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FINITE DOMINATION AND NOVIKOV RINGS. ITERATIVE APPROACH

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ABSTRACT. Suppose C is a bounded chain complex of finitely generated free modules over the LAURENT polynomial ring $L = R[x, x^{-1}]$. Then C is R-finitely dominated, *i.e.*, homotopy equivalent over R to a bounded chain complex of finitely generated projective R-modules, if and only if the two chain complexes $C \otimes_L R((x))$ and $C \otimes_L R((x^{-1}))$ are acyclic, as has been proved by RANICKI. Here $R((x)) = R[[x]][x^{-1}]$ and $R((x^{-1})) = R[[x^{-1}]][x]$ are rings of formal LAURENT series, also known as NOVIKOV rings. In this paper, we prove a generalisation of this criterion which allows us to detect finite domination of bounded below chain complexes of projective modules over LAURENT rings in several indeterminates.

1. FINITELY DOMINATED CHAIN COMPLEXES

Let A denote a ring with unit. We write $\mathrm{Ch}(A)$ for the category of chain complexes of (right) A-modules, and $\mathrm{Ch}^{\mathrm{b}}(A)$ for the full subcategory of bounded chain complexes.

Definition 1.1. Let S be a subring of A; every chain complex of A-modules is then, by restriction, also a chain complex of S-modules. We say that the chain complex $C \in \mathrm{Ch}(A)$ is

(1) *S-finite* if it is bounded and consists of finitely generated free S-modules;
(2) *homotopy S-finite* if it is homotopy equivalent to an S-finite complex $D \in \mathrm{Ch}^{\mathrm{b}}(S)$;
(3) *strict S-perfect* if it is bounded and consists of finitely generated projective S-modules;
(4) *S-finitely dominated* if it is homotopy equivalent to a strict S-perfect complex $D \in \mathrm{Ch}^{\mathrm{b}}(S)$.

Given an S-finitely dominated complex $C \in \mathrm{Ch}(A)$ there exists a strict S-perfect complex $D \in \mathrm{Ch}(S)$ homotopy equivalent to C. The *finiteness obstruction of C* is defined to be

$$\chi(C) = \sum_{j \in \mathbb{Z}} (-1)^j [D_j] \in \tilde{K}_0(S) \ ;$$

it is independent of the choice of D. *The complex C is homotopy S-finite if and only if its finiteness obstruction is trivial*; see [Ros94, Theorem 1.7.12]

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for a textbook proof. In this sense, algebraic K-theory detects homotopy finiteness of finitely dominated chain complexes.

To find out whether a given complex $C \in \mathrm{Ch}(A)$ is homotopy S-finite one should thus first determine whether it is S-finitely dominated. In the special case $S = R$ and $A = R[x, x^{-1}]$, RANICKI has given the following homological characterisation:

Theorem 1.2 (RANICKI [Ran95, Theorem 2])**.** *Let C be a bounded chain complex of finitely generated free $R[x, x^{-1}]$-modules. The following conditions are equivalent:*

(1) *The complex C is R-finitely dominated.*
(2) *Both the following chain complexes are acyclic:*

$$C \otimes_{R[x,x^{-1}]} R((x)) \quad \textit{and} \quad C \otimes_{R[x,x^{-1}]} R((x^{-1})) \ .$$

Here we denote by $R[[x]]$ the ring of formal power series in the indeterminate x, and write $R((x))$ for the localisation of $R[[x]]$ by x. That is, $R((x))$ is the ring of formal LAURENT series

$$\sum_{j=k}^{\infty} a_j x^j \ , \qquad k \in \mathbb{Z} \ ,$$

also known as the NOVIKOV ring of R in x. Similarly, $R[[x^{-1}]]$ is the ring of formal power series in the indeterminate x^{-1}, and the NOVIKOV ring $R((x^{-1}))$ is its localisation by x^{-1}. Elements of the latter can be written as formal LAURENT series of the type

$$\sum_{j=-\infty}^{k} a_j x^j \ , \qquad k \in \mathbb{Z} \ .$$

As it stands this result is not adapted to iteration. In more detail, suppose that R itself is a LAURENT ring $R = K[y, y^{-1}]$, over some ring K; one would want then to be able to apply RANICKI's theorem twice: first to $R \subset R[x, x^{-1}]$, then to $K \subset K[y, y^{-1}] = R$. One difficulty here is that the first application leaves us with a chain complex which consists of projective rather than free modules. In addition, the LAURENT variables are dealt with in a specific order which, intuitively speaking, should have no bearing on the question of finite domination. Both issues are addressed in our main result below.

Write R_n for the ring of LAURENT polynomials in n indeterminates with coefficients in R,

$$R_n = R[x_1,\, x_1^{-1},\, x_2,\, x_2^{-1},\, \cdots\,,\, x_n,\, x_n^{-1}] \ ,$$

so that $R_0 = R$ and $R_k = R_{k-1}[x_k, x_k^{-1}]$ for $k \geq 1$. We will prove the following generalisation of Theorem 1.2 to many variables:

Theorem 1.3. *Let $n \geq 1$. For a bounded below complex C of projective R_n-modules (not necessarily finitely generated) the following four conditions are equivalent:*

(1) *The complex C is R-finitely dominated.*

(2) *The complex C is R-finitely dominated, and for all $n!$ re-numberings of the variables $x_1, x_2, \cdots, x_n$, the complex C is homotopy R_j-finite for $j = 1, 2, \cdots, n$.*

(3) *C is R_n-finitely dominated, and for all $n!$ re-numberings of the variables $x_1, x_2, \cdots, x_n$ the following chain complexes are acyclic:*

$$C \otimes_{R_j} R_{j-1}((x_j)) \quad \textit{and} \quad C \otimes_{R_j} R_{j-1}((x_j^{-1})) \ , \quad 1 \leq j \leq n \ .$$

(4) *C is R_n-finitely dominated, and for some re-numbering of the variables $x_1, x_2, \cdots, x_n$ the following chain complexes are acyclic:*

$$C \otimes_{R_j} R_{j-1}((x_j)) \quad \textit{and} \quad C \otimes_{R_j} R_{j-1}((x_j^{-1})) \ , \quad 1 \leq j \leq n \ .$$

Note that this theorem says in particular that an R-finitely dominated chain complex of R_n-modules is automatically homotopy equivalent over R_k, $1 \leq k \leq n$, to an R_k-finite complex consisting of free rather than projective modules. Nevertheless the proof forces us to work with chain complexes of modules which *a priori* consist of projective modules.

We start by fixing our sign conventions for some constructions from homological algebra, together with a collection of standard results which will be used repeatedly in the sequel. We then develop the relevant theory of mapping tori, and apply all this in the proof of the main theorem. We finish the paper by giving a concrete non-trivial example of a finitely dominated chain complex over a LAURENT ring in finitely many indeterminates, and by discussing finite domination over fields, which essentially reduces to an exercise in linear algebra.

The methods used here borrow heavily from those of RANICKI [Ran95], modified to allow for the presence of several indeterminates and non-free modules. It is possible to approach finite domination over LAURENT rings in several indeterminates from the point of view of toric geometry; this perspective yields a completely different set of conditions, and will be presented in a forthcoming paper.

2. MAPPING CONES AND MAPPING TORI

Chain complexes and mapping cones. We begin with listing some conventions. We will consider arbitrary chain complexes of (right) modules over some ring with unit A; we think of chain complexes as being "vertical". The kth suspension ($k \in \mathbb{Z}$) of a chain complex C is the chain complex $C[k]$ defined by $C[k]_\ell = C_{\ell-k}$ with differential changed by the sign $(-1)^k$.

A twofold chain complex is a chain complex in the category of chain complexes, that is, a family $(D_{p,q})_{p,q\in\mathbb{Z}}$ of R-modules together with "horizontal" and "vertical" differential

$$\partial_h \colon D_{p,q} \longrightarrow D_{p-1,q} \quad \text{and} \quad \partial_v \colon D_{p,q} \longrightarrow D_{p,q-1}$$

satisfying $\partial_h^2 = 0$, $\partial_v^2 = 0$ and $\partial_h\partial_v = \partial_v\partial_h$. The *total complex* of the twofold chain complex D is a chain complex $\mathrm{Tot}(D)$. In chain degree n we have, by definition,

$$\mathrm{Tot}(D)_n = \bigoplus_{p+q=n} D_{p,q} \ ,$$

and the differential is induced by

$$\partial_h \colon D_{p,q} \longrightarrow D_{p-1,q} \quad \text{and} \quad (-1)^p \partial_v \colon D_{p,q} \longrightarrow D_{p,q-1} \ .$$

A map of chain complexes $f \colon C \longrightarrow B$ can be considered as a twofold chain complex with B in column $p = 0$ and C in column $p = 1$, and horizontal differential given by f. Its total complex is known as the *mapping cone of f*, denoted $\mathrm{Cone}\,(f)$. We have $\big(\mathrm{Cone}(f)\big)_k = C_{k-1} \oplus B_k$. There is a natural long exact homology sequence associated to this construction:

$$\ldots \xrightarrow{f} H_k B \longrightarrow H_k \mathrm{Cone}(f) \longrightarrow H_{k-1} C \xrightarrow{f} H_{k-1} B \longrightarrow \ldots \quad (1)$$

In particular, application of the Five Lemma shows that the mapping cone construction is invariant under quasi-isomorphism of maps of chain complexes. That is, given a commutative diagram of chain complexes

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ \simeq \downarrow & & \downarrow \simeq \\ D & \xrightarrow{g} & E \end{array}$$

where the vertical maps are quasi-isomorphisms, the induced map

$$\mathrm{Cone}\,(f) \longrightarrow \mathrm{Cone}\,(g)$$

is a quasi-isomorphism as well. — Let $f \colon C \longrightarrow B$ be a map of chain complexes as before. The canonical projection from the B-summands assemble to a natural map $\mathrm{Cone}\,(f) \longrightarrow \mathrm{coker}\,(f)$.

Lemma 2.1. *If $f \colon C \longrightarrow B$ is an injective map of chain complexes, the natural map $\mathrm{Cone}\,(f) \longrightarrow \mathrm{coker}\,(f)$ is a quasi-equivalence.*

Proof. The long exact sequence in (1) and the long exact sequence associated to the short exact sequence

$$0 \longrightarrow C \xrightarrow{f} B \longrightarrow \mathrm{coker}\,(f) \longrightarrow 0$$

assemble into a commutative ladder diagram, with two out of three maps the identity. By the Five Lemma, the remaining maps (which are induced by the map under investigation) are isomorphisms. □

We have defined the mapping cone by totalising a twofold chain complex. Conversely, one can describe totalisation by iterating the mapping cone construction. For us, the following special case will be sufficient:

Lemma 2.2. *Suppose we have maps of chain complexes $f \colon C \longrightarrow B$ and $g \colon B \longrightarrow A$ with $gf = 0$. Let D denote the twofold chain complex having C, B and A in columns 2, 1 and 0, with horizontal differential given by f and g. The map f induces an inclusion $C[1] \longrightarrow \mathrm{Cone}\,(g)$, and we have an equality of chain complexes $\mathrm{Cone}\,\big(C[1] \longrightarrow \mathrm{Cone}\,(g)\big) = \mathrm{Tot}(D)$.* □

Corollary 2.3. *Suppose that $0 \longrightarrow C \xrightarrow{f} B \xrightarrow{g} A \longrightarrow 0$ is a short exact sequence of chain complexes. Then there is a quasi-isomorphism*

$$C \longrightarrow \big(\mathrm{Cone}\,(g)\big)[-1] \ .$$

Proof. By the previous Lemma we have a map $\mu\colon C[1] \longrightarrow \mathrm{Cone}\,(g)$, and this map is a quasi-isomorphism if and only if its mapping cone is acyclic. But its mapping cone is $\mathrm{Tot}(D)$, using the notation of that Lemma. There is a convergent spectral sequence

$$E^1_{p,q} = H_q D_{*,p} \Longrightarrow H_{q+p}\,\mathrm{Tot}(D) \ ,$$

cf. [ML95, §XI.6]; by exactness, its E^1-term is trivial, hence $\mathrm{Tot}(D)$ is acyclic. It follows that $\mu[-1]\colon C \longrightarrow \big(\mathrm{Cone}\,(g)\big)[-1]$ is a quasi-isomorphism. □

Proposition 2.4. *Suppose C is an R-finitely dominated complex of projective R-modules. Then for any self map $f\colon C \longrightarrow C$ the complex* $\mathrm{Cone}\,(f)$ *is homotopy R-finite.*

Proof. It is enough to show that the finiteness obstruction of C in $\tilde{K}_0(R)$ vanishes: since K-theory doesn't detect differentials, we have

$$[\mathrm{Cone}\,(f)] = [C[1] \oplus C] = -[C] + [C] = 0 \in \tilde{K}_0(R) \ .$$

If C is strict perfect one can easily give an explicit proof: for each C_n choose a finitely generated projective module D_n such that $C_n \oplus D_n$ is free; choose $D_n = 0$ if $C_n = 0$. Then attaching the contractible two-step chain complexes $D_n \xrightarrow{=} D_n$ (concentrated in degrees $n+1$ and n) to $\mathrm{Cone}\,(f)$ results in a bounded chain complex of finitely generated free R-modules which is homotopy equivalent, via the projection, to $\mathrm{Cone}\,(f)$. □

Algebraic mapping tori.

Definition 2.5. Let C be an arbitrary R-module chain complex, and let $h\colon C \longrightarrow C$ be any chain map. The *algebraic mapping torus $T(h)$ of h* is defined as

$$T(h) = \mathrm{Cone}\,\big(C \otimes_R R[x,x^{-1}] \xrightarrow{h\otimes 1 - 1\otimes x} C \otimes_R R[x,x^{-1}]\big) \ .$$

By construction, $T(h)$ is an $R[x,x^{-1}]$-module chain complex which is bounded if C is bounded. If C consists of finitely generated (*resp.* projective, *resp.* free) R-modules, then $T(h)$ consists of finitely generated (*resp.* projective, *resp.* free) $R[x,x^{-1}]$-modules.

The mapping torus construction is functorial on the category of self maps of R-module chain complexes. That is, a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & C \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ D & \xrightarrow{g} & D \end{array} \tag{2}$$

induces an $R[x,x^{-1}]$-linear chain map $(\alpha,\beta)_*\colon T(f) \longrightarrow T(g)$, and this assignment is compatible with vertical composition (vertical stacking of square diagrams). Moreover, if both α and β are quasi-isomorphisms then so is $(\alpha,\beta)_*$. Indeed, the long exact sequences of mapping cones yield a commutative ladder diagram

$$\begin{array}{ccccccc}
\cdots H_{n+1}T(f) & \longrightarrow & H_n\big(C\otimes_R R[x,x^{-1}]\big) & \xrightarrow{\eta} & H_n\big(C\otimes_R R[x,x^{-1}]\big) & \longrightarrow & H_nT(f)\cdots \\
\downarrow{\scriptstyle (\alpha,\beta)_*} & & \downarrow{\scriptstyle \alpha} & & {\scriptstyle \beta}\downarrow & & {\scriptstyle (\alpha,\beta)_*}\downarrow \\
\cdots H_{n+1}T(g) & \longrightarrow & H_n\big(D\otimes_R R[x,x^{-1}]\big) & \xrightarrow{\zeta} & H_n\big(D\otimes_R R[x,x^{-1}]\big) & \longrightarrow & H_nT(g)\cdots
\end{array}$$

(where $\eta = f\otimes 1 - 1\otimes x$ and $\zeta = g\otimes 1 - 1\otimes x$) with exact rows; since $R[x,x^{-1}]$ is a free R-module, the two middle vertical maps are isomorphisms. It follows from the Five Lemma that $(\alpha,\beta)_*$ is a quasi-isomorphism as claimed.

Lemma 2.6. *Let $h\colon C \longrightarrow C$ be a self map of an arbitrary chain complex C of R-modules. The map $(h,h)_*\colon T(h) \longrightarrow T(h)$ is chain homotopic to x, the "multiplication by x" map. In particular, $(h,h)_*$ is a quasi-isomorphism.*

Proof. The homotopy is essentially given by projection on the second summand followed by inclusion into the first summand,

$$\begin{aligned} T(h)_n &= C_{n-1}\otimes_R R[x,x^{-1}] \,\oplus\, C_n\otimes_R R[x,x^{-1}] \\ &\xrightarrow{(\mathrm{pr}_2,0)} C_n\otimes_R R[x,x^{-1}] \,\oplus\, C_{n+1}\otimes_R R[x,x^{-1}] = T(h)_{n+1}\ . \end{aligned}$$

The map x is an isomorphism, hence $(h,h)_*$ is a quasi-isomorphism. □

Proposition 2.7 (MATHER's mapping torus trick)**.** *Suppose $f\colon C \longrightarrow D$ and $g\colon D \longrightarrow C$ are chain maps of R-module chain complexes. Then the two maps*

$$(f,f)_*\colon T(gf) \longrightarrow T(fg) \quad \textit{and} \quad (g,g)_*\colon T(fg) \longrightarrow T(gf)$$

are quasi-isomorphisms. If both C and D are bounded below complexes of projective modules, both maps are homotopy equivalences.

Proof. The composition $(g,g)_*\circ(f,f)_* = (gf,gf)_*\colon T(gf) \longrightarrow T(gf)$ is a quasi-isomorphism by the previous Lemma; consequently, $(f,f)_*$ induces an injective map on homology, and $(g,g)_*$ induces a surjective map on homology. Swapping the rôles of f and g proves the claim. □

Lemma 2.8. *Let C be a chain complex of $R[x,x^{-1}]$-modules (possibly unbounded). Then there is an $R[x,x^{-1}]$-linear quasi-isomorphism $T(x) \longrightarrow C$ where x is short for the R-module chain self map of C given by "multiplication by x". The quasi-isomorphism is natural in C.*

Proof. First we claim that for any $R[x,x^{-1}]$-module M there is an exact sequence of $R[x,x^{-1}]$-modules

$$0 \longrightarrow M\otimes_R R[x,x^{-1}] \xrightarrow{x\otimes 1 - 1\otimes x} M\otimes_R R[x,x^{-1}] \xrightarrow{\epsilon} M \longrightarrow 0\ . \tag{3}$$

Here the map denoted ϵ is given by $m\otimes p \mapsto mp$. To begin with, $x\otimes 1 - 1\otimes x$ is injective and ϵ is surjective, so it remains to prove exactness in the middle. First,

$$\epsilon\circ(x\otimes 1 - 1\otimes x)(m\otimes p) = \epsilon(mx\otimes p - m\otimes px) = mxp - mpx = 0$$

since x is in the centre of $R[x,x^{-1}]$. This shows $\mathrm{Im}\,(x\otimes 1 - 1\otimes x) \subseteq \ker\epsilon$. We will prove the converse inclusion in a slightly indirect manner. We can consider the sequence (3) as a sequence of R-modules and check exactness in the middle in the category of R-modules. The point is that ϵ has an R-linear section σ given by $m \mapsto m \otimes 1$. Consequently, there is an isomorphism $M \otimes_R R[x, x^{-1}] \cong \ker \epsilon \oplus \operatorname{Im} \sigma$ of R-modules, and every element in $\ker \epsilon$ is of the form $m - \sigma\epsilon m$, for some $m \in M \otimes_R R[x, x^{-1}]$. We can write m uniquely as a finite sum of the form $m = \sum_{k \in \mathbb{Z}} m_k \otimes x^k$ with certain $m_k \in M$ (almost all of which are zero); the associated element in $\ker \epsilon$ is

$$m - \sigma\epsilon(m) = \sum_{k \in \mathbb{Z}} m_k \otimes x^k - \sum_{k \in \mathbb{Z}} m_k x^k \otimes 1 \ .$$

We want to demonstrate that this is in the image of $x \otimes 1 - 1 \otimes x$; it is certainly enough to prove this for each individual summand $b_k = m_k \otimes x^k - m_k x^k \otimes 1$. This is trivial for $k = 0$ as $b_0 = 0$. For $k > 0$ we obtain b_k as the image of

$$-\big(m_k x^{k-1} \otimes 1 + m_k x^{k-2} \otimes x + \ldots + m_k \otimes x^{k-1}\big)$$

under the map $x \otimes 1 - 1 \otimes x$; similarly, b_{-k} is the image of

$$m_{-k} x^{-k} \otimes x^{-1} + m_{-k} x^{-(k-1)} \otimes x^{-2} + \ldots + m_{-k} x^{-1} \otimes x^{-k}$$

under the same map. This proves exactness of (3).

Applying this result in each chain level proves that we have a similar exact sequence with M replaced by the chain complex C. It follows from Lemma 2.1 that the canonical map $\operatorname{Cone}(x \otimes 1 - 1 \otimes x) \longrightarrow C$ is a quasi-isomorphism. ☐

3. Proof of Theorem 1.3

(1) ⇒ (2) Suppose C is R-finitely dominated. We can then find a strict R-perfect complex D of R-modules, together with mutually inverse R-linear chain homotopy equivalences $f \colon C \longrightarrow D$ and $g \colon D \longrightarrow C$. Let x denote the R-linear self map of C given by "multiplication by x", as before. The commutative square diagram of R-module chain complexes

$$\begin{array}{ccc} C & \xrightarrow{\ xgf\ } & C \\ {\scriptstyle gf}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ C & \xrightarrow{\ x\ } & C \end{array}$$

yields a quasi-isomorphism of $R[x, x^{-1}]$-module complexes

$$(gf, \mathrm{id})_* \colon T(xgf) \longrightarrow T(x) \ ,$$

cf. beginning of §2. By Mather's mapping torus trick Proposition 2.7 there is an $R[x, x^{-1}]$-linear quasi-isomorphism $(f, f)_* \colon T(xgf) \longrightarrow T(fxg)$. Finally, there is a quasi-isomorphism $T(x) \longrightarrow C$, by Lemma 2.8. We thus have quasi-isomorphisms

$$C \longleftarrow T(x) \longleftarrow T(xgf) \longrightarrow T(fxg) \ .$$

Now the chain complex $T(fxg)$ is strict perfect over $R_1 = R[x, x^{-1}]$ since D is strict perfect over R; in addition, its finiteness obstruction is trivial by Proposition 2.4, applied to the defining mapping cone of the mapping torus, so that $T(fxg)$ is homotopy equivalent to a bounded complex of finitely generated free $R[x,x^{-1}]$-modules. Moreover, all other chain complexes are bounded below and consist of projective $R[x,x^{-1}]$-modules, hence the quasi-isomorphisms are in fact homotopy equivalences. It follows that C is homotopy R_1-finite.

We can iterate the argument, replacing R by R_k and R_1 by R_{k+1}, proving that C is indeed homotopy R_j-finite for $1 \leq j \leq n$.

This argument works for any re-numbering of the variable in precisely the same way. We have thus shown that condition (2) holds.

(2) $\Rightarrow$ (3) For $1 \leq j \leq n$ there is a bounded complex D^j of finitely generated free R_j-modules which is homotopy equivalent (over R_j) to C, by hypothesis. It follows that there are homotopy equivalences

$$\begin{aligned} C \otimes_{R_j} R_{j-1}((x_j)) &\simeq D^j \otimes_{R_j} R_{j-1}((x_j)) \qquad \text{and} \\ C \otimes_{R_j} R_{j-1}((x_j^{-1})) &\simeq D^j \otimes_{R_j} R_{j-1}((x_j^{-1})) \ . \end{aligned} \tag{4}$$

Now we can apply RANICKI's Theorem 1.2 iteratively to the chain complexes D^j, $1 \leq j \leq n$, noting that by the previous step (or the hypothesis, for $j = 1$) we know D^j to be R_{j-1}-finitely dominated. It follows that the chain complexes in (4) are acyclic as claimed.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1): First we may assume that C is a strict R_n-perfect chain complex. Since a finitely generated projective module is a direct summand of a finitely generated free one, there exists a strict R_n-perfect complex $C' \in \mathrm{Ch}^{\mathrm{b}}(R_n)$ with trivial differentials such that $D = C \oplus C'$ consists of finitely generated free modules. By algebraic transversality [Ran95, Proposition 1] there exist chain complexes

$$D^+ \in \mathrm{Ch}^{\mathrm{b}}\big(R_{n-1}[x_n]\big) \ , \quad D^- \in \mathrm{Ch}^{\mathrm{b}}\big(R_{n-1}[x_n^{-1}]\big) \quad \text{and} \quad L \in \mathrm{Ch}^{\mathrm{b}}(R_{n-1})$$

consisting of finitely generated free modules over their respective rings, together with module homomorphisms forming a short exact sequence

$$0 \longrightarrow L \longrightarrow D^+ \oplus D^- \xrightarrow{f^+ - f^-} D \longrightarrow 0 \tag{5}$$

of R_{n-1}-modules, such that the adjoint maps

$$D^+ \otimes_{R_{n-1}[x_n]} R_n \longrightarrow D \quad \text{and} \quad D^- \otimes_{R_{n-1}[x_n^{-1}]} R_n \longrightarrow D$$

are isomorphisms of R_n-module chain complexes.

By Corollary 2.3 the sequence (5) yields a quasi-isomorphism

$$L \xrightarrow{\simeq} \mathrm{Cone}\,\big(D^+ \oplus D^- \xrightarrow{f^+ - f^-} D\big)[-1] \tag{6}$$

which is actually a homotopy equivalence since all constituent chain complexes consist of projective R_{n-1}-modules.

We will now replace the right-hand side of (6) by a quasi-isomorphic complex which contains the chain complex C as a summand up to homotopy, thereby proving that C is R_{n-1}-finitely dominated. We have a short exact sequence of $R_{n-1}[x_n]$-modules

$$0 \longrightarrow R_{n-1}[x_n] \xrightarrow{(+,+)} R_{n-1}[[x_n]] \oplus R_{n-1}[x_n, x_n^{-1}] \xrightarrow{(+,-)} R_{n-1}((x_n)) \longrightarrow 0 \ ;$$

we thus get, by taking tensor product over $R_{n-1}[x_n]$ with D^+, a short exact sequence of chain complexes

$$0 \longrightarrow D^+ \xrightarrow{(+,+)} D^+[[x_n]] \oplus D^+[x_n, x_n^{-1}] \xrightarrow{(+,-)} D^+((x_n)) \longrightarrow 0 \ .$$

Here we have used the following abbreviations:

$$\begin{aligned} D^+[[x_n]] &= D^+ \otimes_{R_{n-1}[x_n]} R_{n-1}[[x_n]] \\ D^+((x_n)) &= D^+ \otimes_{R_{n-1}[x_n]} R_{n-1}((x_n)) \\ D^+[x_n,\, x_n^{-1}] &= D^+ \otimes_{R_{n-1}[x_n]} R_{n-1}[x_n,\, x_n^{-1}] = D^+ \otimes_{R_{n-1}[x_n]} R_n \end{aligned}$$

Invocation of Corollary 2.3 gives us a quasi-isomorphism

$$D^+ \xrightarrow{\simeq} \mathrm{Cone}\big(D^+[[x_n]] \oplus D^+[x_n, x_n^{-1}] \xrightarrow{(+,-)} D^+((x_n))\big)[-1] \ . \tag{7}$$

Recall that by construction of D^+ we have isomorphisms $D^+[x_n, x_n^{-1}] \cong D$ and

$$\begin{aligned} D^+((x_n)) &\cong D^+ \otimes_{R_{n-1}[x_n]} R_{n-1}[x_n, x_n^{-1}] \otimes_{R_{n-1}[x_n, x_n^{-1}]} R_{n-1}((x_n)) \\ &\cong D \otimes_{R_n} R_{n-1}((x_n)) \end{aligned}$$

so that (7) gives rise to a quasi-isomorphism

$$D^+ \xrightarrow[g^+]{\simeq} H^+ := \mathrm{Cone}\big(D^+[[x_n]] \oplus D \xrightarrow{(+,-)} D \otimes_{R_n} R_{n-1}((x_n))\big)[-1] \ .$$

Similarly, replacing x_n by x_n^{-1} throughout, we obtain a quasi-isomorphism

$$D^- \xrightarrow[g^-]{\simeq} H^- := \mathrm{Cone}\big(D^-[[x_n^{-1}]] \oplus D \xrightarrow{(+,-)} D \otimes_{R_n} R_{n-1}((x_n^{-1}))\big)[-1] \ ,$$

where we have used the notation

$$D^-[[x_n^{-1}]] = D^- \otimes_{R_{n-1}[x_n^{-1}]} R_{n-1}[[x_n^{-1}]] \ .$$

Projection onto D yields chain complex maps $h^\pm \colon H^\pm \longrightarrow D$ fitting into a commutative diagram of chain complexes

$$\begin{array}{ccccc} D^- & \xrightarrow{f^-} & D & \xleftarrow{f^+} & D^+ \\ \simeq \big\downarrow g^- & & \simeq \big\downarrow \mathrm{id} & & \simeq \big\downarrow g^+ \\ H^- & \xrightarrow{h^-} & D & \xleftarrow{h^+} & H^+ \end{array}$$

which results in a quasi-isomorphism

$$\mathrm{Cone}\big(D^+ \oplus D^- \xrightarrow{f^+ - f^-} D\big)[-1] \xrightarrow{\simeq} \mathrm{Cone}\big(H^+ \oplus H^- \xrightarrow{h^+ - h^-} D\big)[-1] \ . \tag{8}$$

Recall that D splits as $D = C \oplus C'$, and that consequently the tensor product $D \otimes_{R_n} R_{n-1}((x_n))$ splits as a direct sum of

$$C \otimes_{R_n} R_{n-1}((x_n)) \quad \text{and} \quad C' \otimes_{R_n} R_{n-1}((x_n)) \ .$$

The former summand is acyclic by our hypothesis (4) (for $j = n$) so that H^+ maps by a quasi-isomorphism to

$$K^+ := \mathrm{Cone}\big(D^+[[x_n]] \oplus C \oplus C' \xrightarrow{(+,0,-)} C' \otimes_{R_n} R_{n-1}((x_n))\big)[-1] \ ,$$

with C mapping by the zero map as indicated. In fact, this map is induced by the following commutative diagram by taking mapping cones of horizontal maps:

$$\begin{array}{ccccc} D^+[[x_n]] \oplus C \oplus C' & \xrightarrow{(+,-,-)} & C \otimes_{R_n} R_{n-1}((x_n)) & \oplus & C' \otimes_{R_n} R_{n-1}((x_n)) \\ \downarrow \simeq & & & & \simeq \downarrow (0,\mathrm{id}) \\ D^+[[x_n]] \oplus C \oplus C' & \xrightarrow{(+,0,-)} & 0 & \oplus & C' \otimes_{R_n} R_{n-1}((x_n)) \end{array}$$

Similarly, H^- is quasi-isomorphic to

$$K^- := \mathrm{Cone}\left(D^-[[x_n^{-1}]] \oplus C' \oplus C \xrightarrow{(+,-,0)} C' \otimes_{R_n} R_{n-1}((x_n^{-1}))\right)[-1]$$

with C mapping by the zero map. It follows that the target of the quasi-isomorphism (8) maps by a quasi-isomorphism to

$$\mathrm{Cone}\left(K^+ \oplus K^- \xrightarrow{(+,-)} C' \oplus C\right)[-1] \ . \tag{9}$$

But by direct inspection this last complex contains

$$\mathrm{Cone}\left(C \oplus C \xrightarrow{(+,-)} C\right)[-1] \simeq C[-1]$$

as a direct summand; this makes use of the fact that the C-summands in K^+ and K^- only map non-trivially to the C-summand in the target of (8). Thus $C[-1]$ is homotopy equivalent to a summand of the chain complex (9) which is quasi-isomorphic, via (8) and (6), to the finite complex L of R_{n-1}-modules. It follows that $C[-1]$, considered as an R_{n-1}-module complex, is a retract up to homotopy of the chain complex L. Indeed, the complex (9) can be replaced, up to quasi-isomorphism, by a bounded below complex of projective R_{n-1}-modules which is quasi-isomorphic, and hence chain homotopy equivalent, to L. Using the fact that $C[-1]$ is a bounded below complex of projective R_{n-1}-modules as well it is then standard homological algebra to construct the desired maps of complexes $\alpha\colon C[-1] \longrightarrow L$ and $\beta\colon L \longrightarrow C[-1]$ together with a chain homotopy $\beta\alpha \simeq \mathrm{id}$. It now follows from [Ran85, Proposition 3.2 (ii)] that $C[-1]$ (and hence C) is R_{n-1}-finitely dominated.

In case $n = 1$ this finishes the proof of (4) ⇒ (1). For $n > 1$ we observe that C is now homotopy equivalent over R_{n-1} to a strict perfect complex $B \in \mathrm{Ch}^{\mathrm{b}}(R_{n-1})$ which satisfies condition (4) for $j < n$. By induction, B is R-finitely dominated, hence so is C. — This finishes the proof for general n.

4. A non-trivial finitely dominated chain complex

We will now discuss a generalisation of a 1-variable example given by Hughes and Ranicki [HR96, Example 23.19]. This serves to illustrate the existence of non-trivial finitely dominated chain complexes.

Let R be a commutative integral domain, and write R_n for the Laurent polynomial ring in indeterminates $x_1, x_2, \cdots, x_n$ as before. We actually restrict to the case $n = 2$, leaving the easy generalisation for higher n to the reader. Consider the following square diagram:

$$\begin{array}{ccc} R_2 & \xrightarrow{1-x_1x_2} & R_2 \\ {\scriptstyle 1-x_1}\downarrow & & \downarrow{\scriptstyle 1-x_1} \\ R_2 & \xrightarrow{1-x_1x_2} & R_2 \end{array} \tag{10}$$

Let $h\colon D \longrightarrow D$ denote the chain complex obtained by taking mapping cones in vertical direction, and let C be the mapping cone of h.

Clearly the complex C is not acyclic; indeed, the element $x_2 \in R_2$ represents a non-trivial element in the bottom homology of C. However, we claim that the four chain complexes

$$C \otimes_{R_1} R((x_1)) \quad \text{and} \quad C \otimes_{R_1} R((x_1^{-1})) \ ,$$
$$C \otimes_{R_2} R_1((x_2)) \quad \text{and} \quad C \otimes_{R_2} R_1((x_2^{-1}))$$

are all acyclic. This can be seen as follows: First, the vertical maps in the square (10) become isomorphisms after tensoring over R_1 with $R_1((x_1))$ as $1 - x_1$ is a unit in the latter ring. Consequently, by tensoring and taking mapping cones in vertical directions we obtain a map of acyclic chain complexes

$$\mathrm{Cone}\,\big(R_2 \xrightarrow{1-x_1} R_2\big) \longrightarrow \mathrm{Cone}\,\big(R_2 \xrightarrow{1-x_1} R_2\big)$$

whose mapping cone K is acyclic as well. But formation of mapping cones is compatible with taking tensor products so that there is an isomorphism $K \cong C \otimes_{R_1} R((x_1))$. Consequently the latter chain complex is acyclic. The same argument with the roles of x_1 and x_1^{-1} reversed proves that $C \otimes_{R_1} ((x_1^{-1}))$ is acyclic as well. — Tensoring the square diagram (10) over R_2 with $R_1((x_2))$ and taking mapping cones in vertical directions results in a chain complex map $g\colon E \longrightarrow E$ whose mapping cone J is isomorphic to $C \otimes_{R_2} R_1((x_2))$. Now as a map of graded modules (*i.e.*, disregarding differentials) the map g is given by the map

$$R_2[1] \otimes_{R_2} R_1((x_1)) \oplus R_2 \otimes_{R_2} R_1((x_1)) \\ \longrightarrow R_2[1] \otimes_{R_2} R_1((x_1)) \oplus R_2 \otimes_{R_2} R_1((x_1))$$

induced by multiplication by $1 - x_1x_2$. But this polynomial is a unit in the ring $R_1((x_2))$ (as x_1 is a unit in R_1) so that g is in fact an isomorphism of chain complexes. It follows that $C \otimes_{R_2} R_1((x_2)) \cong \mathrm{Cone}\,(g)$ is acyclic. By exchanging x_2 and x_2^{-1} we see that $C \otimes_{R_2} R_1((x_2^{-1}))$ is acyclic as well.

By Theorem 1.3 this shows that the complex C is R-finitely dominated. The Theorem also says that the chain complexes

$$C \otimes_{R[x_2, x_2^{-1}]} R((x_2)) \quad \text{and} \quad C \otimes_{R_2} R[x_2,\, x_2^{-1}]((x_1))$$

are acyclic, but note that this cannot be proved as easily as above (*viz.*, by showing that the horizontal or vertical maps of (10) become isomorphisms after application of a tensor product functor). It appears that the freedom to re-number the variables is relevant for detecting finite domination in practise.

5. Finite domination over fields

We finish the paper by discussing finite domination over fields which is (not surprisingly) much simpler than the general case. Suppose F is a field, and C is a bounded chain complex of finitely generated projective modules over the Laurent ring

$$L = F[z_1,\, z_1^{-1},\, z_2,\, z_2^{-1},\, \cdots,\, z_n,\, z_n^{-1}] \ .$$

Since F is a field, C is F-finitely dominated if and only if $\dim_F H_k C < \infty$ for all k. (See [Ros94, Theorem 1.7.13] for a proof covering the more general situation of a noetherian ground ring. Since there is no difference between free and projective F-modules, C is F-finitely dominated if and only if C is F-homotopy finite.) We obtain the following multi-variable version of [Ran95, §5, Example]:

Theorem 5.1. *The complex C is F-finitely dominated if and only if the induced chain complexes*

$$C \otimes_{F[z_j,z_j^{-1}]} F(z_j) \ , \quad j = 1,\, 2,\, \cdots,\, n \ ,$$

are acyclic. (Here $F(z_j)$ denotes the field of rational functions in z_j.)

Proof. Suppose first that C is finitely dominated. For fixed k and j, the multiplication action of z_j on C determines an endomorphism f_j of the finite-dimensional F-vector space $H_k(C)$. Its characteristic polynomial $p_j(x) = \det(f_j - x \cdot \mathrm{id})$ satisfies $p_j(f_j) = 0$, by Cayley-Hamilton. Note that as a self map of $H_k(C)$, the action of $p_j(f_j)$ coincides with the one given by multiplication with the polynomial $p_j(z_j)$. For any primitive tensor $a \otimes b \in H_k(C) \otimes_{F[z_j,z_j^{-1}]} F(z_j)$ we have the chain of equalities

$$a \otimes b = a \otimes (p_j \cdot b/p_j) = (a \cdot p_j) \otimes (b/p_j) = 0 \otimes (b/p_j) = 0$$

so that $H_k(C) \otimes_{F[z_j,z_j^{-1}]} F(z_j) = 0$. But $F(z_j)$ is a localisation of $F[z_j,\, z_j^{-1}]$ (*viz.*, its quotient field), whence we have an isomorphism

$$H_k\big(C \otimes_{F[z_j,z_j^{-1}]} F(z_j)\big) \cong H_k(C) \otimes_{F[z_j,z_j^{-1}]} F(z_j) = 0 \ .$$

This proves that $C \otimes_{F[z_j,z_j^{-1}]} F(z_j)$ is acyclic as claimed.

To prove the converse suppose that $C \otimes_{F[z_j,z_j^{-1}]} F(z_j)$ is acyclic for all j. Fix k and j. Exactness of localisation allows us to rewrite this hypothesis as

$$H_k(C) \otimes_{F[z_j,z_j^{-1}]} F(z_j) \cong H_k\big(C \otimes_{F[z_j,z_j^{-1}]} F(z_j)\big) = 0 \ .$$

This implies that the image of any element $g \in H_k(C)$ in the tensor product $H_k(C) \otimes_{F[z_j,z_j^{-1}]} F(z_j)$ is trivial. As an abelian group said tensor product is a quotient of $H_k(C) \otimes_{\mathbb{Z}} F(z_j)$ by relations of the form $a \otimes_{\mathbb{Z}} (pb) - (ap) \otimes_{\mathbb{Z}} b$, for $p \in F[z_j,\, z_j^{-1}]$. In other words we find finitely many Laurent polynomials $p_i \in F[z_j,\, z_j^{-1}]$, and elements $a_i \in H_k(C)$ and $b_i \in F(z_j)$, all depending on g, such that

$$g \otimes_{\mathbb{Z}} 1 = \sum_i \big(a_i \otimes_{\mathbb{Z}} (p_i b_i) - (a_i p_i) \otimes_{\mathbb{Z}} b_i\big) \ . \qquad (*)$$

Since $F(z_j)$ is the quotient field of $F[z_j,\, z_j^{-1}]$ we find a LAURENT polynomial $p(g)$, depending on g, such that $b_i p(g) \in F[z_j,\, z_j^{-1}]$.

The ring L is NOETHERian so that $H_k(C)$ is a finitely generated L-module. Let $g_1,\, g_2,\, \cdots,\, g_m$ be a set of generators, and let $q_j = \prod_{\ell=1}^{m} p(g_\ell)$ be the product of the LAURENT polynomials $p(g)$ constructed above from (*), where g is replaced in turn by the g_ℓ. Then, using the right $F[z_j,\, z_j^{-1}]$-module structure on $F(z_j)$, equation (*) for $g = g_\ell$ says that

$$g_\ell \otimes_{\mathbb{Z}} q_j = \sum_i \big(a_i \otimes_{\mathbb{Z}} (p_i b_i q_j) - (a_i p_i) \otimes_{\mathbb{Z}} (b_i q_j)\big)\ .$$

By choice of q_j we have $b_i q_j \in F[z_j,\, z_j^{-1}]$ so that consequently

$$g_\ell q_j \otimes_{[z_j, z_j^{-1}]} 1 = g_\ell \otimes_{[z_j, z_j^{-1}]} q_j = 0 \quad \text{in } H_k(C) \otimes_{F[z_j, z_j^{-1}]} F[z_j,\, z_j^{-1}] \cong H_k(C)\ ,$$

that is, $g_\ell q_j = 0 \in H_k(C)$. Since the g_ℓ generate $H_k(C)$ this implies that multiplication by q_j annihilates $H_k(C)$.

By what we have just shown, $H_k(C)$ is an $L/(q_1,\, q_2,\, \cdots,\, q_n)$-module in a natural way. But $H_k(C)$ is finitely generated as an L-module, hence as a module over the quotient $L/(q_1,\, q_2,\, \cdots,\, q_n)$ which in turn is a finite dimensional F-vector space. It follows that $H_k(C)$ is of finite dimension over F as required. □

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