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Topology 44 (2005) 661–687
TOPOLOGY
www.elsevier.com/locate/top

A^1-representability of hermitian K-theory and Witt groups

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Received 19 January 2004

Abstract

We show that hermitian K-theory and Witt groups are representable both in the unstable and in the stable $\mathbf{A}^1$-homotopy category of Morel and Voevodsky. In particular, Balmer Witt groups can be nicely expressed as homotopy groups of a topological space. The proof includes a motivic version of real Bott periodicity. Consequences include other new results related to projective spaces, blow ups and homotopy purity. The results became part of the proof of Morel's conjecture on certain $\mathbf{A}^1$-homotopy groups of spheres.

Introduction

Hermitian K-theory is the algebraic counterpart of real topological K-theory and more generally of Atiyah's [2] *Real K-theory* of topological K-theory of $\mathbf{Z}/2$-bundles on spaces with involution. Thanks to the work of Voevodsky and Morel (cf. [36,48]), it is now possible to state precisely how the conjectural algebraic analogue of real Bott periodicity [8] looks like: there should be a motivic $(8, 4)$-periodic $\Omega_{\mathbf{P}^1}$-spectrum representing hermitian K-theory. This is true, as we prove in this paper. Our proof relies among others on recent progress in hermitian K-theory [19], Balmer's triangular Witt groups and Karoubi's famous *Fundamental Theorem* [28].

The hermitian K-groups $K_n^h(A)$ of a ring A (with involution, and with 2 invertibe) are isomorphic to $\pi_n(BO(A)^+)$ for $n \geqslant 1$, where $BO(A)^+$ denotes the group completion of the classifying space of the

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doi:10.1016/j.top.2004.10.004

infinite orthogonal group. The group $K_0^h(A) = GW(A)$ is the *Grothendieck–Witt group* of symmetric spaces over A: take the Grothendieck group of the monoid of isomorphism classes of finitely generated projective A-modules equipped with a symmetric bilinear non-degenerate form.

In the theory of quadratic or symmetric bilinear forms, an even more classical object of study is the Witt group $W(A)$. It is the quotient of $GW(A)$ that identifies the hyperbolic objects with 0. Recently, Balmer introduced a graded 4-periodic generalization W_B^* of Witt groups. It is defined for triangulated categories with duality in general. When applied to the bounded derived category $D^b(P(A))$ of finitely generated projective modules over a given ring, he rediscovers the classical Witt group in degree 0 [4]. His theory allows him to prove powerful theorems, see e.g., [3,5]. We will show in this paper that there is a space (and in fact a spectrum) whose homotopy groups coincide with the purely algebraically defined groups W_B^*.

Besides the study of quadratic forms, orthogonal and symplectic groups and L-theory, another reason to study hermitian K-theory stems from $\mathbf{A}^1$-homotopy as introduced by Morel and Voevodsky. In this framework, hermitian K-theory plays the role of real topological K-theory. In particular, it is not an oriented theory (in the sense of Levine and Morel), therefore it should detect many interesting maps, and we will see that it is in fact closely related to the endomorphism ring of the motivic sphere spectrum.

Recall that both real and complex topological K-theory are representable in the classical stable homotopy category (i.e., the homotopy category of spectra) as are cohomology theories in general. Morel and Voevodsky proved that algebraic K-theory is representable in the unstable [36] and stable [48] $\mathbf{A}^1$-homotopy category. In this article, we prove a similar result for hermitian K-theory and Balmer Witt groups: they are representable in the unstable and in the stable $\mathbf{A}^1$-homotopy category as well. Our motivation is at least twofold: on the one hand side, this implies that we have long exact sequences for hermitian K-theory and Balmer Witt groups not only for elementary distinguished (i.e., Nisnevich) squares, but also for blow-ups and Gysin morphisms (cf. Corollaries 6.3 and 6.4) resulting from the corresponding triangles in the $\mathbf{A}^1$-homotopy category [36, 3.2.23, 3.2.29], [48, 4.11, 4.12, 4.13]. On the other hand, we believed that this result should be useful in proving the following beautiful conjecture of Fabien Morel [33,34] (which he now has proved if k is perfect, and the proof uses indeed some of our results, see [35]) which relates the theory of quadratic forms to stable $\mathbf{A}^1$-homotopy groups of spheres:

0.1. Conjecture. Let k be a field, $\mathscr{SH}(k)$ be the stable $\mathbf{A}^1$-homotopy category as defined in [48] and S^0 the sphere spectrum over $Spec(k)$. Then there are isomorphisms

$$Hom_{\mathscr{SH}(k)}(S^0, S^0) \cong K_0^h(k)$$

and for $n > 0$

$$Hom_{\mathscr{SH}(k)}(\mathbf{G}_m^{\wedge n}, S^0) \cong W(k).$$

Recall that the sphere spectrum is given by $(S^0)_n = (\mathbf{P}_k^1)^{\wedge n}$ and $\mathbf{P}_k^1$ is homotopy equivalent to the smash product of the simplicial circle and the multiplicative group [36, pp. 110–112]. Lannes and Morel [34] establish a morphism $K_0^h(k) = GW(k) \to Hom_{\mathscr{SH}(k)}(S^0, S^0)$ which should be the isomorphism in Conjecture 0.1 above. The main evidence for the part of the conjecture about $K_0^h(k)$ is the fact that the topological Adams spectral sequences has a motivic counterpart converging to a certain completion of $K_0^h(k)$, see [33] for the details.

The proof of unstable $\mathbf{A}^1$-representability of hermitian K-theory and Witt groups relies on homotopy invariance and the Mayer–Vietoris property for Nisnevich squares which we prove in the first two sections of this paper.

To conclude that ordinary algebraic K-theory is representable not only in the unstable $\mathbf{A}^1$-homotopy category $\mathscr{H}(k)$ but also in $\mathscr{SH}(k)$, Voevodsky proves a periodicity theorem [48, Theorem 6.8] which is essentially Quillen's projective bundle theorem [39, 8.2]. This is the algebraic counterpart of Bott periodicity in complex topological K-theory.

The strategy we apply to prove stable representability for hermitian K-theory and Witt groups in Section 5 relies on the study of $K^h_*(R[t,t^{-1}])$ instead of $K^h_*(\mathbf{P}^1_R)$. Our results also give a variant of Voevodsky's proof [48] for the representability of algebraic K-theory (see Remark 5.9). We obtain a periodicity theorem for hermitian K-theory corresponding to the one in real topological K-theory and more generally to topological K-theory of spaces with involution (see Corollary 5.4). Thus, we can construct an $\Omega_{\mathbf{P}^1}$-spectrum **KO** which follows the same periodicity pattern as real topological K-theory. For instance, the motivic bidegree $(4,2)$ is given by antisymmetric forms, see Section 5 for more details. The unit map $S^0 \to \mathbf{KO}$ of the ring spectrum **KO** together with the representability of hermitian K-theory yields a morphism $Hom_{\mathscr{SH}(k)}(S^0,S^0) \to Hom_{\mathscr{SH}(k)}(S^0,\mathbf{KO}) \cong K^h_0(k)$ which should be the inverse of the morphism constructed above. As hermitian K-theory is a non-orientable theory, it is expected to detect other elements of higher $\mathbf{A}^1$-homotopy groups of spheres which are not detected by orientable theories like motivic cohomology and algebraic K-theory. See Section 6 (in particular 6.8 and 6.9 for more details.

Another application due to S. Yagunov of our $\mathbf{A}^1$-representability theorem is the possibility to construct certain transfer maps and to deduce rigidity theorems for KO and W over algerbraically closed fields [55]. There is work in progress by Yagunov and the author [20] about general base fields.

We now present a more detailed overview of this article.

We prove homotopy invariance and Nisnevich–Mayer–Vietoris for hermitian K-theory of regular affine schemes in Section 1, using some techniques of my joint work with Marco Schlichting [19] and the corresponding results for K-theory and Balmer Witt groups. Hermitian K-theory K^h as defined in [18] is not sufficiently well developed yet to prove such results for non-affine schemes.

In Section 2, we extend the definition of hermitian K-theory from affine schemes to non-affine schemes using techniques of [24,47,54] in a way that this new theory KO automatically fulfills Nisnevich–Mayer–Vietoris and homotopy invariance. We do not prove that $K^h(X) \simeq KO(X)$ for non-affine X, although we conjecture that this is true if X is regular. By the techniques of [10], we can also deduce the exactness of the Cousin complex for a regular local ring containing an infinite field (Corollary 2.9).

After some discussions with Paul Balmer and Fabien Morel, it became clear that one can deduce unstable representability of algebraic K-theory from two geometric facts: Nisnevich–Mayer–Vietoris [36, "Brown–Gersten property" 3.1.13] and homotopy invariance. Then one can work with any fibrant replacement of the K-theory presheaf instead of using the explicit construction of Thomason [46] as done in [36, Proposition 4.3.9]. The situation is similar to topology where the behavior of a cohomology theory (fulfilling homotopy invariance and Mayer–Vietoris) for cofibrant spaces (e.g., CW-complexes) is determined by its values on a point. The proof of the representability of hermitian K-theory we give in Section 3 also follows from these two properties of hermitian K-theory. It works in a much more general setting, see Theorem 3.1 for this general $\mathbf{A}^1$-representability theorem. In general, once we have a fibrant presheaf on affine schemes fulfilling Nisnevich–Mayer–Vietoris and homotopy invariance, we can always extend it to non-affine schemes as done in Section 2 and prove representability by some fibrant

replacement of its sheafification as in Theorem 3.1. See Remark 3.8 for a discussion concerning the role of orthogonal Grassmannians.

Balmer Witt groups are defined purely algebraically. In order to prove unstable representability of Balmer Witt groups W^*_{B}, we first show that they are isomorphic to the homotopy groups of the homotopy colimit of the Karoubi tower (Lemma 4.6). The proof that the simplicial sheaf obtained this way represents W^*_{B} is then similar to the proof of the analogous result for hermitian K-theory. That is, we need the Nisnevich–Mayer–Vietoris property for W^*_{B} to show representability in the simplicial homotopy category of Joyal and Jardine [22] and homotopy invariance to show representability in the unstable $\mathbf{A}^1$-homotopy category of Morel and Voevodsky [36]. All this is carried out in Section 4.

In Section 5, we show that computations of the hermitian K-theory of $R[t, t^{-1}]$ when combined with Karoubi's Fundamental Theorem can be reinterpreted as periodicity theorems in the $\mathbf{A}^1$-setting. We then construct $\Omega_{\mathbf{P}^1}$-spectra representing hermitian K-theory and (Balmer) Witt groups in the stable homotopy category $\mathscr{SH}(k)$.

In Section 6, we compute the hermitian K-theory and Witt groups of the projective line and state a result on projective spaces. We also prove the Thom isomorphism (also called *homotopy purity*) and the blow up isomorphism both for Witt groups and hermitian K-theory. We then discuss the relation of our representability theorems and Morel's conjecture on stable $\mathbf{A}^1$-homotopy groups of spheres. Comparing this to a topological version of such a periodicity theorem for K-theory of spaces with involution (cf. the work of Atiyah [2] on *Real K-theory*), we conjecture a link with the Hopf map also in the algebraic setting (Conjecture 6.6).

The appendix (Section 7) contains a dictionary between L-theory, Balmer Witt groups and hermitian K-theory.

Throughout the whole article, we assume that 2 is invertible in all our rings R if not stated explicitly otherwise. Moreover, *scheme* means a separated scheme of finite type (thus quasi-compact, hence noetherian) over a fixed base field k in which 2 is invertible. Working with a noetherian regular ring in which 2 is invertible as a base would not change anything.

I would like to thank Paul Balmer, Christian Häsemeyer, Max Karoubi, Fabien Morel, Marco Schlichting and Bruce Williams for some comments and discussions.

1. Nisnevich–Mayer–Vietoris and homotopy invariance for hermitian K-theory of regular rings

Throughout this article, we will always assume that A is a unitary ring and that 2 is invertible in A. Recall that an additive category is called idempotent complete if any idempotent map $p = p^2$ has an image.

1.1. Definition. A *category with duality* is a triple $(\mathscr{C}, *, \eta)$ consisting of a category $\mathscr{C}$, a functor $* : \mathscr{C} \to \mathscr{C}^{op}$ and $\eta : id_{\mathscr{C}} \Rightarrow **$ a natural equivalence such that for all objects A of $\mathscr{C}$ we have $1_{A^*} = \eta^*_A \circ \eta_{A^*}$.

1.2. Definition. Given a category with duality $(\mathscr{C}, *, \eta)$, its associated hermitian category $\mathscr{C}_h$ is defined as follows: An object (M, ϕ) is an isomorphism $\phi : M \xrightarrow{\cong} M^*$ such that $\phi = \phi^*\eta$. A morphism $\alpha : (M, \phi) \to (N, \psi)$ is a morphism $\alpha : M \to N$ in $\mathscr{C}$ such that $\alpha^*\psi\alpha = \phi$.

1.3. Examples. (1) Let A be a ring with unit, and let $^-\colon A \to A^{op}$ be an involution, i.e., $\overline{a+b} = \bar{a} + \bar{b}$, $\overline{ba} = \bar{a}\bar{b}$ and $\bar{\bar{a}} = a$. Let $P(A)$ be the category of finitely generated projective right A-modules. We define a duality on $P(A)$ by setting $M^* = \{f \in Hom_{\mathbf{Z}}(M, A) | f(ma) = \bar{a} f(m)\}$ which is a right A-module via $fa(m) = f(m)a$. For example, on the group ring $A = RG$, we have the involution $rg \mapsto rg^{-1}$. Then $(P(A), Hom_A(\ , A), \eta)$ is an additive category with duality, where $\eta(m)(f) = \overline{f(m)}$ for all $m \in M$ and f in M^*.

(2) Let X be a scheme. Then the category $Vect(X)$ of locally free $\mathcal{O}_X$-sheaves of finite rank is an exact category with $\underline{Hom}_{\mathcal{O}_X}(\ , \mathcal{O}_X)$ as duality functor. This generalizes Example 1 for commutative rings considered as affine schemes.

1.4. For any category $\mathscr{C}$ we write $B\mathscr{C} := |N\mathscr{C}|$ for the topological space given by the geometric realization of the nerve of $\mathscr{C}$. We write $i\mathscr{C}$ for the category which has the same objects as $\mathscr{C}$ and whose morphisms are the isomorphisms of $\mathscr{C}$. If $\mathscr{C}$ is a symmetric monoidal category in which any morphism is an isomorphism (i.e., $\mathscr{C} = i\mathscr{C}$), Quillen [15] constructs a new category $\mathscr{C}^{-1}\mathscr{C}$, which we abbreviate by $\mathscr{C}^+$, and a functor $\mathscr{C} \to \mathscr{C}^{-1}\mathscr{C}$ such that $B\mathscr{C} \to B\mathscr{C}^{-1}\mathscr{C}$ is a group completion under very mild hypotheses [15, p. 222] which are always satisfied for all categories constructed in this article. For an additive category with duality $(\mathscr{A}, *)$, we observe that the orthogonal sum $(A, \alpha) \perp (B, \beta) := (A \oplus B, \alpha \oplus \beta)$ makes $(\mathscr{A}_h, \oplus)$ into a symmetric monoidal category. Hence we can use Quillen's $S^{-1}S$-construction [15] to define its K-theory:

1.5. Definition. Let $(\mathscr{A}, *, \eta)$ be an additive category with duality. Then its hermitian K-theory space is defined by

$$K^h(\mathscr{A}) := B(i\mathscr{A}_h)^+$$

and its hermitian K-groups are defined by

$$K^h_n(\mathscr{A}) := \pi_n K^h(\mathscr{A}), \quad n \geqslant 0.$$

Using explicit deloopings, we can also define negative hermitian K-groups [19, Section 2] just as one defines ordinary negative K-groups (cf. for example [38]).

1.6. Definition. An object (M, ϕ) of $\mathscr{A}_h$ is called *hyperbolic* if there is an object L in $\mathscr{A}$ together with an isomorphism $(M, \phi) \cong (L \oplus L^*, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)) =: (H(L), \mu_L)$.

1.7. If 2 is invertible, there is an isometry $(M, \phi) \oplus (M, -\phi) \overset{1}{\cong} \to (H(M), \mu_M)$ given by $\begin{pmatrix} 1 & -1 \\ \frac{1}{2}\phi & \frac{1}{2}\phi \end{pmatrix}$ for any (M, ϕ) in $\mathscr{C}_h$. This implies that the full subcategory of hyperbolic objects is cofinal in $\mathscr{A}_h$. Moreover, if $\mathscr{A} = P(A)$ as in Example 1.3.1, the free hyperbolic modules are cofinal in $P(A)_h$ and hence the connected component of 0 of $K^h(P(A))$ is homotopy equivalent to the plus construction applied to $BO(A) = Bcolim_n \operatorname{Aut} H(R^n)$ [25, Théorème 1.6]. We will often write $K^h(A)$ and $K^h_n(A)$ instead of $K^h(P(A))$ and $K^h_n(P(A))$.

1.8. Recall the following definition of Witt groups suggested by Balmer [4] for a triangulated category $\mathscr{T}$ with translation functor T and with an exact duality functor $*$ (i.e., which preserves the distinguished triangles). For any such triangulated category with duality $(\mathscr{T}, T, *, \eta)$, we define its *nth derived Witt group*

$W^n_{\mathrm{B}}(\mathscr{T})$ to be the monoid of isomorphism classes of objects of $\mathscr{T}_h$ relative to the duality functor $T^n \circ *$, modulo the equivalence relation given by identifying the *metabolic objects* (see Definition 1.9 below) with zero. As any object is a direct summand of a metabolic object, $W^n_{\mathrm{B}}(\mathscr{T})$ is actually a group. Balmer proves [4, Theorem 4.3] that there is a natural isomorphism with the classical Witt group $W(\mathscr{A}) \xrightarrow{\cong} W^0_{\mathrm{B}}(D^b(\mathscr{A}))$ if $\mathscr{A}$ is an idempotent complete additive category with duality in which 2 is invertible and $D^b(\mathscr{A})$ is the homotopy category of bounded chain complexes with its standard triangulation. These groups are of period four, i.e., $W^n_{\mathrm{B}} = W^{n+4}_{\mathrm{B}}$. If $\mathscr{A}$ is idempotent complete, we write $W^n_{\mathrm{B}}(\mathscr{A})$ for $W^n_{\mathrm{B}}(D^b(\mathscr{A}))$.

1.9. Definition. Given a triangulated category with duality $(\mathscr{T}, T, *, \eta)$, a symmetric bilinear nondegenerate object (M, ϕ) is called *metabolic* if it posseses a Lagrangian (L, α, z). This means by definition that we have an exact triangle $T^{-1}(L^*) \xrightarrow{z} L \xrightarrow{\alpha} M \xrightarrow{\alpha^*\phi} L^*$ and that $T^{-1}(z^*) = \eta \circ z$ (in other words, z is symmetric with respect to $T^{-1} \circ *$).

Keep in mind that our main example is $\mathscr{T} = D^b(Vect(X))$ with T the shift of chain complexes and duality $*$ induced by $\underline{Hom}_{\mathscr{O}_X}(\ , \mathscr{O}_X)$. This generalizes the classical Witt group as we assume 2 to be invertible (see [4]).

The following results are known only for additive categories, but not for general exact categories (having possibly not split short exact sequences). In [18], we gave a definition of hermitian K-theory K^h of an exact category $\mathscr{E}$ with duality which generalizes Definition 1.5 if all short exact sequences in $\mathscr{E}$ split. But we cannot prove Corollaries 1.12 and 1.14 below for hermitian K-theory of schemes defined this way. Hence we will give another variant of hermitian K-theory for schemes (cf. Definitions 2.2 and 2.4), called KO, which will allow us to generalize these Corollaries to non-affine schemes.

1.10. Theorem. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor between idempotent complete categories with duality. Assume that $K_{-n}(\mathscr{A}) \cong 0 \cong K_{-n}(\mathscr{B})$ for $n = 1, 2$. Assume furthermore that $K_n(\mathscr{A}) \xrightarrow{\cong} K_n(\mathscr{B})$ for all $n \geqslant 0$ and $W^n_{\mathrm{B}}(\mathscr{A}) \xrightarrow{\cong} W^n_B(\mathscr{B})$ for all $n \in \mathbf{Z}$. Then we also have isomorphisms $K^h_n(\mathscr{A}) \xrightarrow{\cong} K^h_n(\mathscr{B})$ for all $n \geqslant -2$.*

Proof. This is a special case of Karoubi's induction principle [19, Lemma 5.13]. □

1.11. Corollary. *Consider a commutative square*

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{f} & \mathscr{B} \\ \downarrow & & \downarrow \\ \mathscr{C} & \xrightarrow{g} & \mathscr{D} \end{array}$$

of additive idempotent complete categories with dualities which becomes homotopy cartesian after applying K, and all negative K-groups of the four categories above vanish. Assume moreover that $D^b(f)$ and $D^b(g)$ are localizations of triangulated categories with dualities and the map $\mathscr{A} \to \mathscr{C}$ induces an equivalence of their kernel categories $\mathscr{T}$. Then the square becomes homotopy cartesian after applying K^h.

Proof. Let $C(f)$ and $C(g)$ be the push-out of additive categories of $C\mathscr{A} \leftarrow A \rightarrow \mathscr{B}$ and $C\mathscr{C} \leftarrow \mathscr{C} \rightarrow \mathscr{D}$, resp. as defined in [19, Section 3.5]. By Theorem 1.10, it suffices to show that the natural map $C(f) \rightarrow C(g)$ induces an isomorphism on Balmer Witt groups W^* to deduce a homotopy equivalence $K^h(C(f)) \rightarrow K^h(C(g))$ and thus the desired homotopy cartesian square. We know that the maps $D^b(\mathscr{A}) \rightarrow D^b(C\mathscr{A})$ and $D^b(\mathscr{B}) \rightarrow D^b(C(f))$ are full inclusions with equivalent cokernels. This follows as by [19, Lemma 2.8] $\mathscr{B} \rightarrow C(f)$ is also filtering, hence the quotients $C\mathscr{A}/\mathscr{A}$ and $C(f)/\mathscr{B}$ are equivalent additive categories and we can apply [44, Theorem 10.1] to get two short exact sequences of triangulated categories. The same argument applies to $g : \mathscr{C} \rightarrow \mathscr{D}$. We will prove that the natural maps $D^b(C\mathscr{A})/\mathscr{T} \rightarrow D^b(C(f))$ and $D^b(C\mathscr{C})/\mathscr{T} \rightarrow D^b(C(g))$ induce isomorphisms on W^*_{B} which suffices by the five lemma as $W^*(C\mathscr{A}) \cong 0 \cong W^*(C\mathscr{C})$ by the usual Eilenberg swindle. But the map $W^*(D^b(C\mathscr{A})/\mathscr{T}) \rightarrow W^*(D^b(C(f)))$ fits into a commutative (check this!) ladder of long exact sequences of Witt groups [3] associated to the two short exact sequences of triangulated categories $D^b(\mathscr{B}) \rightarrow D^b(C(f)) \rightarrow D^b(S\mathscr{A})$ and $\mathscr{T} \rightarrow D^b(\mathscr{A}) \rightarrow D^b(\mathscr{B})$. Hence the claim follows by the five lemma and the identical argument for $W^*(D^b(C\mathscr{C})/\mathscr{T}) \rightarrow W^*(D^b(C(g)))$. □

Remark. The category of finitely generated projective modules over a given ring $P(A)$ is idempotent complete, and it fulfills the above hypothesis $K_n(P(A)) = 0$ for all $n < 0$ if A is regular [6, p. 685].

Theorem 1.10 allows us to prove homotopy invariance:

1.12. Corollary. *Let A be a regular ring. Then we have a homotopy equivalence*

$$K^h(A) \xrightarrow{\simeq} K^h(A[t]).$$

Proof. Use [39, p. 122], [5, Theorem 3.1] and Theorem 1.10. □

More generally, any map of regular rings yielding an affine vector bundle torsor as discussed in the next section induces a K^h-equivalence by Theorem 1.10 and the corresponding result for K-theory [39, p. 128] and Witt groups [13]. Homotopy invariance could probably have been proved in a more classical way, but there seems to be no reference for this.

Next, we establish Nisnevich–Mayer–Vietoris by applying Corollary 1.11. First we recall the definition of an elementary distinguished square [36, Definition 3.1.3]:

1.13. Definition. An *elementary distinguished square* (or a *Nisnevich square* for short) is given by a commutative diagram of schemes

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{j} & X \end{array}$$

such that p is étale, j is an open embedding and $p^{-1}(X - U) \rightarrow X - U$ is an isomorphism of the associated reduced schemes.

Recall that a presheaf F is a Nisnevich sheaf if and only if for any elementary square applying F yields a cartesian square of sets [36, 3.1.4].

We now show that the presheaf KO fulfills the Nisnevich–Mayer–Vietoris property (also called "Brown–Gersten property" or "Nisnevich excision"):

1.14. Corollary. *For any elementary distinguished square as in Definition* 1.13 *consisting of regular affine schemes, applying K^h yields a homotopy cartesian square.*

Proof. When applying K to such a square, we get a homotopy cartesian square by excision [47, Proposition 3.19] and localization [47, Theorem 7.4]. Applying D^b, the hypotheses of Corollary 1.11 are met by excision and localization [5, 1.6, 2.3] as well. Now apply Corollary 1.11. □

2. The definition of *KO*

Recall that throughout this paper, *scheme* means a separated scheme of finite type over a fixed base field k in which 2 is invertible. We first extend the definition of the presheaf K^h on affine schemes to a presheaf KO on quasi-compact schemes using Jouanolou's device (cf. Definition 2.2). Then we observe that we can define KO using affine covers as well (see Definition 2.4, Lemma 2.5) following essentially Thomason [47,54]. All this works just as well for any other fibrant presheaf on affine schemes, provided that it fulfills homotopy invariance and Nisnevich–Mayer–Vietoris.

First, we recall the following result of Jouanolou.

2.1. Lemma. *Let X be a regular scheme. Then there exists an affine vector bundle torsor W over X.*

Proof. See [24, Lemme 1.5] if X is quasiprojective or more generally [54, Proposition 4.3] if X has an ample family of line bundles; recall that any regular noetherian scheme has an ample family of line bundles ([7, II, 2.2.7.1] or [47, p. 284]). □

Moreover, taking the fiber product of two affine torsors W and W' over X as in [24, Proposition 1.6], we see that the homotopy type of $K^h(W)$ is independent of the choice of the torsor W. As K^h is homotopy invariant, this justifies the following definition:

2.2. Definition. Let X be a regular scheme. Then we define $KO(X) := K^h(W)$ where W is an affine torsor over X as in Lemma 2.1.

2.3. Remark. Observe that this is a definition only *up to homotopy*. See [54, Appendix] for techniques to make it functorial by considering all torsors simultaneously, and observe that [54, Lemma A.2] carries over as K^h commutes with colimits. Hence we may assume from now on that KO is a simplicial presheaf on Sm/k, as it will be necessary in the sequel. More generally, given any presheaf P on affine schemes (commuting with colimits) fulfilling Mayer–Vietoris and homotopy invariance for vector bundle torsors, the techniques of [54] allow us to extend it to a presheaf on Sm/k fulfilling these two properties for non-affine schemes as well. So in this section the reader might replace KO by a presheaf P

fulfilling the properties mentioned above as this is all the proofs we will give in this section require (except the Nisnevich–Mayer–Vietoris property which requires the corresponding statement for affine Nisnevich–Mayer–Vietoris squares). Observe also that an affine vector bundle torsor over a scheme smooth over k is itself smooth over k.

2.4. Definition. Let X be a scheme and $U = (U_i)_{i\in I}$ an affine open cover of X. Then we define $KO(X, U) := holim\, KO(U_{i_1} \cap \cdots \cap U_{i_j})$ where the *holim* is taken over the poset over all finite intersections of open sets of U.

In the notation of Thomason [46, Definition 1.9], one would write $\check{\mathbf{H}}^{\bullet}(U, KO)$ instead of $KO(X, U)$.

2.5. Lemma. *Let X be a regular scheme and U, V be two affine covers of X. Then we have a homotopy equivalence $KO(X, U) \simeq KO(X, V)$. In particular, there are natural homotopy equivalences $K^h(X) \xrightarrow{\simeq} KO(X) \xrightarrow{\simeq} KO(X, \{X\})$ for X affine where $\{X\}$ is the trivial cover of X.*

Proof. The arguments of Thomason–Weibel carry over to *KO*: We can first prove Mayer–Vietoris for quasi-compact separated schemes as in [54, Theorem 5.1] using Corollary 1.14. Then we deduce *Čech descent* from [54, Theorem 6.3]. From this, the Lemma follows by arguments similar to those used in [54, Proposition 6.6]. □

2.6. Remark. One might try to define *KO* directly from K^h via this *holim*-construction without Jouanolou's device. But the problem (besides functoriality) is that we need Čech descent and thus the Mayer–Vietoris property of *KO* for regular schemes too prove Lemma 2.5, and proving the general Mayer–Vietoris property from the one for affine schemes requires Jouanolou's device. Nevertheless, once Mayer–Vietoris is established, we can deduce theorems on *KO* for non-affine schemes once we know they are true for affine schemes. Compare [47, Section 9] and [24] for possible applications. We can also define $KO(X)$ for X not quasi-compact this way.

2.7. Proposition. *For any regular scheme X, the projection $\mathbf{A}^1_k \to Spec(k)$ induces a natural homotopy equivalence*

$$KO(X) \xrightarrow{\simeq} KO(X \times \mathbf{A}^1).$$

Proof. This follows from Definition 2.2. □

2.8. Theorem. *For any Nisnevich square as in Definition* 1.13, *the following square*

$$\begin{array}{ccc} KO(X) & \xrightarrow{KO(U)} & KO(U) \\ \downarrow{\scriptstyle KO(p)} & & \downarrow \\ KO(V) & \longrightarrow & KO(U\times_X V) \end{array}$$

is homotopy cartesian.

Proof. For ordinary Mayer–Vietoris (i.e., both p and j are open inclusions), the proof is the same as in [54, Theorem 5.1]. Considering Nisnevich squares in general, the idea is roughly to choose affine

vector bundle torsors for X, U, V and $U\times_X V$ and then proceed similarly to Corollary 1.11. Here are the details: First, choose affine vector bundle torsors $Spec(A)$ and $Spec(C)$ over X and V in a functorial way (look either at Remark 2.3 or replace X by $Spec(A)$ and then V by an affine vector bundle torsor $Spec(C)$ over $V\times_X Spec(A)$). Define $\mathscr{T}_A$ and $\mathscr{T}_C$ to be the kernel categories of the functors $D^b(P(A)) \to D^b(Vect(Spec(A)\times_X U))$ and $D^b(P(C)) \to D^b(Vect(Spec(C)\times_X U))$. Then the map $A \to C$ induces a map between the short exact sequences of triangulated categories $\mathscr{T}_A \to D^b(P(A)) \to D^b(Vect(Spec(A)\times_X U))$ and $\mathscr{T}_C \to D^b(P(C)) \to D^b(Vect(Spec(C)\times_X U))$ which yields isomorphisms between the long exact localization sequences for Witt groups [3] by Jouanolou's device for Witt groups [13] and the five lemma. Now functorially replace $Spec(A)\times_X U$ and $Spec(C)\times_X U$ by affine vector bundle torsors $Spec(B)$ and $Spec(D)$. Writing f and g for the maps $P(A) \to P(B)$ and $P(C) \to P(D)$ defined by composition, one can proceed similar to the proof of Corollary 1.11 in order to prove that the maps $D^b(CP(A)/\mathscr{T}_A) \to D^b(C(f))$ and $D^b(CP(C)/\mathscr{T}_C) \to D^b(C(g))$ induced by universal properties induce isomorphisms on Witt groups. As in Corollary 1.11, it follows that $C(f) \to C(g)$ induces anisomorphism also on K^h_* by Karoubi induction (Theorem 1.10) which applies as K-theory also fulfills strong homotopy invariance (that is, for affine vector bundle torsors, see [24]) and the Nisnevich–Mayer–Vietoris property [47, Proposition 3.9, Theorem 7.4]. □

2.9. Corollary. *If k is infinite, the Cousin complex* (*see e.g.* [10, Section 1]) *in hermitian K-theory yields a resolution of the Zariski sheaf associated to $U \mapsto KO_n(U)$. In particular, if R is local and smooth over k, then the complex*

$$0 \to KO_n(R) \to KO_{n,x}(R) \to \coprod_{x\in X^{(1)}} KO_{n-1,x}(X) \to \dots$$

is exact where $KO_{n,x}(X) := colim_{U\ni x} KO_{n,\bar{x}\cap U}(U)$, $KO_Z(X)$ is the homotopy fiber of $KO(X) \to KO(X - Z)$ and $K_{n,Z}(X) = \pi_n(KO_Z(X))$.

Proof. This follows from Theorem 2.8 and Proposition 2.7 which imply that *KO* is a cohomology theorem fulfilling $COH1$ and $COH3$ and thus is *strictly effaceable* [10, Theorem 5.1.10], and the claim follows [10, Corollary 5.1.11]. □

2.10. If one tries to identify the Cousin complex to the classical Gersten complex, the latter will start with $KO(R) \to KO(Quot(R))$ and then continue with the four theories U, $_-KO$, $_-U$ and KO in a four-periodic pattern. Panin's trick [37] to generalize to the equicharacteristic case once the geometric case is established will carry over to our situation, doing the induction step for all four types of Gersten complexes simultaneously.

2.11. Remark. In the next section (Theorem 3.1), we will see how the Nisnevich–Mayer–Vietoris property and homotopy invariance of a presheaf leads to a representability result in $\mathscr{H}(k)$. One easily checks that the converse is also true: given a simplicially fibrant sheaf F, the theory it represents fulfills the Nisnevich–Mayer–Vietoris property and homotopy invariance, and thus the Gersten conjecture for the Cousin complex holds for the theory represented by F and for rings as in Corollary 2.9, using [10] as above.

3. Unstable $\mathbf{A}^1$-representability of hermitian K-theory

We first prove a general representability theorem for a certain class of functors from smooth schemes to graded abelian groups. For any presheaf of simplicial sets P on the big Nisnevich site $(Sm/k)_{\mathrm{Nis}}$, we denote by $aP \in \Delta^{op}Shv$ its sheafification (with respect to the Nisnevich topology). If F is a sheaf, we write F_{f} for a fibrant replacement with respect to the simplicial model structure of [22, Corollary 2.7]. Recall that in this model structure, a map $F \to G$ is a weak equivalence if and only if it induces a weak equivalence of simplicial sets for all stalks. In particular, the stalks of F and F_{f} are weakly equivalent simplicial sets. The map $F \to G$ is a cofibration if $F(U) \to G(U)$ is a monomorphism for any scheme U, hence any object is cofibrant. Recall further that this simplicial structure yields the (pointed) simplicial homotopy category $\mathscr{H}^s_\bullet(k)$ which then leads to the (pointed) $\mathbf{A}^1$-homotopy category $\mathscr{H}(k)$ of [36] by inverting the $\mathbf{A}^1$-equivalences.

We denote by X_+ the scheme X with an added disjoint base point. The Yoneda embedding sends any (pointed) scheme to a (pointed) presheaf which is an étale sheaf [45, p. 347]. We will often consider it as a simplicially constant sheaf and still denote it by X (resp. X_+).

The model category of simplicial sheafs $\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}$ is actually a simplicial model category [36, Remark 1.9]. In particular, we have a bifunctor $\mathbf{Hom}_{\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}} : \Delta^{op}Shv(Sm/k)^{op}_{\mathrm{Nis}} \times \Delta^{op}Shv(Sm/k)_{\mathrm{Nis}} \to \Delta^{op}Sets$.

We can now establish our general $\mathbf{A}^1$-representability result:

3.1. Theorem. *For any presheaf $P : (Sm/k)_{\mathrm{Nis}} \to \Delta^{op}Sets$ fulfilling the Nisnevish–Mayer–Vietoris property and homotopy invariance and for any regular scheme X, we have a natural isomorphism*

$$\pi_n(P(X)) \cong Hom_{\mathscr{H}(k)}(S^n \wedge X_+, aP_f).$$

Proof. We have $\pi_n(P(X)) \cong \pi_n(aP_f(X))$ by applying [36, 3.1.18] to $\mathscr{A}$= *all regular noetherian schemes*. Furthermore, we have $\pi_n(aP_f(X)) \cong \pi_n(\mathbf{Hom}_{\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}}(X, aP_{\mathrm{f}}))$ because by Yoneda we have $\mathbf{Hom}_{\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}}(X, F)_k := Hom_{\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}}(X \times \Delta^k, F) \cong F(X)_k$ and hence $\mathbf{Hom}_{\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}}(X, F) \cong F(X)$ for any simplicial sheaf F. By [22, p. 73], $\pi_n(\mathbf{Hom}_{\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}}(X, aP_{\mathrm{f}}))$ is isomorphic to $Hom_{\Delta^{op}Shv}(X, \Omega^n aP_{\mathrm{f}})/\sim$ where $\sim$ stands for the smallest equivalence relation generated by simplicial equivalence. We write $Hom_\bullet$ for pointed morphisms. As adding a base point is left adjoint to forgetting the base points, we get $Hom_{\Delta^{op}Shv}(X, \Omega^n aP_{\mathrm{f}})/\sim \cong Hom_{\Delta^{op}Shv\,\bullet}(X_+, \Omega^n aP_{\mathrm{f}})/\sim$. As aP_{f} is fibrant, it follows [22, p. 73] that $\Omega^n aP_{\mathrm{f}}$ is also fibrant and hence [22, p. 72] that $Hom_{\Delta^{op}Shv\,\bullet}(X_+, \Omega^n aP_{\mathrm{f}})/\sim \cong Hom_{\mathscr{H}^s_\bullet(k)}(S^n \wedge X_+, aP_{\mathrm{f}})$.To show that the latter one is isomorphic to $Hom_{\mathscr{H}(k)}(S^n \wedge X_+, aP_{\mathrm{f}})$, we have to check [36, 2.2.5] that aP_{f} is $\mathbf{A}^1$-local. By [36, 2.3.19], a fibrant simplicial sheaf is $\mathbf{A}^1$-local if and only if it is homotopy invariant. We also know [36, 3.1.18] that a map of presheaves that is stalkwise a weak equivalence (hence a weak equivalence of the associated sheaves) is a weak equivalence for any section provided both presheaves fulfill the Brown–Gersten property. Applying this to $P \to aP_{\mathrm{f}}$ yields the desired result as P and hence aP_{f} is homotopy invariant. □

3.2. Remark. The part of the proof showing that $\pi_n(P(X)) \cong Hom_{\mathscr{H}^s_\bullet(k)}(S^n \wedge X_+, aP_{\mathrm{f}})$ is not discussed in the corresponding proof of Morel and Voevodsky for the representability of algebraic K-theory [36, Proposition 3.9] because they consider it to be "formal". The above details might allow some more people to understand this part of their proof. More precisely, our proof allows us to slightly simplify their original

proof, replacing Thomason's hypercohomology spectrum $\mathbf{H}^{\bullet}_{\mathrm{Nis}}(\ ,K^B)$ and his Nisnevich descent result [47, Theorem 10.8] by an arbitrary fibrant replacement aK_{f} and the Brown–Gersten property.

3.3. Remark. Theorem 3.1 also holds if $P = \mathbf{P}_0$ where $\mathbf{P}$ is a presheaf of (not-necessarily connective) Ω-spectra fulfilling the Nisnevich–Mayer–Vietoris property. This follows as for simplicial sets using [23, Corollary 1.4].

The above result strongly emphasizes the analogy with classical homology theories for CW-complexes: The behavior of the homotopy groups of any presheaf P satisfying Nisnevich–Mayer–Vietoris and homotopy invariance (read: an Eilenberg–Steenrod cohomology theory) is determined by their behavior on a point. This is not a full analogue of Brown's representability theorem [9] as we already assumed our cohomology theory to be given as homotopy groups of a simplicial presheaf instead of starting with just a presheaf with values in graded abelian groups. This is why our proof for the $\mathbf{A}^1$-representability of Balmer Witt groups and in particular the classical Witt group (cf. Section 4) is slightly more complicated.

3.4. Corollary. *For any regular scheme X, we have a natural isomorphism*

$$KO_n(X) \cong Hom_{\mathscr{H}(k)}(S^n \wedge X_+, aKO_{\mathrm{f}}).$$

Proof. Apply Theorem 3.1, Remark 3.3, Theorem 2.8 and Proposition 2.7. □

This allows us to extend the definition of KO from Sm/k to any simplicial sheaf:

3.5. Definition. For any object F of $\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}$, we set

$$aKO_{\mathrm{f}}(F) := \mathbf{Hom}_{\bullet}(F_+, aKO_{\mathrm{f}}) \cong \mathbf{Hom}(F, aKO_{\mathrm{f}})$$

and

$$KO_n(F) := Hom_{\mathscr{H}}(S^n \wedge F_+, aKO_{\mathrm{f}}).$$

For any object with base point F of $\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}_\bullet}$, we set

$$aKO_{\mathrm{f}}(F) := \mathbf{Hom}(F, aKO_{\mathrm{f}})$$

and

$$KO_n(F) := Hom_{\mathscr{H}}(S^n \wedge F, aKO_{\mathrm{f}}).$$

The object aP_{f} does not look very explicit, so one might look for other descriptions of aKO_{f}. Let $\pi : (Sm/k)_{et} \to (Sm/k)_{\mathrm{Nis}}$ be the obvious morphism from the big étale site to the big Nisnevich site (compare [36, p. 130]). Then we have a pair of adjoint functors $\pi^* : \Delta^{op}Shv(Sm/k)_{\mathrm{Nis}} \to \Delta^{op}Shv(Sm/k)_{et}$ and $\pi_* : \Delta^{op}Shv(Sm/k)_{et} \to \Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}$. In fact, π_* is just the forgetful functor, having the sheafification functor π^* as a right adjoint. Define a model structure on $\Delta^{op}Shv(Sm/k)_{et}$ as in [22], that is the weak equivalences being the (étale) stalkwise weak equivalences of simplicial sets and every object is cofibrant. Writing $\mathscr{H}^s_{et}(k)$ for its homotopy category, following [36, Proposition 1.47] we obtain a pair of (Quillen) adjoint functors $\mathbf{L}\pi^*$ and $\mathbf{R}\pi_*$ between $\mathscr{H}^s(k)$ and $\mathscr{H}^s_{et}(k)$. The functor $\mathbf{L}\pi^*$ is induced by π^* and $\mathbf{R}\pi_*$ by first choosing a fibrant resolution and then applying π_*.

One reason for considering the étale topology is the following.

3.6. Definition. Let A be a ring and A^n be equipped with the diagonal form $1_n := \langle 1, \dots, 1\rangle$. Then we set $O_n(A) = Aut_{P(A)_h}(A^n, 1_n)$.

3.7. Lemma. *Let X be a scheme. Then in the étale topology, any object (E, ϕ) of $Vect(X)_h$ is locally isomorphic to $(A^n, 1_n)$ for some $n \in \mathbf{N}$.*

Proof. In the Zariski topology, any vector bundle E is locally free, and there are elements $a_1, \dots, a_n$ in the local ring A such that locally $(E, \phi) \cong (A^n, \langle a_1, \dots, a_n\rangle)$ where n is the rank of E (cf. [43, Theorem 1.6.4]). To construct an isomorphism $(A^n, \langle a_1, \dots, a_n\rangle) \cong (A^n, \langle 1, \dots, 1\rangle)$, we need to add the square roots of the a_i to A which yields an étale neighbourhood as 2 is invertible [42, Proposition VI.1] (remember that we assume our forms to be non-degenerate). □

3.8. Remark. As in [36, p. 123], we denote by $\mathbf{R}\Omega$ the total right derived functor of Ω, which is right adjoint to $S^1\wedge$ in $\mathscr{H}^s_\bullet$. It is given by first taking a fibrant resolution and then taking loops, thus always yields a fibrant object. Finally, we define $B_{et}O_n := \mathbf{R}\pi_*\pi^* BO_n$, where $\pi : (Sm/k)_{et} \to (Sm/k)_{\mathrm{Nis}}$ is the obvious morphism of sites and $BO(Spec(A))_n$ is the classifying space (i.e., the nerve) of the group $O_n(A)$. The Lemma 3.7 above implies that $\pi^* aKO_{\mathrm{f}}$ and $\pi^*\mathbf{R}\Omega B\coprod_{n\geqslant 0} BO_n$ are locally isomorphic in the étale topology. To study the relationship between aKO_{f} and $\mathbf{R}\Omega B\coprod_{n\geqslant 0} B_{et}O_n$ in $\mathscr{H}(k)$ remains an open problem. Observe that [36, p. 131] implies that $Hom_{\mathscr{H}^s_\bullet(k)}(X, \coprod_{n\geqslant 0} B_{et}O_n)$ is isomorphic to the monoid of isomorphism classes of quadratic bundles on X.

The fact that we cannot use orthogonal Grassmannians to represent hermitian K-theory as we can use ordinary Grassmannians to represent algebraic K-theory will require a slight modification of Voevodsky's argument when constructing spectra representing KO_* and W^*_{B} as we will see in Section 5.

Nevertheless, when working with rational coefficients, there is a geometric model for the space that represents hermitian K-theory. More precisely, one may combine Lemma 3.7, Corollary 3.4, Voevodsky's result [49, Proposition 3.3.2] that motives and étale motives coincide with rational coefficients and Morel's unpublished (compare [35]) result that the category of Voevodsky motives with rational coefficients is equivalent to the rational motivic stable homotopy category provided -1 is a sum of squares in the base field k, and similarly for the étale topology. Then we see that a spectrum built from aKO_{f} is $\mathbf{A}^1$-stably rational equivalent to one built from $\mathbf{R}\Omega B\coprod_{n\geqslant 0} B_{et}O_n$. It is of course even integrally $\mathbf{A}^1$-equivalent by Lemma 3.7 if the base field k contains all square roots. See [36, Section 4.2] for the techniques of how to construct an orthogonal Grassmannian isomorphic (in $\mathscr{H}(k)$) to $\mathbf{R}\Omega B\coprod_{n\geqslant 0} B_{et}O_n$.

We conclude this discussion by recalling that any antisymmetric form is locally isomorphic (in the Zariski topology) to a standard symplectic form (see e.g. [32, p. 7, Corollary I.3.5]). So $\_KO$-theory as introduced in the next section is representable by $\mathbf{R}\Omega B\coprod_{n\geqslant 0} B_{et}Sp_{2n}$, and the argument is exactly as for algebraic K-theory, replacing Gl_n by Sp_{2n} everywhere.

4. The Karoubi tower and unstable $\mathbf{A}^1$-representability of Witt groups

In this section, we will show (Corollary 4.9) that Balmer Witt groups and in particular the classical Witt group are representable in the unstable $\mathbf{A}^1$-homotopy category $\mathscr{H}(k)$. In order to do so, we first need to construct in a functorial way for each regular ring a topological space whose homotopy groups are

the purely algebraically defined Witt *groups*(Lemma 4.6). This result seems to have some independent interest.

We first define *U*- and *V*-theory and Karoubi Witt groups W^{K}.

4.1. Definition. For any ring A with involution $\bar{}$, we define the hyperbolic functor by

$$H : iP(A) \to iP(A)_h,$$

$$M \mapsto (H(M), \mu_M),$$

$$\alpha \mapsto \alpha \oplus \alpha^{*-1}$$

and the forgetful functor by

$$F : P(A)_h \to P(A),$$

$$(N, \psi) \mapsto N,$$

$$\beta \mapsto \beta.$$

We further set

$$\begin{aligned} U(A) &:= hofib(K(A) \xrightarrow{H} K^h(A)),\\ V(A) &:= hofib(K^h(A) \xrightarrow{F} K(A)),\\ U_n(A) &:= \pi_n(U(A)),\\ V_n(A) &:= \pi_n(V(A)),\\ W_n^{\mathrm{K}}(A) &:= coker(K_n(A) \xrightarrow{H} K_n^h(A)), \end{aligned}$$

where the homotopy fiber and homotopy groups are always with respect to the basepoint given by the zero object.

Observe that W_0^{K} is just the classical Witt group. We write ${}_- K^h$ for the hermitian *K*-theory of antisymmetric forms, i.e., when replacing η by $-\eta$, and similarly for *U*- and *V*-theory.

The following Theorem is due to Karoubi. He calls it "Fundamental Theorem", and one should think about it as a first generalization of Bott periodicity to algebraic *K*-theory:

4.2. Theorem. *For any ring with involution A in which* 2 *is a unit, we have a natural homotopy equivalence*

$$\Omega_- U(A) \simeq V(A).$$

Proof. See [28]. □

4.3. In the proof of the Fundamental Theorem 4.2, the following construction of the rings U_A and V_A [28, 1.4] is crucial: For any ring *A*, one defines *CA* to be the ring of those infinite matrices with coefficients in *A* having only a finite number of coefficients different from zero in each row and each column. Then one defines $SA := CA/A$ and shows that one gets a homotopy fibration $K^h(A) \to K^h(CA) \to K^h(SA)$. In particular, as *K* and K^h of *CA* vanish by the Eilenberg swindle, we see that $\Omega K^h(SA) \simeq K^h(A)$ which

allows us to consider hermitian K-theory as a non-connective Ω-spectrum. Everything above, including Karoubi's Fundamental Theorem 4.2, carries over to these non-connective Ω-spectra. In particular, we have negative K^h-, U- and W^K-groups.

In fact, the above construction can be carried out on the level of additive categories with duality in general [19]. We have maps $\Delta : A \to A \times A^{op}$, given by sending a to $(a, \bar{a})$, and $+ : A \times A^{op} \to A$. Strictly speaking, the latter map is only defined on the level of additive categories. For rings, we have a mapping (a, b) to $diag(a, \bar{b}) \in Mat_2(A)$ and Morita equivalence. Now we define $U_A := lim(CA \to SA \leftarrow SA \times SA^{op})$ and $V_A := lim(CA \times CA^{op} \to SA \times SA^{op} \leftarrow SA)$. We deduce that $\Omega K^h(U_A) \xrightarrow{\simeq} U(A)$, $\Omega U(U_A) \xrightarrow{\simeq} {}_- K^h(A)$ and $\Omega K^h(V_A) \xrightarrow{\simeq} V(A)$. See [28] for more details. Observe that the inclusion $A \to CA$ and the zero map $A \to SA \times SA^{op}$ induce a map $\alpha : A \to U_A$. Iterating this construction and writing U_A^n for $U(U_A^{n-1})$, we get the so-called Karoubi tower:

4.4. Definition. For any ring A with involution, its Karoubi tower is by definition the sequence of maps

$$K^h(A) \xrightarrow{\alpha} K^h(U_A) \xrightarrow{U_\alpha} K^h(U_A^2) \xrightarrow{U_\alpha^2} K^h(U_A^3) \to \dots$$

We also set $KT(A) := hocolim(K^h(A) \to K^h(U_A) \to K^h(U_A^2) \to \dots)$.

The idea of considering this tower is due to B. Williams. Some properties of this tower have been studied by Kobal [29]. An easy observation is the following:

4.5. Lemma. *The homotopy groups of $KT(A)$ are 4-periodic*:

$$\pi_n(KT(A)) \cong \pi_{n+4}(KT(A)) \cong \pi_{n+2}({}_- KT(A)).$$

Proof. We first observe that $\Omega^4 K^h(U_A^4) \simeq \Omega^3 U(U_A^3) \simeq \Omega^2_- K^h(U_A^2) \simeq \Omega_- U(U_A) \simeq K^h(A)$ by the above homotopy equivalences. Hence we have $\pi_n(KT(A)) \cong \pi_n(hocolim\, K^h(U_A^{4i})) \cong \pi_n(hocolim\, \Omega^4 K^h(U_A^{4i})) \cong \pi_n(colim\, \Omega^4 K^h(U_A^{4i}))$ where the last isomorphism follows from the mapping telescope (i.e., the *hocolim* is by definition the *colim* of the diagram replacing every map by a cofibration). Replacing every simplicial set by a fibrant one, and using the explicit description of π_n for fibrant simplicial sets, we get $\pi_n(KT(A)) \cong colim\, \pi_n(\Omega^4 K^h(U_A^{4i})) \cong colim\, \pi_{n+4}(K^h(U_A^{4i})) \cong \pi_{n+4}(KT(A))$. The second equivalence is similar. □

For regular rings, we actually recover the 4-periodic Balmer Witt groups (see 1.8 for the definition):

4.6. Lemma. *If A is a regular ring, we have natural isomorphisms*

$$\pi_k(KT(A)) \cong \pi_k(K^h(U_A^{k+1})) \cong W_B^{-k}(A).$$

Proof. As the negative K-theory of regular rings vanishes [6, p. 685], the map $K^h(U_A^n) \to K^h(U_A^{n+1})$ is an isomorphism for π_k, $k < n$ (look at its homotopy fiber $\Omega K^h(U^n(SA \times SA^{op})) \simeq K(S^n(A))$. Thus, $\pi_k(KT(A)) \cong \pi_k(K^h(U_A^n))\ \forall n > k$. Use Proposition 7.4 to identify these homotopy groups with Balmer Witt groups. □

See 7.6, 7.7 for a statement about non-regular rings.

4.7. As all the above constructions can be done in the category of simplicial sets rather than in **Top**, we can consider KT as a simplicial presheaf on regular affine schemes. Using the functorial version of Jouanolou's device ([24, Lemme 1.5] or [54, Proposition 4.3] and Remark 2.3), we can extend KT to a presheaf on regular schemes which we still denote by KT similarly to the way we extended K^h to KO in Section 2. As Gille [13] has recently shown that Balmer Witt groups fulfill strong homotopy invariance (i.e., for bundles having affine spaces as fibers), we know that this extension still coincides with the groups Balmer originally defined: $\pi_n(KT(X)) \cong W_{\mathrm{B}}^{-n}(X)$.

Next, we show that the Nisnevich–Mayer–Vietoris property holds for KT.

4.8. Proposition. *Applying KT to a Nisnevich square as in Definition* 1.13 *yields a homotopy cartesian square*.

Proof. Again, for affine Nisnevish squares we could use Corollay 1.11, and for general Zariski–Mayer–Vietoris squares we could proceed similarly as in [54, Theorem 5.1] once the result for affine Zariski–Mayer–Vietoris squares is established. For a general Nisnevich square as in Definition 1.13 we choose a functorial replacement of this square by affine vector bundle torsors. As both KO and KT fulfill Jouanolou's device by definition, the claim follows by Theorem 2.8 and the following more general argument. Assume that we have a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of rings (or more generally of idempotent complete additive categories) inducing a homotopy fibration for non-connective Ω-spectra when applying K or K^h. Then we also have homotopy fibrations of non-connective Ω-spectra when applying K^h to $SA \to SB \to SC$ and to $SA \times SA^{op} \to SB \times SB^{op} \to SC \times SC^{op}$. Looking at the homotopy fibers, we get a homotopy fibration of non-connective Ω-spectra

$$K^h(U_A) \to K^h(U_B) \to K^h(U_C)$$

and inductively for all the higher $K^h(U^n)$. It remains to show that $KT(A) \simeq F$ where $F := hofib(KT(B) \to KT(C))$. Defining $F_B^n := hofib(K^h(U_B^n) \to KT(B))$, $F_C^n := hofib(K^h(U_C^n) \to KT(C))$ and $X_n := hofib(F_B^n \to F_C^n)$, this yields $\pi_i(F_B^n) \cong \pi_i(F_C^n)$ for $n > i$. Hence $\pi_i(X_n) = 0$ and consequently $\pi_i(K^h(U_A^n)) \cong \pi_i(F)$ for $n > i$, and the proposition follows. The argument for a commutative square is exactly the same, setting $X_n = holim(F_C^n \to F_D^n \leftarrow F_B^n)$. (Alternatively, on may proceed by studying the homotopy fibers given by $C(f)$ and $C(g)$ as it was done in Theorem 2.8.). □

Denote by aKT the sheafification of KT with respect to the Nisnevich topology. Next we choose a fibrant replacement aKT_{f} with respect to the model structure of [22, Corollary 2.7].

We will now prove that the simplicial sheaf aKT_{f} on $(Sm/k)_{\mathrm{Nis}}$ represents Balmer Witt groups in the unstable $\mathbf{A}^1$-homotopy category $\mathscr{H}(k)$. As usual, we denote by X_+ the scheme X with an added disjoint base point. Then the main result of this section is the following:

4.9. Corollary. *For any regular scheme X, we have a natural isomorphism*

$$W_{\mathrm{B}}^{-n}(X) \cong Hom_{\mathscr{H}(k)}(S^n \wedge X_+, aKT_{\mathrm{f}}).$$

Proof. As Balmer Witt groups fulfill the Nisnevich–Mayer–Vietoris property (Proposition 4.8) and homotopy invariance [5], we can apply Theorem 3.1 and Remark 3.3. □

As for *KO*, this allows us to extend the definition of W^*_{B} from Sm/k to any simplicial sheaf:

4.10. Definition. For any object F of $\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}$, we set

$$W^{-n}_{\mathrm{B}}(F) := Hom_{\mathscr{H}(k)}(S^n \wedge F_+, aKT_{\mathrm{f}}).$$

For any object with base point F of $\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}\bullet}$, we set

$$W^{-n}_{\mathrm{B}}(F) := Hom_{\mathscr{H}(k)}(S^n \wedge F, aKT_{\mathrm{f}}).$$

5. Stable representability of *KO* and *W*

Recall [48] that the stable $\mathbf{A}^1$-homotopy category $\mathscr{SH}(k)$ is the homotopy category of $\mathbf{P}^1$-spectra. The structure maps $E_n \wedge \mathbf{P}^1 \to E_{n+1}$ have adjoint maps $t_n : E_n \to \Omega_{\mathbf{P}^1}E_{n+1}$, and as in topology a spectrum is called an $\Omega_{\mathbf{P}^1}$-spectrum if all the t_n are weak equivalences (i.e., isomorphisms in $\mathscr{H}(k)$). Recall that we have an isomorphism $\mathbf{P}^1 \cong S^1 \wedge \mathbf{G}_m$ in $\mathscr{H}(k)$ (see [36, Lemma 2.15 and Corollary 2.18]). More details on stable $\mathbf{A}^1$-homotopy can also be found in [23] (resp. [21]) where the $\mathbf{A}^1$-analogues of symmetric spectra (resp. *S*-modules) are considered.

In this section, we will construct $\Omega_{\mathbf{P}^1}$-spectra **KO** and **KT** with $\mathbf{KO}_0 = aKO_{\mathrm{f}}$ and $\mathbf{KT}_0 = aKT_{\mathrm{f}}$ which represent hermitian *K*-theory KO_* and Balmer Witt groups W^*_{B} of regular schemes in $\mathscr{SH}(k)$. In order to do so, we need to study $KO_*(R[t,t^{-1}])$ and $W^*_{\mathrm{B}}(R[t,t^{-1}])$ for a regular ring R (in which 2 is invertible). We are interested only in the trivial involution on $P(R[t,t^{-1}])$. Using the involution that sends t to t^{-1} yields different results, of course. The following is essentially a special case of the main result of [19].

5.1. Proposition. *For any regular ring R, we have a split homotopy fibration*

$$KO(R) \to KO(R[t,t^{-1}]) \to K^h(U_R).$$

Proof. The first map $f : KO(R) \to KO(R[t,t^{-1}])$ is induced by the composition $R \to R[t] \to R[t,t^{-1}]$, and by homotopy invariance (Corollary 1.12), the map $KO(R) \xrightarrow{\simeq} KO(R[t])$ is a homotopy equivalence. Observe that we have natural splittings of $BO(R)^+ \to BO(R[t,t^{-1}])^+$ induced by sending t to 1. Now applying the localization theorem of [19] to $R[t] \to R[t,t^{-1}]$ we get a homotopy fibration $K^h(R) \to K^h(R[t,t^{-1}]) \to \mathscr{W}(\mathscr{T})$ where $\mathscr{W}(\mathscr{T})$ is a delooping of the *U*-theory of the exact category of $R[t]$-modules of projective dimension 1 which are annihilated by t^n for n large enough. Finally, the dévissage theorem of [19] implies that $U(R) \to U(\mathscr{T})$ is a homotopy equivalence, so the theorem follows using $\Omega KO(U_R) \simeq U(R)$. □

5.2. Remark. Observe that the proof of the general dévissage theorem of [19] can be simplified in our case thanks to the above splitting and the fact that we do not need to start the Karoubi induction in negative degrees as in [19] but we can start in degree 0 and 1: The fact that $U_0(R) \to U_0(\mathscr{T})$ is an isomorphism follows from comparing the cokernel in the (split) localization sequence in [19] with the one from [25, Corollaries 3.12, 3.13]. The isomorphism $W_0(R) \to W_0(\mathscr{T})$ can be deduced from well-known (e.g. [26, p. 139], [12, Theorem 5.6]) isomorphism $W_0(R[t,t^{-1}]) \cong W_0(R) \oplus W_0(R)$.

5.3. Corollary. *For any regular ring R, there are natural isomorphisms*

$$KO_n(R[t,t^{-1}]) \cong KO_n(R) \oplus U_{n-1}(R)$$

and

$$U_n(R[t,t^{-1}]) \cong U_n(R) \oplus {}_-KO_{n-1}(R).$$

Proof. Recall that $\Omega K(U_R) \simeq K(R)$ and $\Omega K^h(U_R) \simeq U(R)$. By Quillen [39, p. 122], we also have a homotopy fibration $K(R) \to K(R[t,t^{-1}]) \to K(U_R)$. Now use Karoubi's Fundamental Theorem 4.2 and recall the above splitting of $KO(R) \to KO(R[t,t^{-1}])$. □

We will now combine Proposition 5.1 with Karoubi's Fundamental Theorem and deduce the desired Periodicity Theorem which then will yield to the definition of the spectrum **KO**.

For an unpointed scheme (or simplicial sheaf) X and the pointed scheme $\mathbf{G}_m$, consider the cofiber sequence in $\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}$

$$X_+ \vee \mathbf{G}_m \to X_+ \times \mathbf{G}_m \to X_+ \wedge \mathbf{G}_m.$$

We define $U(X) := hofibK(X) \xrightarrow{H} KO(X)$ also for non-affine regular X. Writing ${}_-KO$ for the hermitian K-theory of antisymmetric forms as before, we get the following.

5.4. Corollary. *For any regular scheme X, there are homotopy equivalences*

$$\Omega aKO_{\mathrm{f}}(X_+ \wedge \mathbf{G}_m) \simeq aU_{\mathrm{f}}(X)$$

and

$$\Omega aU_{\mathrm{f}}(X_+ \wedge \mathbf{G}_m) \simeq a_-KO_{\mathrm{f}}(X).$$

Proof. As aKO_{f} is fibrant, we obtain a homotopy fibration [14, Proposition II.3.2]

$$aKO_{\mathrm{f}}(X_+ \wedge \mathbf{G}_m) \to aKO_{\mathrm{f}}(X_+ \times \mathbf{G}_m) \to aKO_{\mathrm{f}}(X_+ \vee \mathbf{G}_m)$$

and similar for aK_{f}, aU_{f} and aKT_{f}. Using the fact that $X_+ \times \mathbf{G}_m = X \times \mathbf{G}_m \amalg \mathbf{G}_m$ and $X_+ \vee \mathbf{G}_m = X \amalg \mathbf{G}_m$, we obtain the homotopy fibration

$$aKO_{\mathrm{f}}(X_+ \wedge \mathbf{G}_m) \to aKO_{\mathrm{f}}(X \times \mathbf{G}_m) \to aKO_{\mathrm{f}}(X).$$

Comparing this homotopy fibration with the split one we just constructed in Proposition 5.1 (and reducing to X affine by Jouanolou's device), we obtain the first homotopy equivalence using that $\Omega KO(U_R) \simeq U(R)$. For the second homotopy equivalence, look at the corresponding homotopy fibrations for U and use that $\Omega U(U_R) \simeq {}_-KO(R)$. □

Now we will construct the spectrum **KO**. For all $k \in \mathbf{N}$, we set $\mathbf{KO}_{4k} := aKO_{\mathrm{f}}$, $\mathbf{KO}_{4k+1} := a_-U_{\mathrm{f}}$, $\mathbf{KO}_{4k+2} := a_-KO_{\mathrm{f}}$ and $\mathbf{KO}_{4k+3} := aU_{\mathrm{f}}$. To define the adjoints of the structure maps $\mathbf{KO}_n \to \Omega_{\mathbf{P}^1}\mathbf{KO}_{n+1}$, we first recall (see [11, Proposition 2.8] or [36, Lemma 1.1.16]) that any object F in $\Delta^{op}Shv(Sm/k)_{\mathrm{Nis}}$ can be written as a homotopy colimit of objects of (Sm/k). More precisely, one can construct a simplicial sheaf QF which is the realization of a diagram of coproducts of schemes, that

is $QF = hocolim_I X_i$ with X_i an object of Sm/k for all $i \in I$, together with a map $QF \to F$ which is a weak equivalence of simplicial sets for each section. In particular $hocolim_I X_i \simeq QF \simeq F$ with respect to the model structures [22,36] we care about.

Next, we define the adjoints of the structure maps and show that they are weak equivalences. For $F \simeq hocolim_I X_i$ as above, we have

$\mathbf{Hom}_\bullet(F_+, aU_\mathrm{f}) \simeq \mathbf{Hom}_\bullet(hocolim X_{i+}, aU_\mathrm{f}) \cong holim \mathbf{Hom}_\bullet(X_{i+}, aU_\mathrm{f})$ by [36, Lemma 1.1.19] and the fact that all objects are cofibrant. By Corollary 5.4 and the simplicial Yoneda Lemma we already used in the proof of Theorem 3.1, we have $\mathbf{Hom}_\bullet(X_{i+}, aU_\mathrm{f}) \cong aU_\mathrm{f}(X_{i+}) \simeq \Omega aKO_\mathrm{f}(X_{i+} \wedge \mathbf{G}_m) \cong \mathbf{Hom}_\bullet(X_{i+} \wedge \mathbf{G}_m, \Omega aKO_\mathrm{f}) \cong \mathbf{Hom}_\bullet(X_{i+} \wedge \mathbf{G}_m \wedge S^1, aKO_\mathrm{f}) \cong \mathbf{Hom}_\bullet(X_{i+}, \Omega_{\mathbf{G}_m \wedge S^1} aKO_\mathrm{f})$ and thus $\mathbf{Hom}_\bullet(F_+, aU_\mathrm{f}) \simeq holim \mathbf{Hom}_\bullet(X_{i+}, \Omega_{\mathbf{G}_m \wedge S^1} aKO_\mathrm{f}) \cong \mathbf{Hom}_\bullet(F_+, \Omega_{\mathbf{G}_m \wedge S^1} aKO_\mathrm{f})$.

Applying π_0, the homotopy class of the identity of aU_f then corresponds to a homotopy class of a map t_{4k+3} from aU_f to $\Omega_{\mathbf{P}^1} aKO_\mathrm{f}$ via the natural isomorphisms $Hom_{\mathscr{H}(k)}(aU_\mathrm{f}, aU_\mathrm{f}) \cong Hom_{\mathscr{H}(k)}(aU_\mathrm{f}, \Omega_{\mathbf{G}_m \wedge S^1} aKO_\mathrm{f})$. This $t_{4k+3} : \mathbf{KO}_{4k+3} \to \Omega_{\mathbf{P}^1} \mathbf{KO}_{4(k+1)}$ is one of our four structure maps. The above computation for $F = X$ being already a scheme shows that α is stalkwise a weak equivalence. The three other structure maps are defined in a similar way.

5.5. Theorem. *The spectrum* **KO** *constructed above is an $\Omega_{\mathbf{P}^1}$-spectrum. For any regular scheme X, we have a natural isomorphism*

$$KO_n(X) \cong Hom_{\mathscr{SH}(k)}(S^n \wedge X_+, \mathbf{KO}).$$

Proof. Follows from the above construction and discussion of **KO** and Corollary 3.4. □

Now let us look at Balmer Witt groups.

5.6. Proposition. *For any regular ring R, there are natural homotopy equivalences*

$$KT(R) \simeq \Omega^2_- KT(R) \simeq \Omega^4 KT(R)$$

and

$$KT(R[t, t^{-1}]) \simeq KT(R) \times KT(R).$$

Proof. The first claim follows from Lemma 4.5. The second claim follows by applying *hocolim* to Corollary 5.3. □

Applying homotopy groups to the second statement, we recover the well-known computations of $W(R[t, t^{-1}])$ and more generally $W^n_\mathrm{B}(R[t, t^{-1}]) \cong W^n_\mathrm{B}(R) \oplus W^n_\mathrm{B}(R)$ as computed in [12].

5.7. Corollary. *For any regular scheme X, there is a homotopy equivalence*

$$aKT_\mathrm{f}(X_+ \wedge \mathbf{G}_m) \simeq aKT_\mathrm{f}(X).$$

Proof. Using Proposition 5.6, the proof is similar to Corollary 5.4. □

We now define the spectrum **KT** by setting $\mathbf{KT}_{4k} := aKT_\mathrm{f}, \mathbf{KT}_{4k+1} := \Omega a_- KT_\mathrm{f}, \mathbf{KT}_{4k+2} := a_- KT_\mathrm{f}$ and $\mathbf{KT}_{4k+3} := \Omega aKT_\mathrm{f}$. The structure maps are defined similar to the ones of **KO**. For example, the natural weak equivalences $\mathbf{Hom}_\bullet(F_+, aKTf) \simeq holim \mathbf{Hom}_\bullet(X_{i+} \wedge \mathbf{G}_m, \Omega\Omega^3 aKTf) \simeq$

$holim\mathbf{Hom}_\bullet(X_{i+}, \Omega_{\mathbf{G}_m\wedge S^1}\Omega a_- KTf) \simeq \mathbf{Hom}_\bullet(F, \Omega_{\mathbf{G}_m\wedge S^1}\Omega a_- KTf)$ we have by Proposition 5.6 and Corollary 5.7 yield the structure map t_{4k} which is a weak equivalence.

5.8. Theorem. *The spectrum* **KT** *we just constructed is an* $\Omega_{\mathbf{P}^1}$*-spectrum. For any regular scheme X, we have a natural isomorphism*

$$W_{\mathrm{B}}^{-n}(X) \cong Hom_{\mathscr{SH}(k)}(S^n \wedge X_+, \mathbf{KT}).$$

Proof. Follows from the construction of **KT** and Corollary 4.9. □

5.9. Remark. By the Fundamental Theorem for regular rings [39, p. 122], we have $K(R[t,t^{-1}]) \simeq K(R) \times K(SR)$ where $\Omega K(SR) \simeq K(R)$ and S stands for the algebraic suspension of a ring [50]. Hence we can also use our method above to construct a $(2,1)$-periodic spectrum which represents algebraic K-theory in $\mathscr{SH}(k)$. This gives a variant of the proof of Voevodsky [48, Section 6.2] who uses Quillen's projective bundle theorem [39, Proposition 8.2], instead. Observe also that applying Theorem 5.1 to $(A \times A^{op})$ yields Quillen's [39, p. 122] theorem on $K(A[t,t^{-1}])$. The fact that one can use a colimit of Grassmannians in order to represent K-theory in $\mathscr{H}(k)$ [36, Theorem 3.13] and a colimit argument [48, Lemma 6.7] allows Voevodsky to construct the $\Omega_{\mathbf{P}^1}$-spectrum **BGL** that represents algebraic K-theory in $\mathscr{SH}(k)$. As discussed at the end of Section 3, orthogonal Grassmanians do not behave as well with respect to hermitian K-theory. Hence Voevodsky's approach to construct the structure maps does not carry over to KO and KT. Roughly speaking, we replace his particular colimit argument involving Grassmannians by the more general one of [11]. Voevodsky [48, Section 6.2] does not explicitly say why the spectrum **BGL** he constructs is an $\Omega_{\mathbf{P}^1}$-spectrum and hence represents algebraic K-theory in $\mathscr{SH}(k)$. The material of this section provides the necessary techniques to fill in the missing details in [48].

6. Applications

We first compute the hermitian K-theory of the projective line:

6.1. Proposition. *For any regular ring R, we have a split homotopy fibration*

$$U(R) \to KO(\mathbf{P}^1_R) \to KO(R).$$

Proof. We have a covering of $\mathbf{P}^1_R$ by two copies of $R[t]$ with intersection $R[t,t^{-1}]$. The claim now follows from Proposition 5.1, Corollary 1.12 and Theorem 2.8. □

The computation of the Witt groups can be done in a similar way, but is already known by [12, Theorem 5.4]. Using $\mathbf{A}^1$-representability, one can easily show the following:

6.2. Proposition. *For any ring R smooth over R, we have homotopy fibrations*

$$F_n(R) \to KO(\mathbf{P}^n_R) \to KO(\mathbf{P}^{n-1}_R),$$

$$\Omega^n KT(R) \to KT(\mathbf{P}^n_R) \to KT(\mathbf{P}^{n-1}_R),$$

where $F_{4k+i}(R)$ *equals* $KO(k)$ *if* $i=0$, $U(k)$ *if* $i=1$, $\_KO(k)$ *if* $i=2$ *and* $\_U(k)$ *if* $i=3$.

Proof. From [36, Corollary 3.2.18], we deduce a cofiber sequence $\mathbf{P}^{n-1}_+ \to \mathbf{P}^n_+ \to (\mathbf{P}^1)^{\wedge n}$ in $\mathscr{H}(k)$. Applying KO to the split cofiber sequence $Spec(k)_+ \to \mathbf{P}^1_+ \to \mathbf{P}^1$ and using Proposition 6.1, we see that $KO(\mathbf{P}^1) \simeq U(Spec(k)_+)$. Smashing the preceding cofiber sequence with powers of $\mathbf{P}^1$, using Corollary 5.4 and proceeding inductively, the first claim follows. The second statement is deduced from the first one by applying powers of U and passing to the usual *hocolim*. □

Observe that the maps $KO(\mathbf{P}^n_R) \to KO(\mathbf{P}^{n-1}_R)$ and $KT(\mathbf{P}^n_R) \to KT(\mathbf{P}^{n-1}_R)$ would not split in general, contrary to the corresponding map in K-theory. If the given ring R is such that $W^n_{\mathrm{B}}(R) = 0$ unless n is a multiple of 4 (for instance if R is a field), we easily get some computations for Witt groups, for instance $W^0_{\mathrm{B}}(\mathbf{P}^n_R) \cong W^0_{\mathrm{B}}(R)$ for $0 < n < 4$ as it was already known by Arason [1] for R a field. As Charles Walter [53] recently proved a more general theorem on Witt groups of projective spaces and projective bundles by completely different methods, we do not investigate our approach any further.

Next, we deduce the Thom isomorphism and the blow-up isomorphism. If E is a vector bundle over X, we define the Thom space to be the pointed sheaf $Th(E) := E/(E - i(X))$ (compare [36, Definition 3.2.16]) where $i : X \to E$ is the zero section. Here and in the sequel, quotients are formed in the category of simplicial Nisnevish sheaves.

6.3. Corollary. *Let $j : Z \to X$ be a closed embedding of smooth schemes with $N_{X,Z}$ as normal bundle. Then we have natural homotopy equivalences $KO(X/(X - j(Z))) \simeq KO(Th(N_{X,Z}))$, $KT(X/(X - j(Z))) \simeq KT(Th(N_{X,Z}))$ and long exact sequences*

$$\ldots \to KO_{n+1}(X - j(Z)) \to KO_n(Th(N_{X,Z})) \to KO_n(X) \to KO_n(X - j(Z)) \to \ldots$$

and

$$\ldots \to W^{n-1}_{\mathrm{B}}(X - j(Z)) \to W^n_{\mathrm{B}}(Th(N_{X,Z})) \to W^n_{\mathrm{B}}(X) \to W^n_{\mathrm{B}}(X - j(Z)) \to \ldots$$

in $\mathscr{SH}(k)$.

Proof. Apply Corollaries 3.4, 4.9, Theorems 5.5, 5.8 and [36, Theorem 3.2.23], [48, Proposition 4.12]. □

6.4. Corollary. *Let $j : Z \to X$ be a closed embedding of smooth schemes, $p : X_Z \to X$ be the blow-up of $j(Z)$ in X and $U = X - j(Z) = X_Z - p^{-1}(j(Z))$. Then we have natural homotopy equivalences $KO((X_Z/U)\coprod_{p^{-1}(Z)} Z) \simeq KO(X/U)$, $KT((X_Z/U)\coprod_{p^{-1}(Z)} Z) \simeq KT(X/U)$ and long exact sequences*

$$\ldots \to KO_n(X) \to KO_n(Z) \oplus KO_n(X_Z) \to KO_n(p^{-1}(Z)) \to KO_{n-1}(X) \to \ldots$$

and

$$\ldots \to W^n_{\mathrm{B}}(X) \to W^n_{\mathrm{B}}(Z) \oplus W^n_{\mathrm{B}}(X_Z) \to W^n_{\mathrm{B}}(p^{-1}(Z)) \to W^{n+1}_{\mathrm{B}}(X) \to \ldots$$

in $\mathscr{SH}(k)$.

Proof. Apply Corollaries 3.4, 4.9, Theorems 5.5, 5.8 and [36, Theorem 3.2.29], [48, Proposition 4.13]. □

6.5. Now consider the Hopf map $\mathbf{P}^1_k \wedge \mathbf{G}_m \simeq \mathbf{A}^2 - 0 \to \mathbf{P}^1_k$. It induces a stable map $\eta : \mathbf{G}_m \to S^0$. Morel conjectured (January 2001) that W^*_{B} is represented by $\mathbf{KO}[\eta^{-1}] := hocolim(\mathbf{KO} \xrightarrow{\eta} \mathbf{KO} \wedge \mathbf{G}_m^{-1} \xrightarrow{\eta} \mathbf{KO} \wedge \mathbf{G}_m^{-2} \ldots)$ where **KO** stands for an $\Omega_{\mathbf{P}^1}$-spectrum representing hermitian K-theory. In fact, his conjectural description of **KO** looked more geometric than ours, involving $B_{et}O$, $B_{et}Sp$ etc., similar to real topological K-theory (compare 3.8). We remark that there is also some work of Barge and Lannes (not yet published) which allows them to prove some periodicity theorem with respect to a *naive* notion of homotopy of algebraic varieties. In particular, they seem to have a more geometric proof of Karoubi's Fundamental Theorem (Theorem 4.2) for regular rings.

Morel also conjectured that there is an exact triangle $\mathbf{KO} \wedge \mathbf{G}_m \xrightarrow{\eta} \mathbf{KO} \to \mathbf{BGL}$ in $\mathcal{SH}(k)$. This does not only look similar to Karoubi's Fundamental Theorem, but even more to Atiyah's work on *Real K-theory* where the Hopf map appears precisely in the analogous topological situation, see [2, Propositions 3.2, 3.3].

Comparing this to the Karoubi tower and knowing that its *hocolim KT* represents W^*_{B} by Theorem 5.8, this leads us to the following:

6.6. Question. *The stabilization of the map in the Karoubi tower* $\mathbf{KO}(\ ) \xrightarrow{\alpha} \mathbf{KO}(U\ )$ *does it coincide with the Hopf map* $\mathbf{KO} \xrightarrow{\eta} \mathbf{KO} \wedge \mathbf{G}_m^{-1}$ *in* $\mathcal{SH}(k)$ *at least up to sign*?

Next, recall the following recent theorem of Morel [35] about stable $\mathbf{A}^1$-homotopy groups of spheres whose proof uses Theorem 5.5 of this paper:

6.7. Theorem. *For any perfect field k of characteristic different from* 2 *and for all* $n > 0$, *there are isomorphisms*

$$Hom_{\mathcal{SH}(k)}(\mathbf{G}_m^{\wedge n}, S^0) \cong W^0_{\mathrm{B}}(k),$$

$$Hom_{\mathcal{SH}(k)}(S^0, S^0) \cong K^h_0(k),$$

where $\mathcal{SH}(k)$ *denotes the stable* $\mathbf{A}^1$*-homotopy category of* [48].

6.8. Introducing *Milnor–Witt K-theory* as a quotient of the tensor algebra generated by the units of k and η [34], this Theorem can be extended to $Hom_{\mathcal{SH}(k)}(S^0, \mathbf{G}_m^{\wedge n})$. In order to better understand how these different conjectures are related and what this has to do with $\mathbf{A}^1$-representability of Balmer Witt groups and hermitian K-theory, consider the following diagram:

$$\begin{array}{ccccccc}
K^h_0(k) & \xrightarrow{\cong} & Hom_{\mathcal{H}(k)}\left(Spec(k)_+, aKO_f\right) & \xrightarrow{\cong} & Hom_{\mathcal{SH}(k)}\left(Spec(k)_+, \mathbf{KO}\right) & \xleftarrow[i]{} & Hom_{\mathcal{SH}(k)}\left(S^0, S^0\right) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \eta} \\
W^0_B(k) & \xrightarrow{\cong} & Hom_{\mathcal{H}(k)}\left(Spec(k)_+, aKT_f\right) & \xrightarrow{\cong} & Hom_{\mathcal{SH}(k)}\left(Spec(k)_+, \mathbf{KT}\right) & \underset{j}{\cdots\cdots\!\!>} & Hom_{\mathcal{SH}(k)}\left(S^0, \mathbf{G}_m^{\wedge -1}\right)
\end{array}$$

Here p is the projection, $\alpha : K^h(\ ) \to K^h(U\ )$ is the map from the Karoubi tower, $\tilde{\alpha}$ are the induced maps and i is given by the fact that the sphere spectrum is a unit. More precisely, we choose the map $Hom_{\mathcal{H}(k)}(S^0, aKO_{\mathrm{f}}) \cong K^h_0(k)$ represented by the form $\langle 1 \rangle$. The left-hand side and the middle square

commute. The conjecture is that there is a map j such that the right-hand side square also commutes (up to sign) and that all the horizontal maps are isomorphisms. If the answer to question 6.6 is yes, then j is induced by $\mathbf{G}_m^{\wedge -1} \overset{i\wedge id}{\to} \mathbf{KO} \wedge \mathbf{G}_m^{\wedge -1} \to \mathbf{KO}[\eta^{-1}] \simeq \mathbf{KT}$.

6.9. Let us give some more evidence that the unit map $S^0 \to \mathbf{KO}$ really induces the desired isomorphism $Hom_{\mathscr{SH}(k)}(S^0, S^0) \cong K_0^h(k)$. The group $K_0^h(k) = GW(k)$ is generated by the elements $\langle u\rangle$, $u \in k^*$ which are subject to the relations $\langle u\rangle = \langle uv^2\rangle$ and $\langle u\rangle + \langle v\rangle = \langle u+v\rangle + \langle (u+v)uv\rangle$ $\forall u + v \neq 0$ [43, p. 66]. The identity in $Hom_{\mathscr{H}(k)}(\mathbf{P}^1_k, \mathbf{P}^1_k)$ stabilizes to the identity in $Hom_{\mathscr{SH}(k)}(S^0, S^0)$ and is mapped to $\langle 1\rangle$ in $K_0^h(k)$ under the unit map defined above. This $\langle 1\rangle$ is then mapped under the map of Lannes–Morel (send $\langle u\rangle$ in $K_0^h(k)$ to $\Sigma^\infty_{\mathbf{P}^1}\mu_u$ with μ_u being the endomorphism of $\mathbf{P}^1$ mapping $[x, y]$ to $[ux, y]$) back to the identity. So $id \in Hom_{\mathscr{SH}(k)}(S^0, S^0)$ is mapped to itself under $Hom_{\mathscr{SH}(k)}(S^0, S^0) \to Hom_{\mathscr{SH}(k)}(S^0, \mathbf{KO}) \cong K_0^h(k) \to Hom_{\mathscr{SH}(k)}(S^0, S^0)$.Unfortunately, we cannot prove that the involved maps are maps of ring spectra and $Hom_{\mathscr{SH}(k)}(S^0, S^0)$-modules which would imply that the above maps are isomorphisms. Showing that the above composition is the identity seems in fact to be the harder part, as Morel (November 2001 and before) claims that using his results in [33], he can prove the existence of a quasi-splitting $Hom_{\mathscr{SH}(k)}(S^0, S^0) \to GW(k)^\wedge$ which is a splitting if k is of finite cohomological dimension, $GW(k)^\wedge$ being the completion of $GW(k)$ by the augmentation ideal.

6.10. In topology, the unit map between the sphere spectrum S^0 and the real topological K-theory spectrum $\mathbf{BO}$ induces a map $Hom_{SH}(S^n, S^0) \to Hom_{SH}(S^n, \mathbf{BO})$ which is an isomorphism not only for $n=0$, but also for $n=1$ and $n=2$. The Morel conjectures together with computations of $Hom_{SH(k)}(S^0 \wedge \mathbf{G}_m^{\wedge j}, \mathbf{KO})$ predict isomorphisms $Hom_{SH(k)}(S^0 \wedge \mathbf{G}_m^{\wedge j}, S^0) \to Hom_{SH(k)}(S^0 \wedge \mathbf{G}_m^{\wedge j}, \mathbf{KO})$ for $j \geqslant 0$. So it seems natural to ask whether the maps $Hom_{SH(k)}(S^1, S^0) \to Hom_{SH(k)}(S^1, \mathbf{KO}) \cong KO_1(k) \cong k^*/(k^*)^2 \times \mathbf{Z}/2$ and $Hom_{SH(k)}(S^1 \wedge \mathbf{G}_m, S^0) \to Hom_{SH(k)}(S^1 \wedge \mathbf{G}_m, \mathbf{KO}) \cong U_0(k) \cong \mathbf{Z}/2$ and $Hom_{SH(k)}(S^2, S^0) \to Hom_{SH(k)}(S^2, \mathbf{KO}) \cong KO_2(k)$ are also isomorphisms or not. Topological real K-theory together with Adams operations allow the construction of the spectrum J which detects most of the 2-torsion of the homotopy groups of spheres by a theorem of Mahowald [31, Theorem 1.5]. So one might try to perform similar constructions with KO, hoping to detect more elements of the higher $\mathbf{A}^1$-homotopy groups of spheres.

Appendix A. On the relationship of hermitian K-theory, L-theory and Balmer Witt groups

People study symmetric bilinear forms (and quadratic forms) for different reasons. Depending on their motivation and their background, they end up with different theories.

L-theory has been extensively studied by topologists because quadratic L-groups contain *surgery obstruction classes*, and *higher signatures* have values in symmetric L-groups. The rings involved are integral group rings of the fundamental group of some topological space. Hence they are often non-commutative, and assuming 2 to be invertible is a too strong restriction.

People interested in the theory of quadratic forms for its own sake work in particular over fields and commutative rings, and more and more also over non-affine schemes. Many of them think that Balmer Witt groups are a convenient generalization of the classical Witt group, allowing strong theorems to be proved which were not available before.

Finally, from a purely K-theoretical point of view, it seems most natural to study hermitian K-theory. In fact, this is essentially the mother theory from which all other theories can be deduced, as we have seen in Section 4. One important difference between hermitian K-theory and the other theories is that hermitian K-groups are defined as being the homotopy groups of a space (or spectrum). This tends to make concrete calculations more difficult, but is much more convenient to establish $\mathbf{A}^1$-representability.

Many mathematicians are comfortable with only one of the three theories. We now give a couple of results (and conjectures), some old and some new, comparing these three theories.

Historically, quadratic L-groups were defined first for group rings by Wall (depending on a fixed subgroup of the Whitehead group) and then in general for rings R with involution (depending on a fixed subgroup of $K_1(R)$). Symmetric L-groups were introduced by Mischenko. Later Ranicki gave a definition of L-groups depending on a fixed subgroup X of $K_0(R)$. There are always a quadratic and a symmetric version of L-groups, denoted by L_*^X and by L^*_X. The case $X = K_0(R)$ is often denoted by L_*^p (and also by U_* in some older articles, not to be confused with our U-groups).

Quadratic L-theory is always 4-periodic. Symmetric L-theory coincides with quadratic L-theory only if 2 is invertible in the given ring. Otherwise, symmetric L-theory need not to be 4-periodic. It becomes 4-periodic after passing to a certain colimit, which is good enough for most geometric applications. It also becomes 4-periodic if we neglect the 2-torsion (*i.e.*, after applying $\otimes_{\mathbf{Z}}\mathbf{Z}[1/2]$) because quadratic and symmetric L-theory coincide up to 2-torsion. When we say a certain L-theory is 4-periodic, actually more is true: we have $L^n \cong {}_-L^{n+2}$ where ${}_-L$ denotes the theory of antisymmetric forms. The same periodicity pattern holds for Balmer Witt groups. Neglecting the 2-torsion, it also holds for Karoubi Witt groups as defined in 4.1 (see [27]). Hence it suffices in general to consider only degrees 0 and 1.

Ranicki first gave a definition of L-theory in terms of *forms* and *formations* and later in terms of *algebraic Poincaré complexes*. See [41, Section 5] for a proof that both definitions coincide.

We will now investigate symmetric L-theory of a ring R in which 2 is invertible with respect to $X = K_0(R)$, which we denote by $L^n(R)$ from now on.

A.1. Proposition. *There is a natural isomorphism*

$$W_n^K(R)\otimes_{\mathbf{Z}}\mathbf{Z}[1/2] \cong L^n(R)\otimes_{\mathbf{Z}}\mathbf{Z}[1/2].$$

Proof. This is already stated (without proof) in [30, p. 321]. For $n = 0$, it is true by definition (use forms instead of algebraic Poincaré complexes to describe L^0). For $n = 1$, first observe [51, p. 286] that $W_1^K(R) = L_1^{\widetilde{K_1(R)}}(R) \cong L_1^{0\subset\widetilde{K_0(R)}}(R)$ [40, p. 14] and hence also $L_1^{K_1(R)}(R) \cong L_1^{0\subset K_0(R)}(R)$ as they differ by the same Tate-cohomology group [17, p. 61], [16, p. 138]. By the Rothenberg sequence [41, Proposition 9.1], $L_1^{0\subset K_0(R)}(R)$ and $L_1^{K_0(R)}(R) = L_1^p(R)$ are isomorphic after tensoring with $\mathbf{Z}[1/2]$. The proposition now follows from the periodicity of L-theory and Karoubi's 12-term exact sequence [27,28]. □

Balmer Witt groups follow the same periodicity pattern as L-theory: $W_{\mathrm{B}}^n \cong {}_-W_{\mathrm{B}}^{n+2} \cong W^{n+4}$. The following comparison statement (conjectured by many people) has now been proved by Ch. Walter:

A.2. Theorem. *There is a natural isomorphism*

$$L^n(R) \cong W_{\mathrm{B}}^{-n}(R).$$

By periodicity, we have $L^n(R) \overset{\cong}{\to} colim\, L^{n+4k}(R)$ if 2 is invertible, otherwise we may take this as the definition of $L^n(R)$. As $P(R)$ is split exact and idempotent complete, a map of complexes is a quasi-isomorphism if and only if it is a homotopy equivalence. Both $L^n(R)$ and $W_{\mathrm{B}}^{-n}(R)$ are generated by elements in $D^b(P(A))$ equipped with a non-degenerate symmetric bilinear form with respect to the duality $Ext^n_R(\ \ , R)$. It remains to compare the cobordism relation for L-groups with the metabolic objects for W^*_B. This is done by Walter in [52]. □

Recall from Section 2 that for a scheme X we can extend K^h from rings to schemes using Jouanolou's device or *holims* of affine covers and get by definition $KO(X)$. We conjecture, but cannot prove, that the theory KO obtained this way coincides with K^h as defined in [18] for regular non-affine schemes. We also define $WO_n(X) := coker(K_n(X) \to KO_n(X))$.

One can also prove the following comparison result between Balmer and Karoubi Witt groups:

A.3. Proposition. (i) *There is a natural isomorphism*

$$W_n^K(R)\otimes_{\mathbf{Z}}\mathbf{Z}[1/2] \cong W_{\mathrm{B}}^{-n}(R)\otimes_{\mathbf{Z}}\mathbf{Z}[1/2];$$

(ii) *for any regular scheme X, there is a natural isomorphism*

$$WO_n(X)\otimes_{\mathbf{Z}}\mathbf{Z}[1/2] \cong W_{\mathrm{B}}^{-n}(X)\otimes_{\mathbf{Z}}\mathbf{Z}[1/2].$$

Proof. (i) This is proved in [19] using a cofinality result for Balmer Witt groups (and is true even if R is not regular). To prove (ii), use strong homotopy invariance of K, KO and W^*_{B}. □

Observe that Proposition A.3 is false even for fields k if we do not tensor with $\mathbf{Z}[1/2]$. In this case, we have $W_1^K(k) = \mathbf{Z}/2$, but $W_{\mathrm{B}}^{-1}(k) = 0$. On the other hand, we have the following comparison result for regular rings including the 2-torsion if we look at negative hermitian K-theory:

A.4. Proposition. *For any regular ring R, there are natural isomorphisms for* $n > 0$

$$W^n_{\mathrm{B}}(R) \cong W^K_{-n}(R) \cong K^h_{-n}(R) \cong U_{-n-1}(R).$$

Proof. See [19, Lemma A.4] and remember that the negative K-theory of a regular ring is trivial. □

A.5. Corollary. *For any regular scheme X, there are natural isomorphisms for* $n > 0$

$$KO_{-n}(X) \cong WO_{-n}(X) \cong W^n_{\mathrm{B}}(X).$$

Proof. Follows from Proposition A.4 and strong homotopy invariance. □

Concerning the comparison of KT as defined in Section 4 and L-theory, Bruce Williams has outlined a proof of the following statement to me. Being unable to cite a reference, I present it here as a conjecture, anticipating his proof in due course.

A.6. Conjecture. (B. Williams) For any (not necessarily regular) ring R in which 2 is invertible, we have a natural isomorphism

$$\pi_n(KT(R)) \cong L^n(R).$$

A.7. Corollary. *If Conjecture* A.6 *is true, we have a natural isomorphism*

$$L^n(R) \cong colim\, K_n^h(U_R^i).$$

Proof. Observe that $\pi_n(KT(R)) \cong colim\, \pi_n(K^h(U_R^i))$. □

References

[1] J.K. Arason, Der wittring der projektiven Räume, Math. Ann. 253 (1980) 205–212.
[2] M.F. Atiyah, K-theory and reality, Quart. J. Math. Oxford 17 (2) (1966) 367–386.
[3] P. Balmer, Triangular Witt groups I, The 12-term localization exact sequence, K-theory 19 (2000) 311–363.
[4] P. Balmer, Triangular Witt groups II, from usual to derived, Math. Z. 236 (2001) 351–382.
[5] P. Balmer, Witt cohomology, Mayer–Vietoris, homotopy invariance, and the Gersten conjecture, K-theorem 23 (2001) 15–30.
[6] H. Bass, Algebraic K-theory, W.A. Benjamin, Inc., New York, Amsterdam, 1968.
[7] P. Berthelot, A. Grothendieck, L. Illusie, Théorie des intersections et théorème de Riemann-Roch, Lecture Notes in Mathematics, vol. 225, Springer, Berlin, 1971.
[8] R. Bott, The stable homotopy of the classical groups, Ann. Math. 70 (1959) 313–337.
[9] E.H. Brown, Cohomology theories, Ann. Math. 75 (1962) 467–484.
[10] J.-L. Colliot-Thélène, R.T. Hoobler, B. Kahn, The Bloch–Ogus–Gabber theorem, Fields Institute Communications 16, American Mathematical Society, Providence, 1997, pp. 31–94.
[11] D. Dugger, Universal homotopy theories, Adv. Math. 164 (2001) 144–176.
[12] S. Gille, On Witt groups with support, Math. Ann. 322 (2002) 103–137.
[13] S. Gille, Homotopy invariance of coherent Witt groups, Math. Z. 244 (2003) 211–233.
[14] P.G. Goerss, J.F. Jardine, Simplicial Homotopy Theory, Progress in Mathematics, vol. 174, Birkhäuser, Verlag, Basel, 1999.
[15] D. Grayson, (after D. Quillen), Higher algebraic K-theory II, Lecture Notes in Mathematics, vol. 551, Springer, Berlin, 1976, pp. 217–240.
[16] I. Hambleton, A. Ranicki, L. Taylor, Round L-theory, J. Pure Appl. Algebra 47 (1987) 131–154.
[17] I. Hambleton, L. Taylor, B. Williams, An Introduction to Maps between Surgery Obstruction Groups, Lecture Notes in Mathematics, vol. 1051, Springer, Berlin, 1984, pp. 49–127.
[18] J. Hornbostel, Constructions and dévissage in hermitian K-theory, K-Theory 26 (2002) 139–170.
[19] J. Hornbostel, M. Schlichting, Localization in Hermitian K-theory of rings, J. LMS 70 (2004) 77–124.
[20] J. Hornbostel, S. Yagunov, Rigidity for Henselian local rings and $\mathbf{A}^1$-representable theories, Preprint 2004, available at http://www.math.uiuc.edu/K-theory/0688/.
[21] P. Hu, S-modules in the category of schemes, Mem. Amer. Math. Soc. 161 (767) (2003).
[22] J.F. Jardine, Simplicial presheaves, J. Pure Appl. Algebra 47 (1987) 35–87.
[23] J.F. Jardine, Motivic symmetric spectra, Doc. Math. 5 (2000) 445–553.
[24] J.-P. Jouanolou, Une suite de Mayer–Vietoris en K-théorie algébrique, Lecture Notes in Mathematics, vol. 341, Springer, Berlin, 1973, pp. 293–316.
[25] M. Karoubi, Localisation des formes quadratiques I, Ann. Sci. de l'ENS t. 7 (1974) 359–404.
[26] M. Karoubi, Localisation des formes quadratiques II, Ann. Sci. de l'ENS t. 8 (1975) 99–155.
[27] M. Karoubi, Théorie de Quillen et homologie du groupe orthogonal, Ann. Math. 112 (1980) 207–257.
[28] M. Karoubi, Le théorème fondamental de la K-théorie hermitienne, Ann. Math. 112 (1980) 259–282.
[29] D. Kobal, Hermitian K-theory and the Karoubi tower, K-theory 17 (1999) 113–140.
[30] J.-L. Loday, Higher Witt groups: a survey, Lecture Notes in Mathematics, vol. 551, Springer, Berlin, 1976, pp. 311–335.
[31] M. Mahowald, The image of J in the EHP sequence, Ann. Math. 116 (1982) 65–112.
[32] J. Milnor, D. Husemoller, Symmetric bilinear forms, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73, Springer, Berlin, 1973.
[33] F. Morel, Suite spectrale d'Adams et invariants cohomologiques des formes quadratiques, C.R. Acad. Sci. Paris, Sér. I 328 (1999) 963–968.

[34] F. Morel, An introduction to $\mathbf{A}^1$-homotopy, Lectures given at the ICTP Trieste summer school, July 2002, ICTP Lecture Notes, vol. 15, 2003.
[35] F. Morel, On the motivic stable π_0 of the sphere spectrum, Proceedings Axiomatic, Enriched and Motivic Homotopy Theories, Kluwer, 2004.
[36] F. Morel, V. Voevodsky, $\mathbf{A}^1$-homotopy theory of schemes, Publications Mathématiques de l'IHES No. 90, 1999, pp. 45–143.
[37] I. Panin, The equi-characteristic case of the Gersten conjecture, Preprint 2000, available at http://www.math.uiuc.edu/K-theory/0389/.
[38] E.K. Pedersen, C.A. Weibel, K-theory homology of spaces, Lecture Notes in Mathematics, vol. 1370, Springer, Berlin, 1989, pp. 346–361.
[39] D. Quillen, Higher algebraic K-theory I, Lecture Notes in Mathematics, vol. 341, Springer, Berlin, 1973, pp. 85–147.
[40] A. Ranicki, Algebraic L-theory I, Twisted Laurent Extensions, Lecture Notes in Mathematics, vol. 343, Springer, Berlin, 1973, pp. 412–463.
[41] A. Ranicki, The algebraic theory of surgery I, Foundations, Proceedings of the London Mathematical Society, vol. 40, 1980, pp. 87–192.
[42] M. Raynaud, Anneaux locaux Henséliens, Lecture Notes in Mathematics, vol. 169, Springer, Berlin, 1970.
[43] W. Scharlau, Quadratic and Hermitian forms, Grundlehren der mathematischen Wissenschaften, vol. 270, Springer, Berlin, 1985.
[44] M. Schlichting, Délaçage de la K-théorie des catégories exactes et K-groupes négatifs, Ph.D. Thesis, Université Paris 7, 2000, available at http://www.uni-essen.de/∼bm0031/.
[45] Théorie des Topos et Cohomologie Etale des Schémas, Tome 2, dirigé par M. Artin, A. Grothendieck, J. L. Verdier, Lecture Notes in Mathematics, vol. 270, Springer, Berlin, 1972.
[46] R.W. Thomason, Algebraic K-theory and Etale cohomology, Ann. Sci. de l'ENS 13 (1985) 437–552.
[47] R.W. Thomason, T. Trobaugh, Higher Algebraic K-theory of Schemes and of Derived Categories, The Grothendieck Festschrift, vol. 3, Birkhäuser PM, vol. 88, 1990, pp. 247–435.
[48] V. Voevodsky, $\mathbf{A}^1$-homotopy theory, Documenta Mathematica, Extra volume ICM, 1998, pp. 579–604.
[49] V. Voevodsky, Triangulated Categories of Motives Over a Field, in Annals of Mathematics Studies, vol. 143, Princeton University Press, Princeton, 2000.
[50] J.B. Wagoner, Delooping spaces in algebraic K-theory, Topology 11 (1972) 349–370.
[51] C.T.C. Wall, Foundations of algebraic L-theory, Lecture Notes in Mathematics, vol. 343, Springer, Berlin, 1973, pp. 266–300.
[52] C. Walter, Grothendieck–Witt groups of triangulated categories, Preprint 2003, available at http://www.math.uiuc.edu/K-theory/643.
[53] C. Walter, Grothendieck–Witt groups of projective bundles, Preprint 2003, available at http://www.math.uiuc.edu/K-theory/644.
[54] C.A. Weibel, Homotopy algebraic K-theory, Contemp. Math. 83 (1989) 461–488.
[55] S. Yagunov, Rigidity II non-orientable case, Doc. Math. 9 (2004) 29–40.