EQUIVARIANT COBORDISM AND HOMOTOPY TYPE

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At the Second Conference on Compact Transformation Groups (University of Massachusetts, Amherst, 1971) Reinhard Schultz posed the question of whether equivariantly homeomorphic G-manifolds are necessarily equivariantly cobordant, G being a compact Lie group. This paper is concerned with the related question in which the assumed equivariant homeomorphism is replaced by an equivariant homotopy equivalence, a weakening of the hypotheses suggested by the well-known fact that unoriented cobordism class is a homotopy-type invariant.

In Section 1 we consider the special case in which the action of G is assumed to be free. Using standard techniques, we are able to prove that free G-manifolds having the same equivariant homotopy type are cobordant as free G-manifolds; this result holds for all compact Lie groups G. The next section considers the question for arbitrary actions of the cyclic group $\mathbf{Z}_2$; here Conner and Stong have shown that the result is true. We give a slightly more explicit proof of their result, which is primarily of interest for its implications concerning semifree actions of odd-order groups and finite abelian groups.

The results in Section 2 suggest that the basic difficulty in generalizing the result of Conner and Stong to other groups is the lack of a decent equivariant transversality theorem. So glaring is this deficiency that one should be led to conjecture that the result is, in general, false; in Section 4 we verify this conjecture by constructing, for each odd prime p, a family of counterexamples. The construction depends upon the discussion in Section 3 and the work of Olum on the homotopy-type of lens spaces.

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1. Free actions. Let G be a compact Lie group and denote by $\hat{\mathfrak{N}}_*^G$ the cobordism ring of (unoriented) manifolds with free G-action. If B_G is a classifying space for principal G-bundles, there is a well-known isomorphism

$$\hat{\mathfrak{N}}_*^G \cong \mathfrak{N}_*(B_G)$$

(with, possibly, a shift in dimension) given by classifying the orbit map; here $\mathfrak{N}_*(B_G)$ denotes the unoriented bordism of B_G. We should remark that a class in $\mathfrak{N}_*(B_G)$ is determined by its Conner-Floyd characteristic numbers [1], since we may choose a model for B_G in which the finite skeleta are honest manifolds.

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These remarks lead immediately to a proof of the following theorem.

THEOREM 1.1. *Let M_1 and M_2 be closed n-dimensional G-manifolds, the action of G being free in each case. If there exists an equivariant homotopy-equivalence $f : M_1 \to M_2$, then $[M_1] = [M_2]$ in $\hat{\mathfrak{N}}_*^G$.*

Proof. Let $\pi_i : M_i \to M_i/G$ denote the orbit map, $i = 1, 2$. Since f is equivariant, there is a unique map $\bar{f}$ making the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1/G & \xrightarrow{\bar{f}} & M_2/G \end{array}$$

commutative, and it is easily verified that $\bar{f}$ is again a homotopy-equivalence. Because the Stiefel-Whitney numbers of a closed manifold are homotopy-type invariants, it follows that $w_k(M_1/G) = \bar{f}^* w_k(M_2/G)$ for all $k \geq 0$. Since the fundamental classes are related by the equation $\bar{f}_*[M_1/G] = [M_2/G]$, a standard argument shows that, given any map $\alpha : M_2/G \to B_G$, the maps α and $\bar{f} \circ \alpha$ have precisely the same Conner-Floyd characteristic numbers; hence $[M_1/G, \bar{f} \circ \alpha] = [M_2/G, \alpha]$ in $\mathfrak{N}_*(B_G)$. In particular, we may apply this result with $\alpha : M_2/G \to B_G$ being a classifying map for G's action on M_2; since, obviously, the map $\bar{f} \circ \alpha$ then classifies the action on M_1, we may conclude that $[M_1] = [M_2]$ in $\hat{\mathfrak{N}}_n^G$. □

2. Actions of $\mathbf{Z}_2$ and related results. We turn now to the (very) special case of manifolds with involution, with fixed-point sets allowed to be non-vacuous.

Let $f : M_1 \to M_2$ be an equivariant homotopy equivalence, where M_1 and M_2 are closed n-dimensional manifolds-with-involution, and let $F(M_i)$ denote the fixed-point set of $\mathbf{Z}_2$ in M_i, $i = 1, 2$. Then $F(M_i)$ is a disjoint union of smoothly-embedded submanifolds of M_i, and the restriction of f to $F(M_i)$ determines a dimension-preserving one-to-one correspondence between the components of $F(M_1)$ and those of $F(M_2)$, with corresponding components being homotopy-equivalent *via* the appropriate (further) restriction of f. It follows that corresponding components are cobordant, but we need a somewhat stronger assertion, which is implied by the next lemma.

LEMMA 2.1. *Let F_1 be any component of $F(M_1)$ and let F_2 be the corresponding component of $F(M_2)$. If ν_i denotes the normal bundle of F_i in M_i, $i = 1, 2$, then $w_k(\nu_1) = (f|_{F_1})^* w_k(\nu_2)$ for all $k \geq 0$.*

Proof [4]. Since both f and $f|_{F_1}$ are homotopy equivalences, we have $w_*(M_1) = f^* w_*(M_2)$ and $w_*(F_1) = (f|_{F_1})^* w_*(F_2)$. Moreover, for the usual reasons, we have $w_*(\tau_{M_2|F_2}) = w_*(F_2) w_*(\nu_2)$ where, in general, τ_Q denotes the tangent bundle to the manifold Q; applying the ring homomorphism $(f|_{F_1})^*$ to this equation, we obtain $(f|_{F_1})^* w_*(\tau_{M_2|F_2}) = w_*(F_1) \cdot (f|_{F_1})^* w_*(\nu_2)$. But $(f|_{F_1})^* w_*(\tau_{M_2|F_2}) = w_*(\tau_{M_1|F_1}) = w_*(F_1) w_*(\nu_1)$ by our previous remarks; thus, $w_*(F_1) w_*(\nu_1) = w_*(F_1)\ (f|_{F_1})^* w_*(\nu_2)$ and the lemma follows easily. □

With the usual abuse of notation, the lemma says that $[F_1, \nu_1] = [F_2, \nu_2]$ in $\mathfrak{N}_m(BO(n-m))$ whenever F_1 and F_2 are corresponding m-dimensional components of the fixed-point sets; this is the stronger assertion mentioned above.

We are now ready to prove the following theorem.

THEOREM 2.2 (Conner–Stong). *If $f : M_1 \to M_2$ is an equivariant homotopy-equivalence between manifolds-with-involution, then M_1 and M_2 are equivariantly cobordant.*

Remark 1. The reader may consult [4] for the original proof. Our proof differs chiefly in that it is slightly more "geometric" so that one might hope to generalize it.

Remark 2. The proof which follows was suggested by [5; Figure 1]. The argument is precisely that needed to show that the unoriented cobordism class of a manifold-pair is an invariant of the homotopy type of the pair.

Proof. We assume for simplicity that $F(M_1)$ and $F(M_2)$ are connected; the modifications necessary to prove the general case will be obvious. With this additional assumption, we may find a manifold W such that $\partial W = F(M_1) \coprod F(M_2)$; moreover, if k is the common dimension of $F(M_1)$ and $F(M_2)$, we may choose W in such a way that there is an $(n-k)$-plane bundle ξ over W satisfying $\xi|_{F(M_1)} \cong \nu_1$, $\xi|_{F(M_2)} \cong \nu_2$, the notation being that previously established. If we provide ξ with the involution given by the antipodal map in the fibres, then the disk bundle $D\xi$ is an equivariant cobordism between $D\nu_1$ and $D\nu_2$. We form a manifold-with-boundary P from the disjoint union $M_1 \times I \coprod D\xi \coprod M_2 \times I$ by identifying $D\nu_i \subset \partial D\xi$ with a $\mathbf{Z}_2$-invariant tubular neighborhood of $F(M_i) \times \{1\}$ in $M_i \times \{1\}$, $i = 1, 2$, and rounding off the resulting corners. This manifold P obviously inherits an involution, providing us with an equivariant cobordism between $M_1 \coprod M_2$ and a certain manifold Q. We could go on to describe Q more precisely, but (for our purposes) it suffices to note that, by construction, the involution on Q is free. Therefore, Q bounds as a manifold-with-involution, e.g., Q is the boundary of the mapping-cylinder of the orbit map $Q \to Q/\mathbf{Z}_2$, provided with the obvious involution. It follows that M_1 and M_2 are equivariantly cobordant, which completes the proof. □

An examination of the above proof leads one immediately to a generalization (of sorts). Let G be a finite group of odd order and suppose that M_1 and M_2 are semifree closed G-manifolds of dimension n. Then, again, any equivariant homotopy-equivalence $f : M_1 \to M_2$ induces a nice one-to-one correspondence between components of the fixed-point sets and we have the following theorem.

THEOREM 2.3. *In this situation, if the normal bundles of corresponding components of the fixed-point sets are cobordant as G-vector-bundles, then M_1 and M_2 are equivariantly cobordant.*

Proof. We proceed, as in the proof of Theorem 2.2, to build an equivariant cobordism between $M_1 \coprod M_2$ and a manifold Q on which G acts freely; we have made precisely the assumption necessary to guarantee that this construction is possible. We then appeal to the fact that the forgetful homomorphism $\mathfrak{N}_*(B_G) \to \mathfrak{N}_*$ is an isomorphism (since G has odd order) to conclude that Q bounds as a free G-manifold iff Q bounds as a manifold. But Q obviously bounds as a manifold, since this is true of $M_1 \coprod M_2$. It follows that M_1 and M_2 are equivariantly cobordant. □

Among the possible applications of this result, we might single out the following corollary.

COROLLARY 2.4. *Let G have odd order and suppose that $f : M_1 \to M_2$ is an equivariant homotopy-equivalence between manifolds with semifree G-action. If f is transverse-regular on $F(M_2)$, then M_1 and M_2 are equivariantly cobordant.*

Proof. Again we may assume without loss of generality that the fixed-point sets are connected. Then the usual manipulations with characteristic numbers show that $[F(M_1), f] = [F(M_2), \mathrm{id}]$ in $\mathfrak{N}_*(F(M_2))$, which obviously implies that $f^*\nu_2$ and ν_2 are cobordant as G-vector-bundles. By the transversality assumption, $f^*\nu_2$ and ν_1 are (equivariantly) isomorphic bundles so that Theorem 2.3 may be applied to give the result. □

We should remark at this point that the examples in Section 4 seem to indicate that these results are, in some sense, the best possible, at least for $\mathbf{Z}_p$-actions, p an odd prime.

As a final application of the techniques of this section, we consider semifree actions of a finite abelian group G, obtaining a result analogous to Theorem 2.3. Specifically, we have the next theorem.

THEOREM 2.5. *Let G be a finite abelian group and let $f : M_1 \to M_2$ be an equivariant homotopy-equivalence between semifree G-manifolds. If the normal bundles of corresponding components of the fixed-point sets are cobordant as G-vector-bundles, then M_1 and M_2 are equivariantly cobordant.*

Proof. If G has odd order, this is a consequence of Theorem 2.3, and so we may as well assume that the order of G is even. Exactly as before, we may construct an equivariant cobordism between $M_1 \coprod M_2$ and a free G-manifold Q and we need only show that Q bounds equivariantly. For this purpose, we appeal to the well-known result that any group of even order contains at least one element of order 2. Choosing such an element of G gives us a free involution on the manifold Q; we may then regard Q as the boundary of the mapping cylinder of the orbit map $Q \to Q/\mathbf{Z}_2$. Finally, since G is abelian, the G-action on Q possesses an obvious extension to an action of G on this mapping cylinder; thus, Q bounds as a G-manifold and the theorem follows. □

Finally we observe that the above argument (suitably reformulated) proves a slightly different result. If the order of the abelian group G is even, a choice of an element of order 2 in G determines an inclusion $\mathbf{Z}_2 \subset G$ and, hence, an action of $\mathbf{Z}_2$ on any G-manifold. Then, if $f : M_1 \to M_2$ is an equivariant homotopy equivalence between semifree G-manifolds such that the normal bundles of corresponding components of the fixed-point sets of $\mathbf{Z}_2$ are cobordant as $G/\mathbf{Z}_2$-bundles, M_1 and M_2 are equivariantly cobordant.

3. A lifting problem. Suppose that G is a finite group and that X is a connected, locally path-connected space on which G operates freely. Then the orbit map $p : X \to X/G$ is a covering map and G appears as the group of covering transformations. If $f : X/G \to X/G$ is a given map, it may or may not be possible to find a map $\bar{f} : X \to X$ which lifts f, and even if a lifting exists, it is not unique. In this section we consider the question of whether f possesses an equivariant lifting (assuming the existence of some lifting).

Assume, then, that $\bar{f}$ is a lifting of f so that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & X \\ p\downarrow & & \downarrow p \\ X/G & \xrightarrow{f} & X/G \end{array}$$

is commutative and choose a basepoint $x_0 \in X$. Then, for any $g \in G$, $p \circ \bar{f}(gx_0) = f \circ p(gx_0) = f \circ p(x_0) = p \circ \bar{f}(x_0)$ which implies that $\bar{f}(gx_0) = \alpha_{\bar{f}}(g)\bar{f}(x_0)$ for a unique element $\alpha_{\bar{f}}(g) \in G$. Since $\bar{f} \circ g$ and $\alpha_{\bar{f}}(g) \circ \bar{f}$ are both liftings of f and since they agree at x_0, we must have $\bar{f}(gx) = \alpha_{\bar{f}}(g)\bar{f}(x)$ for all $x \in X$.

LEMMA 3.1. (1) *For each lifting $\bar{f}$ of f the function $\alpha_{\bar{f}} : G \to G$ is an endomorphism.*

(2) *If $\alpha_{\bar{f}_1}$, $\alpha_{\bar{f}_2} \in$ End (G) correspond to two different liftings of f, then there is some $g \in G$ such that $\alpha_{\bar{f}_2} = i(g) \circ \alpha_{\bar{f}_1}$, where $i(g) : G \to G$ denotes the inner automorphism determined by g.*

Proof. (1) Since G is finite, it suffices to verify that $\alpha_{\bar{f}}(g_1g_2) = \alpha_{\bar{f}}(g_1)\alpha_{\bar{f}}(g_2)$ for all $g_1, g_2 \in G$. But, for any $x \in X$

$$\alpha_{\bar{f}}(g_1g_2)\bar{f}(x) = \bar{f}(g_1g_2x) = \alpha_{\bar{f}}(g_1)\bar{f}(g_2x)$$
$$= \alpha_{\bar{f}}(g_1)\alpha_{\bar{f}}(g_2)\bar{f}(x);$$

since G acts freely on X, the assertion follows.

(2) Because $\bar{f}_1$ and $\bar{f}_2$ are both liftings of f, there is a unique $g_0 \in G$ such that $\bar{f}_2 = g_0 \circ \bar{f}_1$. This implies that for arbitrary $g \in G$ and $x \in X$

$$\alpha_{\bar{f}_2}(g)\bar{f}_2(x) = \bar{f}_2(gx) = g_0\bar{f}_1(gx) = g_0\alpha_{\bar{f}_1}(g)\bar{f}_1(x)$$
$$= g_0\alpha_{\bar{f}_1}(g)g_0^{-1}\bar{f}_2(x);$$

since G acts freely, it follows that $\alpha_{\bar{f}_2} = i(g_0) \circ \alpha_{\bar{f}_1}$. □

Remark. Note that the function

$$G \times \text{End}(G) \to \text{End}(G) : (g, \varphi) \mapsto i(g) \circ \varphi$$

describes an action of G on the set End (G). The lemma implies that, given a map $f : X/G \to X/G$ which can be lifted, the endomorphisms $\alpha_{\bar{f}}$ which correspond to the various liftings of f comprise exactly one orbit of this action. Since a lifting $\bar{f} : X \to X$ is equivariant precisely when $\alpha_{\bar{f}} = \mathrm{id}_G$, one sees that End $(G)/G$ is the natural habitat of the obstruction to finding an equivariant lifting.

There is an alternate (and, perhaps, more useful) description of the endomorphism $\alpha_{\bar{f}}$. With the above notation, let $\varphi : [0; 1] \to X$ be any path satisfying $\varphi(0) = x_0$, $\varphi(1) = \bar{f}(x_0)$. Then $p \circ \varphi : [0; 1] \to X/G$ is a path with $p \circ \varphi(0) = p(x_0)$, $p \circ \varphi(1) = p \circ \bar{f}(x_0) = f(p(x_0))$ and so determines an isomorphism

$$(p \circ \varphi)^* : \pi_1(X/G, f(p(x_0))) \overset{\cong}{\to} \pi_1(X/G, p(x_0)).$$

For simplicity we consider the case in which X is simply-connected, in which case any map can be lifted. Then the above isomorphism is independent of the particular path φ chosen; moreover, there is the usual isomorphism $\lambda : \pi_1(X/G, p(x_0)) \overset{\cong}{\to} G$ obtained by lifting loops at $p(x_0)$ to paths beginning at x_0 and examining the terminus.

LEMMA 3.2. *In the simply-connected case, the diagram*

$$\begin{array}{ccccc} \pi_1(X/G, p(x_0)) & \xrightarrow{f_*} & \pi_1(X/G, f(p(x_0))) & \xrightarrow[\cong]{(p \circ \varphi)^*} & \pi_1(X/G, p(x_0)) \\ \lambda \downarrow \cong & & & & \cong \downarrow \lambda \\ G & & \xrightarrow{\alpha_{\bar{f}}} & & G \end{array}$$

is commutative.

Remark. The result in the general case is analogous. The only real change is that everything must be done modulo the image of the appropriate version of the fundamental group of X.

Proof. This is a routine exercise in applying the definitions involved. Let $\psi : [0; 1] \to X/G$ represent $[\psi] \in \pi_1(X/G, p(x_0))$ and suppose $\bar{\psi} : [0; 1] \to X$ is a lifting of ψ with $\bar{\psi}(0) = x_0$. Then $\bar{\psi}(1) = gx_0$ for a unique $g \in G$, and we obtain that $\lambda[\psi] = g$ so that $\alpha_{\bar{f}} \circ \lambda[\psi] = \alpha_{\bar{f}}(g)$. To compute the other composite, we note that

$$(p \circ \varphi)^* \circ f_*[\psi] = [(p \circ \varphi) * (f \circ \psi) * (p \circ \varphi)^{-1}]$$

where $*$ denotes the usual composition of paths. We next observe that $\varphi * (\bar{f} \circ \bar{\psi}) * (\alpha_{\bar{f}}(g) \circ \varphi^{-1})$ is a path in X which lifts $(p \circ \varphi) * (f \circ \psi) * (p \circ \varphi)^{-1}$ and satisfies

$$\varphi * (\bar{f} \circ \bar{\psi}) * (\alpha_{\bar{f}}(g) \circ \varphi^{-1})(0) = \varphi(0) = x_0$$

and

$$\begin{aligned} \varphi * (\bar{f} \circ \bar{\psi}) * (\alpha_{\bar{f}}(g) \circ \varphi^{-1})(1) &= \alpha_{\bar{f}}(g) \circ \varphi^{-1}(1) \\ &= \alpha_{\bar{f}}(g) \circ \varphi(0) = \alpha_{\bar{f}}(g)x_0 . \end{aligned}$$

Hence, by definition, $\lambda \circ (p \circ \varphi)^* \circ f_*[\psi] = \alpha_{\bar{f}}(g)$ and the lemma is proved. □

The above considerations assume a particularly pleasant form in the case where X is simply-connected and the group G is abelian. In this case, there is a (unique) preferred isomorphism $\beta_{x_1, x_2} : \pi_1(X/G, x_2) \overset{\cong}{\rightrightarrows} \pi_1(X/G, x_1)$ for all $x_1, x_2 \in X$ and one obtains the following proposition.

PROPOSITION 3.3. *If X is a simply-connected, locally path-connected space on which the finite abelian group G acts freely, then a map $f : X/G \to X/G$ possesses an equivariant lifting if and only if the composite*

$$\pi_1(X/G, x_0) \xrightarrow{f_\#} \pi_1(X/G, f(x_0)) \xrightarrow{\beta_{x_0, f(x_0)}} \pi_1(X/G, x_0)$$

is the identity for some (and hence every) choice of the basepoint $x_0 \in X/G$.

We can also take a slightly different view of these results; again, we consider the case of a simply-connected X and a finite abelian G. Suppose that $f : X/G \to X/G$ is a homotopy-equivalence. Then the composite $\beta_{x_0, f(x_0)} \circ f_\# : \pi_1(X/G, x_0) \to \pi_1(X/G, x_0)$ is an isomorphism and so corresponds to an automorphism $\alpha_f : G \to G$. Let $\rho : G \to$ Homeo (X) denote the given free action of G on X; then $\rho \circ \alpha_f : G \to$ Homeo (X) gives a new free action of G on X and we have the result that any lifting $\bar{f}$ of f is an equivariant map $\bar{f} : (X, \rho) \to (X, \rho \circ \alpha_f)$. This observation is particularly interesting in case (X, ρ) happens to be an equivariant CW-complex (in the sense of Illman), since in that case one has Illman's strengthening of a result due to Bredon.

THEOREM 3.4 [2]. *Let G be a compact Lie group and suppose that X and Y are equivariant CW-complexes. Then a G-map $f : X \to Y$ is an equivariant homotopy-equivalence if and only if for each closed subgroup H of G the restriction $f^H : X^H \to Y^H$ induces a one-to-one correspondence between the path-components of X^H and Y^H and isomorphisms $f_\#^H : \pi_k(X^H, x) \to \pi_k(Y^H, f(x))$ for all $k \geq 1$ and every $x \in X^H$.*

Here X^H denotes the fixed-point set of the subgroup H. In our situation, this result has the following consequence.

COROLLARY 3.5. *Suppose G is a finite abelian group and X is a simply-connected free equivariant CW-complex. If $f : X/G \to X/G$ is a homotopy-equivalence and $\alpha_f \in$ Aut (G) is the corresponding automorphism, then any lifting $\bar{f} : X \to X$ of f is an equivariant homotopy-equivalence between (X, ρ) and $(X, \rho \circ \alpha_f)$.*

Proof. It suffices to check that $\bar{f}$ is a weak homotopy-equivalence in the ordinary sense, but this is obvious in view of the commutative diagram

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{\bar{f}_\#} & \pi_n(X, \bar{f}(x_0)) \\ {\scriptstyle p_\#}\downarrow & & \downarrow{\scriptstyle p_\#} \\ \pi_n(X/G, p(x_0)) & \underset{\cong}{\xrightarrow{f_\#}} & \pi_n(X/G, f(p(x_0))) \end{array}$$

in which the vertical maps are isomorphisms for $n \geq 2$. □

The above result holds, in particular, if X is a smooth G-manifold, since Illman has shown that such X are equivariant CW-complexes.

4. Some examples. Our primary goal in this section is to apply the results in Section 3 to prove the following theorem.

THEOREM 4.1. *Let p be an odd prime and let $n \in \mathbf{Z}^+$. Then there is a closed connected $4n$-manifold M^{4n} admitting distinct $\mathbf{Z}_p$-actions $\rho_1, \rho_2 : \mathbf{Z}_p \to \mathrm{Diff}\ (M)$ such that (M^{4n}, ρ_1) and (M^{4n}, ρ_2) are equivariantly homotopy-equivalent but not equivariantly cobordant.*

After establishing this result, we shall indicate briefly some of its extensions.

The proof of Theorem 4.1 depends heavily upon the work of Olum [3]; in this paragraph we summarize the necessary results. We regard S^{2k-1} as the unit sphere in $\mathbf{C}^k$, points of the latter space being denoted by $(z_0, z_1, \cdots, z_{k-1})$ with $z_i \in \mathbf{C}$. We fix a positive integer m and adopt the notation $\zeta = \exp(2\pi i/m)$. Then if $q_1, \cdots, q_{k-1}$ are positive integers less than and relatively prime to m, the map $\gamma : S^{2k-1} \to S^{2k-1}$ defined by $\gamma\ (z_0, z_1, \cdots, z_{k-1}) = (\zeta z_0, \zeta^{q_1} z_1, \cdots, \zeta^{q_{k-1}} z_{k-1})$ generates a free $\mathbf{Z}_m$-action on S^{2k-1}, and the orbit space of this action is, by definition, the lens space $L^{2k-1}(m; q_1, \cdots, q_{k-1})$. Since the fundamental group of this lens space is isomorphic to $\mathbf{Z}_m$, the basepoint is irrelevant and we suppress all mention of it. Then there is a preferred generator $\alpha \in \pi_1(L^{2k-1}(m; q_1, \cdots, q_{k-1}))$ represented by the inclusion of the 1-skeleton in the standard CW-decomposition of this manifold.

THEOREM 4.2 [3; Theorem V]. *Let $L = L^{2k-1}(m; q_1, \cdots, q_{k-1})$ and let $r \in \mathbf{Z}$ satisfy $0 \leq r < m$. If $\vartheta_r : \pi_1(L) \to \pi_1(L)$ is the endomorphism determined by $\vartheta_r(\alpha) = \alpha^r$, then ϑ_r is induced by a self-map of L and the degrees of all maps inducing ϑ_r exhaust the set of integers congruent to r^k (mod m). Moreover, two self-maps of L are homotopic iff they induce the same endomorphism of $\pi_1(L)$ and have the same $\mathbf{Z}$-degree.*

COROLLARY 4.3. *With the same notation, there exists a homotopy-equivalence $f_r : L \to L$ inducing ϑ_r on $\pi_1(L)$ iff $r^k \equiv 1$ (mod m). Note that, in particular, there are self-homotopy-equivalences of $L^{2k-1}(m; q_1, \cdots, q_{k-1})$ inducing nontrivial automorphisms of the fundamental group whenever $(k, \varphi(m)) > 1$.*

Now let p be a fixed odd prime and let $n \in \mathbf{Z}^+$. Because $\varphi(2p) = p - 1$, one has $(2n, \varphi(2p)) \geq 2$ and so there exist nontrivial solutions of the congruence $r^{2n} \equiv 1$ (mod $2p$). Choose a nontrivial solution r_0 satisfying $0 < r_0 < 2p$ and let L^{4n-1} be any of the lens spaces $L^{4n-1}(2p; q_1, \cdots, q_{2n-1})$. Then there is a homotopy-equivalence $f_{r_0} : L^{4n-1} \to L^{4n-1}$ which induces ϑ_{r_0} on $\pi_1(L^{4n-1})$ and which lifts to an equivariant homotopy-equivalence $\bar{f}_{r_0} : (S^{4n-1}, \rho) \to (S^{4n-1}, \rho \circ \vartheta_{r_0})$ where $\rho : \mathbf{Z}_{2p} \to \mathrm{Diff}\ (S^{4n-1})$ is the action which defines $L^{4n-1}(2p; q_1, \cdots, q_{2n-1})$.

We wish to view $\bar{f}_{r_0}$ in a slightly different manner. Note that, because of the usual isomorphism $\mathbf{Z}_{2p} \cong \mathbf{Z}_2 \times \mathbf{Z}_p$, giving a free action of $\mathbf{Z}_{2p}$ on a space X is equivalent to specifying free actions of both $\mathbf{Z}_2$ and $\mathbf{Z}_p$ on X, these actions being required to commute. In our situation, we receive two distinct free actions $\bar{\rho}$, $\bar{\rho} \circ \vartheta_{r_0} : \mathbf{Z}_p \to \text{Diff}\ (S^{4n-1})$, each commuting with the antipodal involution, and an equivariant homotopy-equivalence $\bar{f}_{r_0} : (S^{4n-1}, \bar{\rho}) \to (S^{4n-1}, \bar{\rho} \circ \vartheta_{r_0})$ which also commutes with the antipodal map. If we define $D\lambda$ to be the disk-bundle of the canonical line bundle over $RP(4n - 1)$, then $D\lambda$ is a smooth $4n$-manifold with boundary S^{4n-1} and the above $\mathbf{Z}_p$-actions on S^{4n-1} possess obvious extensions to free actions ${\rho_1}^{(1)}$, ${\rho_2}^{(1)} : \mathbf{Z}_p \to \text{Diff}\ (D\lambda)$; moreover, $\bar{f}_{r_0}$ extends to an equivariant homotopy-equivalence $f_1 : (D\lambda, {\rho_1}^{(1)}) \to (D\lambda, {\rho_2}^{(1)})$. We may also realize S^{4n-1} as the boundary of $D^{4n} \subset \mathbf{C}^{2n}$, extending the actions $\bar{\rho}$, $\bar{\rho} \circ \vartheta_{r_0}$ to (unitary) actions ${\rho_1}^{(2)}$, ${\rho_2}^{(2)}$ on the disk; then $\bar{f}_{r_0}$ gives rise to an equivariant homotopy-equivalence $f_2 : (D^{4n}, {\rho_1}^{(2)}) \to (D^{4n}, {\rho_2}^{(2)})$ in an obvious manner. Finally, we define M^{4n} to be $D\lambda \cup D^{4n}$, where these manifolds are to be identified along their common boundary; piecing together all of the above data, we obtain actions ρ_1, $\rho_2 : \mathbf{Z}_p \to \text{Diff}\ (M)$ and an equivariant map $f : (M^{4n}, \rho_1) \to (M^{4n}, \rho_2)$.

LEMMA 4.4. *The map f is an equivariant homotopy-equivalence.*

Proof. In view of the Bredon–Illman result (Theorem 3.4), it is enough to show that f is a weak homotopy-equivalence in the ordinary sense. To accomplish this, we first note that a straightforward Mayer–Vietoris argument establishes that $f_* : H_*(M; \mathbf{Z}) \to H_*(M; \mathbf{Z})$ is an isomorphism in all dimensions. Then, choosing a basepoint $x_0 \in S^{4n-1}$, we observe that the homomorphism $\pi_k(D\lambda, x_0) \to \pi_k(M, x_0)$ is an isomorphism for $k \leq 4n - 2$; since the diagram

$$\begin{array}{ccc} \pi_k(D\lambda, x_0) & \longrightarrow & \pi_k(M, x_0) \\ (f_1)_\# \downarrow \cong & & \downarrow f_\# \\ \pi_k(D\lambda, f_1(x_0)) & \longrightarrow & \pi_k(M, f(x_0)) \end{array}$$

is commutative, it follows that $f_\# : \pi_k(M, x_0) \to \pi_k(M, f(x_0))$ is an isomorphism for $k \leq 4n - 2$. Because $4n - 2 \geq 2$, the fact that f is a weak homotopy-equivalence now follows in the usual way from the relative Hurewicz theorem, and this proves our assertion. □

Theorem 4.1 is now clearly a consequence of the following lemma.

LEMMA 4.5. *(M, ρ_1) and (M, ρ_2) are not equivariantly cobordant.*

Proof. This is immediate. The fixed point set of any alleged equivariant cobordism between (M, ρ_1) and (M, ρ_2) would have, as one of its components, a smoothly-embedded arc joining the fixed-point in (M, ρ_1) to that in (M, ρ_2) and one could conclude that the normal representations of $\mathbf{Z}_p$ at these fixed-points are equivalent unitary representations. But these representations are exactly the representations ${\rho_1}^{(2)}$, ${\rho_2}^{(2)}$ which are obviously inequivalent. The assertion follows. □

Remark 1. There is another procedure for constructing the manifolds M^{4n}. Let $T : \mathbf{R}^{4n+1} \to \mathbf{R}^{4n+1}$ be the orthogonal involution whose matrix with respect to the standard basis is diag $(-1, -1, \cdots, -1, 1)$ and let $R : \mathbf{R}^{4n+1} \to \mathbf{R}^{4n+1}$ denote reflection in the hyperplane spanned by the first $4n$ standard basis vectors. Then $R \circ T : \mathbf{R}^{4n+1} \to \mathbf{R}^{4n+1}$ restricts to a free involution on the unit-sphere S^{4n} and it is easily verified that the orbit space of this involution is exactly the manifold M^{4n}. One can also describe $\mathbf{Z}_p$-actions on S^{4n} which induce the actions ρ_1, ρ_2 on M^{4n}.

Remark 2. Our construction will also yield $2k$-dimensional examples of the same sort for any k such that the congruence $r^k \equiv 1 \pmod{2p}$ has a nontrivial solution.

Remark 3. If F^m is a connected nonbounding m-manifold and $n \in \mathbf{Z}^+$, the $(4n + m)$-manifold $M^{4n} \times F^m$ obviously admits $\mathbf{Z}_p$-actions which are equivariantly homotopy-equivalent but not equivariantly cobordant. Using this observation, one can obtain examples in every even dimension greater than or equal to 4 and every dimension greater than or equal to 9. Do there exist examples in the remaining dimensions greater than or equal to 2? (Trivially, there can be no 1-dimensional examples.)

Remark 4. The manifolds M^{4n} are not simply-connected (in fact, they are nonorientable) and, therefore, the procedure in Remark 3 cannot produce a simply-connected example. Are there any simply-connected examples?

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