ELSEVIER

Topology and its Applications 112 (2001) 205–213

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

# A new proof of the signature formula for surface bundles

Matthias Hoster

*Ludwig-Maximilians-Universität, Theresienstrasse 39, 80333 München, Germany*

Received 19 April 1999; received in revised form 22 October 1999

**Abstract**

Let $E \to X$ be an oriented surface bundle over a closed surface. Then the signature $\operatorname{sign}(E)$ is determined by the first Chern class of the flat vector bundle $\Gamma$ associated to the monodromy homomorphism $\chi : \pi_1(X) \to \mathrm{Sp}_{2h}(\mathbb{Z})$ of $E$, it is equal to $-4\langle c_1(\Gamma), [X]\rangle$. The aim of this paper is to give an algebro-topological proof of this formula, i.e., one that does not use the Atiyah–Singer index theorem. 

*Keywords:* Signature; Surface bundles; Mapping class group; Symplectic group

*AMS classification:* Primary 55R10, Secondary 55R40; 57T10

## 1. Introduction

Let $E \to X$ be a smooth oriented surface bundle over a surface, more precisely a smooth fibre bundle with fiber $S_h$, an oriented closed surface of genus $h$, and base $X$ an oriented compact surface with boundary $\partial X$. Thus the structure group of $E$ is the group $G := \mathrm{Diffeo}^+(S_h)$ of orientation preserving diffeomorphisms of $S_h$ with the usual $\mathcal{C}^\infty$-topology.

For $h \geqslant 2$ the connected components of $G$ are contractible, so the bundle $E$ is determined by its monodromy homomorphism

$$\rho : \pi_1(X) \to G/G_0 = \mathcal{M}_h$$

into the mapping class group $\mathcal{M}_h$ of $S_h$ ($G_0$ the connected component of the identity). Let $\mathrm{Sp}_{2h}(\mathbb{Z})$ be the Siegel modular group, i.e., the subgroup of the automorphism group $\mathrm{Sp}_{2h}(\mathbb{R})$ of the standard symplectic space $(\mathbb{R}^{2h}, \omega_0)$ consisting of the matrices with integer

*E-mail address:* hoster@rz.mathematik.uni-muenchen.de (M. Hoster).

0166-8641/01/$ – see front matter 
PII: S0166-8641(99)00233-3

coefficients. Because every element of $\mathcal{M}_h$ leaves the intersection form on $H_1(S_h;\mathbb{Z})$ invariant and because $(H_1(S_h;\mathbb{R}),\cdot)\cong(\mathbb{R}^{2h},\omega_0)$, there is a natural homomorphism

$$\sigma:\mathcal{M}_h\to \mathrm{Sp}_{2h}(\mathbb{Z})\subset \mathrm{Sp}_{2h}(\mathbb{R}).$$

Composing these two we get a homomorphism $\chi:=\sigma\circ\rho:\pi_1(X)\to \mathrm{Sp}_{2h}(\mathbb{Z})$. If $X$ has genus $g\geqslant 1$, then it is an Eilenberg–MacLane space $K(\pi_1(X),1)$, thus $X$ is a classifying space for its fundamental group. So the singular cohomology of $X$ is canonically isomorphic to the Eilenberg–MacLane cohomology of the group $\pi_1(X)$ and $\chi$ induces a homomorphism

$$\chi^*:H^*_{EM}\big(\mathrm{Sp}_{2h}(\mathbb{Z});\mathbb{Z}\big)\to H^*(X;\mathbb{Z}).$$

Now Meyer [5] showed, that the signature of the 4-manifold $E$ can be computed in terms of the Leray spectral sequence of $E\to X$ and equals $\mathrm{sign}(E)=\mathrm{sign}(X,\partial X;\mathcal{H}^1(S_h;\mathbb{R}))$. By cutting the base $X$ into spheres with three boundary components each he also gave an explicit method to compute such signatures $\mathrm{sign}(X,\partial X;\Gamma)$ for flat symplectic vector bundles $\Gamma$ in terms of a special cocycle $\tau_h$ of the symplectic group (cf. Section 2). For the case of a surface bundle considered here, this gives the following formula.

**Theorem 1** (Meyer [5,6]). *Let $E\to X$ be a smooth oriented surface bundle over an oriented, closed surface $X$ with genus $g\geqslant 1$, and with fibre $S_h$ an oriented closed surface of genus $h\geqslant 2$. Then the signature of the total space is equal to*

$$\mathrm{sign}(E)=-\big\langle\chi^*[\tau_h],[X]\big\rangle,$$

*where $\chi=\sigma\circ\rho$ is the monodromy map of $E$ followed by the homology representation of $\mathcal{M}_h$ in $\mathrm{Sp}_{2h}(\mathbb{Z})$.*

It follows, for example, that every such bundle $E$ with $h\leqslant 2$ has vanishing signature, because $\mathcal{M}_2$ is $\mathbb{Q}$-acyclic by a result of Igusa [4] (that torus bundles have vanishing signature can be seen in an elementary way).

By using the Atiyah–Singer index theorem for families Meyer derived another formula expressing the signature of a smooth oriented fibre bundle over a closed manifold $X$ in terms of characteristic classes of flat vector bundles on $X$. In the special case of a surface bundle over a closed surface this leads to the following formula.

Let $\Gamma$ be a flat vector bundle with structure group $\mathrm{Sp}_{2h}(\mathbb{R})$. The standard symplectic form $\omega_0$ on $\mathbb{R}^{2h}$ induces a symplectic form on $\Gamma$ and any compatible complex structure $J$ on $\Gamma$ turns $\Gamma$ into a complex vector bundle. Let $c_1(\Gamma)\in H^2(X;\mathbb{Z})$ denote its first Chern class.

**Theorem 2** (Meyer [5]). *Let $E\to X$ be a smooth oriented surface bundle over an oriented, closed surface $X$ with fibre $S_h$. Then the signature of the total space is equal to*

$$\mathrm{sign}(E)=-4\big\langle c_1(\Gamma),[X]\big\rangle,$$

*where $\Gamma=\mathcal{H}^1(S_h;\mathbb{R})$.*

Combining these two theorems, respectively the more general forms of them dealing with signatures $\mathrm{sign}(X,\Gamma)$, identifies the pull back of the signature class $[\tau_h]$ with a characteristic class of a flat vector bundle over $X$.

**Theorem 3.** *For any oriented, closed surface $X$ with genus $g \geqslant 1$, any homomorphism $\chi:\pi_1(X)\to \mathrm{Sp}_{2h}(\mathbb{Z})$ and $\Gamma$ the associated flat symplectic vector bundle there is the equality*

$$\chi^*[\tau_h] = 4c_1(\Gamma)$$

*in $H^2(X;\mathbb{Z})$.*

The aim of this paper is to give an algebro-topological proof of this last theorem, which does not use the Atiyah–Singer index theorem hidden in Theorem 2. From this and Theorem 1 of course we get a new proof for Meyer's Theorem 2. In fact a universal version of Theorem 3 will be proved.

## 2. The signature cocycle

For any topological group $G$ let $G^\delta$ be the underlying discrete group and $D:BG^\delta\to BG$ the continuous map of classifying spaces induced by the identity $G^\delta\to G$. Let $X$ be a compact oriented surface with boundary $\partial X$. Let $\Gamma$ be a flat vector bundle over $X$ with structure group $\mathrm{Sp}_{2h}(\mathbb{R})$. Thus the classifying map of $\Gamma$ factors over $D:B\mathrm{Sp}_{2h}(\mathbb{R})^\delta\to B\mathrm{Sp}_{2h}(\mathbb{R})$.

Then $\Gamma$ is determined by the monodromy homomorphism $\chi:\pi_1(X)\to\mathrm{Sp}_{2h}(\mathbb{R})$, namely $\Gamma=\widetilde{X}\times_{\pi_1(X)}\mathbb{R}^{2h}$, where $\pi_1(X)$ acts on the universal cover $\widetilde{X}$ of $X$ by deck transformation and on $\mathbb{R}^{2h}$ via $\chi$. The standard symplectic form $\omega_0=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ induces a non-degenerate, antisymmetric form on $\Gamma$. Combining this form with the cup product gives a symmetric bilinear map

$$H^1(X,\partial X;\Gamma)\times H^1(X;\partial X;\Gamma)\to H^2(X,\partial X;\mathbb{R})$$

and by evaluation on the orientation class of $X$ a symmetric bilinear form on $H^1(X,\partial X;\Gamma)$. Let $\mathrm{sign}(X,\partial X;\Gamma)$ be the signature of this bilinear form.

Now let $X_0$ be a sphere with three open discs removed. The fundamental group $\pi_1(X_0)$ is a free group on two generators, so that any two elements $\alpha,\beta\in\mathrm{Sp}_{2h}(\mathbb{R})$ define a homomorphism

$$\chi_{\alpha,\beta}:\pi_1(X_0)\to\mathrm{Sp}_{2h}(\mathbb{R})$$

and by this a flat symplectic vector bundle $\Gamma_{\alpha,\beta}$ over $X_0$. Using a triangulation of $(X_0,\partial X_0)$ Meyer [5] showed that there is an isomorphism of $H^1(X_0,\partial X_0;\Gamma_{\alpha,\beta})$ onto

$$H_{\alpha,\beta}:=\bigl\{x=(x_1,x_2)\in\mathbb{R}^{2h}\oplus\mathbb{R}^{2h}\mid(\alpha^{-1}-1)x_1+(\beta-1)x_2=0\bigr\}$$

and that under this isomorphism the bilinear form on $H^1(X_0,\partial X_0;\Gamma_{\alpha,\beta})$ becomes $-\langle\cdot,\cdot\rangle_{\alpha,\beta}$, where for $x,y\in H_{\alpha,\beta}$

$$\langle x,y\rangle_{\alpha,\beta}=\omega_0\bigl(x_1+x_2,(1-\beta)y_2\bigr).$$

Finally let $\tau_h$ be defined by

$$\tau_h : \mathrm{Sp}_{2h}(\mathbb{R}) \times \mathrm{Sp}_{2h}(\mathbb{R}) \to \mathbb{Z},$$
$$\tau_h(\alpha, \beta) := -\operatorname{sign}(X_0, \partial X_0; \Gamma_{\alpha,\beta}) = \operatorname{sign}\big(H_{\alpha,\beta}, \langle\cdot,\cdot\rangle_{\alpha,\beta}\big).$$

Then by cutting a sphere with four open discs removed into two copies of $X_0$ in two different ways and using the Novikov additivity of the signature it is easy to see, that for every $\alpha, \beta, \gamma \in \mathrm{Sp}_{2h}(\mathbb{R})$ the equality

$$\tau_h(\alpha, \beta) + \tau_h(\alpha\beta, \gamma) = \tau_h(\alpha, \beta\gamma) + \tau_h(\beta, \gamma)$$

holds. Thus $\tau_h$ is a cocycle of the symplectic group and defines a class $[\tau_h]$ in the Eilenberg–MacLane cohomology $H^2_{EM}(\mathrm{Sp}_{2h}(\mathbb{R}); \mathbb{Z})$ of this group, which of course is canonically isomorphic to the singular cohomology $H^2(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta; \mathbb{Z})$ of the classifying space of $\mathrm{Sp}_{2h}(\mathbb{R})^\delta$.

## 3. Identification of the signature class

The maximal compact subgroup of $\mathrm{Sp}_{2h}(\mathbb{R})$ is $\mathrm{U}(h)$. So we can consider the first universal Chern class as an element $c_1 \in H^2(B\mathrm{Sp}_{2h}(\mathbb{R}); \mathbb{Z})$. Let $X$ be as in Theorem 3,

$$\chi : \pi_1(X) \to \mathrm{Sp}_{2h}(\mathbb{R})$$

a homomorphism and $\Gamma$ the associated flat vector bundle. Then $c_1(\Gamma) = \chi^* D^* c_1$. Thus to prove Theorem 3 it would be sufficient to prove $[\tau_h] = 4D^*c_1$ in $H^2(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta; \mathbb{Z}) = H^2_{EM}(\mathrm{Sp}_{2h}(\mathbb{R}); \mathbb{Z})$. To do this we use the results of Milnor on the Friedlander conjecture.

**Proposition 1** (Milnor [7]). *Let $G$ be a Lie group with finitely many components and $A$ a finite Abelian group. If the connected component $G_0$ of the identity is solvable or $G$ is a Chevalley group, then $D^* : H^2(BG; A) \to H^2(BG^\delta; A)$ and $D_* : H_2(BG^\delta; A) \to H_2(BG; A)$ are isomorphisms.*

We will use this result for the Abelian group $G = \mathrm{U}(1)$ and the Chevalley group $G = \mathrm{Sp}_{2h}(\mathbb{R})$. This proposition will enable us in Section 4 to prove the following.

**Lemma 1.** *Let $h \geqslant 1$ and $n \in \mathbb{N}$. Then there is the equality $[\tau_h] = 4D^*c_1$ in $H^2(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta; \mathbb{Z}_n)$.*

Then in fact the same is true with integer coefficients. The following theorem is our main result.

**Theorem 4.** *In $H^2_{EM}(\mathrm{Sp}_{2h}(\mathbb{R}); \mathbb{Z}) = H^2(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta; \mathbb{Z})$ the equality $[\tau_h] = 4D^*c_1$ holds for $h \geqslant 1$.*

**Proof.** Let $X$ be an oriented, closed surface with genus $g \geqslant 1$ and $\chi : \pi_1(X) \to \mathrm{Sp}_{2h}(\mathbb{R})^\delta$ a homomorphism. Look at the long exact cohomology sequence induced by the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$ for $n \in \mathbb{N}$:

$$\begin{tikzcd}
H^2(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta;\mathbb{Z}) \arrow[r] \arrow[d, "\chi^*"'] & H^2(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta;\mathbb{Z}) \arrow[r] \arrow[d, "\chi^*"'] & H^2(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta;\mathbb{Z}_n) \arrow[d, "\chi^*"'] \\
H^2(X;\mathbb{Z}) \arrow[r, "n"] & H^2(X;\mathbb{Z}) \arrow[r] & H^2(X;\mathbb{Z}_n)
\end{tikzcd}$$

Because of Lemma 1 and the exactness of the lower row $\chi^*([\tau_h]-4D^*c_1)$ is in $\bigcap_{n\in\mathbb{N}} nH^2(X;\mathbb{Z})$. But because $H^2(X;\mathbb{Z})$ is finitely generated, this intersection is trivial. So $\chi^*[\tau_h]=4c_1(\Gamma)$ in $H^2(X;\mathbb{Z})$.

Suppose now that $[\tau_h]-4D^*c_1\neq 0$ in $H^2(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta;\mathbb{Z})$. $\mathrm{Sp}_{2h}(\mathbb{R})$ is perfect, i.e.,

$$H_1\bigl(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta;\mathbb{Z}\bigr)=H_1^{EM}\bigl(\mathrm{Sp}_{2h}(\mathbb{R});\mathbb{Z}\bigr)=0.$$

Thus

$$H^2\bigl(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta;\mathbb{Z}\bigr)=\mathrm{Hom}\bigl(H_2\bigl(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta;\mathbb{Z}\bigr),\mathbb{Z}\bigr).$$

So there is a class $x\in H_2(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta;\mathbb{Z})$ with $\langle[\tau_h]-4D^*c_1,x\rangle\neq 0$. But by Hopf's theorem [2] every class in $H_2^{EM}(\mathrm{Sp}_{2h}(\mathbb{R});\mathbb{Z})$ can be represented by a surface $X$ as above. So $x=\chi_*[X]$ and $\langle[\tau_h]-4D^*c_1,x\rangle=\langle\chi^*[\tau_h]-4\chi^*D^*c_1,[X]\rangle=0$. That is a contradiction. □

Hence to show Theorem 3 it remains to prove Lemma 1.

## 4. Proof of Lemma 1

**Lemma 2.** *If $[\tau_1]=4D^*c_1$ in $H^2(B\mathrm{U}(1)^\delta;\mathbb{Z}_n)$ then Lemma 1 is true for all $h\geqslant 1$ in $H^2(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta;\mathbb{Z}_n)$.*

**Proof.** The natural inclusion $\mathbb{C}\subset\mathbb{C}^h : z\mapsto(z,0,\ldots,0)^t$ induces an embedding $\iota:\mathrm{U}(1)\hookrightarrow\mathrm{U}(h)$. Then $\iota^*:H^*(B\mathrm{U}(h);\mathbb{Z})=\mathbb{Z}[c_1,\ldots,c_h]\to H^*(B\mathrm{U}(1);\mathbb{Z})=\mathbb{Z}[c_1]$ is the projection $\iota^*(c_1)=c_1$, $\iota^*(c_i)=0$ for $i\geqslant 2$. So $\iota$ induces an isomorphism

$$H^2\bigl(B\mathrm{Sp}_{2h}(\mathbb{R});\mathbb{Z}\bigr)=H^2\bigl(B\mathrm{U}(h);\mathbb{Z}\bigr)\to H^2\bigl(B\mathrm{U}(1);\mathbb{Z}\bigr).$$

Furthermore $\iota$ induces an isomorphism $H^2(B\mathrm{Sp}_{2h}(\mathbb{R});\mathbb{Z}_n)\to H^2(B\mathrm{U}(1);\mathbb{Z}_n)$. Because of Proposition 1 this descends to an isomorphism $H^2(B\mathrm{Sp}_{2h}(\mathbb{R})^\delta;\mathbb{Z}_n)\to H^2(B\mathrm{U}(1)^\delta;\mathbb{Z}_n)$.

Now looking at the explicit presentation of the cocycle $\tau_h$ given by $\tau_h(\alpha,\beta)=\mathrm{sign}(H_{\alpha,\beta},\langle\cdot,\cdot\rangle_{\alpha,\beta})$ shows that $\tau_h(\iota(z_1),\iota(z_2))=\tau_1(z_1,z_2)$ for $z_1,z_2\in\mathrm{U}(1)$. Thus $\iota^*[\tau_h]=[\iota^*\tau_h]=[\tau_1]=4D^*c_1=\iota^*(4D^*c_1)$. But $\iota^*$ is an isomorphism, so Lemma 1 follows. □

Thus it is enough to show that $[\tau_1]=4D^*c_1$ in $H^2_{EM}(\mathrm{U}(1);\mathbb{Z}_n)$.

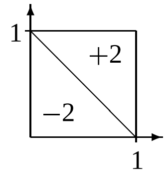
For $\alpha,\beta\in\mathrm{U}(1)$, say $\alpha=\mathrm{e}^{\mathrm{i}a}$, $\beta=\mathrm{e}^{\mathrm{i}b}$, the vector space $H_{\alpha,\beta}$ becomes

$$H_{\alpha,\beta}=\bigl\{x=(x_1,x_2)\in\mathbb{C}\oplus\mathbb{C}\mid\bigl(\mathrm{e}^{-\mathrm{i}a}-1\bigr)x_1+\bigl(\mathrm{e}^{\mathrm{i}b}-1\bigr)x_2=0\bigr\}$$

and $\langle x, y\rangle_{\alpha,\beta}$ is computed to be

$$\langle x, y\rangle_{\alpha,\beta} = \frac{\sin(a+b) - \sin a - \sin b}{1-\cos b}\,\mathrm{re}(\bar{x}_1 y_1).$$

Thus $\tau_1(\mathrm{e}^{\mathrm{i}a}, \mathrm{e}^{\mathrm{i}b}) = 2\,\mathrm{sgn}(\sin(a+b) - \sin a - \sin b)$. So $\tau_1(\mathrm{e}^{2\pi\mathrm{i}\cdot}, \mathrm{e}^{2\pi\mathrm{i}\cdot})$ takes the following values on $[0,1]\times[0,1]$:




And $\tau_1 \equiv 0$ on the lines.

The easiest way to compare $[\tau_1]$ and $D^*c_1$ would be to evaluate both classes on a generator of $H_2^{EM}(\mathrm{U}(1);\mathbb{Z}_n) = \mathbb{Z}_n$, for $H_1(B\mathrm{U}(1)^\delta;\mathbb{Z}_n) = 0$ implies that

$$H^2_{EM}\bigl(\mathrm{U}(1);\mathbb{Z}_n\bigr) = \mathrm{Hom}\bigl(H_2^{EM}(\mathrm{U}(1);\mathbb{Z}_n),\mathbb{Z}_n\bigr).$$

Let $C_*(G)$ be the Eilenberg–MacLane complex of $G := \mathrm{U}(1)$ (cf. [1, II.3]). So $C_k(G)$ is the free Abelian group generated by tuples $[g_1|\cdots|g_k]$, $g_i \in G$, and the differential $d : C_k(G) \to C_{k-1}(G)$ is defined as $\sum_{i=0}^k (-1)^i d_i$, where

$$d_i\bigl[g_1|\cdots|g_k\bigr] = \begin{cases} \bigl[g_2|\cdots|g_k\bigr] & i = 0,\\ \bigl[g_1|\cdots|g_i g_{i+1}|\cdots|g_k\bigr] & 0 < i < k,\\ \bigl[g_1|\cdots|g_{k-1}\bigr] & i = k.\end{cases}$$

**Lemma 3.** *Let $\omega_n := \mathrm{e}^{2\pi\mathrm{i}/n}$. Then the class of the 2-cycle $Q_n := \sum_{j=1}^n [\omega_n|\omega_n^j]$ satisfies $\langle[\tau_1],[Q_n]\rangle \equiv 4 \bmod n$.*

**Proof.** It is $dQ_n = \sum_{j=1}^n([\omega_n^j] - [\omega_n^{j+1}] + [\omega_n]) = n[\omega_n] \equiv 0 \bmod n$. So $Q_n$ is indeed a $\mathbb{Z}_n$-2-cycle of $\mathrm{U}(1)$. Furthermore

$$\bigl\langle[\tau_1],[Q_n]\bigr\rangle = \sum_{j=1}^n \tau_1\bigl(\omega_n,\omega_n^j\bigr) = (n-1)\cdot(-2) + 0 + 2 \equiv 4 \bmod n. \qquad \square$$

Considering the fact that

$$H_2^{EM}\bigl(\mathrm{U}(1);\mathbb{Z}_n\bigr) = H_2\bigl(B\mathrm{U}(1)^\delta;\mathbb{Z}_n\bigr) \xrightarrow{D_*} H_2\bigl(B\mathrm{U}(1);\mathbb{Z}_n\bigr) = H_2\bigl(\mathbb{C}P^\infty;\mathbb{Z}_n\bigr) = H_2\bigl(\mathbb{C}P^1;\mathbb{Z}_n\bigr)$$

is an isomorphism and $\langle c_1, [\mathbb{C}P^1]\rangle = 1$, it remains to show that $D_*[Q_n] = [\mathbb{C}P^1]$. This will be done in two steps in Lemmas 4 and 5. Define the singular 2-cycle $P_n$ of $\mathbb{C}P^1$ by $P_n := \sum_{j=1}^n (P_n^j - P_\times)$, where

$$P_n^j : \Delta^2 \to \mathbb{C}P^1 : (t_0,t_1,t_2) \mapsto \bigl[\sqrt{t_0} + \sqrt{t_1}\,\omega_n : \sqrt{t_2}\,\omega_n^{j+1}\bigr],$$
$$P_\times : \Delta^2 \to \mathbb{C}P^1 : (t_0,t_1,t_2) \mapsto [1:0].$$

**Lemma 4.** *The image under the map* $H_2(\mathbb{C}P^1;\mathbb{Z}) \to H_2(\mathbb{C}P^1;\mathbb{Z}_n)$ *of the class* $[P_n] \in H_2(\mathbb{C}P^1;\mathbb{Z})$ *is* $D_*[Q_n]$.

**Proof.** Let $F_*(G)$ be the standard resolution (cf. [1, I.5]) of $\mathbb{Z}$ over $\mathbb{Z}G$, $G := \mathrm{U}(1)$. So the tuples $(g_0,\dots,g_k)$, $g_i \in G$, build a $\mathbb{Z}$-basis of $F_k(G)$ and the $\mathbb{Z}G$-module structure is given by $g \cdot (g_0,\dots,g_k) = (gg_0,\dots,gg_k)$. The differential $d: F_k(G) \to F_{k-1}(G)$ is defined to be $\sum_{i=0}^k (-1)^i d_i$, where

$$d_i(g_0,\dots,g_k) = (g_0,\dots,g_{i-1},g_{i+1},\dots,g_k).$$

Then the Eilenberg–MacLane complex is $C_k(G) = F_k(G)_G$. The $\mathbb{Z}$-basis elements of $C_k(G)$ are given by the $\mathbb{Z}G$-basis elements of $F_k(G)$ of the form $[g_1|g_2|\cdots|g_k] = (1, g_1, g_1g_2, \dots, g_1g_2\cdots g_k)$.

Take the Milnor model of the universal $G^\delta$-bundle $EG^\delta \to BG^\delta$ (cf. [3, 4.11]), $EG^\delta = G^\delta * G^\delta * \cdots$ and write an element $y \in EG^\delta$ as $y = (g_0,t_0;g_1,t_1;\dots)$, where $g_i \in G$, only finitely many $t_i \neq 0$ and $\sum_{i=1}^\infty t_i = 1$. Then define $\mathbb{Z}G$-homomorphisms $\Phi_i : F_i(G) \to S_i(EG^\delta)$, $i = 0,1,2$, into the singular chain complex of $EG^\delta$ by

$$\Phi_0(g_0) := (g_0,t_0;1,0;\dots),$$

$$\Phi_1(g_0,g_1) := (g_0,t_0;g_1,t_1;1,0;\dots) - (g_1,t_0;g_1,t_1;1,0;\dots),$$

$$\begin{aligned}\Phi_2(g_0,g_1,g_2) := {} & (g_0,t_0;g_1,t_1;g_2,t_2;1,0;\dots) - (g_0,t_0;g_2,t_1;g_2,t_2;1,0;\dots) \\ & + (g_1,t_0;g_2,t_1;g_2,t_2;1,0) - (g_1,t_0;g_1,t_1;g_2,t_2;1,0;\dots).\end{aligned}$$

Here the terms depending on the $t_i$ represent singular simplices in $EG^\delta$. Then $d\Phi_i = \Phi_{i-1}d$ for $i = 1,2$. Because $F_*(G^\delta)$ is free and $S_*(EG^\delta)$ is acyclic, there is a chain map $\Phi_* : F_*(G) \to S_*(EG^\delta)$, unique up to homotopy, extending the given $\Phi_i$'s [1, Lemma I.7.4]. Because this $\Phi_*$ is an augmentation-preserving chain map between free resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$, it is a homotopy equivalence [1, Theorem I.7.5]. Finally $(-)_G$ is an additive functor, so $\Phi_*$ induces a homotopy equivalence

$$\overline{\Phi}_* : C_*(G) \to S_*(BG^\delta).$$

Computing $\overline{\Phi}_*(Q_n)$ gives

$$\begin{aligned}\overline{\Phi}_*(Q_n) = {} & \sum_{j=1}^n \bigl(\bigl[1,t_0;\omega_n,t_1;\omega_n^{j+1},t_2;1,0;\dots\bigr] \\ & \quad - \bigl[1,t_0;\omega_n^{j+1},t_1;\omega_n^{j+1},t_2;1,0;\dots\bigr] \\ & \quad + \bigl[\omega_n,t_0;\omega_n^{j+1},t_1;\omega_n^{j+1},t_2;1,0;\dots\bigr] \\ & \quad - \bigl[\omega_n,t_0;\omega_n,t_1;\omega_n^{j+1},t_2;1,0;\dots\bigr]\bigr) \\ = {} & \sum_{j=1}^n \bigl(\bigl[1,t_0;\omega_n,t_1;\omega_n^{j+1},t_2;1,0;\dots\bigr] - \bigl[1,t_0;1,t_1;\omega_n^{j},t_2;1,0;\dots\bigr]\bigr).\end{aligned}$$

Pulling it back to $\mathbb{C}P^\infty$ via the homeomorphism

$$\mathbb{C}P^\infty \to B\mathrm{U}(1) : [z_0 : z_1 : \cdots] \mapsto \left[\frac{z_0}{|z_0|}, |z_0|^2; \frac{z_1}{|z_1|}, |z_1|^2; \dots\right]$$

we get a cycle $P_n'$ in $\mathbb{C}P^2$, namely

$$P_n' = \sum_{j=1}^{n} \left( \left[\sqrt{t_0} : \sqrt{t_1}\omega_n : \sqrt{t_2}\omega_n^{j+1}\right] - \left[\sqrt{t_0} : \sqrt{t_1} : \sqrt{t_2}\omega_n^{j+1}\right] \right).$$

Because of

$$\begin{aligned} &d\left[\sqrt{t_0} : \sqrt{t_1} + \sqrt{t_2}\omega_n : \sqrt{t_3}\omega_n^{j+1}\right] \\ &\quad = \left[0 : \sqrt{t_0} + \sqrt{t_1}\omega_n : \sqrt{t_2}\omega_n^{j+1}\right] - \left[\sqrt{t_0} : \sqrt{t_1}\omega_n : \sqrt{t_2}\omega_n^{j+1}\right] \\ &\qquad + \left[\sqrt{t_0} : \sqrt{t_1} : \sqrt{t_2}\omega_n^{j+1}\right] - \left[\sqrt{t_0} : \sqrt{t_1} + \sqrt{t_2}\omega_n : 0\right], \end{aligned}$$

$P_n'$ is homologous to

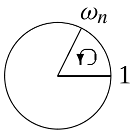
$$\begin{aligned} P_n' &\sim \sum_{j=1}^{n} \left( \left[0 : \sqrt{t_0} + \sqrt{t_1}\omega_n : \sqrt{t_2}\omega_n^{j+1}\right] - \left[\sqrt{t_0} : \sqrt{t_1} + \sqrt{t_2}\omega_n : 0\right] \right) \\ &= \sum_{j=1}^{n} \left[0 : \sqrt{t_0} + \sqrt{t_1}\omega_n : \sqrt{t_2}\omega_n^{j+1}\right] \bmod n. \end{aligned}$$

Using that $\mathbb{C}P^2 \to \mathbb{C}P^2 : [z_1 : z_2 : z_3] \mapsto [z_2 : z_3 : z_1]$ induces the identity in homology and adding $-nP_\times$ to make the last cocycle become an integer cocycle completes the proof. $\square$

**Lemma 5.** $[P_n] = [\mathbb{C}P^1]$ *in* $H_2(\mathbb{C}P^1; \mathbb{Z})$.

**Proof.** Look at the relative homeomorphism $f : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (\mathbb{C}P^1, [1:0]) : z \mapsto [z : 1 - |z|]$. Let $T_n : \mathbb{D}^2 \to \mathbb{D}^2 : z \mapsto \omega_n z$ be the rotation with angle $\omega_n$ and $\sigma : \Delta^2 \to \mathbb{D}^2$ the following simplex, homeomorphic onto its image:

$$\sigma(t_0, t_1, t_2) := \frac{\sqrt{t_0} + \sqrt{t_1}\omega_n}{|\sqrt{t_0} + \sqrt{t_1}\omega_n| + \sqrt{t_2}}$$




Then the cycle $C := \sum_{j=1}^{n} T_n^j(\sigma)$ represents the fundamental class of $(\mathbb{D}^2, \partial\mathbb{D}^2)$. But

$$\begin{aligned} f_*C &= \sum_{j=1}^{n} \left[ \omega_n^j \frac{\sqrt{t_0} + \sqrt{t_1}\omega_n}{|\sqrt{t_0} + \sqrt{t_1}\omega_n| + \sqrt{t_2}} : 1 - \frac{|\sqrt{t_0} + \sqrt{t_1}\omega_n|}{|\sqrt{t_0} + \sqrt{t_1}\omega_n| + \sqrt{t_2}} \right] \\ &= \sum_{j=1}^{n} \left[\sqrt{t_0} + \sqrt{t_1}\omega_n : \sqrt{t_2}\omega_n^{-j}\right]. \end{aligned}$$

So in $H_2(\mathbb{C}P^1; \mathbb{Z}) = H_2(\mathbb{C}P^1, \{[1:0]\}; \mathbb{Z})$ we have $[\mathbb{C}P^1] = f_*[C] = [P_n]$. $\square$

Combining Lemmas 4 and 5 we get $D_*[Q_n] = [\mathbb{C}P^1]$. So by Lemma 3 we have $\langle [\tau_1], [Q_n] \rangle = 4 = 4\langle c_1, [\mathbb{C}P^1] \rangle = 4\langle c_1, D_*[Q_n] \rangle = \langle 4D^*c_1, [Q_n] \rangle$. Thus $[\tau_1] = 4D^*c_1$ because $[Q_n]$ generates $H_2^{EM}(\mathrm{U}(1); \mathbb{Z}_n)$. By Lemma 2 this proves Lemma 1. That completes the proof of Theorem 4.

## References

[1] K.S. Brown, Cohomology of Groups, GTM, Vol. 87, Springer, Berlin, 1994.
[2] H. Hopf, Fundamentalgruppe und zweite Bettische Gruppe, Comment. Math. Helv. 14 (1942) 257–309.
[3] D. Husemoller, Fibre Bundles, GTM, Vol. 20, Springer, Berlin, 1994.
[4] J.I. Igusa, Arithmetic variety of moduli for genus two, Ann. of Math. 72 (1960) 612–649.
[5] W. Meyer, Die Signatur von lokalen Koeffizientensystemen und Faserbündeln, Bonner Mathematische Schriften, Vol. 53, 1972.
[6] W. Meyer, Die Signatur von Flächenbündeln, Math. Ann. 201 (1973) 239–264.
[7] J. Milnor, On the homology of Lie groups made discrete, Comment. Math. Helv. 58 (1983) 72–85.