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A NEW PROOF OF LEBESGUE'S COVERING LEMMA

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Introduction. Lebesgue's covering lemma states that, given an open covering $U_1, \cdots, U_n$ of a compact metric space X, ρ, there is a positive number δ such that if $\rho(x, y) < \delta$ then both x and y belong to some U_i. The purpose of this short note is to enlarge upon this conclusion and thereby provide a more interesting proof of the lemma than the usual ones.

We first explain how we arrive at the new result. Figure 1 shows a compact metric space covered by two open subsets U and V. If δ is the distance between $U-V$ and $V-U$, then any two points whose distance apart is less than δ both lie in U or V; further, no number greater than δ will ensure this. Figure 2 shows a compact metric space X, ρ covered by a finite number of open subsets $U_1, \cdots, U_n$ and one may suspect that the same idea holds. The lines of the figure divide the set X up into a number of "compartments" (those white bits crossed by no lines) and by analogy one may suspect that two of these compartments A and B, at a positive distance apart, have the properties

(i) if $\rho(x, y) < \rho(A, B)$ then both x and y belong to some U_i,

(ii) no number greater than $\rho(A, B)$ has this property. Except in a trivial case this is so, and it is the extension of Lebesgue's lemma that we shall prove.

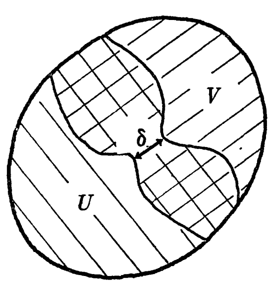

FIG. 1

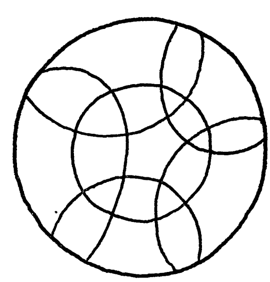

FIG. 2

The trivial exception. When each pair of points is contained in some U_i, no pair of compartments satisfies (ii), because every positive number satisfies (i). In this case, however, Lebesgue's lemma is trivial.

DEFINITION OF A COMPARTMENT. *Let X be a set covered by a finite number of subsets $X_1, \cdots, X_n$. A compartment (of the covering $X_1, \cdots, X_n$) is a nonempty set expressible as the intersection of n distinct sets, consisting of X_i's and complements of X_i's.*

It follows from the definition that the compartments of a finite covering of X form a finite, disjoint covering of X. Where no confusion arises, we simply speak of compartments, instead of compartments of a particular covering. We do this below.

THEOREM. *If* $U_1, \cdots, U_n$ *is an open covering of a compact metric space* X, ρ, *and some pair of points is contained in no* U_i, *then there are two compartments* A *and* B, *a positive distance apart, such that*

(i) *if* $\rho(x, y) < \rho(A, B)$ *then both* x *and* y *belong to some* U_i,

(ii) *no number greater than* $\rho(A, B)$ *has property* (i).

Proof. The two points contained in no U_i belong to a pair of compartments contained in no U_i. Thus we may define

$$\delta = \min \rho(E, F),$$

where E and F are any compartments contained in no U_i. Then δ is attained as the distance between some pair of compartments A and B, and it satisfies the requirements of the theorem.

First, $\delta > 0$. For, let E and F be compartments such that $\rho(E, F) = 0$. Then from E and F we can select sequences $\{x_i\}$ and $\{y_i\}$ such that $\rho(x_i, y_i) \to 0$. By compactness, there is a point x and a subsequence $\{x_{N_i}\}$ of $\{x_i\}$ such that $x_{N_i} \to x$. Since $y_{N_i} \to x$ as well, x belongs to both $\overline{E}$ and $\overline{F}$. But x belongs to some open U_k. Thus U_k meets both E and F and so, by the definition of compartment, contains both E and F.

δ also satisfies (i) and (ii). For, let $\rho(x, y) < \delta$. x and y belong to compartments E and F. If $E = F$ then both x and y necessarily belong to some U_i, because each compartment is contained in some U_i. If $E \neq F$ then $\rho(E, F) < \delta$ and some U_i contains both E and F. Thus some U_i contains both x and y. On the other hand, if $\delta_1 > \delta$ then there are two compartments E and F, contained in no U_i, such that $\rho(E, F) < \delta_1$. In E and F we can select points x and y such that $\rho(x, y) < \delta_1$. Then x and y belong to no U_i since otherwise U_i would contain both E and F.

AN INTERESTING DUAL GALOIS CORRESPONDENCE

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Let P and Q be complete lattices and let $\sigma: P \to Q$ and $\tau: Q \to P$ be isotone (i.e., order preserving) mappings. Define for $x \in P$, $\bar{x} = x^{\sigma\tau}$ and for $y \in Q$ define $y^* = y^{\tau\sigma}$. Suppose that $\bar{x} \leqq x$ and $y \leqq y^*$ for each $x \in P$ and $y \in Q$, respectively. Then it is a standard exercise to verify that $x \to \bar{x}$ is a dual closure operation, $y \to y^*$ is a closure operation, and that σ and τ are inverse isomorphisms (after suitable restriction) between the complete lattices of closed elements of P and Q [1], [2]. The mappings σ, τ are said to establish a *dual Galois correspondence* between P and Q. It is hoped that those with a knowledge of group theory will be able to find interesting applications for the following example of a dual Galois correspondence.