Equivariant Cobordism and Duality

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EQUIVARIANT COBORDISM AND DUALITY([1])

BY

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ABSTRACT. We consider equivariant cobordism theory, defined by means of an equivariant Thom spectrum; in particular, we investigate the relationship between this theory and the more geometric equivariant bordism theory, showing that there is a Poincaré-Lefschetz duality theorem which is valid in this setting.

This paper is an attempt to present further evidence for the proposition that the stable equivariant bordism theories introduced in [2] are the "correct" equivariant bordism theories, in the sense that stabilization enables one to prove many desirable theorems which are probably false prior to stabilization. As examples of the kind of result we have in mind, we may take all theorems about bordism whose proofs rely on transversality arguments.

In [2], we were concerned with the equivariant analogue of the Pontrjagin-Thom isomorphism theorem; as our present test case, we ask whether there is a Poincaré-Lefschetz duality theorem which is valid in this framework, eventually finding that there is such a result. (The author is indebted to Professor R. E. Stong for the crucial observation that "suspension introduces as much transverse-regularity as one needs", as well as many other helpful comments.)

We begin, after recalling the major results in [2], by outlining the definition of equivariant cobordism theory as the cohomology theory with coefficients in the equivariant Thom spectrum MO^G and then noting the existence of cup and cap products of the usual sort. These considerations enable us, in §2, to define Thom homomorphisms in equivariant cobordism which we then show to be isomorphisms, deducing as a corollary the existence of a Gysin sequence. These Thom isomorphisms also play a vital role in §3, where they occur in our initial definition of the first of the duality isomorphisms. Finally, we show that this isomorphism is actually given by cap product with the stable bordism fundamental class, from which we are able to deduce painlessly the existence of the other duality isomorphism.

1. **Preliminaries.** Throughout, we shall use the definitions and results of [2], so a brief summary of that paper seems in order. We present such a summary

in the next few paragraphs and then proceed to the definition of equivariant cobor-
dism, followed by some elementary remarks concerning this theory.

Let G be a compact Lie group, and let $\mathbf{R}^{\infty}(G)$ be the direct sum of countably
many copies of each of the irreducible finite-dimensional orthogonal representations
of G. There is an obvious action of G on $\mathbf{R}^{\infty}(G)$ which induces an action of G
on $BO_n(\mathbf{R}^{\infty}(G))$, where in general $BO_n(W)$ is the space of all n-dimensional sub-
spaces of the representation W. There is also the "tautological" n-plane bundle
γ^n over $BO_n(\mathbf{R}^{\infty}(G))$ which we may view as a G-vector bundle; it is, in fact, the
universal equivariant n-plane bundle. Finally, since G's action on γ^n is fiber-
wise orthogonal, we receive an action of G on the Thom space $MO_n^G = D\gamma^n/S\gamma^n$,
fixing the obvious basepoint.

Now let V be any finite-dimensional orthogonal representation, let $D(V)$ be
the unit disk in V, $S(V)$ the unit sphere and define $\Sigma(V) = D(V)/S(V)$, provided
with the evident action of G. By appealing to the universal property of the bun-
dles γ^n, we obtain G-maps

$$m_{n,|V|}: \quad \Sigma(V) \wedge MO_n^G \to MO_{n+|V|}^G$$

where $|V|$ denotes the dimension of V. The spaces MO_n^G $(n \geq 0)$, together with
all of these "binding" maps, constitute the (orthogonal) Thom spectrum for the
group G, $\mathbf{MO}^G$. Once provided with $\mathbf{MO}^G$, we may define a homology theory with
coefficients in $\mathbf{MO}^G$ by imitating the usual construction. Briefly, being given a
pair (X, A) of G-spaces, and an integer k, we consider the sets of basepoint-
preserving homotopy classes

$$[\Sigma(V), (X/A) \wedge MO_{|V|-k}^G]$$

where V is a G-representation; these sets form a directed system over the di-
rected set $M(G)$ consisting of the isomorphism classes of finite-dimensional G-
representations, the maps of the system being defined using suspension and the
maps $m_{r,s}$. We define $H_k(X, A; \mathbf{MO}^G)$ to be the direct limit of this system; in
[2], it was shown that these groups determine an equivariant homology theory in
the sense there defined. This theory is homotopy-theoretic equivariant bordism.

There is also a geometric version of equivariant bordism, the details of whose
construction may be found in [3]. This is an equivariant homology theory in the
sense of Bredon [1] but not in the sense of [2], in view of the paucity of suspen-
sion isomorphisms in this theory. There are, however, natural suspension homo-
morphisms and, in [2], we detailed the procedure by which one can use these
homomorphisms to "stabilize" equivariant bordism. In particular, we showed
the existence of an equivariant homology theory $\mathfrak{N}_*^{G:S}(\ )$ which is the stabilization

of $\mathfrak{N}_*^G(\ )$, the bordism theory of "singular G-manifolds" in a space with no restrictions on the isotropy subgroups.

Proceeding by analogy with the development of ordinary bordism theory, we next proved the existence of a Pontrjagin-Thom construction

$$\Phi^S: \mathfrak{N}_*^{G:S}(\ ) \to H_*(\ ; \mathbf{MO}^G);$$

this is a natural transformation of equivariant homology theories, preserving the $\mathfrak{N}_*$-module structures which are present. (Here, as elsewhere, $\mathfrak{N}_*$ denotes Thom's unoriented cobordism ring.) If we define a pair (X, A) of G-spaces to be admissible if X is Hausdorff and $A \hookrightarrow X$ is an equivariant cofibration, then the major result of [2] may be stated as follows:

Theorem 1.1. *For any admissible pair (X, A), the homomorphism*

$$\Phi^S_{(X,A)}: \mathfrak{N}_*^{G:S}(X, A) \to H_*(X, A; \mathbf{MO}^G)$$

is an $\mathfrak{N}_*$-module isomorphism.

We ·shall make repeated use of this theorem (as well as some of the observations necessary for its proof) in §3.

The above result suggests that we define equivariant cobordism to be the cohomology theory with coefficients $\mathbf{MO}^G$, and this is the course that we follow. Suppose we are given a pair (X, A) of G-spaces and an integer k, and let V, W be finite-dimensional orthogonal G-representations. Then we define a function

$$\phi^{V,W}: \quad [\Sigma(V) \wedge (X/A), MO^G_{|V|+k}] \to [\Sigma(W \oplus V) \wedge (X/A), MO^G_{|W|+|V|+k}]$$

to be the composite

$$[\Sigma(V) \wedge (X/A), MO^G_{|V|+k}]$$

$$\xrightarrow[(m_{|V|+k,|W|})^\#]{} \quad [\Sigma(W) \wedge \Sigma(V) \wedge (X/A), \Sigma(W) \wedge MO^G_{|V|+k}]$$

$$[\Sigma(W \oplus V) \wedge (X/A), MO^G_{|W|+|V|+k}],$$

where the first map is given by suspension. We note that if V contains a two-dimensional trivial representation, then all of the sets above are abelian groups and $\phi^{V,W}$ is a group homomorphism. It is readily verified that we obtain in this way a directed system over $M(G)$ and we define $H^k(X, A; \mathbf{MO}^G)$ to be the abelian group which is the direct limit of this system. It is clear that this construction is functorial on $P(G)$ (the category of pairs of G-spaces) and, letting $\mathfrak{A}$ denote the category of abelian groups, we have

Theorem 1.2. *The sequence of contravariant functors $\{H^k(\ ; \mathbf{MO}^G): P(G) \to \mathfrak{A} \mid k \in \mathbf{Z}\}$ determines an equivariant cohomology theory, i.e. the following statements are true:*

(1) *if* f_0, f_1: $(X, A) \to (Y, B)$ *are equivariantly homotopic as maps of pairs, then*

$$f_0^* = f_1^*: H^k(Y, B; \mathbf{MO}^G) \to H^k(X, A; \mathbf{MO}^G)$$

for all $k \in \mathbf{Z}$;

(2) *if* (X, A) *is admissible, then there is an exact sequence*

$$H^k(X, A; \mathbf{MO}^G) \xrightarrow{i^*} H^k(X; \mathbf{MO}^G) \xrightarrow{i^*} H^k(A; \mathbf{MO}^G)$$

for each $k \in \mathbf{Z}$, *the homomorphisms being induced by the inclusions*;

(3) *for each* $[P] \in M(G)$ *and each pointed G-space X there is a natural suspension isomorphism*

$$\hat{\sigma}(V): \tilde{H}^k(\Sigma(P) \wedge X; \mathbf{MO}^G) \to \tilde{H}^{k-|P|}(X; \mathbf{MO}^G),$$

for all $k \in \mathbf{Z}$, *where* $\tilde{H}^*(\ ; \mathbf{MO}^G)$ *denotes the reduced theory determined by* $H^*(\ ; \mathbf{MO}^G)$.

Proof. The validity of (1) is an evident consequence of the definitions. To verify (2), note that this sequence is the direct limit of a collection of corresponding exact sequences of homotopy sets. Finally (3) is immediate: there is an obvious identification

$$[\Sigma(V) \wedge (\Sigma(P) \wedge X), MO^G_{|V|+k}] \cong [\Sigma(V \oplus P) \wedge X, MO^G_{|V|+|P|+(k-|P|)}],$$

which gives us the desired isomorphism. □

We should remark that the equivariant Puppe sequence, together with (1)–(3) above, enables us to construct a long exact cohomology sequence for any admissible pair (X, A). In the sequel we will assume this construction to have been carried out when the need arises.

We wish now to discuss an important property of the spectrum $\mathbf{MO}^G$ which we have not yet mentioned and then apply our observations to define cup and cap products. We proceed by straightforward imitation of the usual definitions.

Let γ^n (resp. γ^m) denote the universal G-vector bundle over $BO_n(\mathbf{R}^\infty(G))$ (resp. $BO_m(\mathbf{R}^\infty(G))$). We may form the bundle $\gamma^n \times \gamma^m$ in the usual way, obtaining an equivariant $(n + m)$-plane bundle over $BO_n(\mathbf{R}^\infty(G)) \times BO_m(\mathbf{R}^\infty, G)$. Classifying this bundle gives us a G-map

$$\theta_{n,m}: BO_n(\mathbf{R}^\infty(G)) \times BO_m(\mathbf{R}^\infty(G)) \to BO_{n+m}(\mathbf{R}^\infty(G));$$

as always, this map gives rise to a G-map of Thom spaces

$$\vartheta_{n,m}: M(\gamma^n \times \gamma^m) \to MO^G_{n+m}.$$

But $M(\gamma^n \times \gamma^m) = M\gamma^n \wedge M\gamma^m$, so we actually have maps

$$\vartheta_{n,m}: MO^G_n \wedge MO^G_m \to MO^G_{n+m}, \qquad n, m \in \mathbf{Z}^+;$$

and it is these maps which allow us to define the necessary products.

Let X be a compact G-space and let A, B be G-stable subsets of X with the property that all of the pairs (X, A), (X, B), and $(X, A \cup B)$ are admissible. Note that the diagonal map $\Delta_X: X \to X \times X$ induces a G-map

$$\bar{\Delta}: X/(A \cup B) \to (X/A) \wedge (X/B).$$

Now, suppose we are given $\alpha \in H^n(X, A; \mathbf{MO}^G)$, $\beta \in H^m(X, B; \mathbf{MO}^G)$; choose representative maps $f: \Sigma(V_1) \wedge (X/A) \to MO^G_{|V_1|+n}$, $g: \Sigma(V_2) \wedge (X/B) \to MO^G_{|V_2|+m}$, and consider the composite

$$\Sigma(V_1) \wedge \Sigma(V_2) \wedge (X/A \cup B) \xrightarrow{1 \wedge 1 \wedge \bar{\Delta}} \Sigma(V_1) \wedge \Sigma(V_2) \wedge (X/A) \wedge (X/B)$$

$$\xrightarrow{\cong} \Sigma(V_1) \wedge (X/A) \wedge \Sigma(V_2) \wedge (X/B)$$

$$\xrightarrow{f \wedge g} MO^G_{|V_1|+n} \wedge MO^G_{|V_2|+m} \xrightarrow{\vartheta_{|V_1|+n, |V_2|+m}} MO^G_{|V_1|+|V_2|+n+m}.$$

The homotopy class of this map represents an element of $H^{n+m}(X, A \cup B; \mathbf{MO}^G)$, which is easily seen to depend only upon α and β; we shall call this element the cup product of α and β and denote it by $\alpha \cup \beta$. This product has all of the expected properties, as the reader may verify; we content ourselves with mentioning only one of these properties, which we shall need later.

Proposition 1.3. *Let* $f: (Y; Y_1, Y_2) \to (X; X_1, X_2)$ *be an equivariant map, and let*

$$f^*: H^*(X, X_1 \cup X_2; \mathbf{MO}^G) \to H^*(Y, Y_1 \cup Y_2; \mathbf{MO}^G),$$

$$f_i^*: H^*(X, X_i; \mathbf{MO}^G) \to H^*(Y, Y_i; \mathbf{MO}^G) \quad (i = 1, 2)$$

be induced by f. *If* $\alpha \in H^*(X, X_1)$ *and* $\beta \in H^*(X, X_2)$, *then*

$$f^*(\alpha \cup \beta) = f_1^*(\alpha) \cup f_2^*(\beta).$$

Proof. This is immediate from the definitions. $\square$

With the assumptions above, we may also define a cap product. This time, let $\alpha \in H^n(X, A; \mathbf{MO}^G)$ and $\beta \in H_k(X, A \cup B; \mathbf{MO}^G)$ be given; then there is an element $\alpha \cap \beta \in H_{k-n}(X, B; \mathbf{MO}^G)$, called the cap product of α and β. To define this element, we once again choose representative maps $f: \Sigma(V_1) \wedge (X/A) \to MO^G_{|V_1|+n}$, $g: \Sigma(V_2) \to (X/A \cup B) \wedge MO^G_{|V_2|-k}$; then a representative map for $\alpha \cap \beta$ is defined by the composite

$$\Sigma(V_1) \wedge \Sigma(V_2) \xrightarrow{1 \wedge g} \Sigma(V_1) \wedge (X/A \cup B) \wedge MO^G_{|V_2|-k}$$

$$\xrightarrow{1 \wedge \bar{\Delta} \wedge 1} \Sigma(V_1) \wedge (X/A) \wedge (X/B) \wedge MO^G_{|V_2|-k}$$

$$\cong (X/B) \wedge \Sigma(V_1) \wedge (X/A) \wedge MO^G_{|V_2|-k}$$

$$\xrightarrow{1 \wedge f \wedge 1} (X/B) \wedge MO^G_{|V_1|+n} \wedge MO^G_{|V_2|-k}$$

$$\xrightarrow{1 \wedge \vartheta_{|V_1|+n, |V_2|-k}} (X/B) \wedge MO^G_{|V_1|+|V_2|-(k-n)}.$$

This is easily seen to be well defined; we leave to the reader an investigation of the properties of this product, as its mere existence is sufficient for our present purposes.

2. **The Thom isomorphism.** We wish to show that the considerations of the preceding section enable us to define Thom homomorphisms for the theory $H^*(\; ; MO^G)$. Having done this, we will prove that these are, in fact, isomorphisms in all of the cases which we consider. The proof of this assertion will be modelled on the proof in [2] of the corresponding statement concerning stable bordism.

Let ξ^k be an equivariant vector bundle over the compact Hausdorff G-space X. Then ξ^k is classified by a G-map

$$u: X \to BO_k(\mathbf{R}^\infty(G))$$

which is unique up to equivariant homotopy. This map is covered by a G-map

$$\bar{u}: (D\xi, S\xi) \to (D\gamma^k, S\gamma^k),$$

where $D\xi$ is the disk bundle of ξ, $S\xi$ the boundary sphere bundle, and similarly for γ^k. Collapsing the sphere bundles, we receive a map $\hat{u}: M\xi \to MO^G_k$, whose G-homotopy class represents an element $U_\xi \in H^k(D\xi, S\xi; MO^G)$, to which we shall refer as the Thom class of the bundle ξ. This class we have constructed is natural in an appropriate sense. Specifically, let $f: Y \to X$ be a G-map, so that we may form the bundle $f^*\xi$ over Y. Then there is a G-map $\hat{f}: (D(f^*\xi), S(f^*\xi)) \to (D\xi, S\xi)$ covering f and it is immediate from the definitions involved that $U_{f^*\xi} = \hat{f}^*(U_\xi)$, where

$$\hat{f}^*: H^k(D\xi, S\xi; MO^G) \to H^k(D(f^*\xi), S(f^*\xi); MO^G).$$

is the homomorphism induced by $\hat{f}$. Note that we could just as well use the above procedure to define a relative Thom class.

The other property of the Thom class which we shall need requires somewhat more preparation. Suppose that ξ^k, η^q are G-vector bundles over X and form

$\xi \oplus \eta$, their Whitney sum. There is a commutative diagram (in fact, a pullback diagram)

$$
\begin{array}{ccc}
D(\xi \oplus \eta) & \xrightarrow{\ \hat{\pi}_1\ } & D\eta \\
{\scriptstyle \hat{\pi}_2}\downarrow & \searrow^{\pi} & \downarrow{\scriptstyle \pi_2} \\
D\xi & \xrightarrow{\ \pi_1\ } & X
\end{array}
$$

in which all the maps are bundle projections and we have homomorphisms

$$\hat{\pi}_2^*: H^*(D\xi, S\xi; \mathbf{MO}^G) \to H^*(D(\xi \oplus \eta), D(\pi_1^*\eta|_{S\xi}); \mathbf{MO}^G),$$

$$\hat{\pi}_1^*: H^*(D\eta, S\eta; \mathbf{MO}^G) \to H^*(D(\xi \oplus \eta), D(\pi_2^*\xi|_{S\eta}); \mathbf{MO}^G).$$

Note that $D(\pi_1^*\eta|_{S\xi}) \cup D(\pi_2^*\xi|_{S\eta}) = S(\xi \oplus \eta)$, so that the following assertion at least makes sense.

Lemma 2.1. $U_{\xi \oplus \eta} = \hat{\pi}_2^*(U_\xi) \cup \hat{\pi}_1^*(U_\eta)$.

Proof. Choose G-maps $f_1: X \to BO_k(\mathbf{R}^\infty(G))$, $f_2: X \to BO_q(\mathbf{R}^\infty(G))$ classifying ξ and η respectively. Then the composite

$$X \xrightarrow{\ \Delta_X\ } X \times X \xrightarrow{\ f_1 \times f_2\ } BO_k(\mathbf{R}^\infty(G)) \times BO_q(\mathbf{R}^\infty(G))$$

$$\xrightarrow{\ \theta_{k,q}\ } BO_{k+q}(\mathbf{R}^\infty(G))$$

classifies $\xi \oplus \eta$. The corresponding map of $D(\xi \oplus \eta)$ into $D\gamma^{k+q}$ factors as the composite

$$D(\xi \oplus \eta) \xrightarrow{(\hat{\pi}_2, \hat{\pi}_1)} D\xi \times D\eta \xrightarrow{\ \bar{f}_1 \times \bar{f}_2\ } D\gamma^k \times D\gamma^q \to D\gamma^{k+q};$$

hence the map of $M(\xi \oplus \eta)$ into MO_{k+q}^G which represents $U_{\xi \oplus \eta}$ is easily seen to factor as the composite

$$\frac{D(\xi \oplus \eta)}{S(\xi \oplus \eta)} \xrightarrow{\ \bar{\Delta}\ } \left(\frac{D(\xi \oplus \eta)}{D(\pi_1^*\eta|_{S\xi})}\right) \wedge \left(\frac{D(\xi \oplus \eta)}{D(\pi_2^*\xi|_{S\eta})}\right)$$

$$\xrightarrow{\ \hat{\pi}_2 \wedge \hat{\pi}_1\ } \frac{D\xi}{S\xi} \wedge \frac{D\eta}{S\eta} \xrightarrow{\ \hat{f}_1 \wedge \hat{f}_2\ } MO_k^G \wedge MO_q^G \xrightarrow{\ \vartheta_{k,q}\ } MO_{k+q}^G.$$

Since this map obviously represents $\hat{\pi}_2^*(U_\xi) \cup \hat{\pi}_1^*(U_\eta)$, the lemma follows. $\square$

We now define the Thom homomorphism for the G-vector bundle ξ^k over X in the way which suggests itself. Let $\pi: D\xi \to X$ be the projection, which is an equivariant homotopy equivalence for the usual reasons. Then, if $X_0 \hookrightarrow X$ is a G-cofibration, define a homomorphism

$$\tau^\xi: H^n(X, X_0; \mathbf{MO}^G) \to H^{n+k}(D\xi, D(\xi|_{X_0}) \cup S\xi; \mathbf{MO}^G), \quad n \in \mathbf{Z},$$

to be the composite

$$H^n(X, X_0; \mathrm{MO}^G) \xrightarrow{\pi^*} H^n(D\xi, D(\xi|_{X_0}); \mathrm{MO}^G)$$

$$\xrightarrow{-\cup U_\xi} H^{n+k}(D\xi, D(\xi|_{X_0}) \cup S\xi; \mathrm{MO}^G).$$

We shall refer to this homomorphism as the (cobordism) Thom homomorphism for the bundle ξ^k and the pair (X, X_0); it is natural in an obvious sense, because of the above-mentioned naturality of the Thom class.

We isolate the crucial properties of the Thom homomorphism in our next result.

Lemma 2.2. (a) *If ξ is a product bundle over X, with fibre the finite-dimensional representation Q, then τ^ξ coincides with the suspension isomorphism $\hat{\sigma}(Q)$.*

(b) *If ξ^k and η^q are bundles over X and $\pi_1: D\xi \to X$ is the projection, then $\tau^{\xi \oplus \eta} = \tau^{\pi_1^* \eta} \circ \tau^\xi$.*

Proof. (a) This is a straightforward computation (b) This statement requires a bit of interpretation. Specifically, we wish to show that, for any $n \in \mathbf{Z}$,

$$\tau^{\xi \oplus \eta}: H^n(X, X_0) \to H^{n+k+q}(D(\xi \oplus \eta), D(\xi \oplus \eta|_{X_0}) \cup S(\xi \oplus \eta))$$

may be computed as the composite of

$$\tau^\xi: H^n(X, X_0) \to H^{n+k}(D\xi, D(\xi|_{X_0}) \cup S\xi)$$

and

$$\tau^{\pi_1^* \eta}: H^{n+k}(D\xi, D(\xi|_{X_0}) \cup S\xi) \to H^{n+k+q}(D(\xi \oplus \eta), D(\xi \oplus \eta|_{X_0}) \cup S(\xi \oplus \eta))$$

where we have dropped all mention of MO^G. To this end we shall need the result of Lemma 2.1, so we use the notation introduced for the proof of that assertion. Let $x \in H^n(X, X_0)$ be given. Then

$$\begin{aligned}
\tau^{\xi \oplus \eta}(x) &= \pi^*(x) \cup U_{\xi \oplus \eta} = \pi^*(x) \cup \hat{\pi}_2^*(U_\xi) \cup \hat{\pi}_1^*(U_\eta) \\
&= \hat{\pi}_2^* \pi_1^*(x) \cup \hat{\pi}_2^*(U_\xi) \cup \hat{\pi}_1^*(U_\eta) = \hat{\pi}_2^*(\pi_1^*(x) \cup U_\xi) \cup \hat{\pi}_1^*(U_\eta) \\
&= \hat{\pi}_2^*(\tau^\xi(x)) \cup \hat{\pi}_1^*(U_\eta).
\end{aligned}$$

Since $\hat{\pi}_2: D(\xi \oplus \eta) \to D\xi$ is the projection of the disk bundle associated to $\pi_1^*\eta$ and since $\hat{\pi}_1^*(U_\eta) = U_{\pi_1^*\eta}$ (by naturality), we have

$$\hat{\pi}_2^*(\tau^\xi(x)) \cup \hat{\pi}_1^*(U_\eta) = \tau^{\pi_1^* \eta}(\tau^\xi(x)),$$

which proves (b). □

With this result in hand, we can easily prove

Theorem 2.3. *Let* (X, X_0) *be an admissible pair with* X *a compact Hausdorff* G*-space, and let* ξ^k *be an equivariant vector bundle over* X. *Then, for any* $n \in \mathbf{Z}$,

$$\tau^\xi : H^n(X, X_0; \mathbf{MO}^G) \longrightarrow H^{n+k}(D\xi, D(\xi|_{X_0}) \cup S\xi; \mathbf{MO}^G)$$

is an isomorphism.

Proof. Since X is compact and Hausdorff, the bundle ξ is stably invertible, i.e. there is a G-vector bundle η over X such that $\xi \oplus \eta$ is isomorphic to a product bundle. Choose such a bundle η and suppose that the fibre of the corresponding product bundle is the finite-dimensional representation Q. Then by the above lemma, $\hat{\sigma}(Q) = \tau^{\xi \oplus \eta} = \tau^{\pi^* \mathbf{1}^\eta} \circ \tau^\xi$. Since $\hat{\sigma}(Q)$ is an isomorphism, τ^ξ is at least monic. By the same argument, $\tau^{\pi^* \mathbf{1}^\eta}$ is monic. We immediately deduce that both τ^ξ and $\tau^{\pi^* \mathbf{1}^\eta}$ must be isomorphisms, concluding the proof. $\square$

As an application of this result, we show that there is a Gysin sequence in equivariant cobordism. If ξ^k is a G-vector bundle over the compact Hausdorff G-space X, we define the Euler class of ξ, $e(\xi)$, by the requirement that

$$e(\xi) = \sigma^* i^*(U_\xi) \in H^k(X; \mathbf{MO}^G)$$

where $\sigma : X \to D\xi$ is the zero-section and $i : D\xi \to (D\xi, S\xi)$ is the inclusion. Then we have

Theorem 2.4. *There is a (natural) long exact sequence*

$$\cdots \longrightarrow H^{n-1}(S\xi) \longrightarrow H^{n-k}(X) \xrightarrow{E} H^n(X) \longrightarrow H^n(S\xi) \longrightarrow \cdots$$

where all coefficients lie in $\mathbf{MO}^G$ *and the homomorphism* E *is given by* $E(x) = x \cup e(\xi)$.

Proof. We begin with the long exact cobordism sequence for the pair $(D\xi, S\xi)$:

$$\cdots \longrightarrow H^{n-1}(S\xi) \xrightarrow{\partial^*} H^n(D\xi, S\xi) \xrightarrow{i^*} H^n(D\xi) \xrightarrow{j^*} H^n(S\xi) \xrightarrow{\partial^*} \cdots .$$

We have isomorphisms

$$\tau^\xi : H^{n-k}(X) \longrightarrow H^n(D\xi, S\xi), \qquad \sigma^* : H^n(D\xi) \longrightarrow H^n(X)$$

where $\sigma : X \to D\xi$ is the zero-section as before; it is then immediate that there is an exact sequence of the desired sort which is clearly natural. Hence it only remains to identify the homomorphism $E : H^{n-k}(X) \to H^n(X)$, and this is a straightforward computation. Let $x \in H^{n-k}(X)$; then by definition,

$$E(x) = \sigma^* i^*(\tau^\xi(x)) = \sigma^* i^*(\pi^*(x) \cup U_\xi)$$

$$= \sigma^*(\pi^*(x) \cup i^* U_\xi) = \sigma^* \pi^*(x) \cup \sigma^* i^* U_\xi = x \cup e(\xi),$$

as desired. $\square$

3. **The duality theorems.** Our goal in this section is to prove that there is a Poincaré duality isomorphism relating the stable G-bordism of a closed G-manifold M and the G-cobordism of M. More generally, we prove a Poincaré-Lefschetz duality theorem in this setting. Our proof is modelled on one of the standard proofs of the corresponding theorem concerning ordinary bordism and cobordism, but there are some complications which arise from the lack of a decent equivariant S-duality theory. Overcoming these difficulties will occupy our attention for much of the remainder of this paper.

We need a preliminary definition and a lemma. Let $f: N^n \to M^m$ be a smooth G-map between G-manifolds. An equivariant imbedding $\alpha: N \to D(W) \times M([W] \in M(G))$ is said to be "an imbedding over f" if and only if the diagram

$$
\begin{array}{ccc}
 & & D(W) \times M \\
 & \nearstack{\alpha} & \downarrow \text{ pr}_2 \\
N & \xrightarrow{\quad f \quad} & M
\end{array}
$$

is commutative.

We remark that given f as above, there always exist imbeddings over f. We simply choose some imbedding $e: N \to D(W)$ and define $\alpha: N \to D(W) \times M$ by $\alpha(n) = (e(n), f(n))$.

Lemma 3.1. *Let $f: N^n \to M^m$ be a smooth G-map and let*

$$\alpha_1: N \to D(V_1) \times M, \qquad \alpha_2: N \to D(V_2) \times M$$

be imbeddings over f. Then there is a level-preserving G-imbedding

$$A: N \times I \to D(V_1 \oplus V_2) \times M \times I$$

such that $A(n, 0) = (\text{pr}_1 \circ \alpha_1(n), 0, f(n), 0)$, $A(n, 1) = (0, \text{pr}_1 \circ \alpha_2(n), f(n), 1)$, $\text{pr}_M \circ A(n, t) = f(n)$.

Proof. Define $A(n, t) = ((1 - t)\text{pr}_1 \circ \alpha_1(n), t\text{pr}_1 \circ \alpha_2(n), f(n), t)$. This map has the desired properties. □

Now suppose that M^n is a G-manifold; choose an equivariant imbedding $\phi: M^n \to D(W)$ for some $[W] \in M(G)$ and let ν_ϕ denote the normal bundle of this imbedding. We shall define, for each $k \in \mathbf{Z}$, homomorphisms

$$D_k: \mathfrak{N}_k^{G:S}(M^n) \to H^{|W|-k}(D\nu_\phi, D(\nu_\phi|_{\partial M}) \cup S\nu_\phi)$$

and

$$\overline{D}_k: H^{|W|-k}(D\nu_\phi, D(\nu_\phi|_{\partial M}) \cup S\nu_\phi) \to \mathfrak{N}_k^{G:S}(M^n)$$

which are mutually inverse.

We first define D_k. Let $f: (N^{|Q|+k}, \partial N) \to (D(Q) \times M, S(Q) \times M)$ represent an element $x \in \mathfrak{N}_k^{G:S}(M)$ and let $\bar{f}$ denote the composite of f and the homotopy equivalence $(D(Q) \times M, S(Q) \times M) \to (D(Q) \times D\nu_\phi, S(Q) \times D\nu_\phi)$ given by the zero-section. Choose an imbedding over $\bar{f}$, say

$$\alpha_f: (N, \partial N) \to (D(V) \times D(Q) \times D\nu_\phi, D(V) \times S(Q) \times D\nu_\phi);$$

there is then an equivariant collapsing map

$$\kappa: \Sigma(V) \wedge \Sigma(Q) \wedge (D\nu_\phi/D(\nu_\phi|_{\partial M}) \cup S\nu_\phi) \to D\nu_f/\partial\nu_f$$

where ν_f is the normal bundle of α_f. Classifying ν_f gives us a map

$$\psi_1: (D\nu_f, S\nu_f) \to (D\gamma^{|V|+|W|-k}, S\gamma^{|V|+|W|-k}),$$

and we have a map

$$\psi_2: (D\nu_f, D(\nu_f|_{\partial N})) \to (D(Q), S(Q))$$

obtained in evident fashion from the map $f: N \to D(Q) \times M$; these combine to give a map

$$\psi: (D\nu_f, \partial D\nu_f) \to (D(Q) \times D\gamma^{|V|+|W|-k}, \partial(D(Q) \times D\gamma^{|V|+|W|-k})).$$

Finally, there is the usual map

$$\sigma: (D(Q) \times D\gamma^{|V|+|W|-k}, \partial(D(Q) \times D\gamma^{|V|+|W|-k}))$$
$$\to (D\gamma^{|Q|+|V|+|W|-k}, S\gamma^{|Q|+|V|+|W|-k});$$

the composite of the last two maps induces a basepoint-preserving G-map

$$D\nu_f/\partial D\nu_f \to MO^G_{|Q|+|V|+|W|-k}$$

which, when composed with κ, yields a basepoint-preserving G-map

$$\widetilde{f}: \Sigma(V) \wedge \Sigma(Q) \wedge (D\nu_\phi/D(\nu_\phi|_{\partial M}) \cup S\nu_\phi) \to MO^G_{|Q|+|V|+|W|-k}.$$

The G-homotopy class of $\widetilde{f}$ represents an element $D_k(x) \in H^{|W|-k}(D\nu_\phi, D(\nu_\phi|_M) \cup S\nu_\phi)$, and it is easily verified that we have defined in this way a function

$$D_k: \mathfrak{N}_k^{G:S}(M) \to H^{|W|-k}(D\nu_\phi, D(\nu_\phi|_{\partial M}) \cup S\nu_\phi).$$

(One first checks that, given the map f, the above construction is independent of the particular imbedding α_f which is used; this is an application of Lemma 3.1. Then a standard argument shows that a cobordism leads to a homotopy of the resulting maps, the imbeddings being carefully chosen.)

In order to define $\bar{D}_k$, we first note that there is a collapsing map

$$\tau:\ \Sigma(W)\ \to Dv_\phi/D(v_\phi|_{\partial M})\ \cup Sv_\phi;$$

furthermore, there is the map $\pi\times 1:Dv_\phi\ \to M\times Dv_\phi$ which collapses to give a map

$$\mu:Dv_\phi/D(v_\phi|_{\partial M})\ \cup Sv_\phi\ \to M^+\wedge Dv_\phi/D(v_\phi|_{\partial M})\ \cup Sv_\phi.$$

Now, given an element of $H^{|W|-k}(Dv_\phi,\ D(v_\phi|_{\partial M})\ \cup Sv_\phi)$, choose a representative map

$$g:\ \Sigma(V)\wedge Dv_\phi/D(v_\phi|_{\partial M})\ \cup Sv_\phi\ \to MO^G_{|V|+|W|-k}.$$

Consider the composite

$$\Sigma(V)\ \wedge\ \Sigma(W)\ \xrightarrow{1\wedge\tau}\ \Sigma(V)\ \wedge\ Dv_\phi/D(v_\phi|_{\partial M})\ \cup Sv_\phi$$

$$\xrightarrow{1\wedge u}\ \Sigma(V)\ \wedge\ M^+\ \wedge\ Dv_\phi/D(v_\phi|_{\partial M})\ \cup Sv_\phi$$

$$\xrightarrow{\cong}\ M^+\ \wedge\ \Sigma(V)\ \wedge\ Dv_\phi/D(v_\phi|_{\partial M})\ \cup Sv_\phi$$

$$\xrightarrow{1\wedge g}\ M^+\ \wedge\ MO^G_{|V|+|W|-k};$$

the homotopy class of this map represents an element of $H_k(M)$ and hence an element of $\mathfrak{N}^{G:S}_k(M)$. This construction defines a function

$$\overline{D}_k:H^{|W|-k}(Dv_\phi,\ D(v_\phi|_{\partial M})\ \cup Sv_\phi)\to\mathfrak{N}^{G:S}_k(M);$$

the necessary verifications are trivial.

It is reasonably clear that D_k and $\overline{D}_k$ are group homomorphisms, so that we shall not verify this in proving

Theorem 3.2. *For each* $k\in\mathbf{Z}$, D_k *and* $\overline{D}_k$ *are inverse isomorphisms.*

Remarks. (1) In case $G=\{1\}$, so that we are dealing with ordinary bordism and cobordism, the isomorphism

$$H_k(M;\mathbf{MO})\cong H^{|W|-k}(Dv_\phi,\ D(v_\phi|_{\partial M})\ \cup Sv_\phi;\mathbf{MO})$$

is an immediate consequence of Atiyah's observation that $M/\varnothing$ and $Mv_\phi/M(v_\phi|_{\partial M})$ are S-dual.

(2) The following proof is a straightforward generalization of the proof we gave for Theorem 4.1 in [2].

Proof. We compute the composites $\overline{D}_k\circ D_k$ and $D_k\circ\overline{D}_k$, showing them to be the relevant identity automorphisms.

First, consider $\overline{D}_k\circ D_k$. Let $f:(N^{|Q|+k},\ \partial N)\to(D(Q)\times M,\ S(Q)\times M)$ represent an element of $\mathfrak{N}^{G:S}_k(M)$, and choose an imbedding

$$\alpha_f:(N,\ \partial N)\to(D(V)\times D(Q)\times Dv_\phi,\ D(V)\times S(Q)\times Dv_\phi)$$

as in the definition of D_k. We shall first construct the element of $H_k(M)$ which corresponds to $\overline{D}_k \circ D_k[N, f]$ under the Pontrjagin-Thom isomorphism. We claim that this element may be obtained in the following fashion, the notation being that previously established. We have a map $\hat{f} = \mathrm{pr}_2 \circ f \circ \pi \colon D\nu_f \to M$ and hence a map $(\hat{f}, 1) \colon D\nu_f \to M \times D\nu_f$; in defining $D_k[N, f]$ we produced a map $\psi \colon D\nu_f \to D(Q) \times Dy^{|V|+|W|-k}$. It is easily checked that the composite

$$D\nu_f \xrightarrow{(\hat{f},1)} M \times D\nu_f \xrightarrow{1 \times \psi} M \times D(Q) \times Dy^{|V|+|W|-k}$$

$$\xrightarrow{1 \times \sigma} M \times Dy^{|Q|+|V|+|W|-k}$$

sends $\partial D\nu_f$ into $M \times Sy^{|Q|+|V|+|W|-k}$, so that there is an induced map

$$D\nu_f / \partial D\nu_f \to M^+ \wedge MO^G_{|Q|+|V|+|W|-k}.$$

We also have a collapsing map

$$\Sigma(V) \wedge \Sigma(Q) \wedge \Sigma(W) \xrightarrow{\kappa \circ (1 \wedge 1 \wedge \tau)} D\nu_f / \partial D\nu_f;$$

composing the last two maps mentioned yields a map

$$F \colon \Sigma(V) \wedge \Sigma(Q) \wedge \Sigma(W) \to M^+ \wedge MO^G_{|Q|+|V|+|W|-k}$$

representing an element of $H_k(M)$. It is immediate from the definitions involved that this element is the desired one, so we need only produce the corresponding bordism element. For this purpose note that we have, in the above construction, imbedded N in $D(V) \times D(Q) \times D(W)$ and that the normal bundle of this imbedding is just ν_f. It is easily checked that, if we apply the Pontrjagin-Thom construction to the bordism element $[N, f]$ and the given imbedding of N into $D(V) \times D(Q) \times D(W)$, the resulting element $\Phi[N, f] \in H_{|Q|+k}(D(Q) \times M, S(Q) \times M)$ is represented by the composite

$$\Sigma(V) \wedge \Sigma(Q) \wedge \Sigma(W) \to D\nu_f / \partial D\nu_f \to \Sigma(Q) \wedge M^+ \wedge MO^G_{|V|+|W|-k}$$

where the first map is the collapse mentioned previously and the second is induced by the composite

$$D\nu_f \xrightarrow{(\hat{f},1)} M \times D\nu_f \xrightarrow{1 \times \psi} M \times D(Q) \times Dy^{|V|+|W|-k}$$

$$\xrightarrow{\cong} D(Q) \times M \times Dy^{|V|+|W|-k},$$

when we perform the appropriate collapses. But it is then clear that $\sigma(Q)^{-1}\Phi[N, f] = [F] \in H_k(M)$ where

$$\sigma(Q)^{-1} \colon H_{|Q|+k}(D(Q) \times M, S(Q) \times M) \to H_k(M)$$

is the inverse of the suspension isomorphism for the representation Q. (See [2] for the construction of $\sigma(Q)^{-1}$ which we use here.) By the definition of the stable Pontrjagin-Thom construction, we then have that $\overline{D}_k \circ D_k[N, f] = (\Phi^S)^{-1}[F] = [N, f]$. Thus $\overline{D}_k \circ D_k$ is the identity automorphism.

In order to compute $D_k \circ \overline{D}_k$, let

$$g: \Sigma(V) \wedge (D\nu_\phi/D(\nu_\phi|_{\partial M}) \cup S\nu_\phi) \to MO^G_{|V|+|W|-k}$$

represent an element of $H^{|W|-k}(D\nu_\phi, D(\nu_\phi|_{\partial M}) \cup S\nu_\phi)$. Then, for some $[Q] \in M(G)$, we may view g as a map of pairs

$$g: (D(V) \times D\nu_{\phi'}, \partial(D(V) \times D\nu_\phi)) \to (M\gamma^{|V|+|W|-k}(Q), *),$$

and we may find a submanifold $L^{|V|+|W|} \subset D(V) \times D\nu_\phi$ such that $g|_L$ is a map of pairs

$$g|_L: (L, \partial L) \to (D\gamma^{|V|+|W|-k}(Q), S\gamma^{|V|+|W|-k}(Q)).$$

Furthermore, if we vary g within its G-homotopy class, we may assume that there is a commutative diagram

$$
\begin{array}{c}
D(V) \times D\nu_\phi/\partial(D(V) \times D\nu_\phi) \\
\downarrow \qquad \qquad \searrow^{g} \\
L/\partial L \xrightarrow{\hspace{3cm}} M\gamma \\
\uparrow \qquad \quad \quad \uparrow \\
(L, \partial L) \xrightarrow{\quad g|_L \quad} (D\gamma, S\gamma)
\end{array}
$$

in which the vertical maps are all collapsing maps and we have adopted the abbreviation $\gamma = \gamma^{|V|+|W|-k}(Q)$.

We have a map

$$D(V) \times D\nu_\phi \to M \times D(V) \times D\nu_\phi: (v, w) \mapsto (\pi(w), v, w)$$

and this restricts to give a map $\widetilde{\pi}: L \to M \times L$. Then the composite

$$L \xrightarrow{\widetilde{\pi}} M \times L \xrightarrow{1 \times g|_L} M \times D\gamma(Q) \to M^+ \wedge M\gamma,$$

in which the final map is the standard quotient map, is seen to send ∂L into the basepoint in $M^+ \wedge M\gamma$; hence there is an induced basepoint-preserving G-map $L/\partial L \to M^+ \wedge M\gamma$. Finally, there is a collapsing map $\Sigma(V) \wedge \Sigma(W) \to L/\partial L$; the last two maps compose to yield a basepoint-preserving G-map

$$H: \Sigma(V) \wedge \Sigma(W) \to M^+ \wedge M\gamma$$

and hence an element of $H_k(M)$. We assert that it is obvious that the corresponding element of $\mathfrak{N}^{G:S}_k(M)$ is just $\overline{D}_k[g]$.

Thus, to compute $\overline{D}_k[g]$, it suffices to apply the inverse of the Pontrjagin-

Thom construction to the map H defined above. For this purpose, consider the G-vector bundle $\xi = \mathrm{pr}_2^* \gamma$ over $M \times BO_{|V|+|W|-k}(Q)$; it is clear that ξ is stably-invertible. In fact, there is a G-vector bundle ξ^- such that $\xi \oplus \xi^-$ is a product bundle with fibre Q. Let $\hat{g}: L \to D\xi$ be defined by $\hat{g} = (1 \times g|_L) \circ \tilde{\pi}$, as above, and let $\rho: D\xi \to B\xi$ be the projection. Then we may form the G-manifold $D(\hat{g}^* \rho^* \xi^-)$, which comes to us provided with a G-map into $D(\rho^* \xi^-) = D(Q) \times B\xi$. Let $h: D(\hat{g}^* \rho^* \xi^-) \to D(Q) \times M$ be the composite of this map and the projection of $D(Q) \times B\xi$ onto $D(Q) \times M$. Then a representative of the element $\overline{D}_k[g] \in \mathfrak{N}_k^{G:S}(M)$ is the bordism element $[D(\hat{g}^* \rho^* \xi^-), h] \in \widetilde{\mathfrak{N}}_{|Q|+k}^G(\Sigma(Q) \wedge M^+)$; a proof of this assertion may be found in [2].

Using this representative for $\overline{D}_k[g]$, it is an easy matter to compute $D_k \circ \overline{D}_k[g]$. Note that $D(\hat{g}^* \rho^* \xi^-)$ is given with an imbedding into $D(Q) \times D(V) \times D\nu_\phi$ and that the normal disk bundle of this imbedding may be identified with $D(Q) \times L$. To make use of this observation, we remark that the composite

$$D(Q) \times L \xrightarrow{1 \times g|_L} D(Q) \times D\gamma^{|V|+|w|-k}(Q) \xrightarrow{\sigma} D\gamma^{|Q|+|V|+|W|-k}$$

obviously induces a basepoint-preserving G-map

$$\Sigma(Q) \wedge (L/\partial L) \to MO_{|Q|+|V|+|W|-k}^G.$$

If we compose the latter map with the collapsing map,

$$\Sigma(Q) \wedge \Sigma(V) \wedge (D\nu_\phi/D(\nu_\phi|_{\partial M}) \cup S\nu_\phi) \to \Sigma(Q) \wedge (L/\partial L),$$

we clearly receive a representative of $D_k \circ \overline{D}_k[g]$. But it is equally clear that the above map is just the "Q-fold" suspension of the map g with which we began. This demonstrates that $D_k \circ \overline{D}_k$ is the identity automorphism, concluding the proof of the theorem. $\square$

As an immediate consequence of this theorem, we have the Poincaré-Lefschetz duality theorem.

Corollary 3.3. *Let M^n be a compact G-manifold. Then, for each $k \in \mathbf{Z}$, there is an isomorphism*

$$\mathcal{P}_k: \mathfrak{N}_k^{G:S}(M^n) \xrightarrow{\cong} H^{n-k}(M^n, \partial M^n; \mathbf{MO}^G).$$

Proof. To define $\mathcal{P}_k$, we choose an equivariant imbedding $\phi: M \to D(W)$ as before. Then we have isomorphisms

$$D_k: \mathfrak{N}_k^{G:S}(M^n) \xrightarrow{\cong} H^{|W|-k}(D\nu_\phi, D(\nu_\phi|_{\partial M}) \cup S\nu_\phi; \mathbf{MO}^G)$$

and

$$(\tau^\nu \phi)^{-1}: H^{|W|-k}(D\nu_\phi, D(\nu_\phi|_{\partial M}) \cup S\nu_\phi; \mathbf{MO}^G) \xrightarrow{\cong} H^{n-k}(M^n, \partial M^n; \mathbf{MO}^G);$$

we let $\mathcal{P}_k$ be the composite of these isomorphisms. □

We can say even more about $\mathcal{P}_k$; in particular, $\mathcal{P}_k$ is actually independent of the imbedding ϕ which we used in its definition. This is a consequence of the following considerations.

First note that for any G-manifold M^n there is a distinguished class $[M, \partial M]$ $\in \mathfrak{N}_n^{G:S}(M, \partial M)$ which is represented by the bordism class of the identity map $(M, \partial M) \to (M, \partial M)$. We shall refer to this class as the stable bordism fundamental class of M. We wish to prove

Theorem 3.4. *Let* M^n *be a G-manifold and let* $x \in H^{n-k}(M, \partial M; \mathbf{MO}^G)$. *Then* $\mathcal{P}_k^{-1}(x) = x \cap [M, \partial M]$.

Proof. This statement deserves some amplification; we are using the Pontrjagin-Thom isomorphism here to replace $[M, \partial M]$ by a certain homology element and we are to show that $\mathcal{P}_k^{-1}$ coincides with taking the cap product with this element.

Our first task is to specify our homology version of $[M, \partial M]$. To produce this element, choose an imbedding $\phi: M \to D(W)$ as usual; then the element we want is that which is represented by the composite

$$\Sigma(W) \xrightarrow{\tau} M\nu_\phi/M(\nu_\phi|_{\partial M}) \xrightarrow{\overline{\mu}} (M/\partial M) \wedge M\nu_\phi/M(\nu_\phi|_{\partial M}) \xrightarrow{1 \wedge u_\phi} (M/\partial M) \wedge MO^G_{|W|-n}.$$

Here the first map is the usual collapsing map, the second map is induced by the obvious map $D\nu_\phi \to M \times D\nu_\phi$ and the third map is constructed using a representative of the Thom class of ν_ϕ over $(M, \partial M)$. Call this composite $F: \Sigma(W) \to (M/\partial M) \wedge MO^G_{|W|-n}$.

Now suppose we are given $x \in H^{n-k}(M, \partial M)$ and choose a representative

$$f: \Sigma(V) \wedge (M/\partial M) \to MO^G_{|V|+(n-k)}$$

for x. A representative map for $x \cap [M, \partial M]$ is then easily constructed; it is just the composite

$$\Sigma(V) \wedge \Sigma(W) \xrightarrow{1 \wedge F} \Sigma(V) \wedge (M/\partial M) \wedge MO^G_{|W|-n}$$

$$\xrightarrow{1 \wedge \overline{\Delta} \wedge 1} \Sigma(V) \wedge M^+ \wedge (M/\partial M) \wedge MO^G_{|W|-n}$$

$$\xrightarrow{\cong} M^+ \wedge \Sigma(V) \wedge (M/\partial M) \wedge MO^G_{|W|-n}$$

$$\xrightarrow{1 \wedge f \wedge 1} M^+ \wedge MO^G_{|V|+(n-k)} \wedge MO^G_{|W|-n}$$

$$\xrightarrow{1 \wedge \vartheta} M^+ \wedge MO^G_{|V|+|W|-k},$$

where the unnamed homeomorphism interchanges $\Sigma(V)$ and M^+ and the other maps are as previously defined.

It is then easy to show that the above composite may also be computed as the composite

$$\Sigma(V) \ \wedge \ \Sigma(W) \xrightarrow{1 \wedge (\bar{\mu} \circ \tau)} \ \Sigma(V) \wedge (M/\partial M) \wedge M\nu_\phi/M(\nu_\phi|_{\partial M})$$

$$\xrightarrow{1 \wedge \bar{\Delta} \wedge 1} \ \Sigma(V) \wedge M^+ \wedge (M/\partial M) \wedge M\nu_\phi/M(\nu_\phi|_{\partial M})$$

$$\xrightarrow{\cong} \ M^+ \wedge \Sigma(V) \wedge (M/\partial M) \wedge M\nu_\phi/M(\nu_\phi|_{\partial M})$$

$$\xrightarrow{1 \wedge f \wedge u_\phi} M^+ \wedge MO^G_{|V|+(n-k)} \wedge MO^G_{|W|-n} \xrightarrow{1 \wedge \vartheta} M^+ \wedge MO^G_{|V|+|W|-k};$$

this follows essentially from the definition of F and the existence of various (obviously) commutative diagrams. The argument is completed by the observation that the composite of the first three maps above may also be computed as the composite

$$\Sigma(V) \ \wedge \ \Sigma(W) \xrightarrow{1 \wedge (\mu \circ \tau)} \ \Sigma(V) \wedge M^+ \wedge M\nu_\phi/M(\nu_\phi|_{\partial M})$$

$$\xrightarrow{\cong} \ M^+ \ \wedge \ \Sigma(V) \wedge M\nu_\phi/M(\nu_\phi|_{\partial M})$$

$$\xrightarrow{1 \wedge 1 \wedge \widetilde{\Delta}} M^+ \wedge \ \Sigma(V) \wedge D\nu_\phi/D(\nu_\phi|_{\partial M}) \wedge M\nu_\phi/M(\nu_\phi|_{\partial M})$$

$$\xrightarrow{1 \wedge 1 \wedge \widetilde{\pi} \wedge 1} M^+ \wedge \Sigma(V) \wedge M/\partial M \wedge M\nu_\phi/M(\nu_\phi|_{\partial M}),$$

where $\widetilde{\Delta}$ is induced by the diagonal map $D\nu_\phi \to D\nu_\phi \times D\nu_\phi$ and $\widetilde{\pi}$ is induced by the bundle projection $\pi: D\nu_\phi \to M$. If we now make the indicated replacement, then it is clear from the definitions involved that the map we have been considering is a representative for $\mathcal{P}_k^{-1}(x)$, and this proves the theorem. $\square$

This theorem allows us to prove the existence of the usual companion isomorphism to $\mathcal{P}_k$ above.

Corollary 3.5. *Let M^n be a compact G-manifold with bordism fundamental class $[M, \partial M] \in \mathfrak{N}_n^{G:S}(M, \partial M)$. Then*

$$- \cap [M, \partial M]: H^{n-k}(M; \mathbf{MO}^G) \to \mathfrak{N}_k^{G:S}(M, \partial M)$$

is an isomorphism for each $k \in \mathbf{Z}$.

Proof. This follows immediately from the existence of the commutative diagram

$$\cdots \xrightarrow{\partial^*} H^{n-k}(M, \partial M) \xrightarrow{j^*} H^{n-k}(M) \xrightarrow{i^*} H^{n-k}(\partial M) \xrightarrow{\partial^*} \cdots$$

$$\Big\downarrow -\cap[M, \partial M] \qquad \Big\downarrow -\cap[M, \partial M] \qquad \Big\downarrow -\cap[\partial M]$$

$$\cdots \xrightarrow{i_*} H_k(M) \xrightarrow{j_*} H_k(M, \partial M) \xrightarrow{\partial_*} H_{k-1}(\partial M) \xrightarrow{j_*} \cdots$$

with exact rows, the above theorem, the Five Lemma and the naturality of the Pontrjagin-Thom construction. □

REFERENCES

1. G. E. Bredon, *Equivariant cohomology theories*, Lecture Notes in Math., no. 34, Springer-Verlag, Berlin and New York, 1967.   MR 35 #4914.

2. T. Bröcker and E. C. Hook, *Stable equivariant bordism* (to appear).

3. R. E. Stong, *Unoriented bordism and actions of finite groups*, Mem. Amer. Math. Soc. No. 103 (1970).   MR 42 #8522.

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