$$\begin{aligned}&(\underset{jnl}{\delta S} - \underset{ljn}{\delta S})\underset{qq}{Sa} + (\underset{qnl}{\delta S} - \underset{qln}{\delta S})\underset{jj}{Sa} + \\ &(\underset{qln}{\delta S} - \underset{qnl}{\delta S})\underset{jq}{Sa} + (\underset{jln}{\delta S} - \underset{jnl}{\delta S})\underset{qj}{Sa} + \\ &S^{\rho}S_{\rho}(\underset{jlqnn}{\delta\,\delta\,a} - \underset{jnlq\,l}{\delta\,\delta\,a} + \underset{qlnj\,n}{\delta\,\delta\,a} - \underset{qnlj\,l}{\delta\,\delta\,a}) = 0,\end{aligned} \qquad (2.18)$$

where δ (subindex $l, j \ldots$) is the Kronecker symbol. If all the indices are distinct, then the left hand side of (2.18) vanishes; if $l = j$ and the remaining indices are distinct, we obtain

$$-\underset{nqq}{SSa} + \underset{nqj}{SSa} = 0. \qquad (2.19)$$

Since S_{λ} is a non-zero vector, then S (subindex n) is not zero for some n and hence (2.19) becomes

$$-\underset{qq}{Sa} + \underset{qj}{Sa} = 0. \qquad (2.20)$$

Now let S (subindex $1 \ldots m$) be the zero components of S_{λ}; and let S (subindex $m + 1 \ldots n$) be the non-zero components of S_{λ}. Take the index q in the last set and then (2.20) yields

$$\underset{q}{a} = \underset{j}{a} = \alpha, \quad q \pm j, j = 1 \ldots n. \qquad (2.21)$$

Equation (2.18) now becomes an identity. When the condition (2.21) is used in (2.16), we find

$$'a_{\lambda\mu} = \alpha a_{\lambda\mu}. \qquad (2.22)$$

Hence, we have the

THEOREM. *If coördinate systems in $V_n(\xi^{\lambda})$ and in $'V_n(\xi^{\lambda})$ can be chosen so that at corresponding points $P(\xi^{\lambda})$, $'P(\xi^{\lambda})$, the connections of these spaces are related by (1.3) and if the principal directions of $'a_{\lambda\mu}$ exist in V_n, then the spaces are conformal.*

HOMOTOPY RELATIONS IN FIBRE SPACES

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If X is a topological space, B a metric space with metric function $\rho(b, b')$, and π a continuous map of X on all of B, we shall say that *X is a fibre space over B relative to π* if there exist a positive ϵ_0 and a continuous function ϕ as follows:

1. $\phi = \phi(x, b)$ is a point of X and is defined for all $x \in X$, $b \in B$ such that $\rho(\pi(x), b) < \epsilon_0$,

2. $\pi\phi(x, b) = b$ wherever ϕ is defined,

3. $\phi(x, \pi(x)) = x$.

The map π is called *the projection* $X \rightarrow B$, and the sets $\pi^{-1}(b)$ in X are called the *fibres*. We refer to ϕ as the slicing function; for, if $x_0 \in X$ is fixed and b ranges over the ϵ_0 neighborhood of $\pi(x_0)$, $\phi(x_0, b)$ is a point of $\pi^{-1}(b)$ near x_0, so that ϕ provides a section through x_0 of the fibres neighboring x_0 which is homeomorphic to the neighborhood of $\pi(x_0)$. The simultaneous continuity of ϕ in x, b insures that this section varies continuously with x. Note that ϕ is defined in a neighborhood U of the graph G of π in the product space $X \times B$, and the correspondence $(x, b) \rightarrow (\phi(x, b), b)$ is a retraction of U into G parallel to X. Conversely if such a retraction exists, the function ϕ is readily constructed. Observe also that π is an interior map (carries open sets into open sets). If we did not insist that $\pi(X)$ be all of B, it would follow that $\pi(X)$ is both open and closed in B.

The class of fibre spaces includes the class of fibre bundles in the sense of Whitney.[1] We shall need the following examples of fibre spaces which are also fibre bundles.

I. The product space $X = B \times A$.—Here $\pi(b, a) = b$, and $\phi((b, a), b') = (b', a)$; i.e., ϕ is defined over all of $X \times B$ and the slices are the sections parallel to the coördinate B.

II. X *a covering space of* B.—The function π attaches to each point of X the point of B it covers. It is supposed there is an $\epsilon_0 > 0$ such that π is a topological map on a neighborhood of x whose image contains all points at a distance $< \epsilon_0$ from $\pi(x)$. Then ϕ attaches to (x, b) that point in the neighborhood of x whose image is b.

III. $\pi(X) = B$ *a non-singular map of one differentiable manifold on another*.—By non-singular we mean that the Jacobian of π has maximum rank at every point of X. Then the fibres are differentiable submanifolds. The function ϕ is defined by introducing a Riemannian geometry in X and sectioning the fibres by perpendicular geodesic planes. An example of this is the Hopf[2] mapping of the 3-sphere on the 2-sphere ($S^3 \rightarrow S^2$), likewise his mappings $S^7 \rightarrow S^4$ and $S^{15} \rightarrow S^8$. The fibres are great spheres of dimensions 1, 3 and 7, respectively.

IV. B *a coset space of the compact Lie group* X.—We consider a closed subgroup H of X, and define B to be the space of left (or right) cosets, and $\pi(x) =$ the coset containing x. We first define $\phi(1, b)$ by means of a coördinate neighborhood of the unit 1 and a plane perpendicular to the tangent plane of H at 1. Then $\phi(x, b) = x\phi(1, x^{-1}b)$. For example, let R_n be the rotation group of the n-sphere S^n. If $b_0 \in S^n$ is fixed and $\pi(r) = r(b_0)$, this projection of R_n in S^n is the projection of R_n into the space of left cosets of the subgroup R_{n-1} leaving b_0 fixed. In this way R_n is a fibre space over S^n.

Our principal tool is embodied in[3]

THEOREM 1. *If X is a fibre space over B, Y a topological space, g a continuous map of Y in X, and $h(y, t)$ $(0 \leqq t \leqq 1)$ a homotopy of the map $h(y, 0) = \pi g(y)$ uniform in the sense that there is a $\delta_0 > 0$ such that $|t - t'| < \delta_0$ implies $\rho(h(y, t), h(y, t')) < \epsilon_0$ for all $y \epsilon Y$, then there exists a homotopy $g(y, t)$ of g in X (called the covering homotopy) such that $\pi g(y, t) = h(y, t)$. In addition, if $h(y, t)$ leaves y fixed, so also does $g(y, t)$.*

Subdivide the interval (0, 1) into the subintervals (t_i, t_{i+1}) of length $< \delta_0$, and define $g(y, t)$ stepwise by $g(y, t) = \phi(g(y, t_i), h(y, t))$, $(t_i < t \leqq t_{i+1})$.

In case Y is compact metric, the uniformity requirement is redundant.

COROLLARY 1. *If B is arcwise connected, any two fibres have the same homotopy type.*

If F_1, F_2 are the fibres over b_1, b_2, respectively, and $h(t)$ is a path from b_1 to b_2. Choose $Y = F_1$, $g =$ identity and h as the homotopy of $\pi(F_1)$, then the covering homotopy deforms F_1 into F_2. Using $h'(t) = h(1 - t)$, we deform F_2 into F_1. Since $h'h$ is homotopic to b_1, the homotopy of F_1 into F_2 back into F_1 is homotopic to a homotopy in F_1. Similarly for F_2.

COROLLARY 2. *The Hopf mappings $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$ are essential (i.e., the images are not homotopic to points).*

Let $Y = S^3$ and $g =$ identity. A contraction of $\pi(S^3)$ to a point b_0 of S^2 would be covered by a contraction of S^3 on itself into the circle over b_0. This is impossible since no homotopy of S^3 on itself can free a point.

In the following theorems, $\pi_i(X, F)$ is the ith homotopy group[4] of X relative to the closed set F. If F is a single point and X is arcwise connected, the group is independent of F and we write $\pi_i(X)$.

THEOREM 2. *If X is a fibre space over B, and F_0 is the fibre over the point b_0, then $\pi_i(X, F_0) = \pi_i(B, b_0)$.*

An element of $\pi_i(X, F_0)$ is represented by a continuous map f of an i-cell E^i carrying the boundary $\dot{E}^i$ into F_0 and a fixed point $y_0 \epsilon \dot{E}^i$ into a fixed point $x_0 \epsilon F_0$. Clearly πf represents an element of $\pi_i(B, b_0)$. This defines a homomorphism $\pi_i(X, F_0) \to \pi_i(B, b_0)$. If πf is homotopic to b_0 leaving $\dot{E}^i$ at b_0, the covering homotopy deforms $f(E^i)$ into F_0 keeping $\dot{E}^i$ in F_0 and y_0 at x_0. The correspondence is therefore an isomorphism into a subgroup. Let $g(E^i)$ represent an element of $\pi_i(B, b_0)$. If $h(y, t)$ contracts E^i on itself into y_0, then gh contracts $g(E^i)$ into $g'(E^i) = b_0$. Define $f'(E^i) = x_0$. Then $\pi f' = g'$, and the homotopy of g' into g is covered by a homotopy of f' into an f such that $\pi f = g$. Since gh leaves y_0 at b_0, we have $f(y_0) = x_0$ and the theorem is proved.

COROLLARY 3. *If X is a covering space of B, then $\pi_i(X) = \pi_i(B)$ for $i \geqq 2$.*

Since F_0 is a discrete set of points, and $\dot{E}^i$ is connected, any map of E^i representing an element of $\pi_i(X, F_0)$ must carry $\dot{E}^i$ into x_0. Hence $\pi_i(X, F_0) = \pi_i(X, x_0)$.

COROLLARY 4. *For the 1-sphere S^1, we have $\pi_i(S^1) = 0$ for $i \geqq 2$.*

The covering space of S^1 is a line L, and $\pi_i(L) = 0$.

THEOREM 3. *If S^n is an n-sphere and F a closed arcwise connected proper subset of S^n, then $\pi_i(S^n, F)$ is the direct sum $\pi_i(S^n) + \pi_{i-1}(F)$.*

Let $x_0 \in F$ be the fixed reference point and choose a fixed homotopy $h(x, t)$ of F in S^n contracting F into x_0 and leaving x_0 fixed. Choose a fixed map g of E^{i-1} on $\dot{E}^i$ carrying $\dot{E}^{i-1}$ into y_0 and being topological on $E^{i-1} - \dot{E}^{i-1}$. If f represents an element of $\pi_i(S^n, F)$, then $f' = fg$ represents an element of $\pi_{i-1}(F)$. This defines a homomorphism $\pi_i(S^n, F) \to \pi_{i-1}(F)$. Using coordinates (y, r) in E^i where r = radius vector, and y is the central projection of the point on $\dot{E}^i$, we define

$$f''(y, r) = \begin{cases} f(y, 2r), & 0 \leqq r \leqq 1/2, \\ h(f(y, 1), 2r - 1), & 1/2 \leqq r \leqq 1, \end{cases}$$

and f'' represents an element of $\pi_i(S^n)$. This defines a homomorphism $\pi_i(S^n, F) \to \pi_i(S^n)$. These two homomorphisms induce a homomorphism into the direct sum. On the other hand if f' and f'' are given, we define

$$f(y, r) = \begin{cases} f''(y, 2r), & 0 \leqq r \leqq 1/2 \\ h(f'g^{-1}(y), 2 - 2r), & 1/2 \leqq r \leqq 1; \end{cases}$$

and this shows that the correspondence with the direct sum is 1–1.

Theorems 2 and 3 together with the Hopf mappings of spheres on spheres give the following results.

THEOREM 4. $\pi_i(S^2) = \pi_i(S^3) + \pi_{i-1}(S^1)$, $\pi_i(S^4) = \pi_i(S^7) + \pi_{i-1}(S^3)$, $\pi_i(S^8) = \pi_i(S^{15}) + \pi_{i-1}(S^7)$.

The first of these relations and Corollary 4 give

COROLLARY 5. $\pi_i(S^2) = \pi_i(S^3)$ *for* $i \geqq 3$.

Since $\pi_i(S^n) = 0$ for $i < n$, and $\pi_n(S^n)$ is the infinite cyclic group, denoted by ∞, we have

COROLLARY 6. $\pi_i(S^4) = \pi_{i-1}(S^3)$ for $i = 2, \ldots, 6$, and $\pi_7(S^4) = \infty + \pi_6(S^3)$, $\pi_i(S^8) = \pi_{i-1}(S^7)$ for $i = 1, \ldots, 14$, and $\pi_{15}(S^8) = \infty + \pi_{14}(S^7)$.

The first parts of these relations are special cases of some results of Freudenthal[5] (obtained in an entirely different manner) which state: $\pi_i(S^n) = \pi_{i-1}(S^{n-1})$ for $i = 2, \ldots, 2n-2$ and $n = 2, 3, \ldots$.

The results of the following theorem are based on the coset mapping $R_n \to S^n$ described in IV.

THEOREM 5. $\pi_i(R_{i+1})$ *is a factor group of* $\pi_i(R_i)$, *and* $\pi_i(R_{i+1}) = \pi_i(R_{i+k})$ *for* $k, i = 1, 2, \ldots$.

Since $\pi_i(S^n) = 0$ for $i < n$, any continuous map of an i-sphere in S^n is homotopically deformable into the point b_0. If this i-sphere is mapped in S^n through R_n, the covering homotopy deforms the i-sphere into R_{n-1}. Hence any i-sphere in R_n, $i < n$, is deformable into R_i. If an i-sphere in R_i is the boundary of an $(i + 1)$-cell in R_n, the image of this $(i + 1)$-cell in S^n is contractible into b_0 leaving the boundary of the cell at b_0, providing $i + 1 < n$. Hence if an i-sphere of R_i is contractible in R_n $(n > i)$, it is contractible in R_{i+1}).

COROLLARY 7. $\pi_1(R_n)$ = the cyclic group of period 2 for $n > 1$, and $\pi_2(R_n) = 0$ for $n > 0$.

Since R_2 = projective 3-space, $\pi_1(R_2)$ = group of period 2. Since R_2 is covered by S^3 and $\pi_2(S^3) = 0$, we have by Corollary 3 that $\pi_2(R_2) = 0$.

The results of Theorem 4 cast considerable light on the problem of finding fibre mappings of spheres on spheres. If S^k is a fibre space over $S^n(k \geqq n)$ with fibre F, then $\pi_i(S^n) = \pi_i(S^k) + \pi_{i-1}(F)$. This implies $\pi_{i-1}(F) = 0$ for $i < n$. Hence if the dimension of F is $\leqq n - 2$, F is contractible on itself to a point. This implies $\pi_i(F) = 0$ for all i, and therefore $\pi_i(S^n) = \pi_i(S^k)$. This can happen only if $k = n$. On the other hand, if the dimension of F is $> n - 2$, we obtain $k \geqq 2n - 1$. Thus S^{2n-1} is the sphere of least dimension which can be a proper fibre space over S^n. If we require in addition that the fibre F be a sphere, we obtain from $\pi_{n-1}(F) = \pi_n(S^n)$ that $F = S^{n-1}$. Hence only S^{2n-1} can be proper a sphere space over S^n.

The following is an example of a fibre space which is not a fibre bundle in the sense of Whitney. Let B = the interval $0 \leqq x \leqq 1$ and let X = the triangle in the (x, y)-plane defined by $0 \leqq y \leqq x \leqq 1$. Define $\pi(x, y) = x$. The slicing function ϕ is then given by

$$\phi((x, y), x') = \begin{cases} (x', x') \text{ for } x' \leqq y \\ (x', y) \text{ for } x' > y. \end{cases}$$

[1] These PROCEEDINGS, **26**, 148–153 (1940); also **21**, 464–468 (1935).

[2] *Fundamenta Math.*, **25**, 427–440 (1935).

[3] An inverse of this theorem has been proved by Dr. R. H. Fox as follows: If B is an absolute neighborhood retract and $\pi(X) = B$ is continuous and there is an $\epsilon_0 > 0$ such that the conclusions of Theorem 1 hold, then X is a fibre space over B relative to π.

[4] See W. Hurewicz, *Proc. Amsterdam Acad.*, **38**, 112, 521 (1935); also **39**, 117, 215 (1936).

[5] *Compositio Math.*, **5**, 299–314 (1937).