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Moore, R. L., Trans. Amer. Math. Soc., 17, 1916 (131-164). Hereafter we will refer to this paper as "Foundations."

<sup>10</sup> "Foundations," theorem 6.

<sup>11</sup> Ibid., theorem 28.

<sup>12</sup> Ibid., theorem 29.

<sup>13</sup> If K is a closed subset of M, M(K) is a continuous curve. See "Arc-Curves," theorem 7.

<sup>14</sup> "Arc-Curves," theorem 8.

<sup>15</sup> If P is a cut-point of M(K), every component of M(K)-P contains at least one point of K. See "Arc-Curves," theorem 9, part (1).

<sup>16</sup> Whyburn, G. T., these Proceedings, 13, 1927 (31-38), theorem 10.

<sup>17</sup> Moore, R. L., Math. Zeit., 15, 1922 (255).

<sup>18</sup> "Arc-Curves," theorem 10, part (5).

<sup>19</sup> Schoenflies, A., *Die Entwicklung der Lehre von den Punktmannigfaltigkeiten*, Zweite Teil, Leipzig, **1908** (237).

<sup>20</sup> Moore, R. L., these Proceedings, 11, 1925 (469-476), footnote 5.

<sup>21</sup> See an abstract of a paper by R. G. Lubben, "The Separation of Mutually Separated Subsets of a Continuum by Curves," *Bull. Amer. Math. Soc.*, **32**, 1926 (114).

<sup>22</sup> Subscripts are reduced modulo n.

ON SOME PROPERTIES OF ONE-VALUED TRANSFORMATIONS OF MANIFOLDS

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1. Some Constants of a Transformation.—Given any set of  $\nu$ -cycles in an *n*-complex  $\mu^n$ , its maximum number of independent cycles with respect to homologies shall be called the "rank" of the set; it is not greater than the  $\nu$ th Betti number  $\pi_{\nu}$  of  $\mu^n$ . Consider another *n*-complex  $M^n$  transformed into  $\mu^n$  by a one-valued continuous transformation f and the set of all  $f(c_i^{\nu})$ , where the  $c_i^{\nu}$  are the  $\nu$ -cycles of  $M^n$ ; then the rank  $r_{\nu}$  of this set is a constant of  $f(\nu = 0, 1, ..., n)$ .

If  $c_1^{\nu}, c_2^{\nu}, \ldots, c_{p_{\nu}}^{\nu}$ , where  $p_{\nu}$  is the  $\nu$ th Betti number of  $M^n$ , form a fundamental set in  $M^n$  and  $\gamma_1^{\nu}, \gamma_2^{\nu}, \ldots, \gamma_{\pi_{\nu}}^{\nu}$  a fundamental set in  $\mu^n$ , then the transformations of the  $\nu$ -cycles define a system of homologies<sup>1</sup>

$$f(c_j^{\nu}) \approx \sum_{k=1}^{\pi_{\nu}} \alpha_{jk}^{\nu} \dot{\gamma}_k^{\nu} \quad (j = 1, 2, \dots, p_{\nu})$$
(1)

and  $r_{\nu}$  is the rank of the matrix

$$A_{\nu} = \left| \left| \alpha_{jk}^{\nu} \right| \right|.$$

When the  $c_i^{\nu}$  and  $\gamma_k^{\nu}$  are replaced by other fundamental sets,  $A_{\nu}$  is trans-

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formed by square matrices whose determinants are equal to  $\pm 1$ . Therefore, also the invariant factors of the matrices  $A_{\nu}$  are constants of f.

If  $p_{\nu} = \pi_{\nu}$ , for instance in the important case where  $\mu^n$  is identical with  $M^n$ , the  $A_{\nu}$  are square matrices and their determinants

$$a_{\nu} = |A_{\nu}| = |\alpha_{jk}^{\nu}| \quad (j, k = 1, 2, ..., p_{\nu})$$

(whose absolute values are the products of the invariant factors), are also constants of f, to be considered.

In this paper we shall deal with geometrical properties of f which are expressible in terms of the constants  $r_{\nu}$  and  $a_{\nu}$ . We shall assume that  $M^n$  and  $\mu^n$  are closed connected manifolds. Therefore,

$$p_0 = p_n = \pi_0 = \pi_n = 1, \quad A_0 = || 1 ||, \quad A_n = || a_n ||,$$

where the constant  $a_n = a$  is the Brouwer degree<sup>2</sup> of f.

2. A Formula of Contragredience.-Let

$$L_{\nu} = \left| \left| \left( c_{i}^{\nu} \cdot c_{j}^{n-\nu} \right) \right| \right|, \quad \Lambda_{\nu} = \left| \left| \left( \gamma_{i}^{\nu} \cdot \gamma_{j}^{n-\nu} \right) \right| \right|$$

be the intersection matrices of the fundamental sets  $c_i^{\nu}$ ,  $c_j^{n-\nu}$  in  $M^n$  and  $\gamma_i^{\nu}$ ,  $\gamma_j^{n-\nu}$  in  $\mu^n$  respectively. Their determinants are  $\pm 1$ ,<sup>3</sup> hence the inverses  $L_{\nu}^{-1}$ ,  $\Lambda_{\nu}^{-1}$  are defined. We shall denote by  $E_m$  the matrix unity of order *m*, and as usual by *B'* the transverse of a matrix *B*. We shall prove below (Nos. 6, 7) the following relation of contragredience

$$(L_{n-\nu}^{-1} A_{n-\nu} \Lambda_{n-\nu})' A_{\nu} = a E_{\pi\nu}, \qquad (2)$$

from which all our results will follow at once.

3. Properties of the Constants  $r_{\nu}$ .—Let first be  $a \neq 0$ . Then  $aE_{\pi_{\nu}}$  is of rank  $\pi_{\nu}$ ; therefore, the rank of no matrix on the left hand of (2) can be  $\langle \pi_{\nu} \rangle$  so that

THEOREM I. If the degree of  $f \neq 0$  then  $r_{\nu} = \pi_{\nu}$  ( $\nu = 0, 1, ..., n$ ).

 $A_{\nu}$  has  $p_{\nu}$  rows; hence

THEOREM Ia. If, for a certain  $\nu$ ,  $p_{\nu} < \pi_{\nu}$ , then the degree of f must be zero.

A special case of Ia is

THEOREM Ib. It is not possible to transform the n-sphere into a manifold of which at least one Betti number  $\pi_{\nu}(\nu = 1, 2, ..., n-1)$  is > 0, with a degree different from zero.

Let us now consider the case a = 0. Since the determinants of  $L_{n-\nu}^{-1}$ and  $\Lambda_{n-\nu}^{-1}$  are  $\neq 0$ , the matrix  $(L_{n-\nu}^{-1} A_{n-\nu} \Lambda_{n-\nu})' = B = || b_{ij} ||$  has the rank  $r_{n-\nu}$ . Hence the system

$$\sum_{j=1}^{p_{\nu}} b_{ij} x_j = 0 \quad (i = 1, 2, ..., \pi_{\nu})$$

has  $\pi_{\nu} - r_{n-\nu}$  linearly independent solutions  $x_{1m}, x_{2m}, \ldots, x_{p_{\nu}m}$   $(m = 1, 2, \ldots, \pi_{\nu} - r_{n-\nu})$ . Since a = 0, according to (2) the elements  $\alpha_{jk}^{\nu}$  of  $A_{\nu}$  satisfy the equations

$$\sum_{j=1}^{p_{\nu}} b_{ij} \alpha_{jk}^{\nu} = 0 \quad (k = 1, 2, ..., \pi_{\nu}).$$

Hence, among the columns  $\alpha_{1k}^{\nu}, \alpha_{2k}^{\nu}, \ldots, \alpha_{p_{\nu}k}^{\nu}$  of  $A_{\nu}$  there are at most  $\pi_{\nu} - r_{n-\nu}$  linearly independent; therefore,

THEOREM II. If f is of degree 0 then  $r_{\nu} + r_{n-\nu} \leq \pi_{\nu}$  ( $\nu = 0, 1, ..., n$ ). From now on we shall always assume that  $M^n$  and  $\mu^n$  are identical, although our results will also hold when merely  $p_{\nu} = \pi_{\nu}$ , even if  $M^n \neq \mu^n$ .

The sets of all  $\nu$ -cycles of  $M^n$  and of all  $(n - \nu)$ -cycles have the common rank  $p_{\nu}$ , so that the sum of their ranks is  $2p_{\nu} = 2\pi_{\nu}$ . Therefore, theorems I and II may now be interpreted in the following manner: If the transformation f of  $M^n$  into itself is of degree  $\neq 0$  then there exist no  $c^{\nu}$  nor  $c^{n-\nu}$ not  $\approx 0$  with an f-transform  $\approx 0$ . On the other hand, when f is of degree zero then for each  $\nu$  there exist  $\nu$  or  $(n - \nu)$ -cycles not  $\approx 0$ , but whose ftransforms are  $\approx 0$ . The sum of the ranks of these "degenerating"  $\nu$ - and  $(n - \nu)$ -cycles is at least half the sum of the ranks of all  $\nu$ - and  $(n - \nu)$ -cycles.

4. A Relation between the Constants  $a_{\nu}$ ; an Application.—The determinants of the matrices L and  $\Lambda$  are always  $\pm 1$ ; under the assumption  $M^n = \mu^n$  we have even  $|L_{n-\nu}^{-1}| = |\Lambda_{n-\nu}| = \pm 1$ , because  $L_{n-\nu} = \Lambda_{n-\nu}$ . Therefore, by computing the determinants in (2) we find:

**THEOREM III.** The constants  $a_r$  and the degree a of a transformation of  $M^n$  into itself are related by the equations

$$a_{\nu} \cdot a_{n-\nu} = a^{\nu} \quad (\nu = 0, 1, \ldots, n).$$

We give an example of an application of this theorem: let  $M^n$  be the complex projective plane P. It is a 4-dimensional closed orientable manifold with the Betti numbers

$$p_0 = p_4 = 1$$
,  $p_1 = p_3 = 0$ ,  $p_2 = 1$ .

Therefore, III gives  $a_2^2 = a$ , so that we have

**THEOREM IVa.** The degree of any transformation of the complex projective plane into itself is a perfect square.

An example of a transformation with the degree  $b^2$  with arbitrary b is given by  $z'_i = z^b_i$  (i = 1, 2, 3), where  $z_1: z_2: z_3$  represents a point of P and  $z'_1: z'_2: z'_3$  its image.

Furthermore, according to the formula of Lefschetz<sup>4</sup> the algebraic number of fixed points under a transformation of P is  $1 + a_2 + a_4$ ; but now we see that this number is equal to  $1 + a_2 + a_2^2 \neq 0$  whatever the integer  $a_2$ . Therefore, in analogy with a well-known property of the real projective plane,

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**THEOREM** IVb. Every transformation of the complex projective plane has at least one fixed point.

5. The Behavior of the Kronecker-Indices under a Transformation.—In the case  $M^n = \mu^n$  (2) may be written

$$(L_{n-\nu}^{-1} A_{n-\nu} L_{n-\nu})' A_{\nu} = a E_{p_{\nu}}$$
(2')

or, since  $L_{n-\nu} = (-1)^{\nu(n+1)} L'_{\nu}$  and (BC)' = C'B',

$$L_{\nu}A'_{n-\nu}L_{\nu}^{-1}A_{\nu} = aE_{p_{\nu}}.$$
 (2")

When  $a \neq 0$  then also  $A_{\nu} \neq 0$  (Th. I). Hence  $A_{\nu}^{-1}$  exists and from (2") follows  $L_{\nu}A'_{n-\nu}L_{\nu}^{-1} = aE_{\rho_{\nu}}A_{\nu}^{-1}$ . Furthermore, since  $E_{\rho_{\nu}}$  and  $A_{\nu}$  are commutative,  $L_{\nu}A'_{n-\nu}L_{\nu}^{-1} = A_{\nu}^{-1}.aE_{\rho_{\nu}}$ ,

$$A_{\nu}L_{\nu}A'_{n-\nu} = aL_{\nu}.$$
(3)

Consider now the transformations of fundamental sets  $c_i^{\nu}$ ,  $c_j^{n-\nu}$  given by (1):

$$f(c_{i}^{\nu}) = \bar{c}_{i}^{\nu} \approx \sum_{k=1}^{\nu_{\mu}} \alpha_{ik}^{\nu} c_{k}^{\nu}$$

$$f(c_{j}^{n-\nu}) = \bar{c}_{j}^{n-\nu} \approx \sum_{l=1}^{p_{\nu}} \alpha_{jl}^{n-\nu} c_{l}^{n-\nu}$$
(1')

and the intersection matrix of the transformed cycles  $\overline{L}_{\nu} = || (\overline{c}_i^{\nu} \cdot \overline{c}_j^{n-\nu}) ||$ . We have from (1')

$$(\overline{c_i}, \overline{c_j}^{n-\nu}) = \sum_{k,l} \alpha_{ik}^{\nu} (c_k^{\nu}, c_l^{n-\nu}) \alpha_{jl}^{n-\nu},$$

hence,

 $\overline{L}_{\nu} = A_{\nu}L_{\nu}A'_{n-\nu}$ 

and in view of (3) the equation

$$\overline{L}_{\nu} = aL_{\nu},\tag{5}$$

from which follows  $(\overline{c}_i^{\nu}, \overline{c}_j^{n-\nu}) = a(c_i^{\nu}, c_j^{n-\nu})$  and generally

$$(\bar{c}^{\nu}.\bar{c}^{n-\nu}) = a(c^{\nu}.c^{n-\nu})$$
(5')

for arbitrary cycles  $c^{\nu}$ ,  $c^{n-\nu}$ . Therefore, we have proved

**THEOREM V.** When M undergoes a one-valued transformation of degree  $\neq$  0, all Kronecker-indices of cycle-pairs are multiplied by the degree.

As a corollary of this theorem we have the fact that the property of a pair of cycles  $c^{\nu}$ ,  $c^{n-\nu}$  to intersect or not to intersect one another in the algebraic sense is not only invariant under homeomorphisms of  $M^n$  but under all transformations of  $M^n$  into itself, which have a degree  $\neq 0$ .

We may point out that the assumption  $a \neq 0$  which we have used in deriving (3) and, therefore, in proving V is necessary; for a simple transformation of a surface of genus 2 shows that the statements of theorem V and of its corollary are not correct in the case a = 0.

6. The Method of the Product Manifold, introduced by Lefschetz, will be used in proving the contragredience formula (2). It may be described as follows:<sup>5</sup>

If x and  $\xi$  are points of  $M^n$  and  $\mu^n$ , respectively, and  $X = x \times \xi$  the point of the product  $M^n \times \mu^n$  which represents the pair x,  $\xi$ , then we write

$$x = P(X), \quad \xi = \Pi(X) \quad [X = x \times \xi] \tag{6}$$

and call P and II the "projections" on  $M^n$  and  $\mu^n$ . We consider an *n*-cycle  $\Gamma^n$  in  $M^n \times \mu^n$  which generally is singular and say that x and  $\xi$  are corresponding with respect to  $\Gamma^n$  if there exists an X on  $\Gamma^n$ , so that (6) hold. If we interpret  $\xi$  as the image of x then this correspondence defines a "transformation" T of  $M^n$  into  $\mu^n$ , which may be symbolically expressed by

$$T(x) = \Pi P^{-1}(x) \tag{7a}$$

and similarly a transformation  $T^{-1}$  of  $\mu^n$  into  $M^n$ , the "inverse" of T:

$$T^{-1}(\xi) = P\Pi^{-1}(\xi). \tag{7b}$$

T and  $T^{-1}$  are both, in general, multiply valued.

The following construction gives the image  $T(c^{\nu})$  of a  $\nu$ -cycle  $c^{\nu}$  of  $M^{n}$ : the "cylinder"  $c^{\nu} \times \mu^{n}$  erected on  $c^{\nu}$  in  $M^{n} \times \mu^{n}$  intersects  $\Gamma^{n}$  in a  $\nu$ -cycle  $\Gamma^{n}.c^{\nu} \times \mu^{n}$  and the projection of this cycle on  $\mu^{n}$  is the image of  $c^{\nu}$ :

$$T(c^{\nu}) = \Pi(\Gamma^{n} . c^{\nu} \times \mu^{n}).$$
(8a)

Similarly, we determine

$$T^{-1}(\gamma^{\nu}) = P(\Gamma^{n}.M^{n} \times \gamma^{\nu})$$
(8b)

for each cycle  $\gamma^{\nu}$  of  $\mu^{n}$ .

Concerning fundamental sets of cycles in  $M^n \times \mu^n$  we have the theorem<sup>6</sup> that, if  $c_i^{\lambda}$ ,  $\gamma_j^{\lambda}$  ( $\lambda = 0, 1, \ldots, n$ ;  $i = 1, 2, \ldots, p_{\lambda}$ ;  $j = 1, 2, \ldots, \pi_{\lambda}$ ), are fundamental sets in  $M^n$  and  $\mu^n$ , the set of all products  $c_i^{\lambda} \times \gamma_j^{\nu-\lambda}$  ( $\lambda = 0, 1, \ldots, \nu$ ) forms a fundamental set of  $\nu$ -cycles in  $M^n \times \mu^n$ . The projection of  $\Delta^{\nu} = c_i^{\lambda} \times \gamma_j^{\nu-\lambda}$  on  $M^n$ 

$$P(\Delta^{\nu}) = P(c_i^{\lambda} \times \gamma_j^{\nu-\lambda}) = c_i^{\lambda}$$

is, considered as v-cycle in  $M^n$ ,  $\approx 0$ , if  $\lambda < \nu$ ; likewise is

$$\Pi(\Delta^{\nu}) = \Pi(c_i^{\lambda} \times \gamma_j^{\nu-\lambda}) = \gamma_j^{\nu-\lambda}$$

 $\approx 0$  on  $\mu^n$ , if  $\lambda > 0$ . Thus, if we have any  $\nu$ -cycle

$$\Gamma^{\nu} \approx \sum_{\lambda=0}^{\nu} \sum_{i,j} \eta_{ij}^{\lambda} . c_i^{\lambda} \times \gamma_j^{\nu-\lambda} \quad (i = 1, 2, \ldots, p_{\lambda}; j = 1, 2, \ldots, \pi_{\lambda}) \quad (9)$$

then

$$P(\Gamma^{\nu}) \approx \sum_{i=1}^{p_{\nu}} \eta_{i1}^{\nu} c_i^{\nu} \quad [\text{on } M^n]$$
(9a)

$$\Pi(\Gamma^{\nu}) \approx \sum_{j=1}^{\pi\nu} \eta_{ij}^{0} \gamma_{j}^{\nu} \quad [\text{on } \mu^{n}].$$
(9b)

Now let the cycle  $\Gamma^n$  which defines the transformation T be

$$\Gamma^{n} \approx \sum_{\lambda=0}^{n} \sum_{i,j} \epsilon_{ij}^{\lambda} . c_{i}^{\lambda} \times \gamma_{j}^{n-\lambda}.$$
 (10)

In calculating  $T(c_k^{\nu})$  and  $T^{-1}(\gamma_k^{\nu})$  by (10), (8*a*), (8*b*) we shall make use of <sup>7</sup>  $(c^{\rho} \times \gamma^r . c^q \times \gamma^s) \approx (-1)^{(n-\rho)(n-s)} (c^{\rho} . c^q) \times (\gamma^r . \gamma^s)$  [on  $M^n \times \mu^n$ ]. (11) From (10), (11) and  $(\gamma^{n-\lambda} . \mu^n) = \gamma^{n-\lambda}$  we find

$$\Gamma^{n}.c_{k}^{\nu} \times \mu^{n} \approx \sum_{\lambda=0}^{n} \sum_{i,j} \epsilon_{ij}^{\lambda} (c_{i}^{\lambda} \times \gamma_{j}^{n-\lambda}.c_{k}^{\gamma} \times \mu^{n})$$
$$\approx \sum_{\lambda=0}^{n} \sum_{i,j} \epsilon_{ij}^{\lambda} (c_{i}^{\lambda}.c_{k}^{\nu}) \times \gamma_{j}^{n-\lambda}.$$

In view of  $(c_i^{\lambda}, c_k^{\nu}) = 0$ , if  $\lambda + \nu < n$ , and of (9b), for II  $(\Gamma^n, c_k^{\nu} \times \mu^n)$  the only terms that are essential are those in which  $(\gamma_j^{n-\lambda}, \mu^n)$  is  $\nu$ -dimensional, i.e.  $\lambda = n - \nu$ ; hence,

$$T(c_k^{\nu}) = \Pi(\Gamma^n . c_k^{\nu} \times \mu^n) \approx \sum_{i,j} \epsilon_{ij}^{n-\nu} . (c_i^{n-\nu} . c_k^{\nu}) . \gamma_j^{\nu}.$$
(12a)

Therefore, if

$$T(c_k^{\nu}) \approx \sum_{j=1}^{\pi_{\nu}} \alpha_{kj}^{\nu} \gamma_j^{\nu}, \qquad (13a)$$

then we get from (12a)

$$\alpha_{kj}^{\nu} = \sum_{i=1}^{p_{\nu}} e_{ij}^{n-\nu} (c_i^{n-\nu} . c_k^{\nu})$$

or in terms of matrices, with  $\epsilon_{n-\nu} = ||e_{ij}^{n-\nu}||$ ,  $A_{\nu} = ||\alpha_{kj}^{\nu}||$ 

$$A_{\nu} = L'_{n-\nu} \epsilon_{n-\nu} = (-1)^{\nu(n+1)} L_{\nu} \epsilon_{n-\nu}.^{8}$$
(14a)

Similarly, let be for  $T^{-1}$ 

$$T^{-1}(\gamma_k^{\nu}) \approx \sum_{i=1}^{p_{\nu}} \beta_{ki}^{\nu} c_i^{\nu}.$$

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Then by means of (10), (11) and in view of  $(c^{\lambda} M^{n}) = c^{\lambda}$ :

$$\Gamma^{n}.M^{n} \times \gamma_{k}^{\nu} \approx \sum_{\lambda=0}^{n} \sum_{ij} \epsilon_{ij}^{\lambda}.c_{i}^{\lambda} \times \gamma_{j}^{n-\lambda}.M^{n} \times \gamma_{k}^{\nu}$$
$$\approx \sum_{\lambda=0}^{n} \sum_{ij} \epsilon_{ij}^{\lambda} (-1)^{(n-\lambda)(n-\nu)} c_{i}^{\lambda} \times (\gamma_{j}^{n-\lambda}.\gamma_{k}^{\nu}).$$

According to  $(\gamma^{n-\lambda}, \gamma^{\nu}) = 0$  for  $\lambda > \nu$  and to (9a) we are only interested in those terms on the right hand where  $c_i^{\lambda}$  is of dimensionality  $\nu$ , i.e.,  $\lambda = \nu$ ; hence,

$$T^{-1}(\gamma_k^{\nu}) = P\left(\Gamma^n . M^n \times \gamma_k^{\nu}\right) \approx \sum_{i,j} \epsilon_{ij}^{\nu} (-1)^{n-\nu} (\gamma_j^{n-\nu} . \gamma_k^{\nu}) c_i^{\nu} \quad (12b)$$

and thus from (13b)

$$B_{ki}^{\nu} = (-1)^{n-\nu} \sum_{j=1}^{\pi_{\nu}} \epsilon_{ij}^{\nu} (\gamma_j^{n-\nu}, \gamma_k^{\nu}),$$

i.e., if  $B_{\nu} = \left| \left| \beta_{ki}^{\nu} \right| \right|$ 

$$B_{\nu} = (-1)^{n-\nu} \Lambda'_{n-\nu} \epsilon'_{\nu}. \tag{14b}$$

Now one sees that between the matrices A and B which define the cycle transformations under T and  $T^{-1}$  there must hold a certain relation: On replacing  $\nu$  by  $n - \nu$  in (14*a*) we find  $\epsilon_{\nu} = L_{\nu}^{\prime - 1}A_{n-\nu}$ , and since  $L_{n-\nu} = (-1)^{\nu(n-1)}L_{\nu}^{\prime}$ .

$$\epsilon'_{\nu} = (-1)^{\nu(n+1)} A'_{n-\nu} L'^{-1}_{n-\nu}$$

Therefore, from (14b)

$$B_{\nu} = (-1)^{n(\nu+1)} \Lambda'_{n-\nu} A'_{n-\nu} L'^{-1}_{n-\nu} = (-1)^{n(\nu+1)} (L^{-1}_{n-\nu} A_{n-\nu} \Lambda_{n-\nu})'.$$
(15)

Let now a cycle  $\gamma_k^{\nu}$  of  $\mu^n$  be transformed into  $c^{\nu} = T^{-1}(\gamma_k^{\nu})$  and then back into  $T(c^{\nu}) = TT^{-1}(\gamma_k^{\nu}) = \overline{\gamma}_k^{\nu}$ . We have

$$TT^{-1}(\gamma_k^{\nu}) = \overline{\gamma}_k^{\nu} \approx \sum_{j=1}^{\pi_{\nu}} u_{kj}^{\nu} \gamma_j^{\nu}$$

with a square matrix  $U_{\nu} = || u_{kj}^{\nu} ||$ . Then the homologies

$$T^{-1}(\gamma_k^{\nu}) = c^{\nu} \approx \sum_{i=1}^{p_{\nu}} \beta_{ki}^{\nu} c_i^{\nu}$$
$$T(c^{\nu}) = \overline{\gamma}_k^{\nu} \approx \sum_i \beta_{ki}^{\nu} T(c_i^{\nu}) = \sum_{i,j} \beta_{ki}^{\nu} \alpha_{ij}^{\nu} \gamma_j^{\nu}$$

show that

$$B_{\nu}A_{\nu} = U_{\nu}$$

Hence the matrices  $U_{\nu}$  belonging to the transformation  $TT^{-1}$  of  $\mu^{n}$  into

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itself are according to (15) expressible by the matrices A in the following manner:

$$U_{\nu} = (-1)^{n(\nu+1)} \cdot (L_{n-\nu}^{-1}A_{n-\nu}\Lambda_{n-\nu})'A_{\nu}.$$
(16)

7. Proof of the Contragredience Formula (2).—Up to this point we did not separate one-valued and multiply-valued transformations T. The introduction of  $TT^{-1}$  marks the place where they naturally part. If we go from  $\gamma^{\nu}$  to  $T^{-1}(\gamma^{\nu}) = c^{\nu}$  and from  $c^{\nu}$  back to  $T(c^{\nu}) = TT^{-1}(\gamma^{\nu}) = \overline{\gamma}^{\nu}$ , then in the general case we do not return to  $\gamma^{\nu}$ . For, although the cylinder  $c^{\nu} \times \mu^{n}$  has in common with  $\Gamma^{n}$  the cycle  $\Gamma^{n} . M^{n} \times \gamma^{\nu}$ , of which the projection on  $\mu^{n}$  is  $\gamma^{\nu}$ , this cycle is not the whole intersection  $\Gamma^{n} . c^{\nu} \times \mu^{n}$  and, therefore, the projection  $\Pi(\Gamma^{n} . c^{\nu} \times \mu^{n})$  is different from  $\gamma^{\nu}$ . When, on the contrary, T is one-valued, then for each point x of  $M^{n}$  the product  $x \times \mu^{n}$ has only the point  $X = x \times T(x)$  in common with  $\Gamma^{n}$ ; therefore, for each cycle  $\Delta^{\nu}$  of  $\Gamma^{n}$  the cylinder  $P(\Delta^{\nu}) \times \mu^{n}$  intersects  $\Gamma^{n}$  only in the points of  $\Delta^{\nu}$ ; hence,

$$\Gamma^n . P(\Delta^\nu) \ \times \ \mu^n = \Delta^\nu. \tag{17}$$

If we take  $\Delta^{\nu} = \Gamma^{n} . M^{n} \times \gamma^{\nu}$ , then from (8b) and (17) follows

$$\Gamma^{n}.T^{-1}(\gamma^{\nu}) \times \mu^{n} = \Gamma^{n}.M^{n} \times \gamma^{\nu}$$
(18)

and from (8a)

$$TT^{-1}(\gamma^{\nu}) = \Pi(\Gamma^{n}.M^{n} \times \gamma^{\nu})$$
(19)

for each cycle  $\gamma^{\nu}$  of  $\mu^{n}$ . The right hand of this equation may be determined by the method which has yielded (12*a*); from (10), (11) and  $(c^{\lambda}, M^{n}) = c^{\lambda}$  follows

$$\Gamma^{n}.M^{n} \times \gamma^{\nu} \approx \sum_{\lambda=0}^{n} \sum_{ij} \epsilon_{ij}^{\lambda} (-1)^{(n-\lambda)(n-\nu)} c_{i}^{\lambda} \times (\gamma_{j}^{\lambda-\lambda}.\gamma^{\nu}).$$

For the projection  $\Pi$  we may omit all terms  $(\gamma_j^{n-\mu}, \gamma^{\nu})$  with dimensionality  $\neq \nu$ , i.e., with  $\lambda \neq 0$ ; hence,

$$\Pi(\Gamma^n.M^n\times\gamma^\nu)\approx\sum_{i,j}\epsilon^0_{ij}(-1)^{n(n-\nu)}\gamma^\nu.$$

But because  $p_0 = \pi_0 = 1$ , the matrix  $\| \epsilon_0 \|$  has only one element  $\epsilon_0$ ; from (14*a*) follows  $\epsilon_0 = a_n$ , where  $a_n$  is the only element of the matrix  $A_n$  and this element is by definition (cf. Nr. 1), the *degree a* of *T*. Hence,

$$\Pi(\Gamma^n . M^n \times \gamma^{\nu}) \approx (-1)^{n(\nu+1)} a \gamma^{\nu}$$
(20a)

and from (19)

$$TT^{-1}(\gamma^{\nu}) \approx (-1)^{n(\nu+1)} a \gamma^{\nu}.$$
(20b)

Therefore, the matrix  $U_{\nu}$  defined in Nr. 6 has been determined under the assumption that T is one-valued:

$$U_{\nu} = (-1)^{n(\nu+1)} a E_{\pi_{\nu}} \tag{21}$$

where  $E_{\pi_{\nu}}$  is the matrix unity. From (21) and (16) formula (2) follows immediately.

<sup>1</sup> Cf. S. Lefschetz, (a) Trans. Am. Math. Soc., 28, pp. 1–49; (b) Trans. Am. Math. Soc., 29, pp. 429–462; particularly p. 32 of (a). The sign " $\approx$ " introduced in (a) means " $\sim$ " mod. zero-divisors. The fundamental sets of this paper are all with respect to the operation  $\approx$ .

<sup>2</sup> L. E. Y. Brouwer, Mathem. Annalen, 71, pp. 92-115.

<sup>3</sup> O. Veblen, Trans. Am. Math. Soc., 25, pp. 540-550.

<sup>4</sup> (a), formula 1.1; (b), formula 10.5.

<sup>5</sup> A great deal of Nr. 6 is only a summarizing report on facts which are included in the papers of Lefschetz, quoted above.

<sup>6</sup> Lefschetz, (a) No. 52.

<sup>7</sup> Lefschetz, (a) No. 55. The proof of this formula, not explicitly given there, can be obtained easily by the same considerations as for the formulas of (a) Nos. 53, 54.

<sup>8</sup> Lefschetz, (b) 9.2.

# HARMONY AS A PRINCIPLE OF MATHEMATICAL DEVELOPMENT

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In the preface of volume 6 of the "Collected Works" of Sophus Lie the editor remarked that these works constitute an artistic contribution which is entirely comparable with that of Beethoven. In both cases a few apparently trivial motives dominated the entire creation. In the case of Lie it was mainly the explicit use of the concept of group in domains where its dominance had not been explicitly recognized before his time. This concept was practically refused by the mathematical builders up to the beginning of the nineteenth century but became during this century "the head stone of the corner" largely through the work of Sophus Lie. While the work of Beethoven has reached and probably will continue to reach a much larger number of people than that of Lie, it is questionable whether those to whom Lie's work is actually revealed receive less inspiration therefrom or are less impressed by the marvelous new harmonies which it introduces into a wide range of mathematical developments. As Sophus Lie was a Foreign Associate of this Academy and his work