of those invariants entailed associated to the link $L = \{K_1, \ldots, K_m\}$, a covering link $\tilde{L} = \{\tilde{K}_1, \tilde{K}_{21}, \tilde{K}_{22}, \ldots, \tilde{K}_{2p}, \tilde{K}_{31}, \ldots, \tilde{K}_{m1}, \ldots, \tilde{K}_{mp}\}$ consisting of the lifts of the components of L in a p-fold cyclic cover of S^{2q+1} branched over K_1 (p prime). In case L were a simple homology boundary link with surface system \mathcal{V}, \tilde{L} would be a simple \mathbb{Z}_p -homology boundary link in the $\mathbb{Z}_{(p)}$ -homology sphere S. Then we have the Blanchfield form $B = B(L, \mathcal{V})$ in $L^{\epsilon}(\mathbb{Z}[F \langle x_1, \ldots, x_m \rangle], \Sigma)$ and the Blanchfield form $\tilde{B} = B(\tilde{L}, \tilde{\mathcal{V}})$ in $L^{\epsilon}(\mathbb{Z}_{(p)}[F'], \Sigma')$ where F' is free on 1 + (m-1)p letters. One might then define a \mathbb{Z}_p scheme-cobordism relation on the set of \mathbb{Z}_p -homology boundary links in $\mathbb{Z}_{(p)}$ -homology spheres and see that the operation of forming covering links of the type above carries scheme-cobordism classes to \mathbb{Z}_p scheme cobordism classes. Therefore one expects a functorial relationship between B and \tilde{B} . In fact, since every element of $L^{\epsilon}(\mathbb{Z}F, \Sigma)$ is represented by a simple boundary link, one can geometrically define a *transfer*.

Proposition 7.1. If $\phi : F \langle x_1, \ldots, x_m \rangle \longrightarrow \mathbf{Z}_p$ sends x_1 to 1 and x_i to 0 if i > 1, there is a transfer homomorphism $\operatorname{tr} : L^{\epsilon}(\mathbf{Z}F, \Sigma) \longrightarrow L^{\epsilon}(\mathbf{Z}_{(p)}F', \Sigma')$ where F' is ker ϕ . Moreover, for any simple homology boundary link (L, \mathcal{V}) , and covering link $(\widetilde{L}, \widetilde{\mathcal{V}})$ defined by p-fold branched cover (branching over K_1), $B(\widetilde{L}, \widetilde{\mathcal{V}}) = \operatorname{tr}(B(L, \mathcal{V}))$.

Proof. One way to show this is to note that the free (F) covering space \tilde{X} of X = E(L) associated to \mathcal{V} has precisely the same underlying space as the free (F') covering space of $E(\tilde{L})$ associated to $\tilde{\mathcal{V}}$. Therefore the module on which $B(\tilde{L}, \tilde{\mathcal{V}})$ is defined is merely $H_q(\tilde{X}) \otimes \mathbf{Z}_{(p)}$ considered as a module over $\mathbf{Z}_{(p)}F'$ via $\phi : \mathbf{Z}_{(p)}F' \hookrightarrow \mathbf{Z}_{(p)}F$. The pairing itself therefore admits a purely algebraic definition (which we shall not give here) in terms of $B(L, \mathcal{V})$.

Another way is to define transfer using boundary links, then establish its independence of pattern. Use 4.7 and 5.6 to replace (L, \mathcal{V}) , up to scheme-cobordism, by (L', \mathcal{V}') , the action on a ribbon homology boundary link (R, \mathcal{W}) with identical scheme, by a boundary link (L'', \mathcal{V}'') with $B(L'', \mathcal{V}'') = B(L, \mathcal{V})$. Since L is scheme-cobordant to L', \tilde{L} will be \mathbf{Z}_p scheme-cobordant to \tilde{L}' and hence $B(\tilde{L}) = B(\tilde{L}')$ (neither fact have we proved herein but appeal by analogy to the integral case). Moreover we now argue that the covering link of (L'' acting on R) is the same as the action of the covering link of L'' acting on the covering link of R. This is done by observing that the punctured 2-disk Δ used to decompose L' into two angles will lift to a punctured 2-disk and decompose the covering link. Upon re-doing our additivity theorem, one calculates that $B(\widetilde{L}) = B(\widetilde{R}) \oplus B(\widetilde{L''})$. Since R is scheme-cobordant to 0, $B(\widetilde{R}) = 0$. Finally $B(\widetilde{L''}) = \operatorname{tr}(B(L''))$ by definition of the transfer on boundary links. Thus $B(\widetilde{L}) = \operatorname{tr}(B(L))$ as desired. \Box

Proposition 6.1 was used in [CO2; §3] to calculate our invariants associated to covering links. The invariants there were images of $B(\tilde{L}, \tilde{\mathcal{V}})$ in $L^{\epsilon}(\mathbf{Z}_{(p)}F'_{\text{abelian}}; \Sigma')$, that is, ordinary Blanchfield forms associated to the universal *abelian* covering space of $E(\tilde{L})$ (in fact to successfully compute we always reduced to a \mathbf{Z} covering space, which invariants correspond to the image of $B(\tilde{L}, \tilde{\mathcal{V}})$ in $W^*(\mathbf{Z}_{(p)}[t, t^{-1}]$; determinant = 1).)

References

- [B] R.C. Blanchfield, Intersection Theory of Manifolds with Operators with Applications to Knot Theory, Ann. of Math. 65 (1957), 340–56.
- [CS1] S. Cappell and J.L. Shaneson, *Link cobordism*, Comment. Math. Helv. 55 (1980), 20–49.
- [CS2] _____, The codimension two placement problem and homology equivalent mannifolds, Annals of Math. 99 (1974), 277–348.
- [C1] T.D. Cochran, Derivatives of Links: Milnor's concordance invariants and Massey's products, #427, Memoirs of Amer. Math. Soc., Volume 84, March 1990, Providence, Rhode Island.
- [C2] _____, Link concordance invariants and homotopy theory, Inventiones Math. 90 (1987), 635–645.
- [C3] _____, Slice links in S^4 , Transactions of American Mathematical Society **285** (1984), 389–401.
- [CL] T.D. Cochran and J. Levine, Homology boundary links and the Andrews-Curtis conjecture, Topology 29 (1990), no. 2, 231–239.
- [CO1] T.D. Cochran and K. Orr, Not all links are concordant to boundary links, Bull. Amer. Math. Soc. 23 (July, 1990), no. 1, 99–106.
- [CO2] _____, Not all links are concordant to boundary links, Annals of Math. 138 (1993), 519–554.
- [CO3] _____, Homology Surgery groups and Blanchfield linking forms, in preparation.
- [De] R. DeMeo, Cobordisms of non-boundary links, Ph.D. dissertation, Princeton University, 1980.
- [De2] _____, Link Cobordism, Duke Math. 48 (1981), 23–33.
- [Du] J. DuVal, Form de Blanchfield et cobordisme d'entrelacs bords, Comment. Math. Helv. **61** (1986), 617–635.
- [FV] M. Farber and P. Vogel, in preparation.
- [GH] M.J. Greenberg and J.R. Harper, <u>Algebraic Topology: A first course</u>, revised, Addison-Wesley, Redwood City, California, 1981.
- [H1] J. Hillman, Alexander ideals of links, Springer-Verlag Lecture Notes in Math. 895 Springer-Verlag, Berlin, 1981.
- [H2] _____, Longitudes of a link and principality of an Alexander Ideal, Proc. Amer. Math. Soc. 72 (1978), 370–374.
- [Ka] U. Kaiser, *Homology boundary links and fusion constructions*, preprint, Univ. of Siegen, Germany.
- [Ko] K. Ko, Seifert matrices and boundary link cobordisms, Transactions Amer. Math. Soc. 299 (1987), 657–681.
- [LD] J. Le Dimet, Cobordisme d'Enlacement de Disques, supplement to Bull. Math. France, new series, Memoir 32, 116 (1988).
- [L1] J.P. Levine, *Link invariants via the Eta invariant*, preprint 1992, Brandeis University, Waltham, Massachusetts.
- [L2] _____, Knot modules I, Trans. Amer. Math. Soc. **229** (1977), 1–50.
- [L3] _____, Knot cobordism groups in codimension 2, Comment. Math. Helv. 44 (1969), 229–244.
- [L4] _____, Link concordance and algebraic closure of groups II, Inventiones Math. **96** (1989), 571–592.

- [LMO] J. Levine, W. Mio and K. Orr, Links with vanishing homotopy invariants, Commun. Pure Applied Math. 46 (1993), no. 2, 213–220.
- [Mi] W. Mio, On boundary link cobordism, Math. Proc. Cambridge Phil. Soc. 101 (1987), 259–266.
- [R] A. Ranicki, Exact sequences in the algebraic theory of surgery, Princeton Univ. Press, vol. 26, 1981.
- [Sm] J. Smith, Acyclic Localizations, Journal of Pure and Appl. Algebra 12 (1978), 117–127.
- [S] N. Smythe, Boundary links, Annals of Mathematics Studies 60, ed. R.H. Bing, Princeton University Press, Princeton, N.J. (1966), 69–72.
- [St] J. Stallings, Homology and central series of groups, J. of Algebra 2 (1965), 170–181.
- [V1] P. Vogel, On the Surgery Obstruction Group in Homology Surgery, Inst. Hautes Etudes Sci. Publi. Math 55 (1982), 165–206.
- [V2] _____, Localization non commutative de formes quadratiques, Algebraic K-theory, Proc. Oberwolfach 1980, part II, Lecture Notes in Math 967, Springer-Verlag, Berlin, 376–389.
- [W] C.T.C. Wall, Surgery on Compact Manifolds, Academic Press, London and New York, 1970.
- [Wh] G. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, 1978.

Tim D. Cochran Mathematics Department Rice University P.O.Box 1892 Houston, Texas 77251–1892 Kent E. Orr Mathematics Department Indiana University Bloomington, Indiana 47405