Wolfgang Lück and Holger Reich

Isomorphism Conjectures in K- and L-Theory

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Preface

Comment 1 (By W.): `Here the main preface has to be inserted.`

Comment 2 (By W.): `Shall we insert a general list of criteria why it is worthwhile to study a conjecture.`

A User's Guide

Comment 3 (By W.): `This paragraph has to be adjusted and completed.`

A reader who wants to get specific information or focus on a certain topic should consult the detailed table of contents, the index and the index of notation in order to find the right place in the paper. We have tried to write the text in a way such that one can read small units independently from the rest. Moreover, a reader who may only be interested in the Baum-Connes Conjecture or only in the Farrell-Jones Conjecture for K-theory or for L-theory can ignore the other parts. But we emphasize again that one basic idea of this paper is to explain the parallel treatment of these conjectures.

A reader without much prior knowledge about the Baum-Connes Conjecture or the Farrell-Jones Conjecture should begin with Chapter 1. There, the special case of a torsionfree group is treated, since the formulation of the conjectures is less technical in this case and there are already many interesting applications. The applications are not needed later. A more experienced reader may pass directly to Chapter 2.

Other (survey) articles on the Farrell-Jones Conjecture and the Baum-Connes Conjecture are [126], [143], [163], [226], [246], [335].

We require that the reader is familiar with basic notions in topology (CW-complexes, chain complexes, homology, homotopy groups, manifolds, coverings), functional analysis (Hilbert spaces, bounded operators, differential operators, C^*-algebras), and algebra (groups, modules, elementary homological algebra). **Comment 4** (By W.): `This list is not yet complete.`

Acknowledgments

Comment 5 (By W.): `This has to be completed and adjusted later.`

We want to thank heartily the present and former members of the topology group in Münster who read through the manuscript and made a lot of useful comments, corrections and suggestions. These are **Comment 6** (By W.): `List the people.`

We want to thank the Deutsche Forschungsgemeinschaft which has been and is financing the Sonderforschungsbereich 478 – "Geometrische Strukturen in der Mathematik" and the Graduiertenkolleg "Analytische Topologie und Metageometrie". These institutions made it possible to invite guests and run workshops on the topics of the book. This was very important for its writing. In particular we had fruitful discussions with and obtained a lot of information from **Comment 7** (By W.): `List the people.`

The first author thanks the Max-Planck Institute for Mathematics in Bonn for its hospitality during his stay from April to July 2005 and both authors thank the Mittag-Leffler institut for its hospitality during their stay in May 2006 while parts of this book were written.

Finally the authors want to express their deep gratitude to their families.

Münster, July 2008 *Wolfgang Lück and Holger Reich*

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Contents

0. Introduction

The Isomorphism Conjectures due to Baum-Connes and Farrell-Jones are important conjectures which have many interesting applications and consequences. However, they are not easy to formulate and it is a priori not clear why the actual versions are the most promising ones. They are the final upshot of a longer process which lead step by step to the final versions. It has been influenced and steered by the various new results which have been proven during the last decades and given new insight into the objects, problems and constructions these conjectures aim at. Next we want to explain how one can be lead to these conjectures by general considerations and certain facts.

0.1 Motivation for the Baum-Connes Conjecture

We will start with the easiest and most convenient to handle Isomorphism Conjecture, the Baum-Connes Conjecture for the topological K-theory of the reduced group C^*-algebra, and then pass to the more complicated Farrell-Jones Conjecture for the algebraic K- and L-theory of the group ring.

0.1.1 Topological K-Theory of Reduced Group C^*-Algebras

The target of the Baum-Connes Conjecture is the topological (complex) K-theory of the *reduced C^*-algebra* $C_r^*(G)$ of a topological group G. We will only consider discrete groups G. One defines the *topological K-groups* $K_n(A)$ for any Banach algebra A to be the abelian group $K_n(A) = \pi_{n-1}(GL(A))$ for $n \geq 1$. The famous *Bott Periodicity Theorem* gives a natural isomorphism $K_n(A) \xrightarrow{\cong} K_{n+2}(A)$ for $n \geq 1$. Finally one defines $K_n(A)$ for all $n \in \mathbb{Z}$ so that the Bott isomorphism theorem is true for all $n \in \mathbb{Z}$. It turns out that $K_0(A)$ is the same as the *projective class group* of the ring A which is the Grothendieck group of the abelian monoid of isomorphism classes of finitely generated projective A-modules with the direct sum as addition. The topological K-theory of $\mathbb{C}$ is trivial in odd dimensions and is isomorphic to $\mathbb{Z}$ in even dimensions. More generally, for a finite group G the topological K-theory of $C_r^*(G)$ is the complex representation ring $R_{\mathbb{C}}(G)$ in even dimensions and is trivial in odd dimensions.

If P is an appropriate elliptic differential operator (or more generally an elliptic complex) on a closed n-dimensional Riemannian manifold M, then one can consider its *index* in $K_n(\mathbb{C})$ which is zero for odd n and $\dim_{\mathbb{C}}(\ker(P)) - \dim_{\mathbb{C}}(\operatorname{coker}(P)) \in \mathbb{Z}$ for even n. If M comes with an isometric G-action of a finite group G and P is compatible with the G-action, then $\ker(P)$ and $\operatorname{coker}(P)$ are complex (finite-dimensional) G-representations and one obtains an element in $K_n(C_r^*(G)) = R_{\mathbb{C}}(G)$ by $[\ker(P)] - [\operatorname{coker}(P)]$ for even n. Suppose that G is an arbitrary discrete group and that M is a (not necessarily compact) n-dimensional smooth manifold without boundary with a proper cocompact G-action, a G-invariant Riemannian metric and an appropriate elliptic differential operator P compatible with the G-action. An example is the universal covering $M = \widetilde{N}$ of a closed n-dimensional Riemannian manifold N with $G = \pi_1(N)$ and the lift $\widetilde{P}$ to $\widetilde{N}$ of an appropriate elliptic differential operator P on N. Then one can define an *equivariant index* of P which takes values in $K_n(C_r^*(G))$. Therefore the interest of $K_*(C_r^*(G))$ comes from the fact that it is the natural recipient for indices of certain equivariant operators.

0.1.2 Homological Aspects

A first basic problem is to compute $K_*(C_r^*(G))$ or to identify it with more familiar terms. The key idea comes from the observation that $K_*(C_r^*(G))$ has some homological properties. More precisely, if G is the amalgamated product $G = G_1 *_{G_0} G_2$ for subgroups $G_i \subseteq G$, then there is a long exact sequence

$$\begin{aligned}
\cdots \xrightarrow{\partial_{n+1}} K_n(C_r^*(G_0)) &\xrightarrow{K_n(C_r^*(i_1)) \oplus K_n(C_r^*(i_2))} K_n(C_r^*(G_1)) \oplus K_n(C_r^*(G_2)) \\
&\xrightarrow{K_n(C_r^*(j_1)) - K_n(C_r^*(j_2))} K_n(C_r^*(G)) \xrightarrow{\partial_n} K_{n-1}(C_r^*(G_0)) \\
&\xrightarrow{K_{n-1}(C_r^*(i_1)) \oplus K_{n-1}(C_r^*(i_2))} K_{n-1}(C_r^*(G_2)) \oplus K_{n-1}(C_r^*(G_1)) \\
&\xrightarrow{K_{n-1}(C_r^*(j_1)) - K_{n-1}(C_r^*(j_2))} K_{n-1}(C_r^*(G)) \xrightarrow{\partial_{n-1}} \cdots \qquad (0.1)
\end{aligned}$$

where i_1,i_2, j_1 and j_2 are the obvious inclusions (see [265, Theorem 18 on page 632]). If $\phi\colon G \to G$ is a group automorphism and $G \rtimes_\phi \mathbb{Z}$ the associated semidirect product, then there is a long exact sequence

$$\begin{aligned}
\cdots \xrightarrow{\partial_{n+1}} K_n(C_r^*(G)) &\xrightarrow{K_n(C_r^*(\phi)) - \mathrm{id}} K_n(C_r^*(G)) \xrightarrow{K_n(C_r^*(k))} K_n(C_r^*(G \rtimes_\phi \mathbb{Z})) \\
\xrightarrow{\partial_n} K_{n-1}(C_r^*(G)) &\xrightarrow{K_{n-1}(C_r^*(\phi)) - \mathrm{id}} K_{n-1}(C_r^*(G)) \xrightarrow{K_{n-1}(C_r^*(k))} \cdots \qquad (0.2)
\end{aligned}$$

where k is the obvious inclusion (see [264, Theorem 3.1 on page 151] or more generally [265, Theorem 18 on page 632]).

We compare this with group homology in order to explain the analogy with homology. Recall that the *classifying space* BG of a group G is an aspherical CW-complex whose fundamental group is isomorphic to G and that *aspherical* means that all higher homotopy groups are trivial, or, equivalently, that its universal covering is contractible. It is unique up to homotopy. If one has an amalgamated product $G = G_1 *_{G_0} G_2$, then one can find models for the classifying spaces such that BG_i is a CW-subcomplex of BG and $BG = BG_1 \cup BG_2$ and $BG_0 = BG_1 \cap BG_2$. Thus we obtain a pushout of inclusions of CW-complexes

$$\begin{tikzcd} BG_0 \arrow[r, "Bi_1"] \arrow[d, "Bi_2"'] & BG_1 \arrow[d, "Bj_1"'] \\ BG_2 \arrow[r, "Bj_2"] & BG \end{tikzcd}$$

It yields a long Mayer-Vietoris sequence for the cellular or singular homology

$$\begin{aligned} \cdots \xrightarrow{\partial_{n+1}} H_n(BG_0) &\xrightarrow{H_n(Bi_1)\oplus H_n(Bi_2)} H_n(BG_1) \oplus H_n(BG_2) \\ &\xrightarrow{H_n(Bj_1)-H_n(Bj_2)} H_n(BG) \xrightarrow{\partial_n} H_{n-1}(BG_0) \\ &\xrightarrow{H_{n-1}(Bi_1)\oplus H_{n-1}(Bi_2)} H_{n-1}(BG_2) \oplus H_{n-1}(BG_1) \\ &\xrightarrow{H_{n-1}(Bj_1)-H_{n-1}(Bj_2)} H_{n-1}(BG) \xrightarrow{\partial_{n-1}} \cdots . \end{aligned} \tag{0.3}$$

If $\phi \colon G \to G$ is a group automorphism, then a model for $B(G \rtimes_\phi \mathbb{Z})$ is the mapping torus of $B\phi \colon BG \to BG$ which is obtained from the cylinder $BG \times [0,1]$ by identifying the bottom and the top with the map $B\phi$. Associated to a mapping torus there is the long exact sequence

$$\begin{aligned} \cdots \xrightarrow{\partial_{n+1}} H_n(BG) \xrightarrow{H_n(B\phi)-\mathrm{id}} H_n(BG) &\xrightarrow{H_n(Bk)} H_n(B(G \rtimes_\phi \mathbb{Z})) \\ \xrightarrow{\partial_n} H_{n-1}(BG) &\xrightarrow{H_{n-1}(B\phi)-\mathrm{id}} H_{n-1}(BG) \xrightarrow{H_n(Bk)} \cdots \end{aligned} \tag{0.4}$$

where k is the obvious inclusion of BG into the mapping torus.

0.1.3 The Baum-Connes Conjecture for Torsionfree Groups

There is an obvious analogy between the sequences (0.1) and (0.3) and the sequences (0.2) and (0.4). On the other hand we get for the trivial group $G = \{1\}$ that $H_n(B\{1\}) = H_n(\{\bullet\})$ is $\mathbb{Z}$ for $n = 0$ and trivial for $n \neq 0$ so that the group homology of BG cannot be the same as the topological K-theory of $C_r^*(\{1\})$. But there is a better candidate, namely take the *topological K-homology* of BG instead of the singular homology. Topological K-homology is a homology theory defined for CW-complexes. At least we mention that for a topologist its definition is a routine, namely, it is the homology theory associated to the K-theory spectrum which is in turn the spectrum defining topological K-theory of CW-complexes, i.e. the cohomology theory which comes from considering vector bundles over CW-complexes. In contrast to singular homology the topological K-homology of a point $K_n(\{\bullet\})$ is $\mathbb{Z}$ for even n and is trivial for n odd. So we still get exact sequences (0.3) and (0.4) if we replace H_* by K_* everywhere and we have $K_n(B\{1\}) \cong K_n(C_r^*(\{1\})$ for all $n \in \mathbb{Z}$. This leads to the following conjecture:

Conjecture 0.5 (Baum-Connes Conjecture for torsionfree groups). *Let G be a torsionfree group. Then there is for $n \in \mathbb{Z}$ an isomorphism called* assembly map

$$K_n(BG) \xrightarrow{\cong} K_n(C_r^*(G)).$$

This is indeed a formulation which is equivalent to the Baum-Connes Conjecture provided that G is torsionfree. It cannot hold in general as already the example of a finite group G shows. Namely, if G is finite, then the obvious inclusion induces an isomorphism $K_n(B\{1\}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $n \in \mathbb{Z}$, whereas $K_0(C_r^*(\{1\}) \to K_0(C_r^*(G))$ agrees with the map $R_{\mathbb{C}}(\{1\}) \to R_{\mathbb{C}}(G)$ which is rationally bijective if and only if G itself is trivial. Hence Conjecture 0.5 is not true for non-trivial finite groups.

0.1.4 Formulation of the Baum-Connes Conjecture

What is going wrong? The sequences (0.1) and (0.2) do exist regardless whether the groups are torsionfree or not. More generally, if G acts on a tree, then they can be combined to compute the K-theory $K_*(C_r^*(G))$ of a group G by a certain Mayer-Vietoris sequence from the stabilizers of the vertices and edges (see Pimsner [265, Theorem 18 on page 632]). In the special case, where all stabilizers are finite, one sees that $K_*(C_r^*(G))$ is built by the topological K-theory of the finite subgroups of G in a homological fashion. This leads to the idea that $K_*(C_r^*(G))$ can be computed in a homological way but the building blocks do not only consist of $K_*(C_r^*(\{1\}))$ alone but of $K_*(C_r^*(H))$ for all finite subgroups $H \subseteq G$. This suggest to study *equivariant topological K-theory.* It assigns to every proper G-CW-complex X a sequence of abelian groups $K_n^G(X)$ for $n \in \mathbb{Z}$ such that G-homotopy invariance holds and Mayer-Vietoris sequences exist. A *proper G-CW-complex* is a CW-complex with G-action such that for $g \in G$ and every open cell e with $e \cap g \cdot e \neq \emptyset$ we gave $gx = x$ for all $x \in e$ and all isotropy groups are finite. Two interesting features are that $K_n^G(G/H)$ agrees with $K_n(C_r^*(H))$ for every finite subgroup $H \subseteq G$ and that for a free G-CW-complex X and $n \in \mathbb{Z}$ we have a natural isomorphism $K_n^G(X) \xrightarrow{\cong} K_n(G\backslash X)$. Recall that EG is a free G-CW-complex which is contractible and that $EG \to G\backslash EG = BG$ is the universal covering of BG. We can reformulate Conjecture 0.5 by stating an isomorphism

$$K_n^G(EG) \xrightarrow{\cong} K_n(C_r^*(G)).$$

Now suppose that G acts on a tree T with finite stabilizers. Then the computation of Pimsner [265, Theorem 18 on page 632]) mentioned above can be rephrased that there is an isomorphism

$$K_n^G(T) \xrightarrow{\cong} K_n(C_r^*(G)).$$

In particular the left hand side is independent of the tree T on which G acts by finite stabilizers. This can be explained as follows. It is known that for every finite subgroup $H \subseteq G$ the H-fixed point set T is again a non-empty tree and hence contractible. This implies that two trees T_1 and T_2 on which G acts with finite stabilizers are G-homotopy equivalent and hence have the same equivariant topological K-theory. The same remark applies to $K_n(BG)$ and $K_n(EG)$, namely, two models for BG are homotopy equivalent and to models for EG are G-homotopy equivalent and therefore $K_n(BG)$ and $K_n^G(EG)$ are independent of the choice of a model. This leads to the idea to look for an appropriate proper G-CW-complex $\underline{E}G$ which is characterized by a certain universal property and is unique up to G-homotopy such that for a torsionfree group G we have $EG = \underline{E}G$ and for a tree on which G acts with finite stabilizers we have $\underline{E}G = T$ and that there is an isomorphism

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G)).$$

In particular for a finite group we would like to have $\underline{E}G = G/G = \{\bullet\}$ and then the desired isomorphism above is true for trivial reasons. Recall that EG is characterized up to G-homotopy by the property that it is a G-CW-complex such that EG^H is empty for $H \neq \{1\}$ and is contractible for $H = \{1\}$. Having the case of a tree on which G acts with finite stabilizers in mind, we define the *classifying space for proper G-actions* $\underline{E}G$ to be a G-CW-complex such that EG^H is empty for $|H| = \infty$ and is contractible for $|H| < \infty$. Indeed two models for $\underline{E}G$ are G-homotopy equivalent, a tree on which G acts with finite stabilizers is a model for $\underline{E}G$, we have $EG = \underline{E}G$ if (and only if) G is torsionfree and $\underline{E}G = G/G = \{\bullet\}$ if (and only if) G is finite. This leads to

Conjecture 0.6 (Baum-Connes Conjecture). *Let G be a group. Then there is for all $n \in \mathbb{Z}$ an isomorphism called* assembly map

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G)).$$

This conjecture reduces in the torsionfree case to Conjecture 0.5 and is consistent with the results by Pimsner [265, Theorem 18 on page 632]) for G-acting on a tree with finite stabilizers. It is also true for finite groups G. Pimsner's result does hold more generally for groups acting on trees with not necessarily finite stabilizers. So one should get the analogous result for the left hand side of the isomorphism appearing in the Baum-Connes Conjecture 0.6. Essentially this boils down to the question, whether the analogues of the long exact sequences (0.1) and (0.2) holds for the left side of the isomorphism appearing in the Baum-Connes Conjecture 0.6. This follows for (0.1) from the fact that for $G = G_1 *_{G_0} G_2$ one can find appropriate models for the classifying spaces for proper G-actions such that there is a G-pushout of inclusions of proper G-CW-complexes

$$\begin{array}{ccc} G \times_{G_0} \underline{E}G_0 & \longrightarrow & G \times_{G_1} \underline{E}G_1 \\ \downarrow & & \downarrow \\ G \times_{G_2} \underline{E}G_2 & \longrightarrow & \underline{E}G \end{array}$$

and for a subgroup $H \subseteq G$ and a proper H-CW-complex X there is a natural isomorphism

$$K_n^H(X) \xrightarrow{\cong} K_n^G(G \times_H X).$$

Thus the associated long exact Mayer-Vietoris sequence yields the long exact sequence

$$\begin{aligned} \cdots \xrightarrow{\partial_{n+1}} K_n^{G_0}(\underline{E}G_0) \to K_n^{G_1}(\underline{E}G_1) \oplus K_n^{G_2}(\underline{E}G_2) \to K_n^G(\underline{E}G) \xrightarrow{\partial_n} \\ K_{n-1}^{G_0}(\underline{E}G_0) \to K_{n-1}^{G_1}(\underline{E}G_1) \oplus K_{n-1}^{G_2}(\underline{E}G_2) \to K_{n-1}^{G_0}(\underline{E}G) \to \cdots \end{aligned}$$

which corresponds to (0.1). For (0.2) one uses fact that for a group automorphism $\phi\colon G \xrightarrow{\cong} G$ the $G \rtimes_\phi \mathbb{Z}$-$CW$-complex given by the to both sides infinite mapping telescope of the ϕ-equivariant map $\underline{E}\phi\colon \underline{E}G \to \underline{E}G$ is a model for $\underline{E}(G \rtimes_\phi \mathbb{Z})$.

In general $K_n^G(\underline{E}G)$ is much bigger than $K_n^G(EG) \cong K_n(BG)$ and the canonical map $K_n^G(EG) \to K_n^G(\underline{E}G)$ is rationally injective but not necessarily integrally injective.

0.1.5 Reduced versus Maximal Group C^*-Algebras

All the arguments above do also apply to the *maximal group C^*-algebra* which does even have better functorial properties than the reduced group C^*-algebra. So a priori one may think that one should use the maximal group C^*-algebra instead of the reduced one. However, the version for the maximal group C^*-algebra is not true in general and the version for the reduced group C^*-algebra seems to be the right one. This will be discussed in more detail in Subsection 2.12.2.

0.1.6 Applications of the Baum-Connes Conjecture

The assembly map appearing in the Baum-Connes Conjecture 0.6 has an index theoretic interpretation. An element in $K_0^G(\underline{E}G)$ can be represented by a pair (M, P^*) consisting of a cocompact proper smooth n-dimensional G-manifold M with a G-invariant Riemannian metric together with an elliptic G-complex P^* of differential operators of order 1 on M and its image under the assembly map is a certain equivariant index $\operatorname{ind}_{C_r^*(G)}(M, P^*)$ in $K_n(C_r^*(G))$. There are many important consequences of the Baum-Connes Conjecture such as the *Kadison Conjecture* (see Subsection 1.3.2), the *Zero-Divisor-Conjecture* (see Subsection 1.10.1), the *stable Gromov-Lawson Conjecture* (see Subsection 2.11.2) and the *Novikov Conjecture* (see Section 2.9).

0.2 Motivation for the Farrell-Jones Conjecture for K-Theory

Next we want to deal with the algebraic K-groups $K_n(RG)$ of the group ring RG of a group G with coefficients in an associative ring R with unit.

0.2.1 Algebraic K-Theory of Group Rings

For an associative ring with unit R one defines $K_0(R)$ as the *projective class group* and $K_1(R)$ as the abelianization of $GL(R) = \operatorname{colim}_{n\to\infty} GL_n(R)$. One defines the *higher algebraic K-groups* $K_n(R)$ for $n \geq 1$ as the fundamental groups of a certain *K-theory space* associated to the category of finitely generated projective R-modules. One can define *negative K-groups* $K_n(R)$ for $n \leq -1$ by a certain contracting procedure applied to $K_0(R)$. Finally there exists a *K-theory spectrum* $\mathbf{K}(R)$ such that $\pi_n(\mathbf{K}(R)) = K_n(R)$ for all $n \in \mathbb{Z}$. If $\mathbb{Z} \to R$ is the obvious ring map sending n to $n \cdot 1_R$, then one defines the *reduced K-groups* to be the cokernel of the induced map $K_n(\mathbb{Z}) \to K_n(R)$. The *Whitehead group* $\mathrm{Wh}(G)$ of a group G is the quotient of $K_1(\mathbb{Z}G)$ by elements given by $(1,1)$-matrices of the shape $(\pm g)$ for $g \in G$.

The *reduced projective class group* $\widetilde{K}_0(\mathbb{Z}G)$ is the recipient for the *finiteness obstruction* of a finitely dominated CW-complex X with fundamental group $G = \pi_1(X)$. *Finitely dominated* means that there is a finite CW-complex Y and maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity on X. The Whitehead group $\mathrm{Wh}(G)$ is the recipient of the *Whitehead torsion* of a homotopy equivalence of finite CW-complexes and of a compact h-cobordism over a closed manifold with fundamental group G. An *h-cobordism W over M* consists of a manifold W whose boundary is the disjoint union $\partial W = \partial_0 W \coprod \partial_1 W$ such that both inclusions $\partial_i W \to W$ are homotopy equivalences together with a diffeomorphism $M \xrightarrow{\cong} \partial_0 W$. These are very important topological obstructions whose vanishing has interesting geometric consequences. The vanishing of the finiteness obstruction says that the finitely dominated CW-complex under consideration is homotopy equivalent to a finite CW-complex. The vanishing of the Whitehead torsion of a compact h-cobordism W over M of dimension ≥ 6 implies that the W is trivial, i.e. is diffeomorphic to a cylinder $M \times [0,1]$ relative $M = M \times \{0\}$. This explains why topologists are interested in $K_n(\mathbb{Z}G)$ for groups G.

0.2.2 Appearance of Nil-Terms

The situation for algebraic K-theory of RG is more complicated than the one for the topological K-theory of $C_r^*(G)$. As a special case of the sequence (0.2) we obtain an isomorphism

$$K_n(C_r^*(G \times \mathbb{Z})) = K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)).$$

For K-theory the analogue is the *Bass-Heller-Swan decomposition*

$$K_n(R[\mathbb{Z}]) \;\cong\; K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R),$$

where certain additional terms, the *Nil-terms* $NK_n(R)$ appear. If one replaces R by RG, one gets

$$K_n(R[G \times \mathbb{Z}]) \;\cong\; K_n(RG) \oplus K_{n-1}(RG) \oplus NK_n(RG) \oplus NK_n(RG).$$

Such correction terms in form of Nil-terms appear also, when one wants to get analogues of the sequences (0.1) and (0.2) for algebraic K-theory (see Waldhausen [341] and [342]).

0.2.3 The Farrell-Jones Conjecture for $K_*(RG)$ for Regular Rings and Torsionfree Groups

Suppose that R is a *regular ring*, i.e. it is Noetherian and every R-module possesses a finite-dimensional projective resolution. Any field and $\mathbb{Z}$ are regular rings. Then one can prove in many cases for torsionfree groups that the analogues of the sequences (0.1) and (0.2) do hold for algebraic K-theory (see Waldhausen [341] and [342]). The same reasoning as in the Baum-Connes Conjecture for torsionfree groups leads to

Conjecture 0.7. (Farrell-Jones Conjecture for $K_*(RG)$ for torsion-free groups and regular rings). *Let G be a torsionfree group and let R be a regular ring. Then there is for $n \in \mathbb{Z}$ an isomorphism*

$$H_n(BG; \mathbf{K}(R)) \xrightarrow{\cong} K_n(RG).$$

Here $H_*(-; \mathbf{K}(R))$ is the homology theory associated to the K-theory spectrum of R. It is a homology theory with the property that $H_n(\{\bullet\}; \mathbf{K}(R)) = \pi_n(\mathbf{K}(R)) = K_n(R)$ for $n \in \mathbb{Z}$.

If one drops the condition that G is torsionfree but requires that the order of every finite subgroup of G is invertible in R, then one still can prove in many cases that the analogues of the sequences (0.1) and (0.2) do hold for algebraic K-theory. The same reasoning as in the Baum-Connes Conjecture leads to

Conjecture 0.8. (Farrell-Jones Conjecture for $K_*(RG)$ for regular rings). *Let G be a group. Let R be a regular ring such that $|H|$ is invertible in R for every finite subgroup $H \subseteq G$. Then there is an isomorphism*

$$H_n^G(\underline{E}G; \mathbf{K}_R) \xrightarrow{\cong} K_n(RG).$$

Here $H_n^G(-; \mathbf{K}_R)$ is an appropriate G-homology theory with the property that $H_n^G(G/H; \mathbf{K}_R) \cong H_n^H(\{\bullet\}; \mathbf{K}_R) \cong K_n(RH)$ for every subgroup $H \subseteq G$. Obviously Conjecture 0.8 reduces to Conjecture 0.7 if G is torsionfree.

0.2.4 Formulation of the Farrell-Jones Conjecture for $K_*(RG)$

Conjecture 0.7 can be applied in the case $R = \mathbb{Z}$ what is not true for Conjecture 0.8. So what is the right formulation for arbitrary rings R? The idea is that one does not only need to take all finite subgroups into account but also all virtually cyclic subgroups. A group is called *virtually cyclic* if it is finite or contains $\mathbb{Z}$ as subgroup of finite index. Namely, let $\underline{\underline{E}}G = E_{\mathcal{VCY}}(G)$ be the *classifying space for the family of virtually cyclic subgroups*, i.e. a G-CW-complex $\underline{\underline{E}}G$ such that $\underline{\underline{E}}G^H$ is contractible for every virtually cyclic subgroup $H \subseteq G$ and is empty for every subgroup $H \subseteq G$ which is not virtually cyclic. The G-space $\underline{\underline{E}}G$ is unique up to G-homotopy.

Conjecture 0.9. (Farrell-Jones Conjecture for $K_*(RG)$). *Let G be a group. Let R be an associative ring with unit. Then there is for all $n \in \mathbb{Z}$ an isomorphism called* assembly map

$$H_n^G(\underline{\underline{E}}G; \mathbf{K}_R) \xrightarrow{\cong} K_n(RG).$$

Here $H_n^G(-; \mathbf{K}_R)$ is an appropriate G-homology theory with the property that $H_n^G(G/H; \mathbf{K}_R) \cong H_n^H(\{\bullet\}; \mathbf{K}_R) \cong K_n(RH)$ for every subgroup $H \subseteq G$ and the assembly map is induced by the map $\underline{\underline{E}}G \to \{\bullet\}$.

In this formulation we have absorbed all the Nil-phenomena into the source by replacing $\underline{E}G$ by $\underline{\underline{E}}G$. There is a certain price we have to pay since often there are nice small geometric models for $\underline{E}G$, whereas the spaces $\underline{\underline{E}}G$ are much harder to analyze and are in general huge. There are up to G-homotopy unique G-maps $EG \to \underline{E}G$ and $\underline{E}G \to \underline{\underline{E}}G$ which yield maps

$$H_n(BG; \mathbf{K}(R)) = H_n^G(EG; \mathbf{K}_R) \to H_n^G(\underline{E}G; \mathbf{K}_R) \to H_n^G(\underline{\underline{E}}G; \mathbf{K}_R).$$

We will later see that there is a splitting (see Bartels [20])

$$H_n^G(\underline{\underline{E}}G; \mathbf{K}_R) \cong H_n^G(\underline{E}G; \mathbf{K}_R) \oplus H_n^G(\underline{\underline{E}}G, \underline{E}G; \mathbf{K}_R) \tag{0.10}$$

and $H_n^G(\underline{E}G; \mathbf{K}_R)$ is the easy homological part and $H_n^G(\underline{\underline{E}}G, \underline{E}G; \mathbf{K}_R)$ contains all Nil-type information. If R is regular and the order of any finite subgroup of G is invertible in R, then $H_n^G(\underline{\underline{E}}G, \underline{E}G; \mathbf{K}_R)$ is trivial and hence the natural map $H_n^G(\underline{E}G; \mathbf{K}_R) \xrightarrow{\cong} H_n^G(\underline{\underline{E}}G; \mathbf{K}_R)$ is bijective. Therefore Conjecture 0.9 reduces to Conjecture 0.7 and Conjecture 0.8 when they apply. In the Baum-Connes setting the natural map $K_n^G(\underline{E}G) \xrightarrow{\cong} K_n^G(\underline{\underline{E}}G)$ is always bijective.

0.2.5 Applications of the Farrell-Jones Conjecture for $K_*(RG)$

Since $K_n(\mathbb{Z}) = 0$ for $n \leq -1$ and the maps $\mathbb{Z} \xrightarrow{\cong} K_0(\mathbb{Z})$, which sends n to the class of $\mathbb{Z}^n$, and $\{\pm 1\} \to K_1(\mathbb{Z})$, which sends ± 1 to the class of the $(1,1)$-matrix (± 1), are bijective, an easy spectral sequence argument shows that Conjecture 0.7 implies

Conjecture 0.11. (Farrell-Jones Conjecture $K_*(\mathbb{Z}G)$ in dimension $n \leq 1$). *Let G be a torsionfree group. Then $\widetilde{K}_n(\mathbb{Z}G) = 0$ for $n \in \mathbb{Z}, n \leq 0$ and $\mathrm{Wh}(G) = 0$.*

In particular the finiteness obstruction and the Whitehead torsion are always zero for torsionfree fundamental groups. This implies in particular that every h-cobordism over a simply connected d-dimensional closed manifold for $d \geq 5$ is trivial and thus the Poincaré Conjecture in dimensions ≥ 6 (and with some extra effort also in dimension $d = 5$). This will be explained in Section 1.5.1. The Farrell-Jones Conjecture for K-theory 0.13 implies the *Bass Conjecture* (see Section 2.7). Further applications, e.g. to pseudo-isotopy and to automorphisms of manifolds will be discussed in Subsection 1.7.1 and Subsection 1.9.2.

0.3 Motivation for the Farrell-Jones Conjecture for L-Theory

Next we want to deal with the algebraic L-groups $L_n^\epsilon(RG)$ of the group ring RG of a group G with coefficients in an associative ring R with unit and involution.

0.3.1 Algebraic L-Theory of Group Rings

Let R be an associative ring with unit. An *involution of rings* $R \to R, r \mapsto \overline{r}$ on R is a map satisfying $\overline{r+s} = \overline{r} + \overline{s}$, $\overline{rs} = \overline{s}\,\overline{r}$, $\overline{0} = 0$, $\overline{1} = 1$ and $\overline{\overline{r}} = r$ for all $r, s \in R$. Given a ring with involution, the group ring RG inherits an involution by $\sum_{g \in G} r_g \cdot g = \sum_{g \in G} \overline{r} \cdot g^{-1}$. If the coefficient ring R is commutative, we usually use the trivial involution $\overline{r} = r$. Given a ring with involution, one can associate to it *quadratic L-groups* $L_n^h(R)$ for $n \in \mathbb{Z}$. The abelian group $L_0^h(R)$ can be identified with the *Witt group* of quadratic forms on finitely generated free R-modules, where every hyperbolic quadratic forms represent the zero element and the addition is given by the orthogonal sum of quadratic forms. The abelian group $L_2^h(R)$ is essentially given by the skew-symmetric versions. One defines $L_1^h(R)$ and $L_3^h(R)$ in terms of automorphism of quadratic forms. The L-groups are four-periodic, i.e. there is a natural isomorphism $L_n^h(R) \xrightarrow{\cong} L_{n+4}^h(R)$ for $n \in \mathbb{Z}$. If one uses finitely generated projective R-modules instead of finitely generated free R-modules, one obtains the quadratic L-groups $L_n^p(RG)$ for $n \in \mathbb{Z}$. If one uses finitely generated based free RG-modules and takes the Whitehead torsion into account, then one obtains the quadratic L-groups $L_n^s(RG)$ for $n \in \mathbb{Z}$. For every $j \in \{-\infty\} \amalg \{j \in \mathbb{Z} \mid j \leq 2\}$ there are versions $L_n^{\langle j \rangle}(RG)$, where $\langle j \rangle$ is called *decoration.* The decorations $j = 0, 1$ correspond to the decorations p, h and $j = 2$ is related to the decoration s.

The relevance of the L-groups comes from the fact that they are the recipients for various surgery obstructions. The fundamental *surgery problem* is the following. Consider a map $f\colon M \to X$ from a closed oriented manifold M to a finite Poincaré complex X. We want to know whether we can change it by a process called surgery to a map $g\colon N \to X$ with a closed manifold N as source and the same target such that g is a homotopy equivalence. This can answer the question whether a finite Poincaré complex X is homotopy equivalent to a closed manifold. Notice that a space which is homotopy equivalent to a closed oriented manifold must be a finite Poincaré complex but not every finite Poincaré complex is homotopy equivalent to a closed oriented manifold. If f comes with additional bundle data and has degree 1, we can find g if and only if the so called *surgery obstruction* of f vanishes which takes values in $L_n^h(\mathbb{Z}G)$ for $n = \dim(X)$ and $G = \pi_1(X)$. If we want g to be a simple homotopy equivalence, the obstruction lives in $L_n^s(\mathbb{Z}G)$. We see that analogous to the finiteness obstruction in $\widetilde{K}_0(\mathbb{Z}G)$ and the Whitehead torsion in $\mathrm{Wh}(G)$ the algebraic L-groups are the recipients for important obstructions whose vanishing has interesting geometric consequences. Also the question whether two closed manifolds are diffeomorphic or homeomorphic can be decided via surgery theory of which the L-groups are a part.

0.3.2 The Farrell-Jones Conjecture for $L_*(RG)[1/2]$

If we invert 2, i.e. if we consider the localization $L_n^{\langle -j\rangle}(RG)[1/2]$, then there is no difference between the various decorations and the analogues of the sequences (0.1) and (0.2) are true for L-theory (see Cappell [55]). The same reasoning as for the Baum-Connes Conjecture leads to

Conjecture 0.12. (Farrell-Jones Conjecture for $L_*(RG)[1/2]$). *Let G be a group. Let R be an associative ring with unit and involution. Then there is for all $n \in \mathbb{Z}$ an isomorphism*

$$H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty\rangle})[1/2] \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG)[1/2].$$

Here $H_n^G(-; \mathbf{L}_R^{\langle -\infty\rangle})$ is an appropriate G-homology theory with the property that $H_n^G(G/H; \mathbf{L}_R^{\langle -\infty\rangle}) \cong H_n^H(\{\bullet\}; \mathbf{L}_R^{\langle -\infty\rangle}) \cong L_n^{\langle -\infty\rangle}(RH)$ for every subgroup $H \subseteq G$.

0.3.3 The Farrell-Jones Conjecture for $L_*(RG)$

In general the L-groups $L_n^{\langle j\rangle}(RG)$ do depend on the decoration and often the 2-torsion carries sophisticated information and is hard to handle. Recall that as a special case of the sequence (0.2) we obtain an isomorphism

$$K_n(C_r^*(G \times \mathbb{Z})) = K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)).$$

The L-theory analogues is given by the Shaneson splitting [311]

$$L_n^{\langle j\rangle}(R[\mathbb{Z}]) \cong L_{n-1}^{\langle j-1\rangle}(R) \oplus L_n^{\langle j\rangle}(R).$$

Here for the decoration $j = -\infty$ one has to interpret $j-1$ as $-\infty$. Since S^1 is a model for $B\mathbb{Z}$, we get an isomorphisms

$$H_n(B\mathbb{Z}; \mathbf{L}^{\langle j\rangle}(R)) \ \cong \ L_{n-1}^{\langle j\rangle}(R) \oplus L_n^{\langle j\rangle}(R).$$

Therefore the decoration $-\infty$ shows the right homological behavior and is the right candidate for the formulation of an isomorphism conjecture.

The analogues of the sequences (0.1) and (0.2) do hold not hold for $L_*^{\langle j\rangle}(RG)$, certain correction terms, the *UNil-terms* come in, which are independent of the decoration and are always 2-torsion (see Cappell [54], [55]). As in the algebraic K-theory case this leads to the following

Conjecture 0.13. (Farrell-Jones Conjecture for $L_*(RG)$). *Let G be a group. Let R be an associative ring with unit and involution. Then there is for all $n \in \mathbb{Z}$ an isomorphism called* assembly map

$$H_n^G(\underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty\rangle}) \ \xrightarrow{\cong} \ L_n^{\langle -\infty\rangle}(RG).$$

Here $H_n^G(-; \mathbf{L}_R^{\langle -\infty\rangle})$ is an appropriate G-homology theory with the property that $H_n^G(G/H; \mathbf{L}_R^{\langle -\infty\rangle}) \cong H_n^H(\{\bullet\}; \mathbf{L}_R^{\langle -\infty\rangle}) \cong L_n^{\langle -\infty\rangle}(RH)$ for every subgroup $H \subseteq G$ and the assembly map is induced by the map $\underline{\underline{E}}G \to \{\bullet\}$.

After inverting 2 Conjecture 0.13 is equivalent to Conjecture 0.12.

There is an L-theory version of the splitting (0.10).

0.3.4 Applications of the Farrell-Jones Conjecture for $L_*(RG)$

For applications in geometry the groups $L_n^s(\mathbb{Z}G)$ are the interesting ones. The difference between the various decorations is measured by the so called *Rothenberg sequences* and given in terms of the Tate cohomology of $\mathbb{Z}/2$ with coefficients in $\widetilde{K}_n(\mathbb{Z}G)$ for $n \leq 0$ and $\mathrm{Wh}(G)$ with respect to the involution coming from the involution on the group ring $\mathbb{Z}G$. Hence the decorations do not matter if $\widetilde{K}_n(\mathbb{Z}G)$ for $n \leq 0$ and $\mathrm{Wh}(G)$ vanish. This leads in view of Conjecture 0.11 to the following version of Conjecture 0.13 for torsionfree groups

Conjecture 0.14. (Farrell-Jones Conjecture for $L_*(\mathbb{Z}G)$ for torsionfree groups). *Let G be a torsionfree group. Then there is for $n \in \mathbb{Z}$ and all decorations j an isomorphism*

$$H_n(BG; \mathbf{L}^{\langle j\rangle}(\mathbb{Z})) \ \xrightarrow{\cong} \ L_n^{\langle j\rangle}(RG)$$

and the source, target and the map itself are independent of the decoration j.

Here $H_n(-;\mathbf{L}^{\langle j \rangle}(\mathbb{Z}))$ is the homology theory associated to the L-theory spectrum $\mathbf{L}^{\langle -j \rangle}(\mathbb{Z})$ and satisfies $H_n(\{\bullet\};\mathbf{L}^{\langle j \rangle}(\mathbb{Z})) \cong \pi_n\left(\mathbf{L}^{\langle j \rangle}(\mathbb{Z})\right) \cong L_n^{\langle j \rangle}(\mathbb{Z})$.

The L-theoretic assembly map appearing in Conjecture 0.14 has a geometric meaning. It appears in the so called long *exact surgery sequence.* If N is an aspherical closed oriented manifold with fundamental group G, i.e. a closed oriented manifold homotopy equivalent to BG, then G is torsion-free and the source of the assembly map $H_n(BG;\mathbf{L}^s(\mathbb{Z})\langle 1 \rangle) \xrightarrow{\cong} L_n^s(RG)$ consists of bordism classes of normal maps $M \to N$ with N as target and the assembly map sends it to its surgery obstruction. Here $\mathbf{L}^s(\mathbb{Z})\langle 1 \rangle$ is the 1-connected cover $\mathbf{L}^s(\mathbb{Z})\langle 1 \rangle$ of $\mathbf{L}^s(\mathbb{Z})$. This is analogous to the Baum-Connes setting, where the assembly map assigns to an equivariant index problem its index. The third term in the surgery sequence is given by the so called *structure set* of N. It is the set of equivalence classes of orientation preserving homotopy equivalences $f_0 \colon M_0 \to N$ with a closed oriented topological manifold as source and N as target, where $f_0 \colon M_0 \to N$ and $f_1 \colon M_1 \to N$ are equivalent if there is an orientation preserving homeomorphism $g \colon M_0 \to M_1$ such that $f_1 \circ g$ and f_0 are homotopic. Conjecture 0.14 implies that this structure set is trivial provided that the dimension of N is greater or equal to five. Hence Conjecture 0.14 implies in the orientable case for dimensions ≥ 5 the famous

Conjecture 0.15 (Borel Conjecture). *Let M and N be two closed orientable aspherical topological manifolds whose fundamental groups are isomorphic. Then they are homeomorphic and every homotopy equivalence from M to N is homotopic to a homeomorphism.*

The Borel Conjecture is a topological rigidity theorem for aspherical manifolds and analogous to the *Mostow Rigidity Theorem* which says that two hyperbolic closed Riemannian manifolds with isomorphic fundamental groups are isometrically diffeomorphic. The Borel Conjecture is false if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism. The connection to the Borel Conjecture is one of the main features of the Farrell-Jones Conjecture. The Farrell-Jones Conjecture for L-theory 0.13 implies the *Novikov Conjecture* (see Section 2.9).

0.4 Status of the Baum-Connes and the Farrell-Jones Conjecture

At the time of writing no counterexamples to the Baum-Connes Conjecture 0.6 and the Farrell-Jones Conjecture 0.9 and 0.13 are known to the authors. A detailed report on the groups for which these conjectures are known will be given in Chapter 3. For example the Baum-Connes Conjecture 0.6 is known for a rather large class of groups including amenable groups, word-hyperbolic groups, knot groups and one-relator groups but is open for

$SL(n, \mathbb{Z})$ for $n \geq 3$. The class of groups for which the Farrell-Jones Conjecture 0.9 and 0.13 is known is much smaller, it contains fundamental groups of negatively curved closed Riemannian manifolds for $K_*(RG)$ and $L_*^{\langle j \rangle}(\mathbb{Z}G)$. For $K_n(\mathbb{Z}G)$ for $n \leq 1$ the Farrell-Jones Conjecture 0.9 holds for every subgroup G of the group Γ if Γ is a cocompact discrete subgroup of an almost connected Lie group. **Comment 8** (By W.): `Are there other classes or situations which can or shall be mentioned here?`

0.5 Methods of Proof

We will present three very different methods of proof for the various conjectures.

The main methods of proof for the Baum-Connes Conjecture 0.6 are of analytic nature. In particular the *Dirac-Dual Dirac method* is very important. The main input will be *Kasparov's equivariant KK-theory* and the *Kasparov product.* They allow to define maps such as the assembly map appearing in the Baum-Connes Conjecture 0.6 and its inverse by specifying two elements, the *Dirac element* and the *dual Dirac element*, in the equivariant KK-groups and showing that the Kasparov products of these elements is 1. The construction of these elements requires some input from the group G and its geometry. For instance one needs actions of the groups on Hilbert spaces with certain properties. This will be explained in more detail in Chapter 5.

For the Farrell-Jones Conjecture 0.9 and 0.13 *controlled topology and controlled algebra* is one of the main important tools. Here the basic idea is that geometric objects or algebraic objects come with a reference map to a metric space so that one can measure sizes. For instance for a h-cobordism one wants to measure the size of handles. In algebra one considers geometric modules which assign to each point in a metric space a finitely generated $\mathbb{Z}$-module with a basis such that the non-trivial modules are distributed in a locally finite way. A typical transition from geometry to algebra would be to assign to an h-cobordism the cellular chain complex coming from a handle decomposition but taking into account, where the handle sits. In the sense the assembly map can be viewed as a *forget control map* and to prove the Farrell-Jones Conjecture one needs to get control for instance in the sense that the sizes of the handles become arbitrarily small. In order to get control one needs again some geometric input from the groups, for instance to be the fundamental group of a closed negative curved Riemannian manifold. Chapter 6 is devoted to these techniques.

The third method is of pure homotopy theoretic nature and applies to the Farrell-Jones Conjecture 0.9 for K-theory. The first prototype is the *Dennis-trace* map which allows to detect parts of the algebraic K-theory in Hochschild homology. The Dennis trace map is of linear nature. A much more advanced tool to detect the algebraic K-theory is the *cyclotomic trace* which takes values in *topological cyclic homology.* All these constructions are on the level of spectra and cannot be carried out using chain complexes as in the case of the Dennis trace map and Hochschild homology or related theories such as cyclic homology. These methods can be used to get injectivity results but not surjectivity result about the Farrell-Jones assembly map. In order to apply these methods, one does not need geometric input but homotopy theoretic input from the group G such as certain finiteness conditions about the classifying space $\underline{E}G$ for proper G-actions. These methods will be presented in Chapter 7.

0.6 Structural Aspects

The formulation of the Baum-Connes Conjecture 0.6 and the Farrell-Jones Conjecture 0.9 and 0.13 is very similar in the homotopy theoretic picture. It allows a formulation of a kind of *Metaconjecture* (see 2.1) of which both conjectures are special cases and which has also other very interesting specializations. The main idea is the *assembly principle* which leads to assembly maps in a canonical and universal way by asking for the best approximation of a certain functor by an equivariant homology theory. **Comment 9** (By W.): `Do we treat this aspect?` The basic computational tools and techniques apply to both conjectures. In some sense this parallel treatment of the Baum-Connes Conjecture 0.6 and the Farrell-Jones Conjecture 0.9 and 0.13 and of other variants is one of the topics of this book.

However, the geometric interpretations of the assembly maps in terms of indices, surgery obstructions or forget control are quite different and therefore also the methods of proof use very different input.

0.7 Computational Aspects

In general the target $K_n(C_r^*(G))$ of the assembly map appearing in the Baum-Connes Conjecture 0.6 is very hard to compute, whereas the source $K_n^G(\underline{E}G)$ is much more accessible because one can apply standard techniques from algebraic topology such as spectral sequence and equivariant Chern characters to it and there are often nice small geometric models for $\underline{E}G$. For the Farrell-Jones Conjecture 0.9 and 0.13 this holds for the part $H_n^G(\underline{E}G;\mathbf{K}_R)$ or $H_n^G(\underline{E}G;\mathbf{L}_R)$ respectively appearing in the splitting (0.10). The other part $H_n^G(\underline{\underline{E}}G,\underline{E}G;\mathbf{K}_R)$ or $H_n^G(\underline{\underline{E}}G,\underline{E}G;\mathbf{K}_R)$ is harder to handle since it involves Nil- or UNil-terms respectively and the G-CW-complex $\underline{\underline{E}}$ is not proper and in general huge. Most of the known computations of $K_n(C_r^*(G))$, $K_n(RG)$ and $L_n^{\langle -j\rangle}(RG)$ are based on the Baum-Connes Conjecture 0.6 and the Farrell-Jones Conjecture 0.9 and 0.13. A guide for computations will be given in Chapter 8.

0.8 Notations and Conventions

Here is a briefing on our main notational conventions. Details are of course discussed in the text. The columns in the following table contain our notation for: the spectra, their associated homology theory, the right hand side of the corresponding assembly maps, the functor from groupoids to spectra and finally the G-homology theory associated to these spectra valued functors.

$\mathbf{BU}$	$K_n(X)$	$K_n(C_r^*(G))$	$\mathbf{K}^{\mathrm{top}}$	$H_n^G(X;\mathbf{K}^{\mathrm{top}})$
$\mathbf{K}(R)$	$H_n(X;\mathbf{K}(R))$	$K_n(RG)$	$\mathbf{K}_R$	$H_n^G(X;\mathbf{K}_R)$
$\mathbf{L}^{\langle j\rangle}(R)$	$H_n(X;\mathbf{L}^{\langle j\rangle}(R))$	$L_n^{\langle j\rangle}(RG)$	$\mathbf{L}_R^{\langle j\rangle}$	$H_n^G(X;\mathbf{L}_R^{\langle j\rangle})$

We would like to stress that $\mathbf{K}$ without any further decoration will always refer to the non-connective K-theory spectrum. $\mathbf{L}^{\langle j\rangle}$ will always refer to quadratic L-theory with decoration j. For a C^*- or Banach algebra A the symbol $K_n(A)$ has two possible interpretations but we will mean the topological K-theory.

A ring is always an associative ring with unit, and ring homomorphisms are always unital. Modules are left modules. We will always work in the category of compactly generated spaces, compare [322] and [360, I.4]. For our conventions concerning spectra see Section 4.4.4. Spectra are denoted with boldface letters such as $\mathbf{E}$.

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1. Formulation and Relevance of the Conjectures for Torsionfree Groups

1.1 Introduction

We firstly we discuss the Baum-Connes and Farrell-Jones Conjectures in the case of a torsionfree group since their formulation is less technical than in the general case, but already in the torsionfree case there are many interesting and illuminating conclusions. In fact some of the most important consequences of the conjectures, like for example the Borel Conjecture (see Conjecture 1.49) or the Kadison Conjecture (see Conjecture 1.12), refer exclusively to the torsionfree case. On the other hand in the long run the general case, involving groups with torsion, cannot be avoided. The general formulation yields a clearer and more complete picture, and furthermore there are proofs of the conjectures for torsionfree groups, where in intermediate steps of the proof it is essential to have the general formulation available (compare Section 6.4). **Comment 10** (By W.): `This reference has to be adjusted later.`

We have put some effort into dealing with coefficient rings R other than the integers in connection with the Farrell-Jones Conjecture. A topologist may a priori be interested only in the case $R = \mathbb{Z}$ but other cases are interesting for algebraists and also do occur in computations for integral group rings.

The reader may skip this chapter and may pass immediately to Chapter 2.

1.2 The Baum-Connes Conjecture for Torsionfree Groups

We have already motivated in Subsection 0.1.3 the following

Conjecture 1.1. (Baum-Connes Conjecture for Torsionfree Groups). *Let G be a torsionfree group. Then the Baum-Connes assembly map*

$$K_n(BG) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Recall that complex K-homology $K_*(Y)$ is the homology theory associated to the topological (complex) K-theory spectrum $\mathbf{K}^{\mathrm{top}}$ (which is often denoted $\mathbf{BU}$) and could also be written as $K_*(Y) = H_*(Y;\mathbf{K}^{\mathrm{top}})$. The *co*homology theory associated to the spectrum $\mathbf{K}^{\mathrm{top}}$ is the well known complex K-theory defined in terms of complex vector bundles. Complex K-homology is a 2-periodic theory, i.e. $K_n(Y) \cong K_{n+2}(Y)$. Its coefficients are given by $K_n(\{\bullet\}) \cong \mathbb{Z}$ for even n and by $K_n(\{\bullet\}) \cong \{0\}$ for n odd.

Also the *topological K-groups* $K_n(A)$ of a (complex) Banach algebra A are 2-periodic. Whereas $K_0(A)$ coincides with the algebraically defined K_0-group, the other groups $K_n(A)$ take the topology of the Banach algebra A into account, for instance $K_n(A) = \pi_{n-1}(GL(A))$ for $n \geq 1$.

Let $\mathcal{B}(l^2(G))$ denote the bounded $\mathbb{C}$-linear operators on the complex Hilbert space $l^2(G)$ whose orthonormal basis is G. The *reduced complex group C^*-algebra $C_r^*(G)$* is the closure in the norm topology of the image of the regular representation $\mathbb{C}G \to \mathcal{B}(l^2(G))$, which sends an element $u \in \mathbb{C}G$ to the (left) G-equivariant bounded operator $l^2(G) \to l^2(G)$ given by right multiplication with u. In particular one has natural inclusions

$$\mathbb{C}G \subseteq C_r^*(G) \subseteq \mathcal{B}(l^2(G))^G \subseteq \mathcal{B}(l^2(G)).$$

For information about C^*-algebras and their topological K-theory we refer for instance to [43], [79], [90], [170], [206], [249], [305] and [350].

Remark 1.2 (Rational Computation). The right hand side $K_n(C_r^*(G))$ of the Baum-Connes assembly map appearing in the Baum-Connes Conjecture is very hard to compute, also after rationalization and restricting to torsionfree groups. This is much easier for the left hand side what illustrates the computational aspect of the Baum-Connes Conjecture. Namely, there is an *Atiyah-Hirzebruch spectral sequence* which converges to $K_{p+q}(BG)$ and whose E^2-term is $E^2_{p,q} = H_p(BG;K_q(\{\bullet\}))$ [328, Theorem 15.7 on page 341]. Rationally this spectral sequence collapses and the homological Chern character (see [96]) gives an isomorphism for $n \in \mathbb{Z}$

$$\begin{aligned}\mathrm{ch}\colon \bigoplus_{k\in\mathbb{Z}} H_{n-2k}(BG;\mathbb{Q}) \;&=\; \left(\bigoplus_{p+q=n} H_p(BG;K_q(\{\bullet\}))\right) \otimes_{\mathbb{Z}} \mathbb{Q} \\ &\xrightarrow{\cong} K_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q}. \qquad (1.3)\end{aligned}$$

Example 1.4 (Knot groups). Let G be a *knot group*, i.e. the fundamental group of the complement $S^3 - K$ of a (smooth) knot $K \subseteq S^3$. If we take out the interior or a tubular neighborhood of K, we obtain an orientable compact 3-dimensional M which is homotopy equivalent to $S^3 - K$. By the Alexander-Schoenflies Theorem the manifold M is irreducible (see for instance [220, Lemma 4.4 on page 217]). The Sphere Theorem [161, Theorem 4.3 on page 40] implies that M is a model for BG. In particular G is torsionfree. By Alexander Duality [97, VIII.8.15 on page 301] there is a map $f\colon M \to S^1$ which induces an isomorphism on singular homology. We conclude from the Atiyah-Hirzebruch spectral sequence that f induces an isomorphism $K_n(f)\colon K_n(BG) \to K_n(S^1)$ for all $n \in \mathbb{Z}$. The Baum-Connes Conjecture 1.1 is known to be true for G and implies that

$$K_n(C^r_*(G)) \;\cong\; K_n(S^1) \cong \mathbb{Z}$$

holds for all $n \in \mathbb{Z}$. **Comment 11** (By W.): `Holger should check this example.`

Remark 1.5. (Geometric input is needed). The Example 1.4 illustrates how geometric input about G can be used in combination with the Baum-Connes Conjecture to compute $K_n(C^*_r(G))$. Without some specific information about G such a computation can be arbitrarily complicated. This is illustrated by the Kan-Thurston Theorem which implies that for any CW-complex X there exists a discrete group G such that $K_n(X) \cong K_n(BG)$ holds for all $n \in \mathbb{Z}$ and that $\dim(BG) \leq \dim(X)$. A proof of the Kan-Thurston Theorem is given for instance in [38] and [184].

Remark 1.6 (Torsionfree is Necessary). In the case where G is a finite group the reduced group C^*-algebra $C^*_r(G)$ coincides with the complex group ring $\mathbb{C}G$ and $K_n(C^*_r(G))$ coincides with the complex representation ring $R_{\mathbb{C}}(G)$ of G for all even $n \in \mathbb{Z}$. Since the group homology of a finite group vanishes rationally except in dimension 0, Remark 1.2 shows that we need to assume the group to be torsionfree in Conjecture 1.1.

Remark 1.7 (Real version of the Baum-Connes Conjecture). There is an obvious *real version of the Baum-Connes Conjecture*. It says that for a torsionfree group the real assembly map

$$KO_n(BG) \to KO_n(C^*_r(G;\mathbb{R}))$$

is bijective for $n \in \mathbb{Z}$. We will discuss in Subsection 2.12.1 below that this real version of the Baum-Connes Conjecture is implied by the complex version Conjecture 1.1.

Here $KO_n(C^*_r(G;\mathbb{R}))$ is the topological K-theory of the *real reduced group C^*-algebra* $C^*_r(G;\mathbb{R})$. We use KO instead of K as a reminder that we work with real C^*-algebras. The topological real K-theory $KO_*(Y)$ is the homology theory associated to the spectrum $\mathbf{BO}$, whose associated *co*homology theory is given in terms of real vector bundles. Both, topological K-theory of a real C^*-algebra and KO-homology of a space are 8-periodic and $KO_n(\{\bullet\}) = K_n(\mathbb{R})$ is $\mathbb{Z}$, if $n = 0, 4$ (8), is $\mathbb{Z}/2$ if $n = 1, 2$ (8) and is 0 if $n = 3, 5, 6, 7$ (8).

More information about the K-theory of real C^*-algebras can be found in [307].

1.3 Applications of the Baum-Connes Conjecture for Torsionfree Groups

We now discuss some consequences of the Baum-Connes Conjecture for Torsionfree Groups 1.1.

1.3.1 The Trace Conjecture for Torsionfree Groups

The assembly map appearing in the Baum-Connes Conjecture has an interpretation in terms of index theory. This is important for geometric applications. It is of the same significance as the interpretation of the L-theoretic assembly map as the map σ appearing in the exact surgery sequence discussed in Section 1.9.1. We proceed to explain this.

An element $\eta \in K_0(BG)$ can be represented by a pair (M, P^*) consisting of a cocompact free proper smooth G-manifold M with G-invariant Riemannian metric together with an elliptic G-complex P^* of differential operators of order 1 on M (see [34]). To such a pair one can assign an index $\operatorname{ind}_{C_r^*(G)}(M, P^*)$ in $K_0(C_r^*(G))$ (see [244]) which is the image of η under the assembly map $K_0(BG) \to K_0(C_r^*(G))$ appearing in Conjecture 1.1. With this interpretation the surjectivity of the assembly map for a torsionfree group says that any element in $K_0(C_r^*(G))$ can be realized as an index. This allows to apply index theorems to get interesting information.

Here is a prototype of such an argument. The *standard trace*

$$\operatorname{tr}_{C_r^*(G)} \colon C_r^*(G) \to \mathbb{C} \tag{1.8}$$

sends an element $f \in C_r^*(G) \subseteq \mathcal{B}(l^2(G))$ to $\langle f(1), 1\rangle_{l^2(G)}$. Applying the trace to idempotent matrices yields a homomorphism

$$\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}.$$

Let $\operatorname{pr} \colon BG \to \{\bullet\}$ be the projection. For a group G the following diagram commutes

$$\begin{tikzcd} K_0(BG) \arrow[r, "A"] \arrow[d, "K_0(\operatorname{pr})"'] & K_0(C_r^*(G)) \arrow[r, "\operatorname{tr}_{C_r^*(G)}"] & \mathbb{R} \\ K_0(\{\bullet\}) \arrow[r, "\cong"] & K_0(\mathbb{C}) \arrow[r, "\cong", "\operatorname{tr}_{\mathbb{C}}"'] & \mathbb{Z}. \arrow[u, "i"'] \end{tikzcd} \tag{1.9}$$

Here $i \colon \mathbb{Z} \to \mathbb{R}$ is the inclusion and A is the Baum-Connes assembly map. This non-trivial statement follows from Atiyah's L^2-index theorem [13]. Atiyah's theorem says that the L^2-index $\operatorname{tr}_{C_r^*(G)} \circ A(\eta)$ of an element $\eta \in K_0(BG)$, which is represented by a pair (M, P^*), agrees with the ordinary index of $(G\backslash M; G\backslash P^*)$, which is $\operatorname{tr}_{\mathbb{C}} \circ K_0(\operatorname{pr})(\eta) \in \mathbb{Z}$.

The following conjecture is taken from [32, page 21].

Conjecture 1.10. (Trace Conjecture for Torsionfree Groups). *For a torsionfree group G the image of*

$$\mathrm{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$$

consists of the integers.

The commutativity of diagram (1.9) above implies

Corollary 1.11. *The surjectivity of the Baum-Connes assembly map*

$$K_0(BG) \to K_0(C_r^*(G))$$

implies Conjecture 1.10, the Trace Conjecture for Torsionfree Groups.

1.3.2 The Kadison Conjecture and the Idempotent Conjecture

Conjecture 1.12 (Kadison Conjecture). *If G is a torsionfree group, then the only idempotent elements in $C_r^*(G)$ are 0 and 1.*

Lemma 1.13. *The Trace Conjecture for Torsionfree Groups 1.10 implies the Kadison Conjecture 1.12.*

Proof. Assume that $p \in C_r^*(G)$ is an idempotent different from 0 or 1. From p one can construct a non-trivial projection $q \in C_r^*(G)$, i.e. $q^2 = q$, $q^* = q$, with $\mathrm{im}(p) = \mathrm{im}(q)$ **Comment 12** (By W.): `Shall we state the formula?` and hence with $0 < q < 1$. Since the standard trace $\mathrm{tr}_{C_r^*(G)}$ is faithful, we conclude $\mathrm{tr}_{C_r^*(G)}(q) \in \mathbb{R}$ with $0 < \mathrm{tr}_{C_r^*(G)}(q) < 1$. Since $\mathrm{tr}_{C_r^*(G)}(q)$ is by definition the image of the element $[\mathrm{im}(q)] \in K_0(C_r^*(G))$ under $\mathrm{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$, we get a contradiction to the assumption $\mathrm{im}(\mathrm{tr}_{C_r^*(G)}) \subseteq \mathbb{Z}$.

Recall that a ring R is called an *integral domain* if it has no non-trivial zero-divisors, i.e. if $r, s \in R$ satisfy $rs = 0$, then r or s is 0. Obviously the Kadison Conjecture 1.12 implies for $R \subseteq \mathbb{C}$ the following.

Conjecture 1.14 (Idempotent Conjecture). *Let R be an integral domain and let G be a torsionfree group. Then the only idempotents in RG are 0 and 1.*

The statement in the conjecture above is a purely algebraic statement. If $R \subseteq \mathbb{C}$, it is by the arguments above related to questions about operator algebras, and thus methods from operator algebras can be used to attack it

1.4 The Farrell-Jones Conjecture for Lower and Middle K-Theory for Torsionfree Groups

A ring R is always understood to be associative with unit. We denote by $K_n(R)$ *the algebraic* K*-group* of R for $n \in \mathbb{Z}$. In particular $K_0(R)$ is the projective class group, i.e. the Grothendieck group of isomorphism classes of finitely generated projective R-modules with respect to the direct sum, and elements in $K_1(R)$ can be represented by automorphisms of finitely generated projective R-modules. In this section we are mostly interested in the K-groups $K_n(R)$ with $n \leq 1$. For definitions of these groups we refer to [241], [287], [313], [326], [352] for $n = 0, 1$ and to [27] and [289] for $n \leq 1$.

For a ring R and a group G we denote by

$$A_0 = K_0(i)\colon K_0(R) \to K_0(RG) \tag{1.15}$$

the map induced by the natural inclusion $i\colon R \to RG$. Sending $(g, [P]) \in G \times K_0(R)$ to the class of the RG-automorphism

$$R[G] \otimes_R P \to R[G] \otimes_R P, \quad u \otimes x \mapsto ug^{-1} \otimes x$$

defines a map $\Phi\colon G_{\mathrm{ab}} \otimes_{\mathbb{Z}} K_0(R) \to K_1(RG)$, where G_{ab} denotes the abelianized group. We set

$$A_1 = \Phi \oplus K_1(i)\colon G_{\mathrm{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \to K_1(RG). \tag{1.16}$$

We recall the notion of a regular ring. We think of modules as left modules unless stated explicitly differently. Recall that R is *Noetherian* if every submodule of a finitely generated R-module is again finitely generated. It is called *regular* if it is Noetherian and every R-module has a finite-dimensional projective resolution. Any principal ideal domain such as $\mathbb{Z}$ or a field is regular.

Conjecture 1.17. Farrell-Jones Conjecture for Lower and Middle K-Theory and Torsionfree Groups). *Let G be a torsionfree group and let R be a regular ring. Then*

$$K_n(RG) = 0 \quad \textit{for} \quad n \leq -1$$

and the maps

$$\begin{aligned} K_0(R) &\xrightarrow{A_0} K_0(RG) \quad \textit{and} \\ G_{\mathrm{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) &\xrightarrow{A_1} K_1(RG) \end{aligned}$$

are both isomorphisms.

Every regular ring satisfies $K_n(R) = 0$ for $n \leq -1$ [289, 5.3.30 on page 295] and hence the first statement is equivalent to $K_n(i)\colon K_n(R) \to K_n(RG)$ being an isomorphism for $n \leq -1$. In Remark 1.33 below we explain why we impose the regularity assumption on the ring R.

For a regular ring R and a group G we define $\mathrm{Wh}_1^R(G)$ as the cokernel of the map A_1 and $\mathrm{Wh}_0^R(G)$ as the cokernel of the map A_0. In the important case $R = \mathbb{Z}$ the group $\mathrm{Wh}_1^{\mathbb{Z}}(G)$ coincides with the classical *Whitehead group* $\mathrm{Wh}(G)$ which is the quotient of $K_1(\mathbb{Z}G)$ by the subgroup consisting of the classes of the units $\pm g \in (\mathbb{Z}G)^{\mathrm{inv}}$ for $g \in G$. Moreover, for every ring R we define the *reduced algebraic K-groups* $\widetilde{K}_n(R)$ as the cokernel of the natural map $K_n(\mathbb{Z}) \to K_n(R)$. Obviously $\mathrm{Wh}_0^{\mathbb{Z}}(G) = \widetilde{K}_0(\mathbb{Z}G)$.

Lemma 1.18. *The map A_0 is always injective. If R is commutative and the natural map $\mathbb{Z} \to K_0(R)$, $1 \mapsto [R]$ is an isomorphism, then the map A_1 is injective.*

Proof. The augmentation $\epsilon\colon RG \to R$, which maps each group element g to 1, yields a retraction for the inclusion $i\colon R \to RG$ and hence induces a retraction for A_0. If the map $j\colon \mathbb{Z} \to K_0(R),\ 1 \mapsto [R]$ induces an isomorphism and R is commutative, then we have the commutative diagram

$$\begin{tikzcd}
G_{\mathrm{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \arrow[r, "A_1"] \arrow[dd, "\cong"', "\overline{j} \oplus \mathrm{id}_{K_1(R)}"] & K_1(RG) \arrow[d] \\
& K_1(RG_{\mathrm{ab}}) \arrow[d, "(\det, K_1(\epsilon))"] \\
G_{\mathrm{ab}} \oplus K_1(R) \arrow[r] & RG_{\mathrm{ab}}^{\mathrm{inv}} \oplus K_1(R),
\end{tikzcd}$$

The upper vertical arrow on the right is induced from the map $G \to G_{\mathrm{ab}}$ to the abelianization. The isomorphism $\overline{j}$ is induced by the isomorphism j above in the obvious way. Since RG_{ab} is a commutative ring we have the determinant $\det\colon K_1(RG_{\mathrm{ab}}) \to (RG_{\mathrm{ab}})^{\mathrm{inv}}$. The lower horizontal arrow is induced from the obvious inclusion of G_{ab} into the invertible elements of the group ring RG_{ab} and in particular injective.

In the special case $R = \mathbb{Z}$ Conjecture 1.17 above is equivalent to the following conjecture.

Conjecture 1.19. (Vanishing of Lower and Middle K-Theory for Torsionfree Groups and Integral Coefficients). *For every torsionfree group G we have*

$$K_n(\mathbb{Z}G) = 0 \textit{ for } n \leq -1, \quad \widetilde{K}_0(\mathbb{Z}G) = 0 \quad \textit{and} \quad \mathrm{Wh}(G) = 0.$$

Remark 1.20 (Torsionfree is Necessary). In general $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ do not vanish for finite groups. For example $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/23]) \cong \mathbb{Z}/3$ [241, page 29, 30] and $\mathrm{Wh}(\mathbb{Z}/p) \cong \mathbb{Z}^{\frac{p-3}{2}}$ for p an odd prime [78, 11.5 on page 45]. This shows that the assumption that G is torsionfree is crucial in the formulation of Conjecture 1.17 above.

For more information on $\widetilde{K}_0(\mathbb{Z}G)$ and Whitehead groups of finite groups see for instance [27, Chapter XI], [88], [240], [252], [253] and [326].

1.5 Applications of the Farrell-Jones Conjecture for Lower and Middle *K*-Theory

1.5.1 The s-Cobordism Theorem and the Poincaré Conjecture

The Whitehead group $\mathrm{Wh}(G)$ plays a key role if one studies manifolds because of the so called s-Cobordism Theorem. In order to state it, we explain the notion of an h-cobordism.

Manifold always means smooth manifold unless it is explicitly stated differently. We say that W or more precisely $(W; M^-, f^-, M^+, f^+)$ is an n-dimensional *cobordism* over M^- if W is a compact n-dimensional manifold together with the following: a disjoint decomposition of its boundary ∂W into two closed $(n-1)$-dimensional manifolds $\partial^- W$ and $\partial^+ W$, two closed $(n-1)$-dimensional manifolds M^- and M^+ and diffeomorphisms $f^-\colon M^- \to \partial^- W$ and $f^+\colon M^+ \to \partial^+ W$. The cobordism is called an *h-cobordism* if the inclusions $i^-\colon \partial^- W \to W$ and $i^+\colon \partial^+ W \to W$ are both homotopy equivalences. Two cobordisms $(W; M^-, f^-, M^+, f^+)$ and $(W'; M^-, f'^-, M'^+, f'^+)$ over M^- are *diffeomorphic relative* M^- if there is a diffeomorphism $F\colon W \to W'$ with $F \circ f^- = f'^-$. We call a cobordism over M^- *trivial*, if it is diffeomorphic relative M^- to the trivial h-cobordism given by the cylinder $M^- \times [0,1]$ together with the obvious inclusions of $M^- \times \{0\}$ and $M^- \times \{1\}$. Note that "trivial" implies in particular that M^- and M^+ are diffeomorphic.

The question whether a given h-cobordism is trivial is decided by the Whitehead torsion $\tau(W; M^-) \in \mathrm{Wh}(G)$ where $G = \pi_1(M^-)$. For the details of the definition of $\tau(W; M^-)$ the reader should consult [78], [240] or Chapter 2 in [218]. Compare also [287].

Theorem 1.21 (s-Cobordism Theorem). *Let M^- be a closed connected oriented manifold of dimension $n \geq 5$ with fundamental group $G = \pi_1(M^-)$. Then*

(1) An h-cobordism W over M^- is trivial if and only if its Whitehead torsion $\tau(W, M^-) \in \mathrm{Wh}(G)$ vanishes;
(2) Assigning to an h-cobordism over M^- its Whitehead torsion yields a bijection from the diffeomorphism classes relative M^- of h-cobordisms over M^- to the Whitehead group $\mathrm{Wh}(G)$.

The s-Cobordism Theorem is due to Barden, Mazur and Stallings. There are also topological and PL-versions. Proofs can be found for instance in [191], [194, Essay III], [218] and [295, page 87-90].

The s-Cobordism Theorem tells us that the vanishing of the Whitehead group (as predicted in Conjecture 1.19 for torsionfree groups) has the following geometric interpretation.

Corollary 1.22. *For a finitely presented group G the vanishing of the Whitehead group* $\mathrm{Wh}(G)$ *is equivalent to the statement that each h-cobordism over a closed connected manifold M^- of dimension* $\dim(M^-) \geq 5$ *with fundamental group $\pi_1(M^-) \cong G$ is trivial.*

Knowing that all h-cobordisms over a given manifold are trivial is a strong and useful statement. In order to illustrate this we would like to discuss the case where the fundamental group is trivial.

Since the ring $\mathbb{Z}$ has a Gaussian algorithm, the determinant induces an isomorphism $K_1(\mathbb{Z}) \xrightarrow{\cong} \{\pm 1\}$ (compare [289, Theorem 2.3.2]) and the Whitehead group $\mathrm{Wh}(\{1\})$ of the trivial group vanishes. Hence any h-cobordism over a simply connected closed manifold of dimension ≥ 5 is trivial. As a consequence one obtains the Poincaré Conjecture for high dimensional manifolds.

Theorem 1.23 (Poincaré Conjecture). *Suppose $n \geq 5$. If the closed manifold M is homotopy equivalent to the sphere S^n, then it is homeomorphic to S^n.*

Proof. We only give the proof for $\dim(M) \geq 6$. Let $f\colon M \to S^n$ be a homotopy equivalence. Let $D^n_- \subset M$ and $D^n_+ \subset M$ be two disjoint embedded disks. Let W be the complement of the interior of the two disks in M. Then W turns out to be a simply connected h-cobordism over ∂D^n_-. Hence we can find a diffeomorphism

$$F\colon (\partial D^n_- \times [0,1]; \partial D^n_- \times \{0\}, \partial D^n_- \times \{1\}) \to (W; \partial D^n_-, \partial D^n_+)$$

which is the identity on $\partial D^n_- = \partial D^n_- \times \{0\}$ and induces some (unknown) diffeomorphism $f^+\colon \partial D^n_- \times \{1\} \to \partial D^n_+$. By the *Alexander trick* one can extend $f^+\colon \partial D^n_- = \partial D^n_- \times \{1\} \to \partial D^n_+$ to a homeomorphism $\overline{f^+}\colon D^n_- \to D^n_+$. Namely, any homeomorphism $f\colon S^{n-1} \to S^{n-1}$ extends to a homeomorphism $\overline{f}\colon D^n \to D^n$ by sending $t \cdot x$ for $t \in [0,1]$ and $x \in S^{n-1}$ to $t \cdot f(x)$. Now define a homeomorphism $h\colon D^n_- \times \{0\} \cup_{i_-} \partial D^n_- \times [0,1] \cup_{i_+} D^n_- \times \{1\} \to M$ for the canonical inclusions $i_k\colon \partial D^n_- \times \{k\} \to \partial D^n_- \times [0,1]$ for $k = 0, 1$ by $h|_{D^n_- \times \{0\}} = \mathrm{id}$, $h|_{\partial D^n_- \times [0,1]} = F$ and $h|_{D^n_- \times \{1\}} = \overline{f^+}$. Since the source of h is obviously homeomorphic to S^n, Theorem 1.23 follows.

For $n \leq 2$ the Poincaré Conjecture (see Theorem 1.23) follows from the well-known classification of 1 and 2-dimensional closed manifolds. For

$n > 4$ it was proved by Smale and Newman in the sixties of the last century. **Comment 13** (By W.): `Add references. Maybe there is none for Newman but he was involved.` Freedman [146] solved the case in $n = 4$ in 1982. Recently Perelman announced a proof for $n = 3$, but this proof has still to be checked thoroughly by the experts. **Comment 14** (By W.): `Later adjust this sentence depending on what the status of the proof by Perelman is.` It is essential in its formulation that one concludes M to be homeomorphic (as opposed to diffeomorphic) to S^n. The Alexander trick does not work differentiably. There are *exotic spheres*, i.e. smooth manifolds which are homeomorphic but not diffeomorphic to S^n [238].

More information about the Poincaré Conjecture, the Whitehead torsion and the s-Cobordism Theorem can be found for instance in [56], [78], [98], [146], [147], [156], [191], [218], [239], [240], [242], [287] and [295].

1.5.2 The Finiteness Obstruction

We now discuss the geometric relevance of $\widetilde{K}_0(\mathbb{Z}G)$.

Let X be a CW-complex. It is called *finite* if it consists of finitely many cells, or, equivalently, if it is compact. It is called *finitely dominated* if there is a finite CW-complex Y together with maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity on X. The fundamental group of a finitely dominated CW-complex is always finitely presented.

While studying existence problems for spaces with prescribed properties (like for example group actions), it happens occasionally that it is relatively easy to construct a finitely dominated CW-complex within a given homotopy type, whereas it is not at all clear whether one can also find a homotopy equivalent *finite* CW-complex. *Wall's finiteness obstruction*, a certain obstruction element $\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}\pi_1(X))$, decides the question.

Theorem 1.24 (Properties of the Finiteness Obstruction). *Let X be a finitely dominated CW-complex with fundamental group $\pi = \pi_1(X)$.*

(1) The space X is homotopy equivalent to a finite CW-complex if and only if $\widetilde{o}(X) = 0$ in $\widetilde{K}_0(\mathbb{Z}\pi)$;
(2) Every element in $\widetilde{K}_0(\mathbb{Z}G)$ can be realized as the finiteness obstruction $\widetilde{o}(X)$ of a finitely dominated CW-complex X with $G = \pi_1(X)$, provided that G is finitely presented;
(3) Let Z be a space such that $G = \pi_1(Z)$ is finitely presented. Then there is a bijection between $\widetilde{K}_0(\mathbb{Z}G)$ and the set of equivalence classes of maps $f\colon X \to Z$ with X finitely dominated under the equivalence relation explained below.

The equivalence relation in (iii) is defined as follows: Two maps $f\colon X \to Z$ and $f'\colon X' \to Z$ with X and X' finitely dominated are equivalent if there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[drr, "f"'] & X_1 \arrow[r, "h"] \arrow[dr, "f_1"] & X_2 \arrow[d, "f_2"] & X_3 \arrow[l, "h'"'] \arrow[dl, "f_3"'] & X' \arrow[l, "j'"'] \arrow[dll, "f'"] \\
 & & Z & &
\end{tikzcd},$$

where h and h' are homotopy equivalences and j and j' are inclusions of subcomplexes for which X_1, respectively X_3, is obtained from X, respectively X', by attaching a finite number of cells.

The vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ as predicted in Conjecture 1.19 for torsionfree groups hence has the following interpretation.

Corollary 1.25. *For a finitely presented group G the vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ is equivalent to the statement that any finitely dominated CW-complex X with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.*

For more information about the finiteness obstruction we refer for instance to [140], [141], [214], [245], [278], [287], [336], [346] and [347].

1.5.3 Negative K-Groups and Bounded h-Cobordisms

One possible geometric interpretation of negative K-groups is in terms of bounded h-cobordisms. Another interpretation in terms of pseudoisotopies will be explained in Subsection 1.7.2 below.

We consider *manifolds W parametrized over* $\mathbb{R}^k$, i.e. manifolds which are equipped with a surjective proper map $p\colon W \to \mathbb{R}^k$. We will always assume that the fundamental group(oid) is bounded, compare [260, Definition 1.3]. A map $f\colon W \to W'$ between two manifolds parametrized over $\mathbb{R}^k$ is *bounded* if $\{p' \circ f(x) - p(x) \mid x \in W\}$ is a bounded subset of $\mathbb{R}^k$.

A *bounded cobordism* $(W; M^-, f^-, M^+, f^+)$ is defined just as in Subsection 1.5.1 but compact manifolds are replaced by manifolds parametrized over $\mathbb{R}^k$ and the parametrization for $M^\pm$ is given by $p_W \circ f^\pm$. If we assume that the inclusions $i^\pm\colon \partial^\pm W \to W$ are homotopy equivalences, then there exist deformations $r^\pm\colon W \times I \to W$, $(x,t) \mapsto r_t^\pm(x)$ such that $r_0^\pm = \mathrm{id}_W$ and $r_1^\pm(W) \subset \partial^\pm W$.

A bounded cobordism is called a *bounded h-cobordism* if the inclusions $i^\pm$ are homotopy equivalences and additionally the deformations can be chosen such that the two sets

$$S^\pm = \{p_W \circ r_t^\pm(x) - p_W \circ r_1^\pm(x) \mid x \in W, t \in [0,1]\}$$

are bounded subsets of $\mathbb{R}^k$.

The following theorem (compare [260] and [357, Appendix]) contains the s-Cobordism Theorem 1.21 as a special case, gives another interpretation of elements in $\widetilde{K}_0(\mathbb{Z}\pi)$ and explains one aspect of the geometric relevance of negative K-groups.

Theorem 1.26 (Bounded h-Cobordism Theorem). *Suppose that M^- is parametrized over $\mathbb{R}^k$ and satisfies $\dim M^- \geq 5$. Let π be its fundamental group(oid). Equivalence classes of bounded h-cobordisms over M^- modulo bounded diffeomorphism relative M^- correspond bijectively to elements in $\kappa_{1-k}(\pi)$, where*

$$\kappa_{1-k}(\pi) = \begin{cases} \mathrm{Wh}(\pi) & \text{if } k = 0, \\ \widetilde{K}_0(\mathbb{Z}\pi) & \text{if } k = 1, \\ K_{1-k}(\mathbb{Z}\pi) & \text{if } k \geq 2. \end{cases}$$

The vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ as predicted in Conjecture 1.19 for torsionfree groups hence has the following interpretation.

Corollary 1.27. *For a finitely presented group G the vanishing of $\widetilde{K}_{1-k}(\mathbb{Z}G)$ for $k \geq 1$ is equivalent to the statement that each bounded h-cobordism over a closed connected manifold M^- of dimension $\dim(M^-) \geq 5$ parametrized over $\mathbb{R}^k$ with fundamental group $\pi_1(M^-) \cong G$ is trivial.*

More information about negative K-groups can be found for instance in [9], [27], [63], [64], [128], [232], [259], [260], [273], [280], [289] and [357, Appendix].

1.6 The Farrell-Jones Conjecture for *K*-Theory for Torsionfree Groups

So far we only considered the K-theory groups in dimensions ≤ 1. We now want to explain how Conjecture 1.17 generalizes to higher algebraic K-theory. For the definition of higher algebraic K-theory groups and the (connective) K-theory spectrum see [41], [58], [174], [270], [289], [319], [344] and [352]. We would like to stress that for us $\mathbf{K}(R)$ will always denote the *non-connective algebraic K-theory spectrum* for which $K_n(R) = \pi_n(\mathbf{K}(R))$ holds for all $n \in \mathbb{Z}$. For its definition see [58], [212] and [258].

The Farrell-Jones Conjecture for algebraic K-theory reduces for a torsionfree group to the following conjecture.

Conjecture 1.28. Farrell-Jones Conjecture for K-Theory and Torsionfree Groups). *Let G be a torsionfree group. Let R be a regular ring. Then the assembly map*

$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

Here $H_n(-; \mathbf{K}(R))$ denotes the homology theory which is associated to the spectrum $\mathbf{K}(R)$. It has the property that $H_n(\{\bullet\}; \mathbf{K}(R))$ is $K_n(R)$ for $n \in \mathbb{Z}$, where here and elsewhere $\{\bullet\}$ denotes the space consisting of one point. The space BG is the *classifying space of the group* G, which up to homotopy is characterized by the property that it is a CW-complex with $\pi_1(BG) \cong G$ whose universal covering is contractible. The technical details of the construction of $H_n(-;\mathbf{K}(R))$ and the assembly map will be explained in a more general setting later. **Comment 15** (By W.): `Add reference.`

The point of Conjecture 1.28 is that on the right-hand side of the assembly map we have the group $K_n(RG)$ we are interested in, whereas the left-hand side is a homology theory and hence much easier to compute. For every homology theory associated to a spectrum we have the Atiyah-Hirzebruch spectral sequence, which in our case has $E^2_{p,q} = H_p(BG; K_q(R))$ and converges to $H_{p+q}(BG; \mathbf{K}(R))$.

If R is regular, then the negative K-groups of R vanish and the spectral sequence lives in the first quadrant. Evaluating the spectral sequence for $n = p + q \leq 1$ shows that Conjecture 1.28 above implies Conjecture 1.17.

Remark 1.29. (Rational Computation). Rationally an Atiyah-Hirzebruch spectral sequence collapses always and the homological Chern character gives an isomorphism

$$\mathrm{ch}\colon \bigoplus_{p+q=n} H_p(BG;\mathbb{Q}) \otimes_{\mathbb{Q}} (K_q(R) \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\cong} H_n(BG;\mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The Atiyah-Hirzebruch spectral sequence and the Chern character will be discussed in a much more general setting in Chapter 8.

Remark 1.30 (Separation of Variables). We see that the left-hand side of the isomorphism in the previous remark consists of a group homology part and a part which is the rationalized K-theory of R. (Something similar happens before we rationalize at the level of spectra: The left hand side of Conjecture 1.28 can be interpreted as the homotopy groups of the spectrum $BG_+ \wedge \mathbf{K}(R)$.) So essentially Conjecture 1.28 predicts that the K-theory of RG is built up out of two independent parts: the K-theory of R and the group homology of G. We call this principle *separation of variables.* This principle also applies to other theories such as algebraic L-theory or topological K-theory. See also Remark 8.13.

Remark 1.31. (K-Theory of the Coefficients). Note that Conjecture 1.28 can only help us to explicitly compute the K-groups of RG in cases where we know enough about the K-groups of R. We obtain no new information about the K-theory of R itself. However, already for very simple rings the computation of their algebraic K-theory groups is an extremely hard problem.

It is known that the groups $K_n(\mathbb{Z})$ are finitely generated abelian groups [269]. Due to Borel [45] we know that

$$K_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } n = 0; \\ \mathbb{Q} & \text{if } n = 4k+1 \text{ with } k \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbb{Z}$ is regular we know that $K_n(\mathbb{Z})$ vanishes for $n \leq -1$. Moreover, $K_0(\mathbb{Z}) \cong \mathbb{Z}$ and $K_1(\mathbb{Z}) \cong \{\pm 1\}$, where the isomorphisms are given by the rank and the determinant. One also knows that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$, $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ [207] and $K_4(\mathbb{Z}) \cong 0$ [285].

Finite fields belong to the few rings where one has a complete and explicit knowledge of all K-theory groups [268]. We refer the reader for example to [195], [247], [286], [351] and Soulé's article in [211] for more information about the algebraic K-theory of the integers or more generally of rings of integers in number fields.

Because of Borel's calculation the left hand side of the isomorphism described in Remark 1.29 specializes for $R = \mathbb{Z}$ to

$$H_n(BG;\mathbb{Q}) \oplus \bigoplus_{k=1}^{\infty} H_{n-(4k+1)}(BG;\mathbb{Q}) \tag{1.32}$$

and Conjecture 1.28 predicts that this group is isomorphic to $K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Next we discuss the case where the group G is infinite cyclic.

Remark 1.33 (Bass-Heller-Swan Decomposition). The so called *Bass-Heller-Swan-decomposition*, also known as the *Fundamental Theorem of algebraic K-theory*, computes the algebraic K-groups of $R[\mathbb{Z}]$ in terms of the algebraic K-groups and Nil-groups of R:

$$K_n(R[\mathbb{Z}]) \;\cong\; K_{n-1}(R) \oplus K_n(R) \oplus NK_n(R) \oplus NK_n(R).$$

Here the group $NK_n(R)$ is defined as the cokernel of the split injection $K_n(R) \to K_n(R[t])$. It can be identified with the cokernel of the split injection $K_{n-1}(R) \to K_{n-1}(\mathcal{N}il(R))$. Here $K_n(\mathcal{N}il(R))$ denotes the K-theory of the exact category of nilpotent endomorphisms of finitely generated projective R-modules. For negative n it is defined with the help of Bass' notion of a contracting functor [27] (see also [63]). The groups are known as *Nil-groups* and often denoted $\mathrm{Nil}_{n-1}(R)$.

For proofs of these facts and more information the reader should consult [27, Chapter XII], [30], [150, Theorem on page 236], [270, Corollary in §6 on page 38], [289, Theorems 3.3.3 and 5.3.30], [319, Theorem 9.8] and [327, Theorem 10.1].

If we iterate and use $R[\mathbb{Z}^n] = R[\mathbb{Z}^{n-1}][\mathbb{Z}]$ we see that a computation of $K_n(RG)$ must in general take into account information about $K_i(R)$ for all $i \leq n$. In particular we see that it is important to formulate Conjecture 1.28 with the non-connective K-theory spectrum.

Since S^1 is a model for $B\mathbb{Z}$, we get an isomorphism

$$H_n(B\mathbb{Z};\mathbf{K}(R)) \;\cong\; K_{n-1}(R) \oplus K_n(R)$$

and hence Conjecture 1.28 predicts

$$K_n(R[\mathbb{Z}]) \;\cong\; K_{n-1}(R) \oplus K_n(R).$$

This explains why in the formulation of Conjecture 1.28 the condition that R is regular appears. It guarantees that $NK_n(R) = 0$ [289, Theorem 5.3.30 on page 295]. There are weaker conditions which imply that $NK_n(R) = 0$ but "regular" has the advantage that R regular implies that $R[t]$ and $R[\mathbb{Z}] = R[t^{\pm 1}]$ are again regular, compare the discussion in Section 2 in [28].

The Nil-terms $NK_n(R)$ seem to be hard to compute. For instance $NK_1(R)$ either vanishes or is infinitely generated as an abelian group [110]. In Subsection 2.13.3 we will discuss the Isomorphism Conjecture for NK-groups. For more information about Nil-groups see for instance [81], [83], [162], [353] and [354].

In Example 2.14 we will explain what the Farrell-Jones Conjecture predicts for $K_*(RG)$ if G is a torsionfree word-hyperbolic group and R is a (not necessary regular) ring.

1.7 Applications of the Farrell-Jones Conjecture for *K*-Theory for Torsionfree Groups

1.7.1 The Relation to Pseudoisotopy Theory

Let I denote the unit interval $[0,1]$. A topological *pseudoisotopy* of a compact manifold M is a homeomorphism $h\colon M \times I \to M \times I$, which restricted to $M \times \{0\} \cup \partial M \times I$ is the obvious inclusion. The space $P(M)$ of pseudoisotopies is the (simplicial) group of all such homeomorphisms. Pseudoisotopies play an important role if one tries to understand the homotopy type of the space $\operatorname{Top}(M)$ of self-homeomorphisms of a manifold M. We will see below in Subsection 1.9.2 how the results about pseudoisotopies discussed in this section combined with surgery theory lead to quite explicit results about the homotopy groups of $\operatorname{Top}(M)$.

There is a stabilization map $P(M) \to P(M \times I)$ given by crossing a pseudoisotopy with the identity on the interval I and the stable pseudoisotopy space is defined as $\mathcal{P}(M) = \operatorname{colim}_k P(M \times I^k)$. In fact $\mathcal{P}(-)$ can be extended to a functor on all spaces [159]. The natural inclusion $P(M) \to \mathcal{P}(M)$ induces an isomorphism on the i-th homotopy group if the dimension of M is large compared to i, see [49] and [173].

Waldhausen [343], [344] defines the algebraic K-theory of spaces functor $\mathbf{A}(X)$ and the functor $\mathbf{Wh}^{PL}(X)$ from spaces to spectra (or infinite loop spaces) and a fibration sequence

$$X_+ \wedge \mathbf{A}(\{\bullet\}) \to \mathbf{A}(X) \to \mathbf{Wh}^{PL}(X).$$

Here $X_+ \wedge \mathbf{A}(\{\bullet\}) \to \mathbf{A}(X)$ is an assembly map, which can be compared to the algebraic K-theory assembly map that appears in Conjecture 1.28 via a commutative diagram

$$\begin{array}{ccc} H_n(X;\mathbf{A}(\{\bullet\})) & \longrightarrow & \pi_n(\mathbf{A}(X)) \\ \downarrow & & \downarrow \\ H_n(B\pi_1(X);\mathbf{K}(\mathbb{Z})) & \longrightarrow & K_n(\mathbb{Z}\pi_1(X)). \end{array}$$

In the case where $X \simeq BG$ is aspherical the vertical maps induce isomorphisms after rationalization for $n \geq 1$, compare [343, Proposition 2.2]. Since $\Omega^2 \operatorname{Wh}^{PL}(X) \simeq \mathcal{P}(X)$ (a guided tour through the literature concerning this and related results can be found in [104, Section 9]), Conjecture 1.28 implies rational vanishing results for the groups $\pi_n(\mathcal{P}(M))$ if M is an aspherical manifold. Compare also Remark 2.82.

Corollary 1.34. *Suppose M is a closed aspherical manifold and Conjecture 1.28 holds for $R = \mathbb{Z}$ and $G = \pi_1(M)$, then for all $n \geq 0$*

$$\pi_n(\mathcal{P}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

Similarly as above one defines smooth pseudoisotopies and the space of stable smooth pseudoisotopies $\mathcal{P}^{\mathrm{Diff}}(M)$. There is also a smooth version of the Whitehead space $\operatorname{Wh}^{\mathrm{Diff}}(X)$ and $\Omega^2 \operatorname{Wh}^{\mathrm{Diff}}(M) \simeq \mathcal{P}^{\mathrm{Diff}}(M)$. Again there is a close relation to A-theory via the natural splitting $\mathbf{A}(X) \simeq \Sigma^\infty(X_+) \vee \mathbf{Wh}^{\mathrm{Diff}}(X)$, see [345]. Here $\Sigma^\infty(X_+)$ denotes the suspension spectrum associated to X_+. Using this one can split off an assembly map $H_n(X;\mathbf{Wh}^{\mathrm{Diff}}(\{\bullet\})) \to \pi_n(\mathbf{Wh}^{\mathrm{Diff}}(X))$ from the A-theory assembly map. Since for every space $\pi_n(\Sigma^\infty(X_+)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(X;\mathbb{Q})$ Conjecture 1.28 combined with the rational computation in (1.32) yields the following result.

Corollary 1.35. *Suppose M is a closed aspherical manifold and Conjecture 1.28 holds for $R = \mathbb{Z}$ and $G = \pi_1(M)$. Then for $n \geq 0$ we have*

$$\pi_n(\mathcal{P}^{\mathrm{Diff}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{k=1}^{\infty} H_{n-4k+1}(M;\mathbb{Q}).$$

1.7.2 Negative *K*-Groups and Bounded Pseudoisotopies

We briefly explain a further geometric interpretation of negative K-groups, which parallels the discussion of bounded h-cobordisms in Subsection 1.5.3.

Let $p\colon M \times \mathbb{R}^k \to \mathbb{R}^k$ denote the natural projection. The space $P_b(M;\mathbb{R}^k)$ of bounded pseudoisotopies is the space of all self-homeomorphisms $h\colon M \times \mathbb{R}^k \times I \to M \times \mathbb{R}^k \times I$ such that restricted to $M \times \mathbb{R}^k \times \{0\}$ the map h is the inclusion and such that h is bounded, i.e. the set $\{p \circ h(y) - p(y) \mid y \in M \times \mathbb{R}^k \times I\}$ is a bounded subset of $\mathbb{R}^k$. There is again a stabilization map $P_b(M;\mathbb{R}^k) \to P_b(M \times I;\mathbb{R}^k)$ and a stable bounded pseudoisotopy space $\mathcal{P}_b(M;\mathbb{R}^k) = \operatorname{colim}_j P_b(M \times I^j;\mathbb{R}^k)$. There is a homotopy equivalence $\mathcal{P}_b(M;\mathbb{R}^k) \to \Omega\mathcal{P}_b(M;\mathbb{R}^{k+1})$ [159, Appendix II] and hence the sequence of spaces $\mathcal{P}_b(M;\mathbb{R}^k)$ for $k = 0, 1, \ldots$ is an Ω-spectrum $\mathbf{P}(M)$. Analogously one defines the smooth bounded pseudoisotopies $\mathcal{P}_b^{\mathrm{diff}}(M;\mathbb{R}^k)$ and an Ω-spectrum $\mathbf{P}^{\mathrm{diff}}(M)$. The negative homotopy groups of these spectra have an interpretation in terms of low and negative dimensional K-groups. In terms of unstable homotopy groups this is explained in the following theorem which is closely related to Theorem 1.26 about bounded h-cobordisms.

Theorem 1.36 (Negative Homotopy Groups of Pseudoisotopies). *Let $G = \pi_1(M)$. Suppose n and k are such that $n + k \geq 0$, then for $k \geq 1$ there are isomorphisms*

$$\pi_{n+k}(\mathcal{P}_b(M;\mathbb{R}^k)) = \begin{cases} \mathrm{Wh}(G) & \text{if } n = -1, \\ \widetilde{K}_0(\mathbb{Z}G) & \text{if } n = -2, \\ K_{n+2}(\mathbb{Z}G) & \text{if } n < -2 \end{cases}$$

The same result holds in the smooth case.

Note that Conjecture 1.28 predicts that these groups vanish if G is torsionfree. The result above is due to Anderson and Hsiang [9] and is also discussed in [357, Appendix].

1.8 The Farrell-Jones Conjecture for L-Theory for Torsionfree Groups

We now move on to the L-theoretic version of the Farrell-Jones Conjecture. We will still stick to the case where the group is torsionfree. The conjecture is obtained by replacing K-theory and the K-theory spectrum in Conjecture 1.28 by 4-periodic L-theory and the L-theory spectrum $\mathbf{L}^{\langle -\infty \rangle}(R)$. Explanations will follow below.

Conjecture 1.37. Farrell-Jones Conjecture for L-Theory and Torsionfree Groups). *Let G be a torsionfree group and let R be a ring with involution. Then the assembly map*

$$H_n(BG;\mathbf{L}^{\langle -\infty \rangle}(R)) \to L_n^{\langle -\infty \rangle}(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

To a ring with involution one can associate (decorated) symmetric or quadratic algebraic L-groups, compare [50], [51], [277], [280] and [362]. We will exclusively deal with the quadratic algebraic L-groups and denote them by $L_n^{\langle j \rangle}(R)$. Here $n \in \mathbb{Z}$ and $j \in \{-\infty\} \amalg \{j \in \mathbb{Z} \mid j \leq 2\}$ is the so called *decoration*. The decorations $j = 0, 1$ correspond to the decorations p, h and $j = 2$ is related to the decoration s appearing in the literature. Decorations will be discussed in Remark 1.39 below. The L-groups $L_n^{\langle j \rangle}(R)$ are 4-periodic, i.e. $L_n^{\langle j \rangle}(R) \cong L_{n+4}^{\langle j \rangle}(R)$ for $n \in \mathbb{Z}$.

If we are given an involution $r \mapsto \overline{r}$ on a ring R, we will always equip RG with the involution that extends the given one and satisfies $\overline{g} = g^{-1}$. On $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ we always use the trivial involution and on $\mathbb{C}$ the complex conjugation.

One can construct an *L-theory spectrum* $\mathbf{L}^{\langle j\rangle}(R)$ such that $\pi_n(\mathbf{L}^{\langle j\rangle}(R)) = L_n^{\langle j\rangle}(R)$, compare [279, § 13]. Above and in the sequel $H_n(-;\mathbf{L}^{\langle j\rangle}(R))$ denotes the homology theory which is associated to this spectrum. In particular we have $H_n(\{\bullet\};\mathbf{L}^{\langle j\rangle}(R)) = L_n^{\langle j\rangle}(R)$. We postpone the discussion of the assembly map to where we will construct it in greater generality. **Comment 16** (By W.): `Add reference or drop this sentence.`

Remark 1.38 (The Coefficients in the L-Theory Case). In contrast to K-theory (compare Remark 1.31) the L-theory of the most interesting coefficient ring $R = \mathbb{Z}$ is well known. The groups $L_n^{\langle j\rangle}(\mathbb{Z})$ for fixed n and varying $j \in \{-\infty\} \amalg \{j \in \mathbb{Z} \mid j \leq 2\}$ are all naturally isomorphic (compare Proposition 1.42 below) and we have $L_0^{\langle j\rangle}(\mathbb{Z}) \cong \mathbb{Z}$ and $L_2^{\langle j\rangle}(\mathbb{Z}) \cong \mathbb{Z}/2$, where the isomorphisms are given by the signature divided by 8 and the Arf invariant, and $L_1^{\langle j\rangle}(\mathbb{Z}) = L_3^{\langle j\rangle}(\mathbb{Z}) = 0$, see [47, Chapter III], [277, Proposition 4.3.1 on page 419].

Remark 1.39 (Decorations). L-groups are designed as obstruction groups for surgery problems. The decoration reflects what kind of surgery problem one is interested in. All L-groups can be described as cobordism classes of suitable quadratic Poincaré chain complexes. If one works with chain complexes of finitely generated free based R-modules and requires that the torsion of the Poincaré chain homotopy equivalence vanishes in $\widetilde{K}_1(R)$, then the corresponding L-groups are denoted $L_n^{\langle 2\rangle}(R)$. If one drops the torsion condition, one obtains $L_n^{\langle 1\rangle}(R)$, which is usually denoted $L^h(R)$. If one works with finitely generated projective modules, one obtains $L^{\langle 0\rangle}(R)$, which is also known as $L^p(R)$.

The L-groups with negative decorations can be defined inductively via the Shaneson splitting, compare Remark 1.47 below. Assuming that the L-groups with decorations j have already been defined one sets

$$L_{n-1}^{<j-1>}(R) = \operatorname{coker}(L_n^{<j>}(R) \to L_n^{<j>}(R[\mathbb{Z}])).$$

Compare [280, Definition 17.1 on page 145]. Alternatively these groups can be obtained via a process which is in the spirit of Subsection 1.5.3. One can define them as L-theory groups of suitable categories of modules parametrized over $\mathbb{R}^k$. For details the reader could consult [61, Section 4]. There are forgetful maps $L_n^{\langle j+1\rangle}(R) \to L_n^{\langle j\rangle}(R)$. The group $L_n^{\langle -\infty\rangle}(R)$ is defined as the colimit over these maps. For more information see [275], [280].

For group rings we also define $L_n^s(RG)$ similar to $L_n^{\langle 2\rangle}(RG)$ but we require the torsion to lie in $\operatorname{im} A_1 \subset \widetilde{K}_1(RG)$, where A_1 is the map defined in (1.16).

Observe that $L_n^s(RG)$ really depends on the pair (R,G) and differs in general from $L_n^{\langle 2\rangle}(RG)$.

Remark 1.40 (The Interplay of K- and L-Theory). For $j \le 1$ there are forgetful maps $L_n^{\langle j+1\rangle}(R) \to L_n^{\langle j\rangle}(R)$ which sit inside the following sequence, which is known as the *Rothenberg sequence* [277, Proposition 1.10.1 on page 104], [280, 17.2]

$$\cdots \to L_n^{\langle j+1\rangle}(R) \to L_n^{\langle j\rangle}(R) \to \widehat{H}^n(\mathbb{Z}/2;\widetilde{K}_j(R)) \\ \to L_{n-1}^{\langle j+1\rangle}(R) \to L_{n-1}^{\langle j\rangle}(R) \to \cdots . \quad (1.41)$$

Here $\widehat{H}^n(\mathbb{Z}/2;\widetilde{K}_j(R))$ is the Tate-cohomology of the group $\mathbb{Z}/2$ with coefficients in the $\mathbb{Z}[\mathbb{Z}/2]$-module $\widetilde{K}_j(R)$. The involution on $\widetilde{K}_j(R)$ comes from the involution on R. There is a similar sequence relating $L_n^s(RG)$ and $L_n^h(RG)$, where the third term is the $\mathbb{Z}/2$-Tate-cohomology of $\mathrm{Wh}_1^R(G)$. Note that Tate-cohomology groups of the group $\mathbb{Z}/2$ are always annihilated by multiplication with 2. In particular $L_n^{\langle j\rangle}(R)[\frac{1}{2}] = L_n^{\langle j\rangle}(R) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ is always independent of j.

Let us formulate explicitly what we obtain from the above sequences for $R = \mathbb{Z}G$.

Proposition 1.42. *Let G be a torsionfree group, then Conjecture 1.19 about the vanishing of* $\mathrm{Wh}(G)$*,* $\widetilde{K_0}(\mathbb{Z}G)$ *and* $K_{-i}(\mathbb{Z}G)$ *for* $i \ge 1$ *implies that for fixed* n *and varying* $j \in \{-\infty\} \amalg \{j \in \mathbb{Z} \mid j \le 1\}$ *the* L*-groups* $L_n^{\langle j\rangle}(\mathbb{Z}G)$ *are all naturally isomorphic and moreover* $L_n^{\langle 1\rangle}(\mathbb{Z}G) = L_n^h(\mathbb{Z}G) \cong L_n^s(\mathbb{Z}G)$.

Remark 1.43 (Rational Computation). As in the K-theory case we have an Atiyah-Hirzebruch spectral sequence:

$$E^2_{p,q} = H_p(BG; L_q^{\langle -\infty\rangle}(R)) \quad \Rightarrow \quad H_{p+q}(BG; \mathbf{L}^{\langle -\infty\rangle}(R)).$$

Rationally this spectral sequence collapses and the homological Chern character gives for $n \in \mathbb{Z}$ an isomorphism

$$\mathrm{ch}\colon \bigoplus_{p+q=n} H_p(BG;\mathbb{Q}) \otimes_{\mathbb{Q}} \left(L_q^{\langle -\infty\rangle}(R) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \\ \xrightarrow{\cong} H_n(BG; \mathbf{L}^{\langle -\infty\rangle}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}. \quad (1.44)$$

In particular we obtain in the case $R = \mathbb{Z}$ from Remark 1.38 for all $n \in \mathbb{Z}$ and all decorations j an isomorphism

$$\mathrm{ch}\colon \textstyle\bigoplus_{k=0}^{\infty} H_{n-4k}(BG;\mathbb{Q}) \xrightarrow{\cong} H_n(BG; \mathbf{L}^{\langle j\rangle}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q}. \quad (1.45)$$

This spectral sequence and the Chern character above will be discussed in a much more general setting in Chapter 8.

Remark 1.46 (Torsionfree is Necessary). If G is finite, $R = \mathbb{Z}$ and $n = 0$, then the rationalized left hand side of the assembly equals $\mathbb{Q}$, whereas the right hand side is isomorphic to the rationalization of the real representation ring. Since the group homology of a finite group vanishes rationally except in dimension 0, the previous remark shows that we need to assume the group to be torsionfree in Conjecture 1.37

Remark 1.47 (Shaneson splitting). The Bass-Heller-Swan decomposition in K-theory (see Remark 1.33) has the following analogue for the algebraic L-groups, which is known as the *Shaneson splitting* [311]

$$L_n^{\langle j\rangle}(R[\mathbb{Z}]) \;\cong\; L_{n-1}^{\langle j-1\rangle}(R) \oplus L_n^{\langle j\rangle}(R). \tag{1.48}$$

Here for the decoration $j = -\infty$ one has to interpret $j-1$ as $-\infty$. Since S^1 is a model for $B\mathbb{Z}$, we get an isomorphisms

$$H_n(B\mathbb{Z}; \mathbf{L}^{\langle j\rangle}(R)) \;\cong\; L_{n-1}^{\langle j\rangle}(R) \oplus L_n^{\langle j\rangle}(R).$$

This explains why in the formulation of the L-theoretic Farrell-Jones Conjecture for torsionfree groups (see Conjecture 1.37) we use the decoration $j = -\infty$.

As long as one deals with torsionfree groups and one believes in the low dimensional part of the K-theoretic Farrell-Jones Conjecture (predicting the vanishing of $\mathrm{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$ and of the negative K-groups, see Conjecture 1.19) there is no difference between the various decorations j, compare Proposition 1.42. But as soon as one allows torsion in G, the decorations make a difference and it indeed turns out that if one replaces the decoration $j = -\infty$ by $j = s, h$ or p there are counterexamples for the L-theoretic version of Conjecture 2.5 even for $R = \mathbb{Z}$ [138].

Even though in the above Shaneson splitting (1.48) there are no terms analogous to the Nil-terms in Remark 1.33 such Nil-phenomena do also occur in L-theory, as soon as one considers amalgamated free products. The corresponding groups are the UNil-groups. They vanish if one inverts 2 [55]. For more information about the UNil-groups we refer to [19] [52], [53], [83], [86], [111], [281].

1.9 Applications of the Farrell-Jones Conjecture for *L*-Theory for Torsionfree Groups

1.9.1 The Borel Conjecture

One of the driving forces for the development of the Farrell-Jones Conjectures is still the following topological rigidity conjecture about closed aspherical manifolds, compare [122]. Recall that a manifold, or more generally a

CW-complex, is called *aspherical* if its universal covering is contractible. An aspherical CW-complex X with $\pi_1(X) = G$ is a model for the classifying space BG. If X is an aspherical manifold and hence finite dimensional, then G is necessarily torsionfree.

Conjecture 1.49 (Borel Conjecture). *Let $f\colon M \to N$ be a homotopy equivalence of aspherical closed topological manifolds. Then f is homotopic to a homeomorphism. In particular two closed aspherical manifolds with isomorphic fundamental groups are homeomorphic.*

Closely related to the Borel Conjecture is the conjecture that each aspherical finitely dominated Poincaré complex is homotopy equivalent to a closed topological manifold. The Borel Conjecture 1.49 is false in the smooth category, i.e. if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism [121].

Using surgery theory one can show that in dimensions ≥ 5 and the orientable case the Borel Conjecture is implied by the K-theoretic vanishing Conjecture 1.19 combined with the L-theoretic Farrell-Jones Conjecture.

Theorem 1.50. The Farrell-Jones Conjecture implies the Borel Conjecture). *Let G be a torsionfree group. If $\mathrm{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$ and all the groups $K_{-i}(\mathbb{Z}G)$ with $i \geq 1$ vanish and if the assembly map*

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \to L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$$

is an isomorphism for all n, then the Borel Conjecture holds for all orientable manifolds of dimension ≥ 5 whose fundamental group is G.

The Borel Conjecture 1.49 can be reformulated in the language of surgery theory to the statement that the topological structure set $\mathcal{S}^{\mathrm{top}}(M)$ of an aspherical closed oriented topological manifold M consists of a single point. This set is the set of equivalence classes of orientation preserving homotopy equivalences $f\colon M' \to M$ with a topological closed manifold as source and M as target under the equivalence relation, for which $f_0\colon M_0 \to M$ and $f_1\colon M_1 \to M$ are equivalent if there is a homeomorphism $g\colon M_0 \to M_1$ such that $f_1 \circ g$ and f_0 are homotopic.

The *surgery sequence* of a closed oriented topological manifold M of dimension $n \geq 5$ is the exact sequence

$$\begin{aligned} \cdots \to \mathcal{N}_{n+1}(M \times [0,1], M \times \{0,1\}) \xrightarrow{\sigma} L^s_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial} \mathcal{S}^{\mathrm{top}}(M) \\ \xrightarrow{\eta} \mathcal{N}_n(M) \xrightarrow{\sigma} L^s_n(\mathbb{Z}\pi_1(M)), \end{aligned}$$

which extends infinitely to the left. It is the basic tool for the classification of topological manifolds. (There is also a smooth version of it.) The map σ appearing in the sequence sends a normal map of degree one to its surgery obstruction. This map can be identified with the version of the L-theory assembly map where one works with the 1-connected cover $\mathbf{L}^s(\mathbb{Z})\langle 1 \rangle$ of $\mathbf{L}^s(\mathbb{Z})$.

The map $H_k(M;\mathbf{L}^s(\mathbb{Z})\langle 1\rangle) \to H_k(M;\mathbf{L}^s(\mathbb{Z}))$ is injective for $k=n$ and an isomorphism for $k>n$. Because of the K-theoretic assumptions we can replace the s-decoration with the $\langle -\infty\rangle$-decoration, compare Proposition 1.42. Therefore the Farrell-Jones Conjecture 1.37 implies that the maps $\sigma\colon \mathcal{N}_n(M) \to L_n^s(\mathbb{Z}\pi_1(M))$ and $\mathcal{N}_{n+1}(M\times[0,1], M\times\{0,1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi_1(M))$ are injective respectively bijective and thus by the surgery sequence that $\mathcal{S}^{\mathrm{top}}(M)$ is a point and hence the Borel Conjecture 1.49 holds for M. More details can be found e.g. in [142, pages 17,18,28], [279, Chapter 18].

For more information about surgery theory we refer for instance to [47], [50], [51], [136], [137], [185], [196], [274], [321], [320], and [349].

1.9.2 Automorphisms of Manifolds

If one additionally also assumes the Farrell-Jones Conjectures for higher K-theory, one can combine the surgery theoretic results with the results about pseudoisotopies from Subsection 1.7.1 to obtain the following results.

Theorem 1.51 (Homotopy Groups of $\mathrm{Top}(M)$). *Let M be an orientable closed aspherical manifold of dimension >10 with fundamental group G. Suppose the L-theory assembly map*

$$H_n(BG;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) \to L_n^{\langle -\infty\rangle}(\mathbb{Z}G)$$

is an isomorphism for all n and suppose the K-theory assembly map

$$H_n(BG;\mathbf{K}(\mathbb{Z})) \to K_n(\mathbb{Z}G)$$

is an isomorphism for $n\le 1$ and a rational isomorphism for $n\ge 2$. Then for $1\le i\le (\dim M-7)/3$ one has

$$\pi_i(\mathrm{Top}(M))\otimes_{\mathbb{Z}}\mathbb{Q} = \begin{cases} \mathrm{center}(G)\otimes_{\mathbb{Z}}\mathbb{Q} & \text{if } i=1,\\ 0 & \text{if } i>1\end{cases}$$

In the smooth case one additionally needs to study involutions on the higher K-theory groups. The corresponding result reads:

Theorem 1.52 (Homotopy Groups of $\mathrm{Diff}(M)$). *Let M be an orientable closed aspherical smooth manifold of dimension >10 with fundamental group G. Then under the same assumptions as in Theorem 1.51 we have for $1\le i\le (\dim M-7)/3$*

$$\pi_i(\mathrm{Diff}(M))\otimes_{\mathbb{Z}}\mathbb{Q} = \begin{cases} \mathrm{center}(G)\otimes_{\mathbb{Z}}\mathbb{Q} & \text{if } i=1;\\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M;\mathbb{Q}) & \text{if } i>1 \text{ and } \dim M \text{ odd};\\ 0 & \text{if } i>1 \text{ and } \dim M \text{ even}.\end{cases}$$

See for instance [112], [124, Section 2] and [135, Lecture 5]. For a modern survey on automorphisms of manifolds we refer to [359].

1.10 Other Related Conjectures for Torsionfree Groups

We would now like to mention several conjectures which are not directly implied by the Baum-Connes or Farrell-Jones Conjectures, but which are closely related to the Kadison Conjecture and the Idempotent Conjecture mentioned above.

1.10.1 Zero-Divisor Conjecture

The next conjecture is also called the *Kaplansky Conjecture.*

Conjecture 1.53 (Zero-Divisor-Conjecture). *Let R be an integral domain and G be a torsionfree group. Then RG is an integral domain.*

Obviously the Zero-Divisor-Conjecture 1.53 implies the Idempotent Conjecture 1.14. The Zero-Divisor-Conjecture for $R = \mathbb{Q}$ is implied by the following version of the Atiyah Conjecture (see [220, Lemma 10.5 and Lemma 10.15]).

1.10.2 Atiyah Conjecture

Conjecture 1.54. (Atiyah-Conjecture for Torsionfree Groups). *Let G be a torsionfree group and let M be a closed Riemannian manifold. Let $\overline{M} \to M$ be a regular covering with G as group of deck transformations. Then all L^2-Betti numbers $b_p^{(2)}(\overline{M};\mathcal{N}(G))$ are integers.*

For the precise definition and more information about L^2-Betti numbers and the group von Neumann algebra $\mathcal{N}(G)$ we refer for instance to [220], [223].

This more geometric formulation of the Atiyah Conjecture is in fact implied by the following more operator theoretic version. (The two would be equivalent if one would work with rational instead of complex coefficients below.)

Conjecture 1.55. (Strong Atiyah-Conjecture for Torsionfree Groups). *Let G be a torsionfree group. Then for all (m,n)-matrices A over $\mathbb{C}G$ the von Neumann dimension of the kernel of the induced G-equivariant bounded operator*

$$r_A^{(2)} : l^2(G)^m \to l^2(G)^n$$

is an integer.

The Strong Atiyah-Conjecture for Torsionfree Groups 1.55 implies both the Atiyah-Conjecture for Torsionfree Groups 1.54 [220, Lemma 10.5 on page 371] and the Zero-Divisor-Conjecture 1.53 for $R = \mathbb{C}$ [220, Lemma 10.15 on page 376].

1.10.3 The Embedding Conjecture and the Unit Conjecture

Conjecture 1.56 (Embedding Conjecture). *Let G be a torsionfree group. Then $\mathbb{C}G$ admits an embedding into a skewfield.*

The Embedding Conjecture implies the Zero-Divisor-Conjecture 1.53 for $R = \mathbb{C}$. If G is a torsionfree amenable group, then the Strong Atiyah-Conjecture for Torsionfree Groups 1.55 and the Zero-Divisor-Conjecture 1.53 for $R = \mathbb{C}$ are equivalent [220, Lemma 10.16 on page 376]. For more information about the Atiyah Conjecture we refer for instance to [210], [220, Chapter 10] and [282].

Finally we would like to mention the Unit Conjecture.

Conjecture 1.57 (Unit-Conjecture). *Let R be an integral domain and G be a torsionfree group. Then every unit in RG is trivial, i.e. of the form $r \cdot g$ for some unit $r \in R^{\mathrm{inv}}$ and $g \in G$.*

The Unit Conjecture 1.57 implies the Zero-Divisor-Conjecture 1.53. For a proof of this fact and for more information we refer to [205, Proposition 6.21 on page 95].

1.10.4 L^2-Rho-Invariants and L^2-Signatures

Let M be a closed connected orientable Riemannian manifold. Denote by $\eta(M) \in \mathbb{R}$ the *eta-invariant* of M and by $\eta^{(2)}(\widetilde{M}) \in \mathbb{R}$ the *L^2-eta-invariant* of the $\pi_1(M)$-covering given by the universal covering $\widetilde{M} \to M$. Let $\rho^{(2)}(M) \in \mathbb{R}$ be the *L^2-rho-invariant* which is defined to be the difference $\eta^{(2)}(\widetilde{M}) - \eta(M)$. These invariants were studied by Cheeger and Gromov [72], [73]. They show that $\rho^{(2)}(M)$ depends only on the diffeomorphism type of M and is in contrast to $\eta(M)$ and $\eta^{(2)}(\widetilde{M})$ independent of the choice of Riemannian metric on M. The following conjecture is taken from Mathai [233].

Conjecture 1.58. (Homotopy Invariance of the L^2-Rho-Invariant for Torsionfree Groups). *If $\pi_1(M)$ is torsionfree, then $\rho^{(2)}(M)$ is a homotopy invariant.*

Chang-Weinberger [69] assign to a closed connected oriented $(4k-1)$-dimensional manifold M a Hirzebruch-type invariant $\tau^{(2)}(M) \in \mathbb{R}$ as follows. By a result of Hausmann [160] there is a positive integer r and a closed connected oriented $4k$-dimensional manifold W such that $r \cdot M = \partial W$ holds for the disjoint union $r \cdot M$ of r copies of M and the inclusion of every boundary component of ∂W into W induces an injection on the fundamental groups. Define $\tau^{(2)}(M)$ as $\frac{1}{r} \cdot \mathrm{sign}^{(2)}(\widetilde{W}) - \mathrm{sign}(W)$, where $\mathrm{sign}^{(2)}(\widetilde{W})$ is the L^2-signature of the $\pi_1(W)$-covering given by the universal covering $\widetilde{W} \to W$ and $\mathrm{sign}(W)$ is the signature of W. This is indeed independent of the choice of W. It is reasonable to believe that $\rho^{(2)}(M) = \tau^{(2)}(M)$ is always true. Chang-Weinberger [69] use $\tau^{(2)}$ to prove that if $\pi_1(M)$ is not torsionfree there are infinitely many diffeomorphically distinct manifolds of dimension $4k-1$ with $k \geq 2$, which are tangentially simple homotopy equivalent to M.

Theorem 1.59 (Homotopy Invariance of $\tau^{(2)}(M)$ and $\rho^{(2)}(M)$). *Let M be a closed connected oriented $(4k-1)$-dimensional manifold M such that $G = \pi_1(M)$ is torsionfree.*

(1) If the assembly map $K_0(BG) \to K_0(C^*_{\max}(G))$ for the maximal group C^*-algebra is surjective (see Subsection 2.12.2), then $\rho^{(2)}(M)$ is a homotopy invariant;
(2) Suppose that the Farrell-Jones Conjecture for L-theory 1.37 is rationally true for $R = \mathbb{Z}$, i.e. the rationalized assembly map

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_n^{\langle -\infty \rangle}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism for $n \in \mathbb{Z}$. Then $\tau^{(2)}(M)$ is a homotopy invariant. If furthermore G is residually finite, then $\rho^{(2)}(M)$ is a homotopy invariant.

Proof. 1 This is proved by Keswani [192], [193]. **Comment 17** (By W.): `Shall we also add a reference to the preprint by Piazza-Schick from 2004 if they can repair their gap?`

2 This is proved by Chang [68] and Chang-Weinberger [69] using [229].

Remark 1.60. Let X be a $4n$-dimensional Poincaré space over $\mathbb{Q}$. Let $\overline{X} \to X$ be a normal covering with covering group G. Suppose that the assembly map $K_0(BG) \to K_0(C^*_{\max}(G))$ for the maximal group C^*-algebra is surjective (see Subsection 2.12.2) or suppose that the rationalized assembly map for L-theory

$$H_{4n}(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_{4n}^{\langle -\infty \rangle}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Then the following L^2-signature theorem is proved in Lück-Schick [230, Theorem 2.3]

$$\operatorname{sign}^{(2)}(\overline{X}) = \operatorname{sign}(X).$$

If one drops the condition that G is torsionfree this equality becomes false. Namely, Wall has constructed a finite Poincaré space X with a finite G covering $\overline{X} \to X$ for which $\operatorname{sign}(\overline{X}) \neq |G| \cdot \operatorname{sign}(X)$ holds (see [279, Example 22.28], [348, Corollary 5.4.1]).

Remark 1.61. Cochran-Orr-Teichner give in [77] new obstructions for a knot to be slice which are sharper than the Casson-Gordon invariants. They use L^2-signatures and the Baum-Connes Conjecture 2.4. We also refer to the survey article [76] about non-commutative geometry and knot theory.

1.10.5 The Zero-in-the-Spectrum Conjecture

The following Conjecture is due to Gromov [151, page 120].

Conjecture 1.62 (Zero-in-the-spectrum Conjecture). *Suppose that $\widetilde{M}$ is the universal covering of an aspherical closed Riemannian manifold M (equipped with the lifted Riemannian metric). Then zero is in the spectrum of the minimal closure*

$$(\Delta_p)_{\min}\colon L^2\Omega^p(\widetilde{M}) \supset \operatorname{dom}(\Delta_p)_{\min} \to L^2\Omega^p(\widetilde{M}),$$

for some $p \in \{0, 1, \ldots, \dim M\}$, where Δ_p denotes the Laplacian acting on smooth p-forms on $\widetilde{M}$.

Proposition 1.63. *Suppose that M is an aspherical closed Riemannian manifold with fundamental group G, then the injectivity of the assembly map*

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the Zero-in-the-spectrum Conjecture for $\widetilde{M}$.

Proof. We give a sketch of the proof. More details can be found in [213, Corollary 4]. We only explain that the assumption that in every dimension zero is not in the spectrum of the Laplacian on $\widetilde{M}$, yields a contradiction in the case that $n = \dim(M)$ is even. Namely, this assumption implies that the $C_r^*(G)$-valued index of the signature operator twisted with the flat bundle $\widetilde{M} \times_G C_r^*(G) \to M$ in $K_0(C_r^*(G))$ is zero, where $G = \pi_1(M)$. This index is the image of the class $[S]$ defined by the signature operator in $K_0(BG)$ under the assembly map $K_0(BG) \to K_0(C_r^*(G))$. Since by assumption the assembly map is rationally injective, this implies $[S] = 0$ in $K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q}$. Notice that M is aspherical by assumption and hence $M = BG$. The homological Chern character defines an isomorphism

$$K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q} = K_0(M) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \geq 0} H_{2p}(M; \mathbb{Q})$$

which sends $[S]$ to the Poincaré dual $\mathcal{L}(M) \cap [M]$ of the Hirzebruch L-class $\mathcal{L}(M) \in \bigoplus_{p \geq 0} H^{2p}(M; \mathbb{Q})$. This implies that $\mathcal{L}(M) \cap [M] = 0$ and hence $\mathcal{L}(M) = 0$. This contradicts the fact that the component of $\mathcal{L}(M)$ in $H^0(M; \mathbb{Q})$ is 1.

More information about the Zero-in-the-spectrum Conjecture 1.62 can be found for instance in [213] and [220, Section 12].

1.11 Miscellaneous

A group homomorphism $f\colon G \to H$ induces a map $Bf\colon BG \to BH$ uniquely determined up to homotopy by the property that $\pi_1(BF)$ is up to conjugation f. Thus we get well-defined maps

$$\begin{aligned} K_n(Bf)\colon K_n(BG) &\to K_n(BH); \\ H_n(Bf;\mathbf{K}(R))\colon H_n(BG;\mathbf{K}(R)) &\to H_n(BH;\mathbf{K}(R)); \\ H_n(Bf;\mathbf{L}^{\langle -\infty\rangle}(R))\colon H_n(BG;\mathbf{L}^{\langle -\infty\rangle}(R)) &\to H_n(BH;\mathbf{L}^{\langle -\infty\rangle}(R)). \end{aligned}$$

The group homomorphism f induces a ring homomorphism $Rf\colon RG \to RH$ and thus maps

$$\begin{aligned} K_n(Rf)\colon K_n(RG) &\to K_n(RH); \\ L^{\langle -\infty\rangle}(Rf)\colon L^{\langle -\infty\rangle}(RG) &\to L^{\langle -\infty\rangle}(RH). \end{aligned}$$

We will later see when we have defined the assembly maps in detail that the assembly maps for K- and L-theory are compatible with these maps.

The situation for the reduced group C^*-algebra is more mysterious. If f has a finite kernel, it induces a homomorphism of C^*-algebras $C_r^*(G) \to C_r^*(H)$, but not in general. For instance, the reduced C^*-algebra $C_r^*(\mathbb{Z} * \mathbb{Z})$ of the free group on two letters is simple [266] and hence admits no C^*-homomorphism to the reduced C^*-algebra $\mathbb{C}$ of the trivial group. The Baum-Connes Conjecture predicts that there must always be a map $K_n(C_r^*(G)) \to K_n(C_r^*(H))$ but it is not clear how to define it directly for all group homomorphisms $f\colon G \to H$ without using the Baum-Connes Conjecture.

The assembly maps appearing in this chapter, for instance in the Baum-Connes Conjecture for torsionfree groups (see Conjecture 1.1), the Farrell-Jones Conjecture for K-Theory and Torsionfree Groups (see Conjecture 1.28) and the Farrell-Jones ConjectureFarrell-Jones Conjecture for L-Theory and Torsionfree Groups (see Conjecture 1.37, sometimes are called *classical assembly maps* and are special cases of the assembly maps Chapter 2, namely for the family $\mathcal{TR}$ under the identifications $K_n(BG) = \mathcal{H}_n^G(EG;\mathbf{K}^{\mathrm{top}})$, $H_n(BG;\mathbf{K}(R)) = \mathcal{H}_n^G(EG;\mathbf{K}_R)$, $H_n(BG;^{\langle -\infty\rangle}(R)) = \mathcal{H}_n^G(EG;\mathbf{L}_R^{\langle -\infty\rangle})$ and $EG = E_{\mathcal{TR}}(G)$ (see Remark 2.8).

Exercises

1.1. Compute $K_n(C_r^*(G))$ for $n \in \mathbb{Z}$ and $K_n(\mathbb{Z}G)$ for $n \leq 1$ for G the fundamental group of a closed orientable surface of genus g using the fact that for such G both the Baum-Connes Conjecture 1.1 and the version 1.17 of the Farrell-Jones Conjecture are true.

1.2. Show that the reduced group C^*-algebra $C^*(\mathbb{Z})$ is isomorphic to the C^*-algebra $C(S^1)$ of continuous functions $S^1 \to \mathbb{C}$ with the supremums norm and that under this identification the standard trace sends $f \in C(S^1)$ to the integral $\int_{S^1} f d\mu$, where $d\mu$ is the S^1-invariant Lebesgue measure on S^1 with $d\mu(S^1) = 1$. Prove that the Trace Conjecture 1.10 holds for $\mathbb{Z}$.

1.3. Let R be a ring such that for any integer n the element $n \cdot 1_R$ is invertible in R. Show that the Idempotent Conjecture 1.14 does not hold for a group G which is not torsionfree.

1.4. Show that $1 - t - t^{-1}$ is a unit in $\mathbb{Z}[\mathbb{Z}/5]$. Conclude that $\mathrm{Wh}(\mathbb{Z}/5)$ contains an element of infinite order.

1.5. Let P_* be a finite projective R-chain complex, where finite means that it is finite dimensional and every module is finitely generated and projective means that every chain module is projective. Define its finiteness obstruction $o(P_*) \in K_0(R)$ by $\sum_{n \geq 0} [P_n]$. Its reduced finiteness obstruction $\widetilde{o}(P_*) \in \widetilde{K}_0(R)$ is the image of $o(P_*)$ under the canonical projection. Prove

(1) $o(P_*)$ depends only on the R-chain homotopy class of P_*;
(2) Let $0 \to P_* \to P'_* \to P''_* \to 0$ be an exact sequence of finite projective R-chain complexes. Then

$$o(P_*) - o(P'_*) + o(P''_*) \;=\; 0;$$

(3) The R-chain complex P_* is R-chain homotopy equivalent to a finite free R-chain complex if and only if $\widetilde{o}(P_*) = 0$.

1.6. Let M and N be two closed oriented n-dimensional manifolds whose connected sum $M \sharp N$ is aspherical. Show that $n \leq 2$ or that one of them is homotopy equivalent to a sphere and the other is aspherical.

1.7. Let M be a closed aspherical manifold. Suppose that Conjecture 1.28 holds for $R = \mathbb{Z}$ and $G = \pi_1(M)$, Show that the obvious map $\mathcal{P}^{\mathrm{Diff}}(M) \to \mathcal{P}(M)$ induces a rational isomorphism on π_n if and only if $H_{n-4k+1}(M; \mathbb{Q}) = 0$ holds for all $k \in \mathbb{Z}, k \geq 1$.

1.8. Consider the group G given by the presentation $\langle xy \mid x^3 y^2 \rangle$. Show that G is torsionfree. Compute $L_n^{\langle j \rangle}(\mathbb{Z}G)$ for all j and n provided that Conjecture 1.50 holds for G.

1.9. Let M and N be closed aspherical manifolds. Suppose that the Borel Conjecture 1.49 holds for $\pi_1(M)$. Show that the following assertions are equivalent for a map $f\colon M \to N$ and positive integer n:

(1) f is homotopic to a n-sheeted finite covering;
(2) The group homomorphism $\pi_1(f)$ is injective and its image has index n in $\pi_1(N)$.

1.10. Show that the topological structure set $\mathcal{S}^{\mathrm{top}}(S^n)$ consists of precisely one element in dimensions $n \neq 3$. **Comment 18** (By W.): `Later drop the condition` $n \neq 3$ `depending on what the status of the proof by Perelman is.`

1.11. Let G and H be torsionfree groups. Suppose that the K-theoretic Farrell-Jones Conjecture 1.28 holds for the product $G \times H$. Show that it then must hold for both G and H. Prove the analogous statement for the L-theoretic Farrell-Jones Conjecture 1.37.

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2. Formulation and Relevance of the Conjectures in General

2.1 Introduction

The Baum-Connes and Farrell-Jones Conjectures predict that for every discrete group G the following so called "assembly maps" are isomorphisms:

$$\begin{aligned} H_n^G(E_{\mathcal{FIN}}(G)) &\to K_n(BG); \\ H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) &\to K_n(RG); \\ H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) &\to L_n^{\langle -\infty \rangle}(RG). \end{aligned}$$

Here the targets are the groups one would like to understand, namely the topological K-groups of the reduced group C^*-algebra in the Baum-Connes case and the algebraic K- or L-groups of the group ring RG for R an associative ring with unit. In each case the source is a G-homology theory evaluated on a certain classifying space. In the Baum-Connes Conjecture the G-homology theory is equivariant topological K-theory and the classifying space $E_{\mathcal{FIN}}(G)$ is the classifying space of the family of finite subgroups, which is often called the classifying space for proper G-actions and denoted $\underline{E}G$ in the literature. In the Farrell-Jones Conjecture the G-homology theory is given by a certain K- or L-theory spectrum over the orbit category, and the classifying space $E_{\mathcal{VCY}}(G)$ is the one associated to the family of virtually cyclic subgroups. We often write $\underline{\underline{E}}G$ instead of $E_{\mathcal{VCY}}(G)$.

These conjectures were stated in [33, Conjecture 3.15 on page 254] and [126, 1.6 on page 257]. Our formulations differ from the original ones, but are equivalent. In the case of the Farrell-Jones Conjecture we slightly generalize the original conjecture by allowing arbitrary coefficient rings instead of $\mathbb{Z}$. At the time of writing no counterexample to the Baum-Connes Conjecture 2.4 or the Farrell-Jones Conjecture 2.5 is known to the authors.

In this chapter we will formulate the Baum-Connes and Farrell-Jones Conjectures. We try to emphasize the unifying principle that underlies these conjectures. The point of view taken in this chapter is that all three conjectures are conjectures about specific equivariant homology theories. Some of the technical details concerning the actual construction of these homology theories are deferred to Section 4.4.

2.2 Formulation of the Conjectures

2.2.1 The Metaconjecture

Suppose we are given

- A discrete group G;
- A family $\mathcal{F}$ of subgroups of G;
- A G-homology theory $\mathcal{H}_*^G(-)$.

Then one can formulate the following Meta-Conjecture.

Metaconjecture 2.1. *The* assembly map

$$A_{\mathcal{F}} \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(\{\bullet\})$$

is an isomorphism for $n \in \mathbb{Z}$.

Here are some explanations. A *family $\mathcal{F}$ of subgroups of G* is a set of subgroups of G closed under conjugation, i.e. $H \in \mathcal{F}, g \in G$ implies $g^{-1}Hg \in \mathcal{F}$, and finite intersections, i.e. $H, K \in \mathcal{F}$ implies $H \cap K \in \mathcal{F}$. Throughout the text we will use the notations

$$\mathcal{TR}, \mathcal{FCY}, \mathcal{FIN}, \mathcal{CYC}, \mathcal{VCY}_I, \mathcal{VCY} \quad \text{and} \quad \mathcal{ALL}$$

for the families consisting of the trivial, all finite cyclic, all finite, all (possibly infinite) cyclic, all virtually cyclic of the first kind, all virtually cyclic, respectively all subgroups of a given group G. Recall that a group is called *virtually cyclic* if it is finite or contains an infinite cyclic subgroup of finite index. A group is *virtually cyclic of the first kind* if it admits a surjection onto an infinite cyclic group with finite kernel, compare Lemma 2.18.

We denote by $E_{\mathcal{F}}(G)$ the *classifying space of the family* $\mathcal{F}$. It is a G-CW-complex such that all its isotropy groups belong to $\mathcal{F}$ and $E_{\mathcal{F}}(G)^H$ is contractible for $H \in \mathcal{F}$. Given a G-CW-complex whose isotropy groups belong to $\mathcal{F}$, then there is up to G-homotopy precisely one G-map $X \to E_{\mathcal{F}}(G)$. In particular two models for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent. The G-space $E_{\mathcal{TR}}(G)$ agrees with the G-space EG which is by definition the universal covering of BG. A model for $E_{\mathcal{ALL}}(G)$ is G/G. We sometimes abbreviate $E_{\mathcal{FIN}}(G)$ by $\underline{E}G$ and $E_{\mathcal{VCY}}(G)$ by $\underline{\underline{E}}G$. If $\mathcal{F} \subseteq \mathcal{G}$ is an inclusion of families, there is up to G-homotopy precisely one G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$. More information about classifying spaces for families will be given in Section 4.2.

A *G-homology theory* $\mathcal{H}_*^G$ is the "obvious" G-equivariant generalization of the concept of a homology theory. It assigns to every G-CW-complex X a $\mathbb{Z}$-graded abelian group $\mathcal{H}_*^G(X)$. It is natural in X, satisfies G-homotopy invariance and possesses long exact Mayer-Vietoris sequences. An equivariant homology theory $\mathcal{H}_*^?$ assigns to every discrete group G a G-homology theory $\mathcal{H}_*^G$. These are linked by an induction structure. We will mainly need from the induction structure the following two consequences. Namely, for a subgroup

$H \subseteq G$ and a H-CW-complex X the G-space $G \times_H X$ is a G-CW-complex and there is a natural isomorphism

$$\mathcal{H}_n^H(X) \xrightarrow{\cong} \mathcal{H}_n^G(G \times_H X) \tag{2.2}$$

and for a free G-CW-complex there is an isomorphism

$$\mathcal{H}_n^G(X) \xrightarrow{\cong} \mathcal{H}_n^{\{1\}}(G\backslash X). \tag{2.3}$$

Equivariant homology theories are explained in detail in Section 4.4.

Given a family $\mathcal{F}$ and a G-homology theory $\mathcal{H}_*^G$, the projection $E_{\mathcal{F}}(G) \to G/G$ induces a map of $\mathbb{Z}$-graded abelian groups

$$A_{\mathcal{F}} \colon \mathcal{H}_*^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_*^G(G/G).$$

This is the assembly map appearing in the Metaconjecture 2.1.

Of course the Metaconjecture 2.2.1 is not true for arbitrary G, $\mathcal{F}$ and $\mathcal{H}_*^G(-)$. It is always true for $\mathcal{F} = \mathcal{ALL}$ for the trivial reason that then $E_{\mathcal{ALL}}(G) \to G/G$ is a G-homotopy equivalence. The point is to choose for a given G-homology theory $\mathcal{H}_*^G$ the family $\mathcal{F}$ as small as possible. The Farrell-Jones and Baum-Connes Conjectures state that for specific G-homology theories there is a natural choice of a family $\mathcal{F}(G)$ of subgroups for every group G such that $A_{\mathcal{F}(G)}$ becomes an isomorphism for all groups G.

2.2.2 The Baum-Connes and the Farrell-Jones Conjectures

Let R be a ring (with involution). In Proposition 4.20 we will describe the construction of G-homology theories which will be denoted

$$H_n^G(-;\mathbf{K}_R), \quad H_n^G(-;\mathbf{L}_R^{\langle -\infty \rangle}) \quad \text{and} \quad H_n^G(-;\mathbf{K}^{\mathrm{top}}).$$

If G is the trivial group, these homology theories specialize to the (non-equivariant) homology theories with similar names that appeared in Chapter 1, namely to

$$H_n(-;\mathbf{K}(R)), \quad H_n(-;\mathbf{L}^{\langle -\infty \rangle}(R)) \quad \text{and} \quad K_n(-).$$

Another main feature of these G-homology theories is that evaluated on the one point space $\{\bullet\}$ (considered as a trivial G-space) we obtain the K- and L-theory of the group ring RG, respectively the topological K-theory of the reduced C^*-algebra (see Proposition 4.20 and Theorem 4.7 (3))

$$\begin{aligned} K_n(RG) &\cong H_n^G(\{\bullet\};\mathbf{K}_R), \\ L_n^{\langle -\infty \rangle}(RG) &\cong H_n^G(\{\bullet\};\mathbf{L}_R^{\langle -\infty \rangle}) \quad \text{and} \\ K_n(C_r^*(G)) &\cong H_n^G(\{\bullet\};\mathbf{K}^{\mathrm{top}}). \end{aligned}$$

We are now prepared to formulate in full generality the conjectures around which this book is centered. Recall that $\mathcal{FIN}$ is the family of finite subgroups and that $\mathcal{VCY}$ is the family of virtually cyclic subgroups.

Conjecture 2.4 (Baum-Connes Conjecture). *Let G be a group. Then for all $n \in \mathbb{Z}$ the so called* assembly map

$$A_{\mathcal{FIN}} \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\mathrm{top}}) \to H_n^G(\{\bullet\}; \mathbf{K}^{\mathrm{top}}) \cong K_n(C_r^*(G))$$

induced by the projection $E_{\mathcal{FIN}}(G) \to \{\bullet\}$ is an isomorphism.

Conjecture 2.5 (Farrell-Jones Conjecture for K- and L-theory). *Let R be a ring (with involution) and let G be a group. Then for all $n \in \mathbb{Z}$ the so called* assembly maps

$$A_{\mathcal{VCY}} \colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \to H_n^G(\{\bullet\}; \mathbf{K}_R) \cong K_n(RG);$$
$$A_{\mathcal{VCY}} \colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(\{\bullet\}; \mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(RG),$$

which are induced by the projection $E_{\mathcal{VCY}}(G) \to \{\bullet\}$, are isomorphisms.

The conjecture for the topological K-theory of C^*-algebras is known as the Baum-Connes Conjecture and reads as follows.

We will explain the analytic assembly map $\mathrm{ind}_G \colon K_n^G(X) \to K_n(C_r^*(G))$, which can be identified with the assembly map appearing in the Baum-Connes Conjecture 2.4 in Section 5.3.

Remark 2.6. (Interpretations of the assembly maps). Of course the conjectures really come to life only if the abstract point of view taken in this section is connected up with more concrete descriptions of the assembly maps. We have already discussed a surgery theoretic description in Theorem 1.50 and an interpretation in terms index theory in Subsection 1.3.1 More information about alternative interpretations of assembly maps can be found in Section 5.3 and 6.3. These concrete interpretations of the assembly maps lead to applications. We already discussed many such applications in Chapter 1 and we will see further applications, for instance in connection with the Novikov Conjecture (see Section 2.9) and the Stable Gromov-Lawson Conjecture (see Section 2.11.2).

Remark 2.7 (The choice of the right family). As explained above the Baum-Connes Conjecture 2.4 and the Farrell-Jones 2.5 can be considered as special cases of the Meta-Conjecture 2.1. In all three cases we are interested in a computation of the right hand side $\mathcal{H}_n^G(\{\bullet\})$ of the assembly map, which can be identified with $K_n(RG)$, $L_n^{\langle -\infty \rangle}(RG)$ or $K_n(C_r^*(G))$. The left hand side $\mathcal{H}_n^G(E_{\mathcal{F}}(G))$ of such an assembly map is much more accessible and the smaller $\mathcal{F}$ is, the easier it is to compute $\mathcal{H}_n^G(E_{\mathcal{F}}(G))$ using homological methods like spectral sequences, Mayer-Vietoris arguments and equivariant Chern characters. We have already explained in the Introduction (see Chapter 0) why we pick the family $\mathcal{FIN}$ or $\mathcal{VCY}$ respectively for the Baum-Connes Conjecture 2.4 and the Farrell-Jones 2.5 respectively.

Remark 2.8. (Classical assembly maps). If we take the family $\mathcal{TR}$ consisting of the trivial subgroup only, then $EG = E_{\mathcal{TR}}(G)$ and hence $BG = G\backslash E_{\mathcal{TR}}(G)$ and we get from the induction structure (see Subsection 4.4.2) identifications

$$\begin{aligned} K_n(BG) &= \mathcal{H}_n^G(E_{\mathcal{TR}}(G); \mathbf{K}^{\mathrm{top}}); \\ H_n(BG; \mathbf{K}(R)) &= \mathcal{H}_n(E_{\mathcal{TR}}(G); \mathbf{K}_R); \\ H_n(BG; \mathbf{L}^{\langle -\infty \rangle})(R)) &= \mathcal{H}_n(E_{\mathcal{TR}}(G); \mathbf{L}_R^{\langle -\infty \rangle}). \end{aligned}$$

Under these identification the assembly maps of Chapter 1

$$\begin{aligned} K_n(BG) &\to K_n(C_r^*(G)); \\ H_n(BG; \mathbf{K}(R)) &\to K_n(RG); \\ H_n(BG; \mathbf{L}^{\langle -\infty \rangle})(R)) &\to L_n^{\langle -\infty \rangle}(RG). \end{aligned}$$

which we will often call the *classical assembly maps* and have appeared in the Baum-Connes Conjecture for torsionfree groups (see Conjecture 1.1), the Farrell-Jones Conjecture for K-Theory and Torsionfree Groups (see Conjecture 1.28) and the Farrell-Jones Conjecture for L-Theory and Torsionfree Groups (see Conjecture 1.37, correspond to the assembly maps $A_{\mathcal{TR}}$ associated to $\mathbf{K}^{\mathrm{top}}$, $\mathbf{K}_R$ and $\mathbf{L}^{\langle -\infty \rangle}$.

Example 2.9 (Infinite dihedral group). Let D_∞ be the infinite dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2 \cong \mathbb{Z} \rtimes \mathbb{Z}/2$. Up to conjugation D_∞ contains two non-trivial finite subgroups, namely $\{1\} * \mathbb{Z}/2$ and $\mathbb{Z}/2 * \{1\}$. There is an obvious action of $D_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}/2$ on $\mathbb{R}$, where $\mathbb{Z}$ acts by translation and $\mathbb{Z}/2$ by $-\mathrm{id}_{\mathbb{R}}$. If we take $\{n/2 \mid n \in \mathbb{Z}\}$ as the 0-skeleton, $\mathbb{R}$ becomes a proper D_∞-CW-complex whose H-fixed point set consists of a point if H is a non-trivial finite subgroup of D_∞. Hence $\mathbb{R}$ with this operation is a model for $E_{\mathcal{FIN}}(D_\infty)$. There is an obvious $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$-pushout

$$\begin{array}{ccc} D_\infty \times S^0 & \longrightarrow & D_\infty/(\{1\} * \mathbb{Z}/2) \coprod D_\infty/(\mathbb{Z}/2 * \{1\}) \\ \big\downarrow & & \big\downarrow \\ D_\infty \times D^1 & \longrightarrow & E_{\mathcal{FIN}}(D_\infty) \end{array}$$

Associated to it is the long Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to K_n^{D_\infty}(D_\infty) \to K_n^{D_\infty}(D_\infty/(\{1\} * \mathbb{Z}/2)) \oplus K_n^{D_\infty}(D_\infty/(\mathbb{Z}/2 * \{1\})) \\ \to K_n^{D_\infty}(E_{\mathcal{FIN}}(D_\infty)) \to K_{n-1}^{D_\infty}(D_\infty) \\ \to K_{n-1}^{D_\infty}(D_\infty/(\{1\}*\mathbb{Z}/2)) \oplus K_{n-1}^{D_\infty}(D_\infty/(\mathbb{Z}/2*\{1\})) \to K_{n-1}^{D_\infty}(E_{\mathcal{FIN}}(D_\infty)) \to \cdots . \end{aligned}$$

From the Baum-Connes Conjecture 2.4 and general properties of equivariant topological K-theory we get for every finite subgroup $H \subseteq D_\infty$ identifications

$$K_n^{D_\infty}(E_{\mathcal{FIN}}(D_\infty)) = K_n(C_r^*(D_\infty));$$
$$K_n^{D_\infty}(D_\infty/H) = K_n^H(H/H) \;=\; \begin{cases} R_{\mathbb{C}}(H) & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

Thus we obtain the short exact sequence

$$0 \to R_{\mathbb{C}}(\{1\}) \xrightarrow{i_1 \oplus i_2} R_{\mathbb{C}}(\mathbb{Z}/2) \oplus R_{\mathbb{C}}(\mathbb{Z}/2) \xrightarrow{j_1 \oplus j_2} K_0(C_r^*(D_\infty)) \to 0,$$

where i_1 and i_2 are induced by the inclusion $\{1\} \to \mathbb{Z}/2$ and j_1 and j_2 are induced by the two inclusions of $\mathbb{Z}/2$ into $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$, and

$$K_1(C_r^*(D_\infty) \;=\; \{0\}.$$

(Of course this can also be obtained directly from the long exact sequence (0.1)). This implies

$$K_n(C_r^*(D_\infty)) \;=\; \begin{cases} \mathbb{Z}^3 & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

The Farrell-Jones Conjecture 2.5 gives no information since D_∞ is virtually cyclic. However, certain variations of it such as the versions 2.17 and 2.21 do give by the same argument

$$K_n(\mathbb{C}[D_\infty]) \;\cong\; K_n(\mathbb{C}) \oplus K_n(\mathbb{C}) \oplus K_n(\mathbb{C})$$

and in particular

$$K_n(\mathbb{C}[D_\infty]) \;=\; \{0\} \quad \text{for } n \le -1;\; K_0(\mathbb{C}[D_\infty]) \;=\; \mathbb{Z}^3;$$
$$K_0(\mathbb{C}[D_\infty]) \;=\; \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times,$$

and after inverting 2

$$L_n^{\langle j \rangle}(\mathbb{Z}[D_\infty])[1/2] \;=\; \begin{cases} \mathbb{Z}[1/2]^3 & n = 0 \mod 4; \\ \{0\} & \text{otherwise.} \end{cases}$$

One does know that there is for all $n \in \mathbb{Z}$ an isomorphism (see Waldhausen [341] and [342])

$$\widetilde{K}_n(\mathbb{Z}[D_\infty]) \;\cong\; \widetilde{K}_n(\mathbb{Z}[\mathbb{Z}/2]) \oplus \widetilde{K}_n(\mathbb{Z}(\mathbb{Z}/2]),$$

but the groups $\widetilde{K}_n(\mathbb{Z}[\mathbb{Z}/2])$ are not known for all $n \in \mathbb{Z}$ at the time of writing. At least one concludes that $\widetilde{K}_n(\mathbb{Z}[D_\infty])$ for $n \le 0$ and $\mathrm{Wh}(D_\infty)$ vanish. This implies that $L_n^{\langle j \rangle}(\mathbb{Z}[D_\infty])$ is independent of the decoration. There is a splitting

$$\widetilde{L}_n^{\langle j \rangle}(\mathbb{Z}[D_\infty]) \;\cong\; \widetilde{L}_n^{\langle j \rangle}(\mathbb{Z}[\mathbb{Z}/2]) \oplus \widetilde{L}_n^{\langle j \rangle}(\mathbb{Z}[\mathbb{Z}/2]) \oplus \mathrm{UNil}_n\,.$$

The groups $\widetilde{L}_n^{\langle j \rangle}(\mathbb{Z}[\mathbb{Z}/2])$ are independent of the decoration and well-known but the computation of UNil_n has been completely only a few years ago (see for instance [19], [82], [86]). These groups are always annihilated by 4. They are zero or unpleasantly large, namely contain $(\mathbb{Z}/2)^\infty$ as a direct summand.

This illustrates that the computation of algebraic K- and L-groups is much harder than the one of the topological K-theory of the reduced group C^*-algebra.

Notice that the classical assembly maps

$$\begin{aligned} K_n(BD_\infty) &\to K_n(C_r^*(D_\infty)); \\ H_n(BD_\infty; \mathbf{K}(\mathbb{C})) &\to K_n(\mathbb{C}[D_\infty]); \\ H_n(BD_\infty; \mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) &\to L_n^{\langle -\infty\rangle}(\mathbb{Z}[D_\infty]). \end{aligned}$$

are not injective. This follows from a calculation using the fact that $\widetilde{\mathcal{H}}_*(D_\infty) = \widetilde{\mathcal{H}}_*(\mathbb{Z}/2) \oplus \widetilde{\mathcal{H}}_*(\mathbb{Z}/2)$ holds for any homology theory $\mathcal{H}_*$ Since $\mathcal{H}_*(\{\bullet\}; \mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} = H_*(BD_\infty) \otimes_{\mathbb{Z}} \mathbb{Q}$ for any homology theory $\mathcal{H}_*$, the classical assembly maps above are rationally injective but their images agree rationally with the image of the obvious homomorphisms

$$\begin{aligned} K_n(\{\bullet\}) &\to K_n(C_r^*(D_\infty)); \\ K_n(\mathbb{C}) &\to K_n(\mathbb{C}[D_\infty]); \\ L_n^{\langle -\infty\rangle}(\mathbb{Z}) &\to L_n^{\langle -\infty\rangle}(\mathbb{Z}[D_\infty]). \end{aligned}$$

which are not surjective.

2.3 Varying the Family of Subgroups

Suppose we are given a family of subgroups $\mathcal{F}'$ and a subfamily $\mathcal{F} \subset \mathcal{F}'$. Since all isotropy groups of $E_{\mathcal{F}}(G)$ lie in $\mathcal{F}'$ we know from the universal property of $E_{\mathcal{F}'}(G)$ (see Section 4.2) that there is a G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{F}'}(G)$ which is unique up to G-homotopy. For every G-homology theory $\mathcal{H}_*^G$ we hence obtain a *relative assembly map*

$$A_{\mathcal{F}\to\mathcal{F}'} \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{F}'}(G)).$$

If $\mathcal{F}' = \mathcal{ALL}$, then $E_{\mathcal{F}'}(G) = \{\bullet\}$ and $A_{\mathcal{F}\to\mathcal{F}'}$ specializes to the assembly map $A_{\mathcal{F}}$ we discussed in the previous section. If we now gradually increase the family, we obtain a factorization of the classical assembly $A_{\mathcal{TR}}$ into several relative assembly maps. We obtain for example from the inclusions

$$\mathcal{TR} \subset \mathcal{FCY} \subset \mathcal{FIN} \subset \mathcal{VCY} \subset \mathcal{ALL}$$

for every G-homology theory $\mathcal{H}_n^G(-)$ the following commutative diagram.

$$\begin{tikzcd} \mathcal{H}_n^G(EG) \arrow[rr, "A_{\mathcal{TR}}"] \arrow[d] & & \mathcal{H}_n^G(\{\bullet\}) \\ \mathcal{H}_n^G(E_{\mathcal{FCY}}(G)) \arrow[r] & \mathcal{H}_n^G(E_{\mathcal{FIN}}(G)) \arrow[r] \arrow[ur, "A_{\mathcal{FIN}}"] & \mathcal{H}_n^G(E_{\mathcal{VCY}}(G)). \arrow[u, "A_{\mathcal{VCY}}"'] \end{tikzcd} \tag{2.10}$$

Such a factorization is extremely useful because one can study the relative assembly map $A_{\mathcal{F}\to\mathcal{F}'}$ in terms of absolute assembly maps corresponding to groups in the bigger family as explained next. For a family $\mathcal{F}$ of subgroups of G and a subgroup $H \subset G$ we define a family of subgroups of H

$$\mathcal{F} \cap H = \{K \cap H \mid K \in \mathcal{F}\}.$$

Theorem 2.11 (Transitivity Principle). *Let $\mathcal{H}^?_*(-)$ be an equivariant homology theory in the sense Subsection 4.4.2. Suppose $\mathcal{F} \subset \mathcal{F}'$ are two families of subgroups of G. Suppose that $K \cap H \in \mathcal{F}$ for each $K \in \mathcal{F}$ and $H \in \mathcal{F}'$ (this is automatic if $\mathcal{F}$ is closed under taking subgroups). Let N be an integer. If for every $H \in \mathcal{F}'$ and every $n \leq N$ the assembly map*

$$A_{\mathcal{F}\cap H\to\mathcal{ALL}} \colon \mathcal{H}^H_n(E_{\mathcal{F}\cap H}(H)) \to \mathcal{H}^H_n(\{\bullet\})$$

is an isomorphism, then for every $n \leq N$ the relative assembly map

$$A_{\mathcal{F}\to\mathcal{F}'} \colon \mathcal{H}^G_n(E_{\mathcal{F}}(G)) \to \mathcal{H}^G_n(E_{\mathcal{F}'}(G))$$

is an isomorphism.

Proof. If we equip $E_{\mathcal{F}}(G) \times E_{\mathcal{F}'}(G)$ with the diagonal G-action, it is a model for $E_{\mathcal{F}}(G)$. Now apply Lemma 4.16 in the special case $Z = E_{\mathcal{F}'}(G)$.

At the level of spectra this transitivity principle is due to Farrell and Jones [126, Theorem A.10]. Next we discuss applications of the Transitivity Principle 2.11.

2.3.1 The General Versions Specialize to the Torsionfree Versions

If G is a torsionfree group, then the family $\mathcal{FIN}$ obviously coincides with the trivial family $\mathcal{TR}$. Since a nontrivial torsionfree virtually cyclic group is infinite cyclic we also know that the family $\mathcal{VCY}$ reduces to the family of all cyclic subgroups, denoted $\mathcal{CYC}$.

Proposition 2.12. *Let G be a torsionfree group.*

(1) If R is a regular ring, then the relative assembly map

$$A_{\mathcal{TR}\to\mathcal{CYC}} \colon H^G_n(E_{\mathcal{TR}}(G); \mathbf{K}_R) \to H^G_n(E_{\mathcal{CYC}}(G); \mathbf{K}_R)$$

is an isomorphism;

(2) For every ring R the relative assembly map

$$A_{\mathcal{TR}\to\mathcal{CYC}} \colon H^G_n(E_{\mathcal{TR}}(G); \mathbf{L}^{\langle-\infty\rangle}_R) \to H^G_n(E_{\mathcal{CYC}}(G); \mathbf{L}^{\langle-\infty\rangle}_R)$$

is an isomorphism.

Proof. Because of the Transitivity Principle 2.11 it suffices in both cases to prove that the classical assembly map $A = A_{\mathcal{TR}\to\mathcal{ALL}}$ is an isomorphism in the case where G is an infinite cyclic group. For regular rings in the K-theory case and with the $-\infty$-decoration in the L-theory case this is true as we discussed in Remark 1.33 respectively Remark 1.47.

As an immediate consequence from Remark 2.8 and Proposition 2.12 we obtain

Corollary 2.13. *(1) For a torsionfree group the Baum-Connes Conjecture 2.4 is equivalent to its "torsionfree version" Conjecture 1.1;*
(2) For a torsionfree group the Farrell-Jones Conjecture 2.5 for algebraic K- is equivalent to the "torsionfree version" Conjecture 1.28, provided R is regular;
(3) For a torsionfree group the Farrell-Jones Conjecture 2.5 for algebraic L-theory is equivalent to the "torsionfree version" Conjecture 1.37.

Example 2.14. (Algebraic K-theory for RG for torsionfree word hyperbolic groups and arbitrary rings). In Chapter 1 we have formulated the Farrell-Jones Conjecture for torsionfree groups and algebraic K-theory (see Conjecture 1.37) only for regular rings. What does the Farrell-Jones Conjecture 2.5 implies if R is not regular. We want to illustrate this in a special case taken from [224, Remark 8.14].

Let G be a torsionfree group with the property that any infinite cyclic group has finite index in its centralizer. Examples are torsionfree word-hyperbolic groups. Then every infinite cyclic subgroups is contained in a unique maximal infinite cyclic subgroup. Let I be the set of conjugacy classes of maximal infinite cyclic subgroups. Then the Farrell-Jones Conjecture 2.5 predicts for all $n \in \mathbb{Z}$

$$K_n(RG) \;\cong\; H_n(BG; \mathbf{K}(R) \oplus \left(\bigoplus_I NK_n(R)\right).$$

In general the additional term coming from the Nil-terms is even more complicated.

Comment 19 (By W.): `Add later reference to the place, where we deal with Nil-groups and` $\underline{\underline{E}}G$ `explicitly. I think that I can prove a more general statement.`

2.3.2 The Baum-Connes Conjecture and the Families $\mathcal{FCY}$ and $\mathcal{VCY}$

Replacing the family $\mathcal{FIN}$ of finite subgroups by the family $\mathcal{VCY}$ of virtually cyclic subgroups would not make any difference in the Baum-Connes Conjecture 2.4. The Transitivity Principle 2.11 and the fact that the Baum-Connes Conjecture 2.4 is known for virtually cyclic groups implies the following.

Proposition 2.15. *For every group G and every $n \in \mathbb{Z}$ the relative assembly map for topological K-theory*

$$A_{\mathcal{FIN} \to \mathcal{VCY}} \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\mathrm{top}}) \to H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}^{\mathrm{top}})$$

is an isomorphism.

The following result is proven in [234] by considering Euler classes. A different proof using the Atiyah-Segal Completion Theorem for families, which in contrast to the one in [234] applies also to the real case, is presented in [24, Theorem 0.4].

Proposition 2.16. *For every group G and every $n \in \mathbb{Z}$ the relative assembly map for topological K-theory*

$$A_{\mathcal{FCY} \to \mathcal{FIN}} \colon H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}^{\mathrm{top}}) \to H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\mathrm{top}})$$

is an isomorphism.

In particular the Baum-Connes Conjecture predicts that the $\mathcal{FCY}$-assembly map

$$A_{\mathcal{FCY}} \colon H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}^{\mathrm{top}}) \to K_n(C_r^*(G))$$

is always an isomorphism.

2.3.3 Algebraic K-Theory for Special Coefficient Rings

In the algebraic K-theory case we can reduce to the family of finite subgroups if we assume special coefficient rings.

Proposition 2.17. *Suppose R is a regular ring in which the orders of all finite subgroups of G are invertible. Then for every $n \in \mathbb{Z}$ the relative assembly map for algebraic K-theory*

$$A_{\mathcal{FIN} \to \mathcal{VCY}} \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \to H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$$

is an isomorphism.

In particular the Farrell-Jones Conjecture 2.5 for K-theory implies the Farrell-Jones Conjecture for $K_*(RG)$ for regular rings 0.8.

Proof. We first show that RH is regular for a finite group H. Since R is Noetherian and H is finite, RH is Noetherian. It remains to show that every RH-module M has a finite dimensional projective resolution. By assumption M considered as an R-module has a finite dimensional projective resolution. If one applies $RH \otimes_R -$ this yields a finite dimensional RH-resolution of $RH \otimes_R \operatorname{res} M$. Since $|H|$ is invertible, the RH-module M is a direct summand of $RH \otimes_R \operatorname{res} M$ and hence has a finite dimensional projective resolution.

Because of the Transitivity Principle 2.11 we need to prove that the $\mathcal{FIN}$-assembly map $A_{\mathcal{FIN}}$ is an isomorphism for virtually cyclic groups V. Because of Lemma 2.18 we can assume that either $V \cong H \rtimes \mathbb{Z}$ or $V \cong K_1 *_H K_2$ with finite groups H, K_1 and K_2. From [341] we obtain in both cases long exact sequences involving the algebraic K-theory of the constituents, the algebraic K-theory of V and also additional Nil-terms. However, in both cases the Nil-terms vanish if RH is a regular ring (compare Theorem 4 on page 138 and the Remark on page 216 in [341]). Thus we get long exact sequences

$$\cdots \to K_n(RH) \to K_n(RH) \to K_n(RV) \to K_{n-1}(RH) \to K_{n-1}(RH) \to \cdots$$

and

$$\begin{aligned}\cdots \to K_n(RH) \to K_n(RK_1) \oplus K_n(RK_2) \to K_n(RV) \\ \to K_{n-1}(RH) \to K_{n-1}(RK_1) \oplus K_{n-1}(RK_2) \to \cdots\end{aligned}$$

One obtains analogous exact sequences for the sources of the various assembly maps from the fact that the sources are equivariant homology theories and one can find specific models for $E_{\mathcal{FIN}}(V)$. These sequences are compatible with the assembly maps. The assembly maps for the finite groups H, K_1 and K_2 are bijective. Now a Five-Lemma argument shows that also the one for V is bijective.

In the proof above we used the following important fact about virtually cyclic groups.

Lemma 2.18. *If G is an infinite virtually cyclic group then we have the following dichotomy.*

(I) Either G admits a surjection with finite kernel onto the infinite cyclic group $\mathbb{Z}$, or
(II) G admits a surjection with finite kernel onto the infinite dihedral group $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2 = \mathbb{Z} \rtimes \mathbb{Z}/2$.

Proof. The proof is not difficult and can be found [128, Lemma 2.5].

2.3.4 Splitting off Nil-Terms and Rationalized Algebraic *K*-Theory

Recall that the Nil-terms, which prohibit the classical assembly map from being an isomorphism, are direct summands of the K-theory of the infinite cyclic group (see Remark 1.33). Something similar remains true in general [20].

Proposition 2.19. *(1) For every group G, every ring R and every $n \in \mathbb{Z}$ the relative assembly map*

$$A_{\mathcal{FIN} \to \mathcal{VCY}} \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \to H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$$

is split-injective;

(2) *Suppose R is such that $K_{-i}(RV) = 0$ for all virtually cyclic subgroups V of G and for sufficiently large i (for example $R = \mathbb{Z}$ will do, compare Proposition 2.28). Then the relative assembly map*

$$A_{\mathcal{FIN}\to\mathcal{VCY}} \colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \to H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle})$$

is split-injective.

Combined with the Farrell-Jones Conjectures we obtain that the homology group $H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R)$ is a direct summand in $K_n(RG)$. It is much better understood (compare Chapter 8) than the remaining summand which is isomorphic to $H_n^G(E_{\mathcal{VCY}}(G), E_{\mathcal{FIN}}(G);\mathbf{K}_R)$. This remaining summand is the one which plays the role of the Nil-terms for a general group. It is known that for $R = \mathbb{Z}$ the negative dimensional Nil-groups which are responsible for virtually cyclic groups vanish [128]. They vanish rationally, in dimension 0 by [85] and in higher dimensions by [200]. For more information see also [84]. Analogously to the proof of Proposition 2.17 we obtain the following proposition.

Proposition 2.20. *We have*

$$H_n^G(E_{\mathcal{VCY}}(G), E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathbb{Z}}) = 0 \quad \text{for } n < 0 \quad \text{and}$$
$$H_n^G(E_{\mathcal{VCY}}(G), E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0 \quad \text{for all } n \in \mathbb{Z}.$$

In particular the Farrell-Jones Conjecture 2.5 for the algebraic K-theory of the integral group ring predicts that the map

$$A_{\mathcal{FIN}} \colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always an isomorphism.

2.3.5 Inverting 2 in *L*-Theory

Proposition 2.21. *For every group G, every ring R with involution, every decoration j and all $n \in \mathbb{Z}$ the relative assembly map*

$$A_{\mathcal{FIN}\to\mathcal{VCY}} \colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle j\rangle})[1/2] \to H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle j\rangle})[1/2]$$

is an isomorphism.

Proof. According to the Transitivity Principle 2.11 it suffices to prove the claim for a virtually cyclic group. Now argue analogously to the proof of Proposition 2.17 using the exact sequences in [54] and the fact that the UNil-terms appearing there vanish after inverting two [54]. Also recall from Remark 1.40 that after inverting 2 there are no differences between the decorations.

In particular the L-theoretic Farrell-Jones Conjecture 2.5 implies that for every decoration j the assembly map

$$A_{\mathcal{FIN}} \colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle j\rangle})[1/2] \to L_n^{\langle j\rangle}(RG)[1/2]$$

is an isomorphism.

2.3.6 *L*-theory and Virtually Cyclic Subgroups of the First Kind

Recall that a group is virtually cyclic of the first kind if it admits a surjection with finite kernel onto the infinite cyclic group. The family of these groups is denoted $\mathcal{VCY}_I$.

Proposition 2.22. *For all groups G, all rings R and all $n \in \mathbb{Z}$ the relative assembly map*

$$A_{\mathcal{FIN} \to \mathcal{VCY}_I} \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(E_{\mathcal{VCY}_I}(G); \mathbf{L}_R^{\langle -\infty \rangle})$$

is an isomorphism.

Proof. The point is that there are no UNil-terms for infinite virtually cyclic groups of the first kind. This follows essentially from [275] and [276] as carried out in [222, Lemma 4.2].

2.3.7 Rationally $\mathcal{FIN}$ Reduces to $\mathcal{FCY}$

We will see later (compare Theorem 8.6, 8.7 and 8.12) that in all three cases, topological K-theory, algebraic K-theory and L-theory, the rationalized left hand side of the $\mathcal{FIN}$-assembly map can be computed very explicitly using the equivariant Chern-Character. As a by-product these computations yield that after rationalizing the family $\mathcal{FIN}$ can be reduced to the family $\mathcal{FCY}$ of finite cyclic groups and that the rationalized relative assembly maps $A_{\mathcal{TR} \to \mathcal{FCY}}$ are injective.

Proposition 2.23. *For every ring R, every group G and all $n \in \mathbb{Z}$ the relative assembly maps*

$$A_{\mathcal{FCY} \to \mathcal{FIN}} \colon H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$A_{\mathcal{FCY} \to \mathcal{FIN}} \colon H_n^G(E_{\mathcal{FCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$A_{\mathcal{FCY} \to \mathcal{FIN}} \colon H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}^{\mathrm{top}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\mathrm{top}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are isomorphisms and the relative assembly maps $A_{\mathcal{TR} \to \mathcal{FCY}}$ and $A_{\mathcal{TR} \to \mathcal{FIN}}$ are all rationally injective.

Recall that the statement for topological K-theory is even known integrally, compare Proposition 2.16. Combining the above with Proposition 2.20 and Proposition 2.21 we see that the Farrell-Jones Conjecture 2.5 predicts in particular that the $\mathcal{FCY}$-assembly maps

$$A_{\mathcal{FCY}} \colon H_n^G(E_{\mathcal{FCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_n^{\langle -\infty \rangle}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$A_{\mathcal{FCY}} \colon H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are always isomorphisms.

2.4 Induction Theorems

2.4.1 The Conjectures as Generalized Induction Theorems

The results above illuminate that one may think of the Farrell-Jones Conjectures 2.5 and the Baum-Connes Conjecture 2.4 as "generalized induction theorems". The prototype of an induction theorem is Artin's Theorem about the representation ring $R_{\mathbb{C}}(G)$ of a finite group G. Let us recall Artin's Theorem. For a proof see for instance [89, 15.4 on page 378] or [309, Theorem 26 in 12.5 on page 97].

For finite groups H and a field F of characteristic zero the representation ring $R_F(H)$ coincides with $K_0(FH)$. Let $\mathbb{Z}[1/|G|] \subseteq \mathbb{Q}$ be the ring obtained from $\mathbb{Z}$ by inverting the order $|G|$ of G.

Theorem 2.24 (Artin's Theorem). *The induction homomorphism*

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{CYC}}(G)} R_{\mathbb{C}}(H) \otimes_{\mathbb{Z}} \mathbb{Z}[1/|G|] \xrightarrow{\cong} R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Z}[1/|G|]$$

is an isomorphism.

Note that this is a very special case of Theorem 8.6 or 8.7, compare Remark 8.11.

Artin's Theorem 2.24 says that rationally one can compute $R_{\mathbb{C}}(G)$ if one knows all the values $R_{\mathbb{C}}(C)$ (including all maps coming from induction with group homomorphisms induced by conjugation with elements in G) for all cyclic subgroups $C \subseteq G$. The idea behind the Farrell-Jones Conjectures 2.5 and the Baum-Connes Conjecture 2.4 is analogous. We want to compute the functors $K_n(RG)$, $L_n(RG)$ and $K_n(C_r^*(G))$ from their values (including their functorial properties under induction) on elements of the family $\mathcal{FIN}$ or $\mathcal{VCY}$.

The situation in the Farrell Jones and Baum-Connes Conjectures is more complicated than in Artin's Theorem 2.24, since we have already seen in Subsection 0.2.2 that a computation of $K_n(RG)$, $L_n^{\langle -\infty \rangle}(RG)$ and $K_n(C_r^*(G))$ does involve also the values $K_p(RH)$, $L_p^{\langle -\infty \rangle}(RH)$ and $K_p(C_r^*(H))$ for $p \leq n$. A degree mixing occurs.

2.4.2 Dress Induction for Infinite Groups

Very important for the computations of K- and L-groups of group rings of finite groups are the induction theorems due to Dress [100] and [101]. They reduce the computations for a finite group to certain families of subgroups, where the family depends on the coefficient ring and the theory one is considering. Artin's Theorem 2.24 is a special case. In Bartels-Lück [23] the induction theorems for finite groups due to Dress have been generalized to infinite groups. Here is a summary of the main results.

Let $\mathcal{F} \subseteq \mathcal{FIN}$ be a subclass of the class of finite groups which is closed under isomorphism of groups and taking subgroups. Define

$$\mathcal{F}' \subseteq \mathcal{VCY}$$

to be the class of groups V for which there exists an extension $1 \to \mathbb{Z} \to V \to F \to 1$ for a group $F \in \mathcal{F}$ or for which $V \in \mathcal{F}$ holds. With this notion we get $\mathcal{VCY} = \mathcal{FIN}'$.

Let p be a prime. A finite group G is called *p-elementary* if it is isomorphic to $C \times P$ for a cyclic group C and a p-group P such that the order $|C|$ is prime to p. A finite group G is called *p-hyperelementary* if it can be written as an extension $1 \to C \to G \to P \to 1$ for a cyclic group C and a p-group P such that the order $|C|$ is prime to p. A finite group G is called *elementary* or *hyperelementary* respectively if it is p-elementary or p-hyperelementary respectively for an appropriate prime p. Let $\mathcal{E}_p$ and $\mathcal{H}_p$ respectively be the class of groups which are p-elementary groups and p-hyperelementary respectively for a prime p. Let $\mathcal{E}$ and $\mathcal{H}$ respectively be the class of groups which are elementary and hyperelementary respectively.

The following results refer to the Farrell-Jones Conjecture 2.5 or to the fibered version 2.84 (see Subsection 2.13.2 and Section 2.14). The phrase in the range $\leq N$ means that the corresponding assembly maps are bijective in dimensions $n \leq N$ for some integer N and no statement is made for $n > N$.

Theorem 2.25 (Induction theorem for algebraic K-theory). *Let G be a group and let N be an integer. Then*

(1) The group G satisfies the (Fibered) Farrell-Jones Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{VCY}$ if and only if G satisfies the (Fibered) Farrell-Jones Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{H}'$;

(2) Let p be a prime. Then G satisfies the (Fibered) Farrell-Jones Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{VCY}$ after applying $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} -$ if and only if G satisfies the (Fibered) Farrell-Jones Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{H}_p'$ after applying $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} -$;

(3) Suppose that R is regular and $\mathbb{Q} \subseteq R$. Then the group G satisfies the Farrell-Jones Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{VCY}$ if and only if G satisfies the Farrell-Jones Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{H}$.
If we assume that R is regular and $\mathbb{C} \subseteq R$, then we can replace $\mathcal{H}$ by $\mathcal{E}$;

(4) Suppose that R is regular and $\mathbb{Q} \subseteq R$. Let p be a prime. Then G satisfies the Farrell-Jones Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{VCY}$ after applying $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} -$ if and only if G satisfies the Farrell-Jones Conjecture (in the range $\leq N$) for

algebraic K-theory with coefficients in R for the family $\mathcal{H}_p$ after applying $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} -$.
If we assume that R is regular and $\mathbb{C} \subseteq R$, then we can replace $\mathcal{H}_p$ by $\mathcal{E}_p$.

Theorem 2.26 (Induction theorem for algebraic L-theory). *Let G be a group. Then*

(1) The group G satisfies the (Fibered) Farrell-Jones Conjecture (in the range $\leq N$) for algebraic L-theory with coefficients in R for the family $\mathcal{VCY}$ if and only if G satisfies the (Fibered) Farrell-Jones Conjecture for algebraic L-theory with coefficients in R for the family $\left(\mathcal{H}_2 \cup \bigcup_{p \text{ prime}, p \neq 2} \mathcal{E}_p\right)'$;
(2) The group G satisfies the (Fibered) Farrell-Jones Conjecture (in the range $\leq N$) for algebraic L-theory with coefficients in R for the family $\mathcal{VCY}$ after applying $\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} -$ if and only if G satisfies the (Fibered) Farrell-Jones Conjecture for algebraic L-theory with coefficients in R for the family $\bigcup_{p \text{ prime}, p \neq 2} \mathcal{E}_p$ after applying $\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} -$.

2.5 Specializations of the Farrell-Jones Conjecture for Lower and Middle K-Theory

As opposed to topological K-theory and L-theory, which are periodic, the algebraic K-theory groups of coefficient rings such as $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$ are known to be bounded below. Using the spectral sequences for the left hand side of an assembly map that will be discussed in Subsection 8.5.1 this leads to vanishing results in negative dimensions and a concrete description of the groups in the first non-vanishing dimension.

We begin with the following conjecture. Explanations about the colimit that appears follow below.

Conjecture 2.27. (The Farrell-Jones Conjecture for $K_n(\mathbb{Z}G)$ for $n \leq -1$). *For every group G we have*

$$K_{-n}(\mathbb{Z}G) = 0 \quad \textit{for } n \geq 2,$$

and the map

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{F}}(G)} K_{-1}(\mathbb{Z}H) \xrightarrow{\cong} K_{-1}(\mathbb{Z}G)$$

is an isomorphism, where $\mathcal{F}$ can be chosen as the family $\mathcal{FIN}$ of finite subgroups or as the family $\mathcal{H}$ of hyperelementary finite subgroups.

Given a family $\mathcal{F}$ of subgroups of G define the category $\mathtt{Sub}_{\mathcal{F}}(G)$ as follows. Objects are subgroups H with $H \in \mathcal{F}$. For $H, K \in \mathcal{F}$ let $\operatorname{conhom}_G(H,K)$ be the set of all group homomorphisms $f \colon H \to K$, for which there exists a group element $g \in G$ such that f is given by conjugation with g. The group of inner automorphism $\operatorname{inn}(K)$ consists of those automorphisms $K \to K$, which are given by conjugation with an element $k \in K$. It acts on $\operatorname{conhom}(H,K)$ from the left by composition. Define the set of morphisms in $\mathtt{Sub}_{\mathcal{F}}(G)$ from H to K to be $\operatorname{inn}(K)\backslash \operatorname{conhom}(H,K)$. Composition of group homomorphisms defines the composition of morphisms in $\mathtt{Sub}_{\mathcal{F}}(G)$. We mention that $\mathtt{Sub}_{\mathcal{F}}(G)$ is a quotient category of the orbit category $\mathtt{Or}_{\mathcal{F}}(G)$ which we will introduce in Section 4.4.5. Note that there is a morphism from H to K only if H is conjugate to a subgroup of K. Clearly $K_n(R(-))$ yields a functor from $\mathtt{Sub}_{\mathcal{F}}(G)$ to abelian groups since inner automorphisms on a group G induce the identity on $K_n(RG)$. Using the inclusions into G, one obtains a map

$$i\colon \operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{F}}(G)} K_n(RH) \to K_n(RG).$$

The colimit can be interpreted as the 0-th Bredon homology group

$$H_0^G(E_{\mathcal{F}}(G); K_n(R(?)))$$

(compare Example 4.12) and the map i is the edge homomorphism in the equivariant Atiyah-Hirzebruch spectral sequence discussed in Subsection 8.5.1. In Conjecture 2.27 we consider the first non-vanishing entry in the lower left hand corner of the E_2-term because of the following vanishing result [128, Theorem 2.1] which generalizes vanishing results for finite groups from [63].

Proposition 2.28. *If V is a virtually cyclic group, then $K_{-n}(\mathbb{Z}V) = 0$ for $n \geq 2$.*

Therefore Conjecture 2.27 is a consequence of the K-theoretic Farrell-Jones Conjecture 2.5 in the case $R = \mathbb{Z}$. Note that by the results discussed in Subsection 2.3.4 we know that in negative dimensions we can replace $\mathcal{VCY}$ by $\mathcal{FIN}$ and then by Theorem 2.25 (1) by $\mathcal{H}$ (see also [23, Section 7]).

If our coefficient ring R is a regular ring in which the orders of all finite subgroups of G are invertible, then we know already from Proposition 2.17 that we can reduce to the family of finite subgroups. In the proof of Proposition 2.17 we have seen that then RH is again regular if $H \subset G$ is finite. Since negative K-groups vanish for regular rings [289, 5.3.30 on page 295], the following is implied by the Farrell-Jones Conjecture 2.5 (see also [23, Section 7]).

Conjecture 2.29. (Farrell-Jones Conjecture for $K_0(RG)$ for regular rings R). *Suppose R is a regular ring in which the orders of all finite subgroups of G are invertible (for example a field of characteristic 0), then*

$$K_{-n}(RG) = 0 \quad \text{for } n \geq 1$$

and the map

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{F}}(G)} K_0(RH) \xrightarrow{\cong} K_0(RG)$$

is an isomorphism, where $\mathcal{F}$ can be chosen to be $\mathcal{FIN}$ or $\mathcal{H}$. If we assume that R is regular and $\mathbb{C} \subseteq R$, then $\mathcal{F}$ can be chosen to be the family $\mathcal{E}$ of elementary subgroups.

The conjecture above holds if G is virtually poly-cyclic. Surjectivity is proven in [248] (see also [75] and Chapter 8 in [256]), injectivity in [293]. We will show in Lemma 2.38 (1) that the map appearing in the conjecture is always rationally injective for $R = \mathbb{C}$.

The conjectures above describe the first non-vanishing term in the equivariant Atiyah-Hirzebruch spectral sequence (see Subsection 8.5.1). Already the next step is much harder to analyze in general because there are potentially non-vanishing differentials. We know however that after rationalizing the equivariant Atiyah-Hirzebruch spectral sequence for the left hand side of the $\mathcal{FIN}$-assembly map collapses. **Comment 20** (By W.): `Add later reference?` As a consequence we obtain that the following conjecture follows from the K-theoretic Farrell-Jones Conjecture 2.5.

Conjecture 2.30. *For every group G, every ring R and every $n \in \mathbb{Z}$ the map*

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_n(RH) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective.

Note that for $K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$ the conjecture above is always true but not very interesting, because for a finite group H it is known that $\widetilde{K}_0(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, compare [325, Proposition 9.1], and hence the left hand side reduces to $K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. However, the full answer for $K_0(\mathbb{Z}G)$ should involve the negative K-groups, compare Example 8.10.

Analogously to Conjecture 2.30 the following can be derived from the K-theoretic Farrell-Jones Conjecture 2.5, compare [227].

Conjecture 2.31. *The map*

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} \operatorname{Wh}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always injective.

In general one does not expect this map to be an isomorphism. There should be additional contributions coming from negative K-groups. Conjecture 2.31 is true for groups satisfying a mild homological finiteness condition, see Theorem 3.28.

2.6 G-Theory

Instead of considering finitely generated projective modules one may apply the standard K-theory machinery to the category of finitely generated modules. This leads to the definition of the groups $G_n(R)$ for $n \geq 0$. For instance $G_0(R)$ is the abelian group whose generators are isomorphism classes $[M]$ of finitely generated R-modules and whose relations are given by $[M_0] - [M_1] + [M_2]$ for any exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of finitely generated modules. One may ask whether versions of the Farrell-Jones Conjectures for G-theory instead of K-theory might be true. The answer is negative as the following discussion explains.

For a finite group H the ring $\mathbb{C}H$ is semisimple. Hence any finitely generated $\mathbb{C}H$-module is automatically projective and $K_0(\mathbb{C}H) = G_0(\mathbb{C}H)$. Recall that a group G is called *virtually poly-cyclic* if there exists a subgroup of finite index $H \subseteq G$ together with a filtration $\{1\} = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots \subseteq H_r = H$ such that H_{i-1} is normal in H_i and the quotient H_i/H_{i-1} is cyclic. More generally for all $n \in \mathbb{Z}$ the forgetful map

$$f\colon K_n(\mathbb{C}G) \to G_n(\mathbb{C}G)$$

is an isomorphism if G is virtually poly-cyclic, since then $\mathbb{C}G$ is regular [298, Theorem 8.2.2 and Theorem 8.2.20] and the forgetful map f is an isomorphism for regular rings, compare [289, Corollary 53.26 on page 293]. In particular this applies to virtually cyclic groups and so the left hand side of the Farrell-Jones assembly map does not see the difference between K- and G-theory if we work with complex coefficients. We obtain a commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \arrow[r] \arrow[d, "\cong"'] & K_0(\mathbb{C}G) \arrow[d, "f"] \\
\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} G_0(\mathbb{C}H) \arrow[r] & G_0(\mathbb{C}G)
\end{tikzcd}$$

where, as indicated, the left hand vertical map is an isomorphism. Conjecture 2.29, which is implied by the Farrell-Jones Conjecture, says that the upper horizontal arrow is an isomorphism. A G-theoretic analogue of the Farrell-Jones Conjecture would say that the lower horizontal map is an isomorphism. However, there are cases where the upper horizontal arrow is known to be an isomorphism, but the forgetful map f on the right is not injective or not surjective:

If G contains a non-abelian free subgroup, then the class $[\mathbb{C}G] \in G_0(\mathbb{C}G)$ vanishes [220, Theorem 9.66 on page 364] and hence the map $f\colon K_0(\mathbb{C}G) \to G_0(\mathbb{C}G)$ has an infinite kernel ($[\mathbb{C}G]$ generates an infinite cyclic subgroup in $K_0(\mathbb{C}G)$). The Farrell-Jones Conjecture for $K_0(\mathbb{C}G)$ (see Conjecture 2.29) is known for non-abelian free groups.

The Farrell-Jones Conjecture is also known for $A = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2$ and hence $K_0(\mathbb{C}A)$ is countable, whereas $G_0(\mathbb{C}A)$ is not countable [220, Example 10.13 on page 375]. Hence the map f cannot be surjective.

At the time of writing we do not know a counterexample to the statement that for an amenable group G, for which there is an upper bound on the orders of its finite subgroups, the forgetful map $f\colon K_0(\mathbb{C}G) \to G_0(\mathbb{C}G)$ is an isomorphism. We do not know a counterexample to the statement that for a group G, which is not amenable, $G_0(\mathbb{C}G) = \{0\}$. We also do not know whether $G_0(\mathbb{C}G) = \{0\}$ is true for $G = \mathbb{Z} * \mathbb{Z}$.

For more information about $G_0(\mathbb{C}G)$ we refer for instance to [220, Subsection 9.5.3].

2.7 Bass Conjectures

Complex representations of a finite group can be studied using characters. We now want to define the Hattori-Stallings rank of a finitely generated projective $\mathbb{C}G$-module which should be seen as a generalization of characters to infinite groups.

Let $\operatorname{con}(G)$ be the set of conjugacy classes (g) of elements $g \in G$. Denote by $\operatorname{con}(G)_f$ the subset of $\operatorname{con}(G)$ consisting of those conjugacy classes (g) for which each representative g has finite order. Let $\operatorname{class}_0(G)$ and $\operatorname{class}_0(G)_f$ be the $\mathbb{C}$-vector space with the set $\operatorname{con}(G)$ and $\operatorname{con}(G)_f$ as basis. This is the same as the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on $\operatorname{con}(G)$ and $\operatorname{con}(G)_f$ with finite support. Define the *universal $\mathbb{C}$-trace* as

$$\operatorname{tr}^u_{\mathbb{C}G}\colon \mathbb{C}G \to \operatorname{class}_0(G), \quad \sum_{g\in G} \lambda_g \cdot g \ \mapsto \ \sum_{g \in G} \lambda_g \cdot (g). \tag{2.32}$$

It extends to a function $\operatorname{tr}^u_{\mathbb{C}G}\colon M_n(\mathbb{C}G) \to \operatorname{class}_0(G)$ on (n,n)-matrices over $\mathbb{C}G$ by taking the sum of the traces of the diagonal entries. Let P be a finitely generated projective $\mathbb{C}G$-module. Choose a matrix $A \in M_n(\mathbb{C}G)$ such that $A^2 = A$ and the image of the $\mathbb{C}G$-map $r_A\colon \mathbb{C}G^n \to \mathbb{C}G^n$ given by right multiplication with A is $\mathbb{C}G$-isomorphic to P. Define the *Hattori-Stallings rank* of P as

$$\operatorname{HS}_{\mathbb{C}G}(P) = \operatorname{tr}^u_{\mathbb{C}G}(A) \in \operatorname{class}_0(G). \tag{2.33}$$

The Hattori-Stallings rank depends only on the isomorphism class of the $\mathbb{C}G$-module P and induces a homomorphism $\operatorname{HS}_{\mathbb{C}G}\colon K_0(\mathbb{C}G) \to \operatorname{class}_0(G)$.

Conjecture 2.34 (Strong Bass Conjecture for $K_0(\mathbb{C}G)$). *The $\mathbb{C}$-vector space spanned by the image of the map*

$$\operatorname{HS}_{\mathbb{C}G}\colon K_0(\mathbb{C}G) \to \operatorname{class}_0(G)$$

is $\operatorname{class}_0(G)_f$.

This conjecture is implied by the surjectivity of the map

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{C} \to K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C}, \tag{2.35}$$

and hence by the Farrell-Jones Conjecture for $K_0(\mathbb{C}G)$ (see Conjecture 2.29). We will see below that the surjectivity of the map (2.35) also implies that the map $K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \to \operatorname{class}_0(G)$, which is induced by the Hattori-Stallings rank, is injective. Hence we do expect that the Hattori-Stallings rank induces an isomorphism

$$K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \operatorname{class}_0(G)_f.$$

There are also versions of the Bass conjecture for other coefficients than $\mathbb{C}$. It follows from results of Linnell [209, Theorem 4.1 on page 96] that the following version is implied by the Strong Bass Conjecture for $K_0(\mathbb{C}G)$ (see Conjecture 2.34).

Conjecture 2.36 (The Strong Bass Conjecture for $K_0(\mathbb{Z}G)$). *The image of the composition*

$$K_0(\mathbb{Z}G) \to K_0(\mathbb{C}G) \xrightarrow{\mathrm{HS}_{\mathbb{C}G}} \operatorname{class}_0(G)$$

is contained in the $\mathbb{C}$-vector space of those functions $f\colon \operatorname{con}(G) \to \mathbb{C}$ which vanish for $(g) \in \operatorname{con}(g)$ with $g \neq 1$.

Finally we mention the following variant of the Bass Conjecture.

Conjecture 2.37 (The Weak Bass Conjecture). *Let P be a finitely generated projective $\mathbb{Z}G$-module. The value of the Hattori-Stallings rank of $\mathbb{C}G \otimes_{\mathbb{Z}G} P$ at the conjugacy class of the identity element is given by*

$$\mathrm{HS}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{Z}G} P)((1)) \;=\; \dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P).$$

Here $\mathbb{Z}$ is considered as a $\mathbb{Z}G$-module by the trivial G-action.

The K-theoretic Farrell-Jones Conjecture implies the Conjectures 2.34, 2.36 and 2.37 above. More precisely we have the following proposition.

Proposition 2.38. *(1) The map*

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always injective. If the map is also surjective (compare Conjecture 2.29) then the Hattori-Stallings rank induces an isomorphism

$$K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \operatorname{class}_0(G)_f$$

and in particular the Strong Bass Conjecture for $K_0(\mathbb{C}G)$ (see Conjecture 2.34) and hence also the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$ (see Conjecture 2.36) hold;

(2) The surjectivity of the assembly map appearing in the Farrell-Jones Conjecture 2.5

$$A_{\mathcal{VCY}}\colon H_0^G(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the Rational $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$ Conjecture 2.44 and hence also the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$ (see Conjecture 2.36);

(3) The Strong Bass Conjecture for $K_0(\mathbb{C}G)$(see Conjecture 2.34) implies the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$. The Strong Bass Conjecture for $K_0(\mathbb{Z}G)$ (see Conjecture 2.36) implies the Weak Bass Conjecture 2.37.

Proof. (1) follows from the following commutative diagram, compare [216, Lemma 2.15 on page 220]

$$\begin{tikzcd}
\operatorname{colim}_{H\in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H)\otimes_{\mathbb{Z}}\mathbb{C} \arrow[rr] \arrow[d, "\cong"'] & & K_0(\mathbb{C}G)\otimes_{\mathbb{Z}}\mathbb{C} \arrow[d] \\
\operatorname{colim}_{H\in \mathtt{Sub}_{\mathcal{FIN}}(G)} \operatorname{class}_0(H) \arrow[r, "\cong"] & \operatorname{class}_0(G)_f \arrow[r, "i"] & \operatorname{class}_0(G).
\end{tikzcd}$$

Here the vertical maps are induced by the Hattori-Stallings rank, the map i is the natural inclusion and in particular injective and we have the indicated isomorphisms.

(2) According to Proposition 2.20 the surjectivity of the map $A_{\mathcal{VCY}}$ appearing in (2) implies the surjectivity of the corresponding assembly map $A_{\mathcal{FIN}}$ (rationalized and with $\mathbb{Z}$ as coefficient ring) for the family of finite subgroups. The map $A_{\mathcal{FIN}}$ is natural with respect to the change of the coefficient ring from $\mathbb{Z}$ to $\mathbb{Q}$. By Theorem 8.7 we know that for every coefficient ring R there is an isomorphism from

$$\bigoplus_{p,q,p+q=0}\ \bigoplus_{(C)\in(\mathcal{FCY})} H_p(BZ_GC;\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\Theta_C\cdot K_q(RC)\otimes_{\mathbb{Z}}\mathbb{Q}$$

to the 0-dimensional part of the left hand side of the rationalized $\mathcal{FIN}$-assembly map $A_{\mathcal{FIN}}$. The isomorphism is natural with respect to a change of coefficient rings. To see that the Rational $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$ Conjecture 2.44 follows, it hence suffices to show that the summand corresponding to $C=\{1\}$ and $p=q=0$ is the only one where the map induced from $\mathbb{Z}\to\mathbb{Q}$ is possibly non-trivial. But $K_q(\mathbb{Q}C)=0$ if $q<0$, because $\mathbb{Q}C$ is semisimple and hence regular, and for a finite cyclic group $C\neq\{1\}$ we have by [216, Lemma 7.4]

$$\Theta_C\cdot K_0(\mathbb{Z}C)\otimes_{\mathbb{Z}}\mathbb{Q}=\operatorname{coker}\left(\bigoplus_{D\subsetneq C} K_0(\mathbb{Z}D)\otimes_{\mathbb{Z}}\mathbb{Q}\ \to\ K_0(\mathbb{Z}C)\otimes_{\mathbb{Z}}\mathbb{Q}\right)=0,$$

since by a result of Swan $K_0(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}\to K_0(\mathbb{Z}H)\otimes_{\mathbb{Z}}\mathbb{Q}$ is an isomorphism for a finite group H, see [325, Proposition 9.1].

(3) As already mentioned the first statement follows from [209, Theorem 4.1 on page 96]. The second statement follows from the formula

$$\sum_{(g)\in\operatorname{con}(G)} \operatorname{HS}_{\mathbb{C}G}(\mathbb{C}\otimes_{\mathbb{Z}}P)(g)=\dim_{\mathbb{Z}}(\mathbb{Z}\otimes_{\mathbb{Z}G}P).$$

The next result is due to Berrick, Chatterji and Mislin [42, Theorem 5.2]. The Bost Conjecture is a variant of the Baum-Connes Conjecture and is explained in Subsection 2.12.3.

Theorem 2.39. *If the assembly map appearing in the Bost Conjecture 2.72 is rationally surjective, then the Strong Bass Conjecture for $K_0(\mathbb{C}G)$ (compare 2.34) is true.*

Remark 2.40. (The Strong Bass Conjecture for fields of characteristic zero). Let F be a field of characteristic zero. Fix an integer $m \geq 1$. Let $F \subseteq F(\zeta_m)$ be the Galois extension given by adjoining the primitive m-th root of unity ζ_m to F. Denote by $\mathrm{Gal}(F \subset F(\zeta_m))$ the Galois group of this extension of fields, i.e. the group of automorphisms of fields $\sigma\colon F(\zeta_m) \to F(\zeta_m)$ which induce the identity on F. It can be identified with a subgroup of $\mathbb{Z}/m^*$ by sending σ to the unique element $u(\sigma) \in \mathbb{Z}/m^*$ for which $\sigma(\zeta_m) = \zeta_m^{u(\sigma)}$ holds.

Let G be a group. Let g_1 and g_2 be two elements of G of finite order. We call them F-conjugate if for some (and hence all) positive integers m with $g_1^m = g_2^m = 1$ there exists an element σ in the Galois group $\mathrm{Gal}(F \subset F(\zeta_m))$ with the property $(g_1^{u(\sigma)}) = (g_2)$. Denote by $\mathrm{con}(G,F)_f$ the set of F-conjugacy classes $(g)_F$ of elements $g \in G$ of finite order. Let $\mathrm{class}_0(G,F)_f$ be the F-vector space with the set $\mathrm{con}(G,F)_f$ as basis, or, equivalently, the F-vector space of functions $\mathrm{con}(G,F)_f \to F$ with finite support. Recall that for a finite group H taking characters yields an isomorphism [310, Corollary 1 on page 96]

$$\chi\colon F \otimes_{\mathbb{Z}} R_F(H) = F \otimes_{\mathbb{Z}} K_0(FH) \xrightarrow{\cong} \mathrm{class}_0(H,F)_f. \tag{2.41}$$

By [219, Theorem 0.4] and(2.41) the map appearing in the version of the Farrell-Jones Conjecture 2.29 for $K_0(FG)$ can be identified with a map

$$\mathrm{class}_0(G,F)_f \to F \otimes_{\mathbb{Z}} K_0(FG).$$

Using the Hattori-Stallings rank one can show that this map is always injective. If the version of the Farrell-Jones Conjecture 2.29 is true, this map is an isomorphism. This generalizes (2.41) for finite groups to infinite groups.

More information and further references about the Bass Conjecture can be found for instance in [29], [42, Section 7], [48], [106], [107], [133], [209] [220, Subsection 9.5.2], and [246, page 66ff].

2.8 Change of Rings for $\mathbb{Z}G$, $\mathbb{Q}G$, $C_r^*(G)$ and $\mathcal{N}(G)$ for K_0

We now discuss some further questions and facts that seem to be relevant in the context of the Bass Conjectures.

Conjecture 2.42. (Integral $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$-Conjecture). *The change of coefficient map*

$$\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Q}G)$$

is trivial.

Note that the Integral $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$-Conjecture above would imply that the following diagram commutes.

$$\begin{tikzcd} K_0(\mathbb{Z}G) \arrow[r] \arrow[d, "p_*"'] & K_0(\mathbb{Q}G) \\ K_0(\mathbb{Z}) \arrow[r, "\dim_{\mathbb{Z}}", "\cong"'] & \mathbb{Z}. \arrow[u, "i"'] \end{tikzcd}$$

Here p_* is induced by the projection $G \to \{1\}$ and i sends $1 \in \mathbb{Z}$ to the class of $\mathbb{Q}G$.

We do not know a proof which shows that the K-theoretic Farrell-Jones Conjecture 2.5 implies the conjecture above although both conjectures are linked. We can only conclude from it a factorization

$$\widetilde{K}_0(\mathbb{Z}G) \to H_1^G(E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathbb{Z}}) \to \widetilde{K}_0(\mathbb{Q}G) \tag{2.43}$$

This follows from the equivariant Atiyah-Hirzebruch spectral sequence and Proposition 2.19 since for finite H the change of rings map $\widetilde{K}_0(\mathbb{Z}H) \to \widetilde{K}_0(\mathbb{Q}H)$ is trivial, $K_n(\mathbb{Z}H) = 0$ for $n \leq -2$ and $|H| < \infty$ and $K_n(\mathbb{Q}V) = 0$ for virtually cyclic V and $n \leq -1$. **Comment 21** (By W.): `Shall we elaborate on this or put this as an exercise?` At least the K-theoretic Farrell-Jones Conjecture 2.5 implies the following rational version

Conjecture 2.44 (Rational $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$-Conjecture). *For every group G the map*

$$\widetilde{K}_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \to \widetilde{K}_0(\mathbb{Q}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

induced by the change of coefficients is trivial.

Recall that the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$ (see Conjecture 2.36) says that for every finitely generated projective $\mathbb{Z}G$-module P the Hattori-Stallings rank of $\mathbb{C}G \otimes_{\mathbb{Z}G} P$ looks like the Hattori-Stallings rank of a free $\mathbb{C}G$-module. Clearly it follows from the Rational $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$-Conjecture 2.44

Remark 2.45. The passage from $\widetilde{K}_0(\mathbb{Z}G)$ to $\widetilde{K}_0(\mathcal{N}(G))$). Let $\mathcal{N}(G)$ denote the group von Neumann algebra of G. It is known that for every group G the composition

$$\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Q}G) \to \widetilde{K}_0(\mathbb{C}G) \to \widetilde{K}_0(C_r^*(G)) \to \widetilde{K}_0(\mathcal{N}(G))$$

is the zero-map (see for instance [220, Theorem 9.62 on page 362] or [302]). Since the group von Neumann algebra $\mathcal{N}(G)$ is not functorial under arbitrary group homomorphisms such as $G \to \{1\}$, this does *not* imply that the diagram

$$\begin{array}{ccc} K_0(\mathbb{Z}G) & \longrightarrow & K_0(\mathcal{N}(G)) \\ {\scriptstyle p_*}\downarrow & & \uparrow{\scriptstyle i} \\ K_0(\mathbb{Z}) & \xrightarrow[\cong]{\dim_{\mathbb{Z}}} & \mathbb{Z} \end{array}$$

commutes. However, commutativity is equivalent to the Weak Bass Conjecture 2.37. For a discussion of these questions see [107].

Finally we mention the following result. Denote by Λ^G the subring of $\mathbb{Q}$ which is obtained from $\mathbb{Z}$ by inverting all orders $|H|$ of finite subgroups H of G, i.e.

$$\Lambda^G = \mathbb{Z}\left[|H|^{-1} \mid H \subset G,\ |H| < \infty\right]. \tag{2.46}$$

Theorem 2.47. (Passage from $K_0(C_r^*(G))$ to $K_0(\mathcal{N}(G))$). *Let G be a group which satisfies the Baum-Connes Conjecture 2.4. Then the image of the map*

$$\Lambda^G \otimes_{\mathbb{Z}} K_0(C_r^*(G)) \to \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$$

is contained in the image of the map

$$\operatorname{colim}_{C \in \mathsf{Sub}(G;\mathcal{FCY})} \left(\Lambda^G \otimes_{\mathbb{Z}} R_{\mathbb{C}}(C)\right) \;\to\; \Lambda^G \otimes_{\mathbb{Z}} K_0(\mathcal{N}(G))$$

which comes from the induction homomorphisms $R_{\mathbb{C}}(C) = K_0(\mathcal{N}(C)) \to K_0(\mathcal{N}(G))$ for the finite cyclic subgroups $C \subseteq G$.

Proof. This follows from [220, Theorem 5.4].

In particular we see that for a torsionfree group G which satisfies the Baum-Connes Conjecture 2.4 the map $\widetilde{K}_0(C_r^*(G)) \to \widetilde{K}_0(\mathcal{N}(G))$ is trivial. This has also been proved by Chatterji-Mislin [70].

If G is finite, then Theorem 2.47 agrees with Artin's Theorem 2.24.

2.9 The Novikov Conjecture

The Novikov Conjecture which we will explain below has been one of the main motivation to formulate and to study the Baum-Connes Conjecture 2.4 and the Farrell-Jones Conjecture 2.5.

2.9.1 The Original Novikov Conjecture

We now explain the Novikov Conjecture in its original formulation.

Let G be a (not necessarily torsionfree) group and $u\colon M \to BG$ be a map from a closed oriented smooth manifold M to BG. Let $\mathcal{L}(M) \in \prod_{k\geq 0} H^k(M;\mathbb{Q})$ be the *L-class of M*, which is a certain polynomial in the Pontrjagin classes and hence depends a priori on the tangent bundle and hence on the smooth structure of M. For $x \in \prod_{k\geq 0} H^k(BG;\mathbb{Q})$ define the *higher signature of M associated to x and u* to be

$$\operatorname{sign}_x(M,u) := \langle \mathcal{L}(M) \cup u^*x, [M]\rangle \qquad \in \mathbb{Q}. \tag{2.48}$$

The Hirzebruch signature formula says that for $x = 1$ the signature $\operatorname{sign}_1(M,u)$ coincides with the ordinary signature $\operatorname{sign}(M)$ of M, if $\dim(M) = 4n$, and is zero, if $\dim(M)$ is not divisible by four. Recall that for $\dim(M) = 4n$ the *signature* $\operatorname{sign}(M)$ of M is the signature of the non-degenerate bilinear symmetric pairing on the middle cohomology $H^{2n}(M;\mathbb{R})$ given by the intersection pairing $(a,b) \mapsto \langle a \cup b, [M]\rangle$. Obviously $\operatorname{sign}(M)$ depends only on the oriented homotopy type of M. We say that sign_x for $x \in H^*(BG;\mathbb{Q})$ is *homotopy invariant* if for two closed oriented smooth manifolds M and N with reference maps $u\colon M \to BG$ and $v\colon N \to BG$ we have

$$\operatorname{sign}_x(M,u) = \operatorname{sign}_x(N,v)$$

if there is an orientation preserving homotopy equivalence $f\colon M \to N$ such that $v \circ f$ and u are homotopic.

Conjecture 2.49 (Novikov Conjecture). *Let G be a group. Then* sign_x *is homotopy invariant for all $x \in \prod_{k\geq 0} H^k(BG;\mathbb{Q})$.*

By Hirzebruch's signature formula the Novikov Conjecture 2.49 is true for $x = 1$.

2.9.2 The Novikov Conjecture Follows from the Farrell-Jones Conjecture or from the Baum-Connes Conjecture

There is the following version of the Novikov Conjecture 2.49.

Conjecture 2.50 (K- and L-theoretic Novikov Conjectures). *For every group G the classical assembly maps*

$$\begin{aligned} H_*(BG;\mathbf{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\to K_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \\ H_*(BG;\mathbf{L}^p(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\to L^p_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \\ K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} &\to K_*(C^*_r(G)) \otimes_{\mathbb{Z}} \mathbb{Q} \end{aligned}$$

are injective.

Observe that, since the $\mathbb{Z}/2$-Tate cohomology groups vanish rationally, there is no difference between the various decorations in L-theory because of the Rothenberg sequence. We have chosen the p-decoration above.

Using surgery theory one can show [281, Proposition 6.6 on page 300] the following **Comment 22** (By W.): `Add reference or explanation?`

Proposition 2.51. *For a group G the original Novikov Conjecture 2.49 is equivalent to the L-theoretic Novikov Conjecture 2.50, i.e. the injectivity of the classical assembly maps*

$$H_*(BG; \mathbf{L}^p(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \to L^p_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Later in Proposition 2.55 we will prove in particular the following statement.

Proposition 2.52. *The Novikov Conjecture 2.50 for topological K-theory, i.e. the injectivity of the classical assembly map*

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_*(C^*_r(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the L-theoretic Novikov Conjecture 2.50 and hence the original Novikov Conjecture 2.49.

We conclude from Proposition 2.19 and Proposition 2.23 together with Proposition 2.51 and Proposition 2.52

Proposition 2.53. *(1) The rational injectivity of the assembly map appearing in L-theoretic Farrell-Jones Conjecture (Conjecture 2.5) implies the L-theoretic Novikov Conjecture (Conjecture 2.50) and hence the original Novikov Conjecture 2.49;*

(2) The rational injectivity of the assembly map appearing the Baum-Connes Conjecture 2.4 implies the injectivity of the rationalized classical assembly map

$$K_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(C^*_r(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

and hence the L-theoretic Novikov Conjecture 2.50 and the original Novikov Conjecture 2.49;

(3) The rational injectivity of the assembly map appearing in the Farrell-Jones Conjecture for algebraic K-theory (Conjecture 2.5) implies the K-theoretic Novikov Conjecture for R as coefficients, i.e. the injectivity of the classical assembly map

$$H_n(BG; \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 2.54 (Integral Injectivity Fails). In general the classical assembly maps $A = A_{\mathcal{TR}}$ themselves, i.e. without rationalizing, are not injective. For example one can use the Atiyah-Hirzebruch spectral sequence (see Subsection 8.5.1) to see that for $G = \mathbb{Z}/5$

$$H_1(BG;\mathbf{K}^{\mathrm{top}}) \quad \text{and} \quad H_1(BG;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z}))$$

contain 5-torsion, whereas for every finite group G the topological K-theory of $\mathbb{C}G$ is torsionfree and the torsion in the L-theory of $\mathbb{Z}G$ is always 2-torsion, compare Proposition 8.1 1 and Proposition 8.3 1.

For more information about the Novikov Conjectures we refer for instance to [44], [58], [61], [91], [135], [142], [197], [279] and [290].

2.10 Relating Topological *K*-Theory and *L*-Theory

For every real C^*-algebra A there is an isomorphism $L_n^p(A)[1/2] \xrightarrow{\cong} K_n(A)[1/2]$ (see [290]). This can be used to compare L-theory to topological K-theory and leads to the following result.

Proposition 2.55. *Let $\mathcal{F}$ be a family of subgroups of G with $\mathcal{F} \subseteq \mathcal{FIN}$. If the topological K-theory assembly map*

$$A_{\mathcal{F}}\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{K}^{\mathrm{top}})[1/2] \to K_n(C_r^*(G))[1/2]$$

is injective, then for an arbitrary decoration j also the map

$$A_{\mathcal{F}}\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{L}_{\mathbb{Z}}^{\langle j\rangle})[1/2] \to L_n^{\langle j\rangle}(\mathbb{Z}G)[1/2]$$

is injective.

Proof. First recall from Remark 1.40 that after inverting 2 there is no difference between the different decorations and we can hence work with the p-decoration. One can construct for any subfamily $\mathcal{F} \subseteq \mathcal{FIN}$ the following commutative diagram [218, Section 7.5]

$$
\begin{array}{ccc}
H_n^G(E_{\mathcal{F}}(G);\mathbf{L}^p_{\mathbb{Z}}[1/2]) & \xrightarrow{A^1_{\mathcal{F}}} & L^p_n(\mathbb{Z}G)[1/2] \\
{\scriptstyle i_1}\downarrow\cong & & {\scriptstyle j_1}\downarrow\cong \\
H_n^G(E_{\mathcal{F}}(G);\mathbf{L}^p_{\mathbb{Q}}[1/2]) & \xrightarrow{A^2_{\mathcal{F}}} & L^p_n(\mathbb{Q}G)[1/2] \\
{\scriptstyle i_2}\downarrow\cong & & {\scriptstyle j_2}\downarrow \\
H_n^G(E_{\mathcal{F}}(G);\mathbf{L}^p_{\mathbb{R}}[1/2]) & \xrightarrow{A^3_{\mathcal{F}}} & L^p_n(\mathbb{R}G)[1/2] \\
{\scriptstyle i_3}\downarrow\cong & & {\scriptstyle j_3}\downarrow \\
H_n^G(E_{\mathcal{F}}(G);\mathbf{L}^p_{C^*_r(?;\mathbb{R})}[1/2]) & \xrightarrow{A^4_{\mathcal{F}}} & L^p_n(C^*_r(G;\mathbb{R}))[1/2] \\
{\scriptstyle i_4}\downarrow\cong & & {\scriptstyle j_4}\downarrow\cong \\
H_n^G(E_{\mathcal{F}}(G);\mathbf{K}^{\mathrm{top}}_{\mathbb{R}}[1/2]) & \xrightarrow{A^5_{\mathcal{F}}} & K_n(C^*_r(G;\mathbb{R}))[1/2] \\
{\scriptstyle i_5}\downarrow & & {\scriptstyle j_5}\downarrow \\
H_n^G(E_{\mathcal{F}}(G);\mathbf{K}^{\mathrm{top}}_{\mathbb{C}}[1/2]) & \xrightarrow{A^6_{\mathcal{F}}} & K_n(C^*_r(G))[1/2]
\end{array}
$$

Here

$$
\begin{gathered}
\mathbf{L}^p_{\mathbb{Z}}[1/2], \quad \mathbf{L}^p_{\mathbb{Q}}[1/2], \quad \mathbf{L}^p_{\mathbb{R}}[1/2], \quad \mathbf{L}_{C^*_r(?;\mathbb{R})}[1/2], \\
\mathbf{K}^{\mathrm{top}}_{\mathbb{R}}[1/2] \quad \text{and} \quad \mathbf{K}^{\mathrm{top}}_{\mathbb{C}}[1/2]
\end{gathered}
$$

are covariant $\mathsf{Or}(G)$-spectra (compare Section 4.4.4 and in particular Proposition 4.20) such that the n-th homotopy group of their evaluations at G/H are given by

$$
\begin{gathered}
L^p_n(\mathbb{Z}H)[1/2], \quad L^p_n(\mathbb{Q}H)[1/2], \quad L^p_n(\mathbb{R}H)[1/2], \quad L^p_n(C^*_r(H;\mathbb{R}))[1/2], \\
K_n(C^*_r(H;\mathbb{R}))[1/2] \quad \text{respectively} \quad K_n(C^*_r(H)[1/2].
\end{gathered}
$$

All horizontal maps are assembly maps induced by the projection $\mathrm{pr}\colon E_{\mathcal{F}}(G) \to \{\bullet\}$. The maps i_k and j_k for $k = 1, 2, 3$ are induced from a change of rings. The isomorphisms i_4 and j_4 come from the general isomorphism for any real C^*-algebra A

$$L^p_n(A)[1/2] \xrightarrow{\cong} K_n(A)[1/2]$$

and its spectrum version [290, Theorem 1.11 on page 350]. The maps i_1, j_1, i_2 are isomorphisms by [277, page 376] and [279, Proposition 22.34 on page 252]. The map i_3 is bijective since for a finite group H we have $\mathbb{R}H = C^*_r(H;\mathbb{R})$. The maps i_5 and j_5 are given by extending the scalars from $\mathbb{R}$ to $\mathbb{C}$ by induction. For every real C^*-algebra A the composition

$$K_n(A)[1/2] \to K_n(A \otimes_{\mathbb{R}} \mathbb{C})[1/2] \to K_n(M_2(A))[1/2]$$

is an isomorphism and hence j_5 is split injective. An $\mathsf{Or}(G)$-spectrum version of this argument yields that also i_5 is split injective.

Remark 2.56. One may conjecture that the right vertical maps j_2 and j_3 are isomorphisms and try to prove this directly. Then if we invert 2 everywhere the Baum-Connes Conjecture 2.4 for the real reduced group C^*-algebra, would be equivalent to the Farrell-Jones Farrell-Jones Conjecture for $L_*(\mathbb{Z}G)[1/2]$.

2.11 Further Applications of the Baum-Connes Conjecture

Next we discuss some further applications of the Baum-Connes Conjecture 2.4.

2.11.1 The Modified Trace Conjecture

The following conjecture generalizes Conjecture 1.10 to the case where the group need no longer be torsionfree. For the standard trace compare (1.8).

Conjecture 2.57 (Modified Trace Conjecture for a group G). *Let G be a group. Let Λ^G be the ring introduced in* (2.46). *Then the image of the homomorphism induced by the standard trace*

$$\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R} \tag{2.58}$$

is contained in Λ^G.

The following result is proved in [221, Theorem 0.3]. It follows from Theorem 2.47.

Theorem 2.59. *Let G be a group. Then the image of the composition*

$$K_0^G(E_{\mathcal{FIN}}(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{A_{\mathcal{FIN}} \otimes_{\mathbb{Z}} \mathrm{id}} K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\operatorname{tr}_{C_r^*(G)}} \mathbb{R}$$

is Λ^G. Here $A_{\mathcal{FIN}}$ is the map appearing in the Baum-Connes Conjecture 2.4. In particular the Baum-Connes Conjecture 2.4 implies the Modified Trace Conjecture.

The original version of the Trace Conjecture which is due to Baum and Connes [32, page 21] makes the stronger statement that the image of $\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$ is the additive subgroup of $\mathbb{Q}$ generated by all numbers $\frac{1}{|H|}$, where $H \subset G$ runs though all finite subgroups of G. Roy has constructed a counterexample to this version in [299] based on her article [300]. The examples of Roy do *not* contradict the Modified Trace Conjecture 2.57 or the Baum-Connes Conjecture 2.4.

2.11.2 The Stable Gromov-Lawson-Rosenberg Conjecture

The Stable Gromov-Lawson-Rosenberg Conjecture is a typical conjecture relating Riemannian geometry to topology. It is concerned with the question when a given manifold admits a metric of positive scalar curvature. To discuss its relation with the Baum-Connes Conjecture we will need the real version of the Baum-Connes Conjecture, compare Subsection 2.12.1.

Let $\Omega_n^{\mathrm{Spin}}(BG)$ be the bordism group of closed Spin-manifolds M of dimension n with a reference map to BG. Let $C_r^*(G;\mathbb{R})$ be the real reduced group C^*-algebra and let $KO_n(C_r^*(G;\mathbb{R})) = K_n(C_r^*(G;\mathbb{R}))$ be its topological K-theory. We use KO instead of K as a reminder that we here use the real reduced group C^*-algebra. Given an element $[u\colon M \to BG] \in \Omega_n^{\mathrm{Spin}}(BG)$, we can take the $C_r^*(G;\mathbb{R})$-valued index of the equivariant Dirac operator associated to the G-covering $\overline{M} \to M$ determined by u. Thus we get a homomorphism

$$\mathrm{ind}_{C_r^*(G;\mathbb{R})}\colon \Omega_n^{\mathrm{Spin}}(BG) \to KO_n(C_r^*(G;\mathbb{R})). \tag{2.60}$$

A *Bott manifold* is any simply connected closed Spin-manifold B of dimension 8 whose $\widehat{A}$-genus $\widehat{A}(B)$ is 8. We fix such a choice, the particular choice does not matter for the sequel. Notice that $\mathrm{ind}_{C_r^*(\{1\};\mathbb{R})}(B) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$ is a generator and the product with this element induces the Bott periodicity isomorphisms $KO_n(C_r^*(G;\mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(G;\mathbb{R}))$. In particular

$$\mathrm{ind}_{C_r^*(G;\mathbb{R})}(M) = \mathrm{ind}_{C_r^*(G;\mathbb{R})}(M \times B), \tag{2.61}$$

if we identify $KO_n(C_r^*(G;\mathbb{R})) = KO_{n+8}(C_r^*(G;\mathbb{R}))$ via Bott periodicity.

Conjecture 2.62. (Stable Gromov-Lawson-Rosenberg Conjecture). *Let M be a closed connected* Spin*-manifold of dimension $n \geq 5$. Let $u_M\colon M \to B\pi_1(M)$ be the classifying map of its universal covering. Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if*

$$\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}([M, u_M]) \;=\; 0 \qquad \in KO_n(C_r^*(\pi_1(M);\mathbb{R})).$$

If M carries a Riemannian metric with positive scalar curvature, then the index of the Dirac operator must vanish by the Bochner-Lichnerowicz formula [288]. The converse statement that the vanishing of the index implies the existence of a Riemannian metric with positive scalar curvature is the hard part of the conjecture. The following result is due to Stolz. A sketch of the proof can be found in [324, Section 3], details are announced to appear in a different paper.

Theorem 2.63. *If the assembly map for the real version of the Baum-Connes Conjecture (compare Subsection 2.12.1) is injective for the group G, then the Stable Gromov-Lawson-Rosenberg Conjecture 2.62 is true for all closed* Spin*-manifolds of dimension ≥ 5 with $\pi_1(M) \cong G$.*

The requirement $\dim(M) \geq 5$ is essential in the Stable Gromov-Lawson-Rosenberg Conjecture, since in dimension four new obstructions, the Seiberg-Witten invariants, occur. The unstable version of this conjecture says that M carries a Riemannian metric with positive scalar curvature if and only if $\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}([M, u_M]) = 0$. Schick [304] constructs counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau (see also [105]). It is not known at the time of writing whether the unstable version is true for finite fundamental groups. Since the Baum-Connes Conjecture 2.4 is true for finite groups (for the trivial reason that $E_{\mathcal{FIN}}(G) = \{\bullet\}$ for finite groups G), Theorem 2.63 implies that the Stable Gromov-Lawson Conjecture 2.62 holds for finite fundamental groups (see also [291]).

The index map appearing in (2.60) can be factorized as a composition

$$\operatorname{ind}_{C_r^*(G;\mathbb{R})} \colon \Omega_n^{\mathrm{Spin}}(BG) \xrightarrow{D} KO_n(BG) \xrightarrow{A} KO_n(C_r^*(G;\mathbb{R})), \qquad (2.64)$$

where D sends $[M, u]$ to the class of the G-equivariant Dirac operator of the G-manifold $\overline{M}$ given by u and $A = A_{\mathcal{TR}}$ is the real version of the classical assembly map. The homological Chern character defines an isomorphism

$$KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \in \mathbb{Z}} H_{n+4p}(BG; \mathbb{Q}).$$

Recall that associated to M there is the $\widehat{A}$-*class*

$$\widehat{\mathcal{A}}(M) \in \prod_{p \geq 0} H^p(M; \mathbb{Q}) \qquad (2.65)$$

which is a certain polynomial in the Pontrjagin classes. The map D appearing in (2.64) sends the class of $u \colon M \to BG$ to $u_*(\widehat{\mathcal{A}}(M) \cap [M])$, i.e. the image of the Poincaré dual of $\widehat{\mathcal{A}}(M)$ under the map induced by u in rational homology. Hence $D([M, u]) = 0$ if and only if $u_*(\widehat{\mathcal{A}}(M) \cap [M])$ vanishes. For $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$ define the *higher* $\widehat{A}$-*genus of* (M, u) *associated to* x to be

$$\widehat{A}_x(M, u) = \langle \widehat{\mathcal{A}}(M) \cup u^* x, [M] \rangle = \langle x, u_*(\widehat{\mathcal{A}}(M) \cap [M]) \rangle \ \in \mathbb{Q}. \qquad (2.66)$$

The vanishing of $\widehat{\mathcal{A}}(M)$ is equivalent to the vanishing of all higher $\widehat{A}$-genera $\widehat{A}_x(M, u)$.

2.11.3 The Homological Gromov-Lawson-Rosenberg Conjecture

The following conjecture is a weak version of the Stable Gromov-Lawson-Rosenberg Conjecture 2.62

Conjecture 2.67. (Homological Gromov-Lawson-Rosenberg Conjecture). *Let G be a group. Then for any closed* Spin*-manifold M, which admits a Riemannian metric with positive scalar curvature, the $\widehat{A}$-genus $\widehat{A}_x(M, u)$ vanishes for all maps $u \colon M \to BG$ and elements $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$.*

From the discussion above we obtain the following result.

Proposition 2.68. *If the classical assembly map*

$$KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to KO_n(C_r^*(G;\mathbb{R})) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective for all $n \in \mathbb{Z}$, then the Homological Gromov-Lawson-Rosenberg Conjecture holds for G.

Remark 2.69. (Aspherical manifolds and positive scalar curvature). The Homological Gromov-Lawson-Rosenberg Conjecture 2.67 implies that an aspherical closed Spin-manifold M cannot carry a Riemannian metric with positive scalar curvature. Namely, we can take $G = \pi_1(M)$, $M = BG$, $u = \mathrm{id}$ and $x \in H^{\dim(M)}(M;\mathbb{Q})$ to be the class uniquely determined by $\langle x, [M]\rangle = 1$. Since the component of $\widehat{A}(M)$ in dimension zero is 1, we get

$$\widehat{A}_x(M, \mathrm{id}) \ = \ \langle x, [M]\rangle \ = \ 1.$$

2.12 Variants of the Baum-Connes Conjecture

In this section we discuss variants of the Baum-Connes Conjecture 2.4.

2.12.1 The Real Version of the Baum-Connes Conjecture

There is an obvious real version of the Baum-Connes Conjecture, which predicts that for all $n \in \mathbb{Z}$ and groups G the assembly map

$$A^{\mathbb{R}}_{\mathcal{FIN}} \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{R}}^{\mathrm{top}}) \to KO_n(C_r^*(G;\mathbb{R}))$$

is an isomorphism. Here $H_n^G(-;\mathbf{K}_{\mathbb{R}}^{\mathrm{top}})$ is an equivariant homology theory whose distinctive feature is that $H_n^G(G/H;\mathbf{K}_{\mathbb{R}}^{\mathrm{top}}) \cong KO_n(C_r^*(H;\mathbb{R}))$. Recall that we write $KO_n(-)$ only to remind ourselves that the C^*-algebra we apply it to is a real C^*-algebra, like for example the real reduced group C^*-algebra $C_r^*(G;\mathbb{R})$. The following result appears in [36].

Proposition 2.70. *The Baum-Connes Conjecture 2.4 implies the real version of the Baum-Connes Conjecture.*

In the proof of Proposition 2.55 we have already seen that after inverting 2 the “real assembly map” is a retract of the complex assembly map. In particular with 2-inverted or after rationalizing also injectivity results or surjectivity results about the complex Baum-Connes assembly map yield the corresponding results for the real Baum-Connes assembly map.

2.12.2 The Version of the Baum-Connes Conjecture for Maximal Group C^*-Algebras

For a group G let $C^*_m(G)$ be its *maximal group C^*-algebra*, compare [263, 7.1.5 on page 229]. The maximal group C^*-algebra has the advantage that every homomorphism of groups $\phi\colon G \to H$ induces a homomorphism $C^*_m(G) \to C^*_m(H)$ of C^*-algebras. This is not true for the reduced group C^*-algebra $C^*_r(G)$. Here is a counterexample: since $C^*_r(F)$ is a simple algebra if F is a non-abelian free group [266], there is no unital algebra homomorphism $C^*_r(F) \to C^*_r(\{1\}) = \mathbb{C}$.

One can construct a version of the Baum-Connes assembly map using an equivariant homology theory $H^G_n(-;\mathbf{K}^{\mathrm{top}}_m)$ which evaluated on G/H yields the K-theory of $C^*_m(H)$ (use Proposition 4.20 and a suitable modification of $\mathbf{K}^{\mathrm{top}}$, compare Section 4.3.1).

Since on the left hand side of a $\mathcal{FIN}$-assembly map only the maximal group C^*-algebras for finite groups H matter, and clearly $C^*_m(H) = \mathbb{C}H = C^*_r(H)$ for such H, this left hand side coincides with the left hand side of the usual Baum-Connes Conjecture. There is always a C^*-homomorphism $p\colon C^*_m(G) \to C^*_r(G)$ (it is an isomorphism if and only if G is amenable [263, Theorem 7.3.9 on page 243]) and hence we obtain the following factorization of the usual Baum-Connes assembly map

$$\begin{tikzcd} & K_n(C^*_m(G)) \arrow[d, "K_n(p)"] \\ H^G_n(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathrm{top}}) \arrow[ur, "A^m_{\mathcal{FIN}}"] \arrow[r, "A_{\mathcal{FIN}}"'] & K_n(C^*_r(G)) \end{tikzcd} \tag{2.71}$$

The *maximal C^*-algebra version of the Baum-Connes Conjecture* says that map $A^m_{\mathcal{FIN}}$ is bijective. However, it is known that the map $A^m_{\mathcal{FIN}}$ is in general not surjective. The Baum-Connes Conjecture would imply that the map is $A^m_{\mathcal{FIN}}$ is always injective, and that it is surjective if and only if the vertical map $K_n(p)$ is injective.

A countable group G is called *K-amenable* if the map $p\colon C^*_m(G) \to C^*_r(G)$ induces a KK-equivalence (compare [87]). This implies in particular that the vertical map $K_n(p)$ is an isomorphism for all $n \in \mathbb{Z}$. Note that for K-amenable groups the Baum-Connes Conjecture holds if and only if the "maximal" version of the assembly map $A^m_{\mathcal{FIN}}$ is an isomorphism for all $n \in \mathbb{Z}$. A-T-menable groups are K-amenable, compare Theorem 3.1. But $K_0(p)$ is not injective for every infinite group which has property (T) such as for example $SL_n(\mathbb{Z})$ for $n \geq 3$, compare for instance the discussion in [180]. There are groups with property (T) for which the Baum-Connes Conjecture is known (compare Subsection 3.2.2 and hence there are counterexamples to the conjecture that $A^m_{\mathcal{FIN}}$ is an isomorphism.

In Theorem 1.59 and Remark 1.60 we have discussed applications of the maximal C^*-algebra version of the Baum-Connes Conjecture.

The left hand side of the assembly map appearing in the Baum-Connes Conjecture 0.6 is natural with respect to arbitrary group homomorphisms $\phi\colon G \to K$. Namely, such a group homomorphism induces a ϕ-equivariant map $\underline{E}G \to \underline{E}K$ and thus a G-map $f\colon K \times_\phi \underline{E}G \to \underline{E}K$. Since $\underline{E}G$ is proper, there is a canonical map $K_n^G(\underline{E}G) \to K_n^K(K \times_\phi \underline{E}G)$. Its composition with the map induced by f yields a homomorphism

$$\phi_*\colon K_n^G(\underline{E}G) \to K_n^K(\underline{E}K).$$

Thus the Baum-Connes Conjecture 0.6 predicts that ϕ induces a natural map

$$\phi_*\colon K_n(C_r^*(G)) \to K_n(C_r^*(K)).$$

Such a map has not been constructed in general without using the Baum-Connes Conjecture for the reduced group C^*-algebra. Notice that such a map obviously exists for the maximal group C^*-algebra and that assembly map for the maximal group C^*-algebra is compatible with these maps. This will follow in our setup from the constructions **Comment 23** (By W.): `Add reference.` and has been proved in the KK-picture by Valette [335, Theorem 1.1]. **Comment 24** (By W.): `We have discussed this in the torsionfree case already in Section 1.11. Shall we add somewhere a systematically discussion of the naturality of the source of the assembly map and its target. If yes, we should add references accordingly and possibly cut the discussion at other places.`

2.12.3 The Bost Conjecture

Some of the strongest results about the Baum-Connes Conjecture are proven using the so called Bost Conjecture (see [204]). The Bost Conjecture is the version of the Baum-Connes Conjecture, where one replaces the reduced group C^*-algebra $C_r^*(G)$ by the Banach algebra $l^1(G)$ of absolutely summable functions on G. Again one can use the spectra approach (compare Subsection 4.4.4 and 4.3.1 and in particular Proposition 4.20) to produce a variant of equivariant K-homology denoted $H_n^G(-;\mathbf{K}_{l^1}^{\mathrm{top}})$ which this time evaluated on G/H yields $K_n(l^1(H))$, the topological K-theory of the Banach algebra $l^1(H)$. Analogously as in Subsection 2.2.2 we obtain an associated assembly map and we believe that it coincides with the one defined using a Banach-algebra version of KK-theory in [204]. **Comment 25** (By W.): `Do Mislin or someone else elaborates on this?`

Conjecture 2.72 (Bost Conjecture). *Let G be a countable group. Then the assembly map*

$$A_{\mathcal{FIN}}^{l^1}\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{l^1}^{\mathrm{top}}) \to K_n(l^1(G))$$

is an isomorphism.

Again the left hand side coincides with the left hand side of the Baum-Connes assembly map because for finite groups H we have $l^1(H) = \mathbb{C}H = C_r^*(H)$ as topological complex algebras. There is always an injective norm decreasing $*$-homomorphism $q\colon l^1(G) \to C_r^*(G)$ and one obtains a factorization of the usual Baum-Connes assembly map

$$\begin{tikzcd}
 & K_n(l^1(G)) \arrow[d, "K_n(q)"] \\
H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\mathrm{top}}) \arrow[ur, "A_{\mathcal{FIN}}^{l^1}"] \arrow[r, "A_{\mathcal{FIN}}"] & K_n(C_r^*(G)).
\end{tikzcd}$$

Every group homomorphism $G \to H$ induces a homomorphism of Banach algebras $l^1(G) \to l^1(H)$. So similar as in the maximal group C^*-algebra case this approach repairs the lack of functoriality for the reduced group C^*-algebra.

The disadvantage of $l^1(G)$ is however that indices of operators tend to take values in the topological K-theory of the group C^*-algebras, not in $K_n(l^1(G))$. Moreover the representation theory of G is closely related to the group C^*-algebra, whereas the relation to $l^1(G)$ is not well understood.

For more information about the Bost Conjecture 2.72 see [204], [315].

2.12.4 The Baum-Connes Conjecture with Coefficients

The Baum-Connes Conjecture 2.4 can be generalized to the Baum-Connes Conjecture with Coefficients. Let A be a separable C^*-algebra with an action of the countable group G. Then there is an assembly map

$$KK_n^G(E_{\mathcal{FIN}}(G); A) \to K_n(A \rtimes G) \tag{2.73}$$

defined in terms of equivariant KK-theory, compare Sections 5.4 and 5.5.

Conjecture 2.74. (Baum-Connes Conjecture with Coefficients). *For every separable C^*-algebra A with an action of a countable group G and every $n \in \mathbb{Z}$ the assembly map* (2.73) *is an isomorphism.*

There are counterexamples to the Baum-Connes Conjecture with Coefficients, compare Remark 3.3. If we take $A = \mathbb{C}$ with the trivial action, the map (2.73) can be identified with the assembly map appearing in the ordinary Baum-Connes Conjecture 2.4.

Remark 2.75 (A Spectrum Level Description). There is a formulation of the Baum-Connes Conjecture with Coefficients in the framework explained in Section 4.4.4. Namely, construct an appropriate covariant functor $\mathbf{K}^{\mathrm{top}}(A \rtimes \mathcal{G}^G(-))\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ such that

$$\pi_n(\mathbf{K}^{\mathrm{top}}(A \rtimes \mathcal{G}^G(G/H)) \cong K_n(A \rtimes H)$$

holds for all subgroups $H \subseteq G$ and all $n \in \mathbb{Z}$, and consider the associated G-homology theory $H_*^G(-;\mathbf{K}^{\mathrm{top}}(A \rtimes \mathcal{G}^G(-)))$. Then the map (2.73) can be identified with the map which the projection $\mathrm{pr}\colon E_{\mathcal{FIN}}(G) \to \{\bullet\}$ induces for this homology theory.

Remark 2.76 (Farrell-Jones Conjectures with Coefficients). One can also formulate a "Farrell-Jones Conjecture with Coefficients". (This should not be confused with the Fibered Farrell-Jones Conjecture discussed in Subsection 2.13.2.) Fix a ring S and an action of G on it by isomorphisms of rings. Construct an appropriate covariant functor $\mathbf{K}(S \rtimes \mathcal{G}^G(-))\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ such that

$$\pi_n(\mathbf{K}(S \rtimes \mathcal{G}^G(G/H))) \cong K_n(S \rtimes H)$$

holds for all subgroups $H \subseteq G$ and $n \in \mathbb{Z}$, where $S \rtimes H$ is the associated twisted group ring. Now consider the associated G-homology theory $H_*^G(-;\mathbf{K}(S \rtimes \mathcal{G}^G(-)))$. There is an analogous construction for L-theory. A *Farrell-Jones Conjecture with Coefficients* would say that the map induced on these homology theories by the projection $\mathrm{pr}\colon E_{\mathcal{VCY}}(G) \to \{\bullet\}$ is always an isomorphism. We do not know whether there are counterexamples to the Farrell-Jones Conjectures with Coefficients, compare Remark 3.3.

2.12.5 The Coarse Baum Connes Conjecture

We briefly explain the Coarse Baum-Connes Conjecture, a variant of the Baum-Connes Conjecture, which applies to metric spaces. Its importance lies in the fact that isomorphism results about the Coarse Baum-Connes Conjecture can be used to prove injectivity results about the classical assembly map for topological K-theory. Compare also Section 6.5.

Let X be a proper (closed balls are compact) metric space and H_X a separable Hilbert space with a faithful nondegenerate $*$-representation of $C_0(X)$, the algebra of complex valued continuous functions which vanish at infinity. A bounded linear operator T has a support $\operatorname{supp} T \subset X \times X$, which is defined as the complement of the set of all pairs (x, x'), for which there exist functions ϕ and $\phi' \in C_0(X)$ such that $\phi(x) \neq 0$, $\phi'(x') \neq 0$ and $\phi' T \phi = 0$. The operator T is said to be a finite propagation operator if there exists a constant α such that $d(x, x') \leq \alpha$ for all pairs in the support of T. The operator is said to be *locally compact* if ϕT and $T\phi$ are compact for every $\phi \in C_0(X)$. An operator is called *pseudolocal* if $\phi T \psi$ is a compact operator for all pairs of continuous functions ϕ and ψ with compact and disjoint supports.

The *Roe algebra* $C^*(X) = C(X, H_X)$ is the operator-norm closure of the $*$-algebra of all locally compact finite propagation operators on H_X. The algebra $D^*(X) = D^*(X, H_X)$ is the operator-norm closure of the pseudolocal finite propagation operators. One can show that the topological K-theory of the quotient algebra $D^*(X)/C^*(X)$ coincides up to an index shift with the analytically defined (non-equivariant) K-homology $K_*(X)$, compare Section 5.2. For a uniformly contractible proper metric space the coarse assembly map $K_n(X) \to K_n(C^*(X))$ is the boundary map in the long exact sequence associated to the short exact sequence of C^*-algebras

$$0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0.$$

For general metric spaces one first approximates the metric space by spaces with nice local behaviour, compare [284]. For simplicity we only explain the case, where X is a discrete metric space. Let $P_d(X)$ the Rips complex for a fixed distance d, i.e. the simplicial complex with vertex set X, where a simplex is spanned by every collection of points in which every two points are a distance less than d apart. Equip $P_d(X)$ with the spherical metric, compare [365].

A discrete metric space has *bounded geometry* if for each $r > 0$ there exists a $N(r)$ such that for all x the ball of radius r centered at $x \in X$ contains at most $N(r)$ elements.

Conjecture 2.77. (Coarse Baum-Connes Conjecture). *Let X be a proper discrete metric space of bounded geometry. Then for $n = 0, 1$ the coarse assembly map*

$$\operatorname{colim}_d K_n(P_d(X)) \to \operatorname{colim}_d K_n(C^*(P_d(X))) \cong K_n(C^*(X))$$

is an isomorphism.

The conjecture is false if one drops the bounded geometry hypothesis. A counterexample can be found in [366, Section 8]. Our interest in the conjecture stems from the following fact, compare [284, Chapter 8].

Proposition 2.78. *Suppose the finitely generated group G admits a classifying space BG of finite type. If G considered as a metric space via a word length metric satisfies the Coarse Baum-Connes Conjecture 2.77, then the classical assembly map $A\colon K_*(BG) \to K_*(C_r^*G)$ which appears in Conjecture 1.1 is injective.*

The Coarse Baum-Connes Conjecture for a discrete group G (considered as a metric space) can be interpreted as a case of the Baum-Connes Conjecture with Coefficients 2.74 for the group G with a certain specific choice of coefficients, compare [370].

Further information about the coarse Baum-Connes Conjecture can be found for instance in [167], [168], [170], [284], [364], [371], [365], [367], and [368].

2.12.6 The Baum-Connes Conjecture for Non-Discrete Groups

Throughout this subsection let T be a locally compact second countable topological Hausdorff group. There is a notion of a classifying space for proper

T-actions $\underline{E}T$ (see [33, Section 1 and 2] [332, Section I.6], [225, Section 1]) and one can define its equivariant topological K-theory $K_n^T(\underline{E}T)$. The definition of a reduced C^*-algebra $C_r^*(T)$ and its topological K-theory $K_n(C_r^*(T))$ makes sense also for T. There is an assembly map defined in terms of equivariant index theory

$$A_{\mathcal{K}} \colon K_n^T(\underline{E}T) \to K_n(C_r^*(T)). \tag{2.79}$$

The Baum-Connes Conjecture for T says that this map is bijective for all $n \in \mathbb{Z}$ [33, Conjecture 3.15 on page 254].

Now consider the special case where T is a connected Lie group. Let $\mathcal{K}$ be the family of compact subgroups of T. There is a notion of a T-CW-complex and of a classifying space $E_{\mathcal{K}}(T)$ **Comment 26** (By W.): `Add reference to the literature or to a section in the book depending on whether we treat topological groups.` The classifying space $E_{\mathcal{K}}(T)$ yields a model for $\underline{E}T$. Let $K \subset T$ be a maximal compact subgroup. It is unique up to conjugation. The space T/K is contractible and in fact a model for $\underline{E}T$ (see [1, Appendix, Theorem A.5], [2, Corollary 4.14], [225, Section 1]). One knows (see [33, Proposition 4.22], [188])

$$K_n^T(\underline{E}T) = K_n^T(T/K) = \begin{cases} R_{\mathbb{C}}(K) & n = \dim(T/K) \mod 2, \\ 0 & n = 1 + \dim(T/K) \mod 2, \end{cases}$$

where $R_{\mathbb{C}}(K)$ is the complex representation ring of K.

Next we consider the special case where T is a totally disconnected group. Let $\mathcal{KO}$ be the family of compact-open subgroups of T. A T-CW-complex and a classifying space $E_{\mathcal{KO}}(T)$ for T and $\mathcal{KO}$ are defined. **Comment 27** (By W.): `Add reference to the literature or to a section in the book depending on whether we treat topological groups.` Then $E_{\mathcal{KO}}(T)$ is a model for $\underline{E}T$ since any compact subgroup is contained in a compact-open subgroup, and the Baum-Connes Conjecture says that the assembly map yields for $n \in \mathbb{Z}$ an isomorphism

$$A_{\mathcal{KO}} \colon K_n^T(E_{\mathcal{KO}}(T)) \to K_n(C_r^*(T)). \tag{2.80}$$

For more information see [35].

2.13 Variants of the Farrell-Jones Conjecture

In this section we discuss variants of the Baum-Connes Conjecture 2.5.

2.13.1 Pseudoisotopy Theory

An important variant of the Farrell-Jones Conjecture deals with the pseudoisotopy spectrum functor $\mathbf{P}$, which we already discussed briefly in Subsection 1.7.1. In fact it is this variant of the Farrell-Jones Conjecture (and its fibered version which will be explained in the next subsection) for which the strongest results are known at the time of writing.

In Proposition 4.22 we will explain that every functor $\mathbf{E}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$, which sends equivalences of groupoids to weak equivalences of spectra, yields a corresponding equivariant homology theory $H_n^G(-;\mathbf{E})$. Now whenever we have a functor $\mathbf{F}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$, we can precompose it with the functor "classifying space" which sends a groupoid $\mathcal{G}$ to its classifying space $B\mathcal{G}$. (Here $B\mathcal{G}$ is simply the realization of the nerve of $\mathcal{G}$ considered as a category.) In particular this applies to the pseudoisotopy functor $\mathbf{P}$. Thus we obtain a homology theory $H_n^G(-;\mathbf{P}\circ B)$ whose essential feature is that

$$H_n^G(G/H;\mathbf{P}\circ B) \cong \pi_n(\mathbf{P}(BH)),$$

i.e. evaluated at G/H one obtains the homotopy groups of the pseudoisotopy spectrum of the classifying space BH of the group H. As the reader may guess there is the following conjecture.

Conjecture 2.81. (Farrell-Jones Conjecture for Pseudoisotopies of Aspherical Spaces). *For every group G and all $n \in \mathbb{Z}$ the assembly map*

$$H_n^G(E_{\mathcal{VCY}}(G);\mathbf{P}\circ B) \to H_n^G(\{\bullet\};\mathbf{P}\circ B) \cong \pi_n(\mathbf{P}(BG))$$

is an isomorphism. Similarly for $\mathbf{P}^{\mathrm{diff}}$, the pseudoisotopy functor which is defined using smooth pseudoisotopies.

A formulation of a conjecture for spaces which are not necessarily aspherical will be given in the next subsection, see in particular Remark 2.86.

Remark 2.82. (Relating K-Theory and Pseudoisotopy Theory). We already outlined in Subsection 1.7.2 the relationship between K-theory and pseudoisotopies. The comparison in positive dimensions described there can be extended to all dimensions. Vogell constructs in [337] a version of A-theory using retractive spaces that are bounded over $\mathbb{R}^k$ (compare Subsection 1.5.3 and Subsection 1.7.1). This leads to a functor $\mathbf{A}^{-\infty}$ from spaces to non-connective spectra. Compare also [62], [338], [339] and [356]. We define $\mathbf{Wh}_{PL}^{-\infty}$ via the fibration sequence

$$X_+ \wedge \mathbf{A}^{-\infty}(\{\bullet\}) \to \mathbf{A}^{-\infty}(X) \to \mathbf{Wh}_{PL}^{-\infty}(X),$$

where the first map is the assembly map. The natural equivalence

$$\Omega^2\mathbf{Wh}_{PL}^{-\infty}(X) \simeq \mathbf{P}(X)$$

seems to be hard to trace down in the literature but should be true. We will assume it in the following discussion.

Precompose the functors above with the classifying space functor B to obtain functors from groupoids to spectra. The pseudoisotopy assembly map which appears in Conjecture 2.81 is an isomorphism if and only if the A-theory assembly map

$$H_{n+2}^G(E_{\mathcal{VCY}}(G); \mathbf{A}^{-\infty} \circ B) \to H_{n+2}^G(\{\bullet\}; \mathbf{A}^{-\infty} \circ B) \cong \pi_{n+2}(\mathbf{A}^{-\infty}(BG))$$

is an isomorphism. This uses a 5-lemma argument and the fact that for a fixed spectrum $\mathbf{E}$ the assembly map

$$H_n^G(E_{\mathcal{F}}(G); B\mathcal{G}^G(-)_+ \wedge \mathbf{E}) \to H_n^G(\{\bullet\}; B\mathcal{G}^G(-)_+ \wedge \mathbf{E})$$

is always bijective. There is a linearization map $\mathbf{A}^{-\infty}(X) \to \mathbf{K}(\mathbb{Z}\Pi(X)_\oplus)$ (see the next subsection for the notation) which is always 2-connected and a rational equivalence if X is aspherical (recall that $\mathbf{K}$ denotes the non-connective K-theory spectrum). For finer statements about the linearization map, compare also [251].

The above discussion yields in particular the following, compare [126, 1.6.7 on page 261].

Proposition 2.83. *The rational version of the K-theoretic Farrell-Jones Conjecture 2.5 is equivalent to the rational version of the Farrell-Jones Conjecture for Pseudoisotopies of Aspherical Spaces 2.81. If the assembly map in the conjecture for pseudoisotopies is (integrally) an isomorphism for $n \leq -1$, then so is the assembly map in the K-theoretic Farrell-Jones Conjecture for $n \leq 1$.*

2.13.2 The Fibered Version of the Farrell-Jones Conjecture

Next we present the more general fibered versions of the Farrell-Jones Conjectures. These fibered versions have better inheritance properties, compare Section 3.5.

In the previous section we considered functors $\mathbf{F}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$, like $\mathbf{P}$, $\mathbf{P}^{\mathrm{diff}}$ and $\mathbf{A}^{-\infty}$, and the associated equivariant homology theories $H_n^G(-; \mathbf{F} \circ B)$ (compare Proposition 4.22). Here B denotes the classifying space functor, which sends a groupoid $\mathcal{G}$ to its classifying space $B\mathcal{G}$. In fact all equivariant homology theories we considered so far can be obtained in this fashion for special choices of $\mathbf{F}$. Namely, let $\mathbf{F}$ be one of the functors

$$\mathbf{K}(R\Pi(-)_\oplus), \quad \mathbf{L}^{\langle -\infty \rangle}(R\Pi(-)_\oplus) \quad \text{or} \quad \mathbf{K}^{\mathrm{top}}(C_r^*\Pi(-)_\oplus),$$

where $\Pi(X)$ denotes the fundamental groupoid of a space, $R\mathcal{G}_\oplus$ respectively $C_r^*\mathcal{G}_\oplus$ is the R-linear respectively the C^*-category associated to a groupoid $\mathcal{G}$ and $\mathbf{K}$, $\mathbf{L}^{\langle -\infty \rangle}$ and $\mathbf{K}^{\mathrm{top}}$ are suitable functors which send additive respectively C^*-categories to spectra, compare the proof of Theorem 4.7. There is a natural equivalence $\mathcal{G} \to \Pi B\mathcal{G}$. Hence, if we precompose the functors above with the classifying space functor B, we obtain functors which are equivalent to the functors we have so far been calling

$$\mathbf{K}_R, \quad \mathbf{L}_R^{\langle -\infty \rangle} \quad \text{and} \quad \mathbf{K}^{\mathrm{top}},$$

compare Theorem 4.7. Note that in contrast to these three cases the pseudoisotopy functor $\mathbf{P}$ depends on more than just the fundamental groupoid. However Conjecture 2.81 above only deals with aspherical spaces.

Given a G-CW-complex Z and a functor $\mathbf{F}$ from spaces to spectra we obtain a functor $X \mapsto \mathbf{F}(Z \times_G X)$ which digests G-CW-complexes. In particular we can restrict it to the orbit category to obtain a functor

$$\mathbf{F}(Z \times_G -) \colon \mathsf{Or}(G) \to \mathsf{SPECTRA}.$$

According to Proposition 4.20 we obtain a corresponding G-homology theory

$$H_n^G(-; \mathbf{F}(Z \times_G -))$$

and associated assembly maps. Note that restricted to the orbit category the functor $EG \times_G -$ is equivalent to the classifying space functor B and so $H_n^G(-; \mathbf{F} \circ B)$ can be considered as a special case of this construction.

Conjecture 2.84 (Fibered Farrell-Jones Conjectures). *Let R be a ring (with involution). Let $\mathbf{F} \colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be one of the functors*

$$\mathbf{K}(R\Pi(-)_\oplus), \quad \mathbf{L}^{\langle -\infty \rangle}(R\Pi(-)_\oplus), \quad \mathbf{P}(-), \quad \mathbf{P}^{\mathrm{diff}}(-) \quad \text{or} \quad \mathbf{A}^{-\infty}(-).$$

Then for every free G-CW-complex Z and all $n \in \mathbb{Z}$ the associated assembly map

$$H_n^G(E_{\mathcal{VCY}}(G); \mathbf{F}(Z \times_G -)) \to H_n^G(\{\bullet\}; \mathbf{F}(Z \times_G -)) \cong \pi_n(\mathbf{F}(Z/G))$$

is an isomorphism.

Remark 2.85 (A Fibered Baum-Connes Conjecture). With the family $\mathcal{FIN}$ instead of $\mathcal{VCY}$ and the functor $\mathbf{F} = \mathbf{K}^{\mathrm{top}}(C_r^* \Pi(-)_\oplus)$ one obtains a *Fibered Baum-Connes Conjecture.*

Remark 2.86 (The Special Case $Z = \widetilde{X}$). Suppose $Z = \widetilde{X}$ is the universal covering of a space X equipped with the action of its fundamental group $G = \pi_1(X)$. Then in the algebraic K- and L-theory case the conjecture above specializes to the "ordinary" Farrell-Jones Conjecture 2.5. In the pseudoisotopy and A-theory case one obtains a formulation of an (unfibered) conjecture about $\pi_n(\mathbf{P}(X))$ or $\pi_n(\mathbf{A}^{-\infty}(X))$ for spaces X which are not necessarily aspherical.

Remark 2.87 (Relation to the Original Formulation). In [126] Farrell and Jones formulate a fibered version of their conjectures for every (Serre) fibration $Y \to X$ over a connected CW-complex X. In our set-up this corresponds to choosing Z to be the total space of the fibration obtained from $Y \to X$ by pulling back along the universal covering projection $\widetilde{X} \to X$. This space is a free G-space for $G = \pi_1(X)$. Note that an arbitrary free G-*CW*-complex Z can always be obtained in this fashion from a map $Z/G \to BG$, compare [126, Corollary 2.2.1 on page 264].

Remark 2.88. (Relating K-Theory and Pseudoisotopy Theory in the Fibered Case). The linearization map $\pi_n(\mathbf{A}^{-\infty}(X)) \to K_n(\mathbb{Z}\Pi(X))$ is always 2-connected, but for spaces which are not aspherical it need not be a rational equivalence. Hence the comparison results discussed in Remark 2.82 apply for the fibered versions only in dimensions $n \leq 1$.

2.13.3 The Isomorphism Conjecture for *NK*-Groups

In Remark 1.33 we defined the groups $NK_n(R)$ for a ring R. They are the simplest kind of Nil-groups responsible for the infinite cyclic group. Since the functor $\mathbf{K}_R$ is natural with respect to ring homomorphism we can define $\mathbf{NK}_R$ as the (objectwise) homotopy cofiber of $\mathbf{K}_R \to \mathbf{K}_{R[t]}$. There is an associated assembly map.

Conjecture 2.89 (Isomorphism Conjecture for NK-groups). *The assembly map*

$$H_n^G(E_{\mathcal{VCY}}(G); \mathbf{NK}_R) \to H_n^G(\{\bullet\}; \mathbf{NK}_R) \cong NK_n(RG)$$

is always an isomorphism.

There is a weak equivalence $\mathbf{K}_{R[t]} \simeq \mathbf{K}_R \vee \mathbf{NK}_R$ of functors from $\mathsf{GROUPOIDS}$ to $\mathsf{SPECTRA}$. This implies for a fixed family $\mathcal{F}$ of subgroups of G and $n \in \mathbb{Z}$ that whenever two of the three assembly maps

$$\begin{aligned} A_{\mathcal{F}} \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{R[t]}) &\to K_n(R[t][G]), \\ A_{\mathcal{F}} \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_R) &\to K_n(R[G]), \\ A_{\mathcal{F}} \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{NK}_R) &\to NK_n(RG) \end{aligned}$$

are bijective, then so is the third (compare [22, Section 7]). Similarly one can define a functor $\mathbf{E}_R$ from the category $\mathsf{GROUPOIDS}$ to $\mathsf{SPECTRA}$ and weak equivalences

$$\mathbf{K}_{R[t,t^{-1}]} \to \mathbf{E}_R \leftarrow \mathbf{K}_R \vee \Sigma\mathbf{K}_R \vee \mathbf{NK}_R \vee \mathbf{NK}_R,$$

which on homotopy groups corresponds to the Bass-Heller-Swan decomposition (see Remark 1.33). One obtains a two-out-of-three statement as above with the $\mathbf{K}_{R[t]}$-assembly map replaced by the $\mathbf{K}_{R[t,t^{-1}]}$-assembly map.

2.13.4 Homotopy *K*-Theory

Homotopy K-Theory $KH_*(R)$ is a variant of algebraic K-theory and was defined by Weibel [355], building on the definition of Karoubi-Villamayor K-theory. The homotopy algebraic K-theory groups of a ring R are denoted by $KH_n(R)$. Their crucial property is homotopy invariance: $KH_n(R) \cong KH_n(R[t])$. In particular, homotopy algebraic K-theory does not contain Nil-groups and we get

$$KH_n(R[\mathbb{Z}]) \cong KH_n(R) \oplus KH_{n-1}(R).$$

One can construct corresponding spectra $\mathbf{K}H(R)$ with $\pi_n(\mathbf{K}H(R)) = KH_n(R)$ and functors $\mathbf{K}H_R\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ yielding equivariant homology theories $H_*(-;\mathbf{K}H)$ which satisfy $H_n^G(G/H;\mathbf{K}H_R) = KH_n(RH)$. This suggests the following conjecture (see Bartels-Lück [24]).

Conjecture 2.90. (Farrell-Jones Conjecture for $KH_*(RG)$). *Let G be a group and let R be a ring. Then for all $n \in \mathbb{Z}$ the map*

$$A_{\mathcal{FIN}}\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}H_R) \to H_n^G(\{\bullet\};\mathbf{K}H_R) \cong HK_n(RG);$$

induced by the projection $E_{\mathcal{FIN}}(G) \to \{\bullet\}$ are isomorphisms.

We think about Conjecture 2.90 as an Isomorphism Conjecture for algebraic K-theory modulo Nil-groups.

There is a natural map $\mathbf{K}(R) \to \mathbf{K}H(R)$. Similarly we obtain a natural transformation $\mathbf{K}_R \to \mathbf{K}H_R$ of functors from $\mathsf{GROUPOIDS}$ to $\mathsf{SPECTRA}$. Thus we obtain a natural transformation of equivariant homology theories $H_*^?(-;\mathbf{K}_R) \to H_*^?(-;\mathbf{K}H_R)$ and a commutative diagram between assembly maps

$$\begin{array}{ccc} H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) & \longrightarrow & K_n(RG) \\ \downarrow & & \downarrow \\ H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}H_R) & \longrightarrow & KH_n(RG) \end{array} \tag{2.91}$$

Thus Conjecture 2.90 for $KH_*(RG)$ is related to the Farrell-Jones Conjecture 2.5 for $K_*(RG)$.

The status and consequences of the Farrell-Jones Conjecture for $KH_*(RG)$ (see Conjecture 2.90) will be discussed in Subsection 3.3.3.

2.13.5 Algebraic K-Theory of the Hecke Algebra

In Subsection 2.12.6 we mentioned the classifying space $E_{\mathcal{KO}}(G)$ for the family of compact-open subgroups and the Baum-Connes Conjecture for a totally disconnected group T. There is an analogous conjecture dealing with the algebraic K-theory of the Hecke algebra.

Let $\mathcal{H}(T)$ denote the *Hecke algebra* of T which consists of locally constant functions $G \to \mathbb{C}$ with compact support and inherits its multiplicative structure from the convolution product. The Hecke algebra $\mathcal{H}(T)$ plays the same role for T as the complex group ring $\mathbb{C}G$ for a discrete group G and reduces to this notion if T happens to be discrete. There is a T-homology theory $\mathcal{H}_*^T$ with the property that for any open and closed subgroup $H \subseteq T$ and all $n \in \mathbb{Z}$ we have $\mathcal{H}_n^T(T/H) = K_n(\mathcal{H}(H))$, where $K_n(\mathcal{H}(H))$ is the algebraic K-group of the Hecke algebra $\mathcal{H}(H)$.

Conjecture 2.92 (Isomorphism Conjecture for the Hecke-Algebra). *For a totally disconnected group T the assembly map*

$$A_{\mathcal{KO}}\colon \mathcal{H}_n^T(E_{\mathcal{KO}}(T)) \to \mathcal{H}^T(\{\bullet\}) = K_n(\mathcal{H}(T)) \qquad (2.93)$$

induced by the projection $\mathrm{pr}\colon E_{\mathcal{KO}}(T) \to \{\bullet\}$ *is an isomorphism for all* $n \in \mathbb{Z}$.

In the case $n = 0$ this reduces to the statement that

$$\operatorname{colim}_{T/H \in \mathsf{Or}_{\mathcal{KO}}(T)} K_0(\mathcal{H}(H)) \to K_0(\mathcal{H}(T)) \qquad (2.94)$$

is an isomorphism. For $n \leq -1$ one obtains the statement that $K_n(\mathcal{H}(G)) = 0$. The group $K_0(\mathcal{H}(T))$ has an interpretation in terms of the smooth representations of T. The G-homology theory can be constructed using an appropriate functor $\mathbf{E}\colon \mathsf{Or}_{\mathcal{KO}}(T) \to \mathsf{SPECTRA}$ and the recipe explained in Section 4.4.4. The desired functor $\mathbf{E}$ is given in [301].

2.14 Miscellaneous

The formulations of the Baum-Connes Conjecture 2.4 and the Farrell-Jones Conjecture 2.5 agree with the original formulations in [33, Conjecture 3.15 on page 254] and [126, 1.6 on page 257]. This is indicated in [92, p.239, p.247-248], a detailed proof can be found in [157, Corollary 9.2].

One can also formulate a fibered Meta-Conjecture for a discrete group G, a family $\mathcal{F}$ of subgroups of G which is closed under taking subgroups, i.e. $H \subseteq K, K \in \mathcal{F} \Rightarrow H \in \mathcal{F}$, and an equivariant homology theory $\mathcal{H}_*^?(-)$.

Metaconjecture 2.95 (Fibered Meta Conjecture). *For each group homomorphism $\phi\colon K \to G$ the assembly map*

$$A_{\phi^*\mathcal{F}}\colon \mathcal{H}_n^K(E_{\phi^*\mathcal{F}}(K)) \to \mathcal{H}_n^K(\{\bullet\})$$

is an isomorphism for $n \in \mathbb{Z}$, where $\phi^*\mathcal{F}$ is the family of subgroups of K given by $\{H \subseteq K \mid \phi(H) \in \mathcal{F}\}$.

The Fibered Meta Conjecture 2.95 for $\mathcal{F} = \mathcal{VCY}$ and $\mathcal{H}_*^?(-) = \mathcal{H}_*^?(-;\mathbf{K}_R)$ agrees with the Fibered Farrell-Jones Conjecture 2.84 (see [24, Remark 6.6] and [226, Remark 4.14]. **Comment 28** (By W.): `This is true in general if the functor` $\mathbf{E}$ `factorizes through` $\mathsf{GROUPOIDS}$`. It is not true for pseudo-isotopy in general. Shall we elaborate on this?` The Fibered Meta Conjecture 2.95 satisfies the obvious version of the Transitivity Principle 2.11 (see [24, Theorem 2.4]).

Comment 29 (By W.): `Add here or later references to the recent work of Emerson and Meyer [108], and maybe also to Meyer and Nest [237].`

Comment 30 (By W.): `Add here or later a reference to the papers by Balmer and Matthey [16], [17], [18]?`

Exercises

2.1. Compute $K_n(C_r^*(SL(2,\mathbb{Z}))$.

2.2. Let G be a group such that there is a model for $E_{\mathcal{FIN}}(G)$ such that the quotient $G\backslash E_{\mathcal{FIN}}(G)$ is compact. Show that $K_n(C_r(G))$ is finitely generated as abelian group for all $n \in \mathbb{Z}$, if G satisfies the Baum-Connes Conjecture 2.4.

2.3. Show that a virtually cyclic group G is of the first kind if and only if $H_1(G;\mathbb{Z})$ is infinite.

2.4. Let G be a group which contains a torsionfree normal subgroup T of finite index $[G:T]$. Suppose that $[G:T]$ is odd. Show that for all rings R and all $n \in \mathbb{Z}$ the relative assembly map

$$A_{\mathcal{FIN}\to\mathcal{VCY}}\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \to H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle})$$

is an isomorphism.

2.5. Let S_3 be the symmetric groups of permutations of $\{1,2,3\}$ and let F be a field of characteristic zero. Show that the induction map $\bigoplus_{C\in\mathcal{CYC}} R_F(C) \to R_F(S_3)$ has a cokernel of order 2 if F does not contain a non-trivial third root of unity and is surjective otherwise.

2.6. Let G be a group. Let $A(G)$ be the Grothendieck group associated to the abelian monoid of G-isomorphisms classes of G-sets S which are proper and cofinite, i.e. which are finite unions of homogeneous spaces G/H for finite subgroups H. Let $P^G\colon A(G) \to K_0(\mathbb{Q}G)$ be the homomorphisms sending the class of a cofinite proper G-set to the class of the finitely generated projective $\mathbb{Q}G$-module $\mathbb{Q}(S)$ whose underlying $\mathbb{Q}$-vector space has S as basis. Suppose that the Farrell-Jones Conjecture for $K_0(RG)$ for regular rings R holds for $R = \mathbb{Q}$ (see Conjecture 2.29) holds for G. Using the fact that $\phi^C \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}}$ is surjective for every finite cyclic group C (see [331, Exercise 1 on page 12 and Theorem 4.4.1 on page 80]) prove that

$$\phi^G \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}}\colon A(G)\otimes_{\mathbb{Z}}\mathbb{Q} \to K_0(\mathbb{Q}G)\otimes_{\mathbb{Z}}\mathbb{Q}$$

is surjective.

2.7. Let G be a group. Suppose that for every finitely generated projective $\mathbb{C}G$-module P there exists positive integers m and n such that $P^m \oplus (\mathbb{C}G)^n$ is free where P^m denotes the m-fold direct sum of copies of P. Show that G must be torsionfree.

2.8. Let G be a finite group. Show that the homomorphism

$$j\colon G_0(\mathbb{Z}G) \to K_0(\mathbb{Q}G), \quad [M] \mapsto [\mathbb{Q}G \otimes_{\mathbb{Z}G} M]$$

is well-defined and surjective. Show that the map

$$k\colon K_0(\mathbb{Z}G) \to K_0(\mathbb{Q}G), \quad [M] \mapsto [\mathbb{Q}G \otimes_{\mathbb{Z}G} M]$$

is rationally surjective if and only if G is trivial.

2.9. Let G be a group which satisfies the version 2.29 of the Farrell-Jones Conjecture for $R = \mathbb{C}$ and the Baum-Connes Conjecture 2.4.

(1) Show that the two change of rings homomorphisms

$$K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \Lambda^G \to K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \Lambda^G;$$
$$K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G \to K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \Lambda^G$$

have the same image. (The ring Λ^G has been introduced in (2.46).)

(2) Explain that the counterexample of Roy mentioned in Subsection 2.11 shows that assertion (1) is not true in general without applying $- \otimes_{\mathbb{Z}} \Lambda^G$.

2.10. Let R be a regular ring and let G be a group such that the order of any finite subgroup of G is invertible in R. Show that then the Isomorphism Conjecture 2.89 for NK-groups for RG is equivalent to the statement that $NK_n(RG) = 0$ is zero for all $n \in \mathbb{Z}$.

2.11. Let G be a group. Let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of G which are closed under taking subgroups. Let $\mathcal{H}_*^?$ an equivariant homology theory. Suppose that G satisfies the Fibered Meta Conjecture 2.95 for the family $\mathcal{F}$. Show that then G satisfies the Fibered Meta Conjecture 2.95 for the family $\mathcal{G}$.

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3. Status of the Conjectures

3.1 Introduction

In this chapter we give a report on the status of the conjectures as it is known to the authors at the time of writing.

3.2 Status of the Baum-Connes-Conjecture

3.2.1 Status of the Baum-Connes Conjecture with Coefficients

We begin with the Baum-Connes Conjecture with Coefficients 2.74. It has better inheritance properties than the Baum-Connes Conjecture 2.4 itself and contains it as a special case.

Comment 31 (By W.): `This theorem has to be replaced by the results announced by Yu and Kasparov.`

Theorem 3.1. (Baum-Connes Conjecture with Coefficients and a-T-menable Groups). *The discrete group G satisfies the Baum-Connes Conjecture with Coefficients 2.74 and is K-amenable provided that G is a-T-menable.*

This theorem is proved in Higson-Kasparov [165, Theorem 1.1], where more generally second countable locally compact topological groups are treated (see also [181]).

A group G is *a-T-menable*, or, equivalently, has the *Haagerup property* if G admits a metrically proper isometric action on some affine Hilbert space. *Metrically proper* means that for any bounded subset B the set $\{g \in G \mid gB \cap B \neq \emptyset\}$ is finite. An extensive treatment of such groups is presented in [74]. Any a-T-menable group is countable. The class of a-T-menable groups is closed under taking subgroups, under extensions with finite quotients and under finite products. It is not closed under semidirect products. Examples of a-T-menable groups are countable amenable groups, countable free groups, discrete subgroups of $SO(n,1)$ and $SU(n,1)$, Coxeter groups, countable groups acting properly on trees, products of trees, or simply connected CAT(0) cubical complexes. A group G has Kazhdan's *property*

(T) if, whenever it acts isometrically on some affine Hilbert space, it has a fixed point. An infinite a-T-menable group does not have property (T). Since $SL(n,\mathbb{Z})$ for $n \geq 3$ has property (T), it cannot be a-T-menable.

Using the Higson-Kasparov result Theorem 3.1 and known inheritance properties of the Baum-Connes Conjecture with Coefficients (compare Section 3.5 and [254],[255]) Mislin describes an even larger class of groups for which the conjecture is known [246, Theorem 5.23].

Theorem 3.2. (The Baum-Connes Conjecture with Coefficients and the Class of Groups LH$\mathcal{ETH}$). *The discrete group G satisfies the Baum-Connes Conjecture with Coefficients 2.74 provided that G belongs to the class* **LH**$\mathcal{ETH}$.

The class **LH**$\mathcal{ETH}$ is defined as follows. Let **H**$\mathcal{TH}$ be the smallest class of groups which contains all a-T-menable groups and contains a group G if there is a 1-dimensional contractible G-CW-complex whose stabilizers belong already to **H**$\mathcal{TH}$. Let **H**$\mathcal{ETH}$ be the smallest class of groups containing **H**$\mathcal{TH}$ and containing a group G if either G is countable and admits a surjective map $p\colon G \to Q$ with Q and $p^{-1}(F)$ in **H**$\mathcal{ETH}$ for every finite subgroup $F \subseteq Q$ or if G admits a 1-dimensional contractible G-CW-complex whose stabilizers belong already to **H**$\mathcal{ETH}$. Let **LH**$\mathcal{ETH}$ be the class of groups G whose finitely generated subgroups belong to **H**$\mathcal{ETH}$.

The class **LH**$\mathcal{ETH}$ is closed under passing to subgroups, under extensions with torsionfree quotients and under finite products. It contains in particular one-relator groups and Haken 3-manifold groups (and hence all knot groups). All these facts of the class **LH**$\mathcal{ETH}$ and more information can be found in Mislin [246, Section 5].

Vincent Lafforgue has an unpublished proof of the Baum-Connes Conjecture with Coefficients 2.74 for word-hyperbolic groups. **Comment 32** (By W.): `This sentence has to be dropped or adapted when the results by Kasparov-Yu have been published.`

Remark 3.3. There are counterexamples to the Baum-Connes Conjecture with (commutative) Coefficients 2.74 as soon as the existence of finitely generated groups containing arbitrary large expanders in their Cayley graph is shown [166, Section 7]. The existence of such groups has been claimed by Gromov [153], [154]. Details of the construction are described by Ghys in [149]. At the time of writing no counterexample to the Baum-Connes Conjecture 2.4 (without coefficients) is known to the authors.

Comment 33 (By W.): `Shall we include a mini-survey on groups such as word hyperbolic groups with property T and a-T-menable groups? These mini-surveys can be quite useful.`

3.2.2 Status of Baum-Connes Conjecture (Without Coefficients)

Next we deal with the Baum-Connes Conjecture 2.4 itself. Recall that all groups which satisfy the Baum-Connes Conjecture with Coefficients 2.74 do in particular satisfy the Baum-Connes Conjecture 2.4.

Theorem 3.4 (Status of the Baum-Connes Conjecture). *A group G satisfies the Baum-Connes Conjecture 2.4 if it satisfies one of the following conditions.*

(1) It is a discrete subgroup of a connected Lie groups L, whose Levi-Malcev decomposition $L = RS$ into the radical R and semisimple part S is such that S is locally of the form

$$S = K \times SO(n_1, 1) \times \ldots \times SO(n_k, 1) \times SU(m_1, 1) \times \ldots \times SU(m_l, 1)$$

for a compact group K;

(2) The group G has property (RD) and admits a proper isometric action on a strongly bolic weakly geodesic uniformly locally finite metric space;

(3) G is a subgroup of a word hyperbolic group; **Comment 34** *(By W.):* `When Kasparov-Yu have proven it with coefficients, we shall discard this item and add a reference at another appropriate place.`

(4) G is a discrete subgroup of $Sp(n, 1)$.

Proof. The proof under condition (1) is due to Julg-Kasparov [183]. The proof under condition (2) is due to Lafforgue [201] (see also [315]). Word hyperbolic groups have property (RD) [94]. Any subgroup of a word hyperbolic group satisfies the conditions appearing in the result of Lafforgue and hence satisfies the Baum-Connes Conjecture 2.4 [243, Theorem 20]. The proof under condition (4) is due to Julg [182].

Lafforgue's result about groups satisfying condition (2) yielded the first examples of infinite groups which have Kazhdan's property (T) and satisfy the Baum-Connes Conjecture 2.4. Here are some explanations about condition 2.

A *length function* on G is a function $L\colon G \to \mathbb{R}_{\geq 0}$ such that $L(1) = 0$, $L(g) = L(g^{-1})$ for $g \in G$ and $L(g_1 g_2) \leq L(g_1) + L(g_2)$ for $g_1, g_2 \in G$ holds. The word length metric L_S associated to a finite set S of generators is an example. A length function L on G has *property (RD)* ("rapid decay") if there exist $C, s > 0$ such that for any $u = \sum_{g \in G} \lambda_g \cdot g \in \mathbb{C}G$ we have

$$||\rho_G(u)||_\infty \;\leq\; C \cdot \left(\sum_{g \in G} |\lambda_g|^2 \cdot (1 + L(g))^{2s} \right)^{1/2},$$

where $||\rho_G(u)||_\infty$ is the operator norm of the bounded G-equivariant operator $l^2(G) \to l^2(G)$ coming from right multiplication with u. A group G has *property (RD)* if there is a length function which has property (RD). More information about property (RD) can be found for instance in [71], [202] and [335, Chapter 8]. Bolicity generalizes Gromov's notion of hyperbolicity for metric spaces. We refer to [187] for a precise definition.

Remark 3.5. We do not know whether all groups appearing in Theorem 3.4 satisfy also the Baum-Connes Conjecture with Coefficients 2.74.

Remark 3.6 ($SL_n(\mathbb{Z})$). It is not known at the time of writing whether the Baum-Connes Conjecture is true for $SL_n(\mathbb{Z})$ for $n \geq 3$.

Remark 3.7 (The Status for Topological Groups). We only dealt with the Baum-Connes Conjecture for discrete groups. We already mentioned that Higson-Kasparov [165] treat second countable locally compact topological groups. The Baum-Connes Conjecture for second countable almost connected groups G has been proven by Chabert-Echterhoff-Nest [67] based on the work of Higson-Kasparov [165] and Lafforgue [204]. The Baum-Connes Conjecture with Coefficients 2.74 has been proven for the connected Lie groups L appearing in Theorem 3.4 1 by [183] and for $Sp(n, 1)$ by Julg [182].

3.2.3 The Injectivity Part of the Baum-Connes Conjecture

In this subsection we deal with injectivity results about the assembly map appearing in the Baum-Connes Conjecture 2.4. Recall that rational injectivity already implies the Novikov Conjecture 2.49 (see Proposition 2.53) and the Homological Stable Gromov-Lawson-Rosenberg Conjecture 2.67 (see Proposition 2.68 and 2.23).

Theorem 3.8. (Rational Injectivity of the Baum-Connes Assembly Map). *The assembly map appearing in the Baum-Connes Conjecture 2.4 is rationally injective if G belongs to one of the classes of groups below.*

(1) Groups acting properly isometrically on complete manifolds with non-positive sectional curvature;
(2) Discrete subgroups of Lie groups with finitely many path components;
(3) Discrete subgroups of p-adic groups.

Proof. The proof of assertions (1) and (2) is due to Kasparov [189], the one of assertion (3) to Kasparov-Skandalis [190].

A metric space (X, d) admits a *uniform embedding into Hilbert space* if there exist a separable Hilbert space H, a map $f\colon X \to H$ and non-decreasing functions ρ_1 and ρ_2 from $[0, \infty) \to \mathbb{R}$ such that $\rho_1(d(x, y)) \leq ||f(x) - f(y)|| \leq \rho_2(d(x, y))$ for $x, y \in X$ and $\lim_{r\to\infty} \rho_i(r) = \infty$ for $i = 1, 2$. A metric is *proper* if for each $r > 0$ and $x \in X$ the closed ball of radius r centered at x is compact. The question whether a discrete group G equipped with a proper left G-invariant metric d admits a uniform embedding into Hilbert space is independent of the choice of d, since the induced coarse structure does not depend on d [316, page 808]. For more information about groups admitting a uniform embedding into Hilbert space we refer to [99], [155].

The class of finitely generated groups, which embed uniformly into Hilbert space, contains a subclass A, which contains all word hyperbolic groups, finitely generated discrete subgroups of connected Lie groups and finitely generated amenable groups and is closed under semidirect products [368, Definition 2.1, Theorem 2.2 and Proposition 2.6]. Gromov [153], [154] has announced examples of finitely generated groups which do not admit a uniform embedding into Hilbert space. Details of the construction are described in Ghys [149].

The next theorem is proven by Skandalis-Tu-Yu [316, Theorem 6.1] using ideas of Higson [164].

Theorem 3.9 (Injectivity of the Baum-Connes Assembly Map). *X Let G be a countable group. Suppose that G admits a G-invariant metric for which G admits a uniform embedding into Hilbert space. Then the assembly map appearing in the Baum-Connes Conjecture with Coefficients 2.74 is injective.*

We now discuss conditions which can be used to verify the assumption in Theorem 3.9.

Remark 3.10 (Linear Groups). A group G is called *linear* if it is a subgroup of $GL_n(F)$ for some n and some field F. Guentner-Higson-Weinberger [155] show that every countable linear group admits a uniform embedding into Hilbert space and hence Theorem 3.9 applies.

Remark 3.11. (Groups Acting Amenably on a Compact Space). A continuous action of a discrete group G on a compact space X is called *amenable* if there exists a sequence

$$p_n \colon X \to M^1(G) = \{f \colon G \to [0,1] \mid \sum_{g \in G} f(g) = 1\}$$

of weak-$*$-continuous maps such that for each $g \in G$ one has

$$\lim_{n \to \infty} \sup_{x \in X} ||g * (p_n(x) - p_n(g \cdot x))||_1 \;=\; 0.$$

Note that a group G is amenable if and only if its action on the one-point-space is amenable. More information about this notion can be found for instance in [6], [7].

Higson-Roe [169, Theorem 1.1 and Proposition 2.3] show that a finitely generated group equipped with its word length metric admits an amenable action on a compact metric space, if and only if it belongs to the class A defined in [368, Definition 2.1], and hence admits a uniform embedding into Hilbert space. Hence Theorem 3.9 implies the result of Higson [164, Theorem 1.1] that the assembly map appearing in the Baum-Connes Conjecture with Coefficients 2.74 is injective if G admits an amenable action on some compact space.

Word hyperbolic groups and the class of groups mentioned in Theorem 3.8(2) fall under the class of groups admitting an amenable action on some compact space [169, Section 4].

Remark 3.12. Higson [164, Theorem 5.2] shows that the assembly map appearing in the Baum-Connes Conjecture with Coefficients 2.74 is injective if $\underline{E}G$ admits an appropriate compactification. This is a C^*-version of the result for K-and L-theory due to Carlsson-Pedersen [61], compare Theorem 3.29.

Remark 3.13. We do not know whether the groups appearing in Theorem 3.8 and Theorem 3.9 satisfy the Baum-Connes Conjecture 2.4.

Next we discuss injectivity results about the classical assembly map for topological K-theory.

The *asymptotic dimension* of a proper metric space X is the infimum over all integers n such that for any $R > 0$ there exists a cover $\mathcal{U}$ of X with the property that the diameter of the members of $\mathcal{U}$ is uniformly bounded and every ball of radius R intersects at most $(n+1)$ elements of $\mathcal{U}$ (see [152, page 28]).

The next result is due to Yu [367].

Theorem 3.14. (The C^*-Theoretic Novikov Conjecture and Groups of Finite Asymptotic Dimension). *Let G be a group which possesses a finite model for BG and has finite asymptotic dimension. Then the assembly map in the Baum-Connes Conjecture 1.1*

$$K_n(BG) \to K_n(C_r^*(G))$$

is injective for all $n \in \mathbb{Z}$.

3.2.4 The Coarse Baum-Connes Conjecture

The coarse Baum-Connes Conjecture was explained in Section 2.12.5. Recall the descent principle (Proposition 2.78): if a countable group can be equipped with a G-invariant metric such that the resulting metric space satisfies the Coarse Baum-Connes Conjecture, then the classical assembly map for topological K-theory is injective.

Recall that a discrete metric space has bounded geometry if for each $r > 0$ there exists a $N(r)$ such that for all x the ball of radius $N(r)$ centered at $x \in X$ contains at most $N(r)$ elements.

The next result is due to Yu [368, Theorem 2.2 and Proposition 2.6].

Theorem 3.15. (Status of the Coarse Baum-Connes Conjecture). *The Coarse Baum-Connes Conjecture 2.77 is true for a discrete metric space X of bounded geometry if X admits a uniform embedding into Hilbert space.*

In particular a countable group G satisfies the Coarse Baum-Connes Conjecture 2.77 if G equipped with a proper left G-invariant metric admits a uniform embedding into Hilbert space.

Also Yu's Theorem 3.14 is proven via a corresponding result about the Coarse Baum-Connes Conjecture.

3.3 Status of the Farrell-Jones Conjecture

Next we deal with the Farrell-Jones Conjecture.

3.3.1 The Fibered Farrell-Jones Conjecture

The Fibered Farrell-Jones Conjecture 2.84 was discussed in Subsection 2.13.2. Recall that it has better inheritance properties (compare Section 3.5) and contains the ordinary Farrell-Jones Conjecture 2.5 as a special case.

Theorem 3.16. (Status of the Fibered Farrell-Jones Conjecture).

(1) Let G be a discrete group which satisfies one of the following conditions.
  - *(a) There is a Lie group L with finitely many path components and G is a cocompact discrete subgroup of L;*
  - *(b) The group G is virtually torsionfree and acts properly discontinuously, cocompactly and via isometries on a simply connected complete nonpositively curved Riemannian manifold.*

  Then
  - *(1) The version of the Fibered Farrell-Jones Conjecture 2.84 for the topological and the smooth pseudoisotopy functor is true for G;*
  - *(2) The version of the Fibered Farrell-Jones Conjecture 2.84 for K-theory and $R = \mathbb{Z}$ is true for G in the range $n \leq 1$, i.e. the assembly map is bijective for $n \leq 1$.*

Moreover we have the following statements.

(ii) The version of the Fibered Farrell-Jones Conjecture 2.84 for K-theory and $R = \mathbb{Z}$ is true in the range $n \leq 1$ for braid groups;

(iii) The L-theoretic version of the Fibered Farrell-Jones Conjecture 2.84 with $R = \mathbb{Z}$ holds after inverting 2 for elementary amenable groups.

Proof. (i) For assertion (1) see [126, Theorem 2.1 on page 263], [126, Proposition 2.3] and [134, Theorem A]. Assertion (2) follows from (1) by Remark 2.88.

(ii) See [134].

(iii) is proven in [132, Theorem 5.2]. For crystallographic groups see also [363].

A surjectivity result about the Fibered Farrell-Jones Conjecture for Pseudoisotopies appears as the last statement in Theorem 3.20.

The rational comparison result between the K-theory and the pseudoisotopy version (see Proposition 2.83) does not work in the fibered case, compare Remark 2.88. However, in order to exploit the good inheritance properties one can first use the pseudoisotopy functor in the fibered set-up, then specialize to the unfibered situation and finally do the rational comparison to K-theory.

Remark 3.17. The version of the Fibered Farrell-Jones Conjecture 2.84 for L-theory and $R = \mathbb{Z}$ seems to be true if G satisfies the condition (a) appearing in Theorem 3.16. Farrell and Jones [126, Remark 2.1.3 on page 263] say that they can also prove this version without giving the details.

Remark 3.18. Let G be a virtually poly-cyclic group. Then it contains a maximal normal finite subgroup N such that the quotient G/N is a discrete cocompact subgroup of a Lie group with finitely many path components [361, Theorem 3, Remark 4 on page 200]. Hence by Subsection 3.5.3 and Theorem 3.16 the version of the Fibered Farrell-Jones Conjecture 2.84 for the topological and the smooth pseudoisotopy functor, and for K-theory and $R = \mathbb{Z}$ in the range $n \leq 1$, is true for G. Earlier results of this type were treated for example in [115], [120].

3.3.2 Status of the (Unfibered) Farrell-Jones Conjecture

Here is a sample of some results one can deduce from Theorem 3.16.

Theorem 3.19. (The Farrell-Jones Conjecture and Subgroups of Lie groups). *Suppose H is a subgroup of G, where G is a discrete cocompact subgroup of a Lie group L with finitely many path components. Then*

(1) The version of the Farrell-Jones Conjecture for K-theory and $R = \mathbb{Z}$ is true for H rationally, i.e. the assembly map appearing in Conjecture 2.5 is an isomorphism after applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$;

(2) The version of the Farrell-Jones Conjecture for K-theory and $R = \mathbb{Z}$ is true for H in the range $n \leq 1$, i.e. the assembly map appearing in Conjecture 2.5 is an isomorphism for $n \leq 1$.

Proof. The results follow from Theorem 3.16, since the Fibered Farrell-Jones Conjecture 2.84 passes to subgroups [126, Theorem A.8 on page 289] (compare Section 3.5.2) and implies the Farrell-Jones Conjecture 2.5.

We now discuss results for torsionfree groups. Recall that for $R = \mathbb{Z}$ the K-theoretic Farrell-Jones Conjecture in dimensions ≤ 1 together with the L-theoretic version implies already the Borel Conjecture 1.49 in dimension ≥ 5 (see Theorem (1.50)).

A complete Riemannian manifold M is called *A-regular* if there exists a sequence of positive real numbers A_0, A_1, A_2, $\ldots$ such that $||\nabla^n K|| \leq A_n$, where $||\nabla^n K||$ is the supremum-norm of the n-th covariant derivative of the curvature tensor K. Every locally symmetric space is A-regular since ∇K is identically zero. Obviously every closed Riemannian manifold is A-regular.

Theorem 3.20. (Status of the Farrell-Jones Conjecture for Torsion-free Groups). *Consider the following conditions for the group G.*

(1) $G = \pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature which is A-regular.
(2) $G = \pi_1(M)$ for a closed Riemannian manifold M with non-positive sectional curvature.
(3) $G = \pi_1(M)$ for a complete Riemannian manifold with negatively pinched sectional curvature.
(4) G is a torsionfree discrete subgroup of $GL(n, \mathbb{R})$.
(5) G is a torsionfree solvable discrete subgroup of $GL(n, \mathbb{C})$.
(6) $G = \pi_1(X)$ for a non-positively curved finite simplicial complex X.
(7) G is a strongly poly-free group in the sense of Aravinda-Farrell-Roushon [11, Definition 1.1]. The pure braid group satisfies this hypothesis.

Then

(a) Suppose that G satisfies one of the conditions (1) *to* (7)*. Then the K-theoretic Farrell-Jones Conjecture is true for $R = \mathbb{Z}$ in dimensions $n \leq 1$. In particular Conjecture 1.19 holds for G.*
(b) Suppose that G satisfies one of the conditions (1), (2), (3) *or* (4)*. Then G satisfies the Farrell-Jones Conjecture for Torsionfree Groups and L-Theory 1.37 for $R = \mathbb{Z}$.*
(c) Suppose that G satisfies (2)*. Then the Farrell-Jones Conjecture for Pseudoisotopies of Aspherical Spaces 2.81 holds for G.*
(d) Suppose that G satisfies one of the conditions (1), (3) *or* (4)*. Then the assembly map appearing in the version of the Fibered Farrell-Jones Conjecture for Pseudoisotopies 2.84 is surjective, provided that the G-space Z appearing in Conjecture 2.84 is connected.*

Proof. Note that condition (2) is a special case of condition (1) because every closed Riemannian manifold is A-regular. If M is a pinched negatively curved complete Riemannian manifold, then there is another Riemannian metric for which M is negatively curved complete and A-regular. This fact is mentioned in [130, page 216] and attributed there to Abresch [3] and Shi [312]. Hence also condition (3) can be considered as a special case of condition (1). The manifold $M = G\backslash GL(n, \mathbb{R})/O(n)$ is a non-positively curved complete locally symmetric space and hence in particular A-regular. So condition (4) is a special case of condition (1).

Assertion (a) under the condition (1) is proven by Farrell-Jones in [130, Proposition 0.10 and Lemma 0.12]. The earlier work [125] treated the condition (2). Under condition (5) assertion (a) is proven by Farrell-Linnell [132, Theorem 1.1]. The result under condition (6) is proven by Hu [172], under condition (7) it is proven by Aravinda-Farrell-Roushon [11, Theorem 1.3].

Assertion (b) under condition (1) is proven by Farrell-Jones in [130]. The condition (2) was treated earlier in [127].

Assertion (c) is proven by Farrell-Jones in [125] and assertion (d) by Jones in [178].

Remark 3.21. As soon as certain collapsing results (compare [129], [131]) are extended to orbifolds, the results under (4) above would also apply to groups with torsion and in particular to $SL_n(\mathbb{Z})$ for arbitrary n.

3.3.3 The Farrell-Jones Conjecture for Arbitrary Coefficients

So far all positive results about the Farrell-Jones Conjecture dealt only with the case $R = \mathbb{Z}$. The following result due to Bartels-Reich [25] deals with algebraic K-theory for arbitrary coefficient rings R. It extends Bartels-Farrell-Jones-Reich [22].

Theorem 3.22. *Suppose that G is the fundamental group of a closed Riemannian manifold with negative sectional curvature. Then the K-theoretic part of the Farrell-Jones Conjecture 2.5 is true for any ring R, i.e. the assembly map*

$$A_{\mathcal{VCY}}\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) \;\to\; K_n(RG)$$

is an isomorphism for all $n \in \mathbf{Z}$.

Theorem 2.14 can be restated because of Example 2.14 in the following way since G is word-hyperbolic. Let I be the set of conjugacy classes of maximal infinite cyclic subgroups. Then we get for all $n \in \mathbb{Z}$ an isomorphism

$$K_n(RG) \;\cong\; H_n(BG;\mathbf{K}(R) \oplus \left(\bigoplus_I NK_n(R)\right).$$

Suppose additionally that R is regular. Then $NK_n(R) = \{0\}$ for all $n \in \mathbb{Z}$ and $K_n(R) = 0$ for $n \leq -1$. Thus we get (see also Conjecture 1.17)

$$K_n(RG) \;\cong\; \begin{cases} \{0\} & n \leq -1; \\ K_0(R) & n = 0; \\ G_{\mathrm{ab}} \otimes_{\mathbb{Z}} K_1(R) \oplus K_1(R) & n = 1; \\ H_n(BG;\mathbf{K}(R) & n \in \mathbb{Z}. \end{cases}$$

Certain classes of groups $\mathcal{C}_0$ and $\mathcal{CL}$ are defined by Bartels-Lück [24] and by Waldhausen [342, Definition 19.2]. The class $\mathcal{C}_0$ is the smallest class of groups which contains all virtually cyclic groups, is closed under taking directed unions of subgroups and if G acts on a tree such that all stabilizers belong to $\mathcal{C}_0$, then G belongs to $\mathcal{C}_0$.

Groups in $\mathcal{C}_0$ may contain torsion, whereas groups appearing in $\mathcal{CL}$ are always torsionfree.

Theorem 3.23. *(1) The class $\mathcal{C}_0$ contains one-relator groups and poly-free groups. The class $\mathcal{CL}$ is contained in $\mathcal{C}_0$. The class $\mathcal{CL}$ contains torsionfree one-relator groups. The group $G = \pi_1(M)$ belongs to $\mathcal{CL}$ and hence to $\mathcal{C}_0$, if M is a finite connected sum of irreducible Haken 3-manifolds with infinite fundamental groups or if M is a compact 2-dimensional manifold or if M is a submanifold of S^3.*

(2) The Fibered Version of the Farrell-Jones Conjecture 2.90 for $KH_*(RG)$ holds for groups G in the class $\mathcal{C}_0$ for all rings R;

(3) If R is a regular ring with $\mathbb{Q} \subset R$, then for every group $G \in \mathcal{C}_0$ the injectivity part of Farrell-Jones Conjecture for $K_*(RG)$ for regular rings 0.8 is true, i.e. for every $n \in \mathbb{Z}$ the assembly map for algebraic K-theory

$$A_{\mathcal{FIN}}\colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \to H_n^G(\{\bullet\}; \mathbf{K}_R) = K_n(RG)$$

is injective;

(4) If G belongs to $\mathcal{C}_0$, then the injectivity part of the Farrell-Jones Conjecture 2.5 for $K_*(\mathbb{Z}G)$ is true rationally, or, by Proposition 2.20 equivalently, the assembly map

$$A_{\mathcal{FIN}}\colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathbb{Z}}) \to H_n^G(\{\bullet\}; \mathbf{K}_{\mathbb{Z}}) = K_n(\mathbb{Z}G)$$

is rationally injective;

(5) For every group G in $\mathcal{C}_0$ the L-theoretic Farrell-Jones Conjecture 2.5 is true after inverting 2, or equivalently (see Proposition 2.21), for every ring R with involution, every decoration j and all $n \in \mathbb{Z}$ the relative assembly map

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{\langle j \rangle})[1/2] \to L_R^{\langle j \rangle}(\mathbb{Z}G)[1/2]$$

is an isomorphism;

(6) If G belongs to $\mathcal{CL}$, then the K-theoretic part of the Farrell-Jones Conjecture 2.5 is true for any regular ring R, i.e. the classical assembly map

$$H_n^G(BG; \mathbf{K}(R)) \;\to\; K_n(RG)$$

is an isomorphism for all $n \in \mathbf{Z}$

Proof. (1) is proved in [24, Proposition 0.9] and [342, Theorem 17.5 on page 250].

(2) is proved in [24, Theorem 0.5]. The same is true for $\mathcal{CL}$ if G is additionally torsionfree;

(3) see [24, Theorem 0.8].

(4) see [24, Theorem 0.8].

(5) see [24, Theorem 0.13].

(6) is proven in the fundamental papers by Waldhausen [341] and [342, Theorem 19.4 on page 249].

Results related to Theorem 3.23 can be found in [179] and [296].

3.3.4 Injectivity Part of the Farrell-Jones Conjecture

The next result about the classical K-theoretic assembly map is due to Bökstedt-Hsiang-Madsen [44].

Theorem 3.24. (Rational Injectivity of the Classical K-Theoretic Assembly Map). *Let G be a group such that the integral homology $H_j(BG;\mathbb{Z})$ is finitely generated for each $j \in \mathbb{Z}$. Then the rationalized assembly map*

$$A\colon H_n(BG;\mathbf{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n^G(E_{\mathcal{TR}}(G);\mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective for all $n \in \mathbf{Z}$.

Because of the homological Chern character (see Remark 1.29 we obtain for the groups treated in Theorem 3.24 an injection

$$\bigoplus_{s+t=n} H_s(BG;\mathbb{Q}) \otimes_{\mathbb{Q}} (K_t(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}) \to K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{3.25}$$

Next we describe a generalization of Theorem 3.24 above from the trivial family $\mathcal{TR}$ to the family $\mathcal{FIN}$ of finite subgroups due to Lück-Reich-Rognes-Varisco [227]. Let $\mathbf{K}_{\mathbb{Z}}^{\mathrm{con}}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ be the connective version of the functor $\mathbf{K}_{\mathbb{Z}}$ of (4.8). In particular $H_n(G/H;\mathbf{K}_{\mathbb{Z}}^{\mathrm{con}})$ is isomorphic to $K_n(\mathbb{Z}H)$ for $n \geq 0$ and vanishes in negative dimensions. For a prime p we denote by $\mathbb{Z}_p$ the p-adic integers. Let $K_n(R;\mathbb{Z}_p)$ denote the homotopy groups $\pi_n(\mathbf{K}^{\mathrm{con}}(R)\widehat{_p})$ of the p-completion of the connective K-theory spectrum of the ring R.

Theorem 3.26. (Rational Injectivity of the Farrell-Jones Assembly Map for Connective K-Theory). *Suppose that the group G satisfies the following two conditions:*

(H) *For each finite cyclic subgroup $C \subseteq G$ and all $j \geq 0$ the integral homology group $H_j(BZ_GC;\mathbb{Z})$ of the centralizer Z_GC of C in G is finitely generated;*

(K) *There exists a prime p such that for each finite cyclic subgroup $C \subseteq G$ and each $j \geq 1$ the map induced by the change of coefficients homomorphism*

$$K_j(\mathbb{Z}C;\mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_j(\mathbb{Z}_pC;\mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective.

Then the rationalized assembly map

$$A_{\mathcal{VCY}}\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}}^{\mathrm{con}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an injection for all $n \in \mathbb{Z}$.

Remark 3.27. The methods of Chapter 8 apply also to $\mathbf{K}_{\mathbb{Z}}^{\mathrm{con}}$ and yield under assumption (H) and (K) an injection

$$\bigoplus_{s+t=n,\ t\geq 0} \ \bigoplus_{(C)\in(\mathcal{FCY})} H_s(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot K_t(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \to \ K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Notice that in the index set for the direct sum appearing in the source we require $t \geq 0$. This reflects the fact that the result deals only with the connective K-theory spectrum. If one drops the restriction $t \geq 0$ the Farrell-Jones Conjecture 2.5 predicts that the map is an isomorphism, compare Proposition 2.19 and Theorem 8.7. If we restrict the injection to the direct sum given by $C = 1$, we rediscover the map (3.25) whose injectivity follows already from Theorem 3.24.

The condition (K) appearing in Theorem 3.26 is conjectured to be true for all primes p (compare [306], [317] and [318]) but no proof is known. The weaker version of condition (K), where C is the trivial group is also needed in Theorem 3.24. But that case is known to be true and hence does not appear in its formulation. The special case of condition (K), where $j = 1$ is implied by the Leopoldt Conjecture for abelian fields (compare [250, IX, § 3]), which is known to be true [250, Theorem 10.3.16]. This leads to the following result.

Theorem 3.28. (Rational Contribution of Finite Subgroups to $\mathrm{Wh}(G)$**).** *Let G be a group. Suppose that for each finite cyclic subgroup $C \subseteq G$ and each $j \leq 4$ the integral homology group $H_j(BZ_GC)$ of the centralizer Z_GC of C in G is finitely generated. Then the map*

$$\operatorname{colim}_{H\in \mathtt{Sub}_{\mathcal{FIN}}(G)} \mathrm{Wh}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \ \to \ \mathrm{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

is injective, compare Conjecture 2.31.

The result above should be compared to the result which is proven using Fuglede-Kadison determinants in [228, Section 5], [220, Theorem 9.38 on page 354]: for *every* (discrete) group G and every finite normal subgroup $H \subseteq G$ the map $\mathrm{Wh}(H) \otimes_{\mathbb{Z}G} \mathbb{Z} \to \mathrm{Wh}(G)$ induced by the inclusion $H \to G$ is rationally injective.

The next result is taken from Rosenthal [293] and [292], where the techniques and results of Carlsson-Pedersen [61] are extended from the trivial family $\mathcal{TR}$ to the family of finite subgroups $\mathcal{FIN}$. The statement about word hyperbolic groups is proven by Rosenthal-Schütz[294].

Theorem 3.29. *Let G be a group. Suppose there exists a model E for the classifying space $E_{\mathcal{FIN}}(G)$ which admits a metrizable compactification $\overline{E}$ to which the group action extends. Suppose $\overline{E}^H$ is contractible and E^H is dense in $\overline{E}^H$ for every finite subgroup $H \subset G$. Suppose compact subsets of E become small near $\overline{E} - E$. Then*

(1) For every ring R the assembly map

$$A_{\mathcal{FIN}} \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \;\to\; K_n(RG)$$

is split injective;

(2) Let R be a ring with involution such that there exists $i \in \mathbb{Z}$ for which $K_n(R) = \{0\}$ holds for all $n \leq i$. Then the assembly map

$$A_{\mathcal{FIN}} \colon H_n^G(E_{\mathcal{FIN}}(G); L_R^{\langle -\infty \rangle}) \;\to\; L_n^{\langle -\infty \rangle}(RG)$$

is split injective;

(3) Word hyperbolic groups satisfy the assumptions about $E_{\mathcal{FIN}}(G)$.

A compact subset $K \subset E$ is said to become *small near* $\overline{E} - E$ if for every neighbourhood $U \subset \overline{E}$ of a point $x \in \overline{E} - E$ there exists a neighbourhood $V \subset \overline{E}$ such that $g \in G$ and $gK \cap V \neq \emptyset$ implies $gK \subset U$.

We finally discuss injectivity results about assembly maps for the trivial family. The following result is due to Ferry-Weinberger [144, Corollary 2.3] extending earlier work of Farrell-Hsiang [114].

Theorem 3.30. *Suppose $G = \pi_1(M)$ for a complete Riemannian manifold of non-positive sectional curvature. Then the L-theory assembly map*

$$A \colon H_n(BG; \mathbf{L}_{\mathbb{Z}}^{\epsilon}) \to L_n^{\epsilon}(\mathbb{Z}G)$$

is injective for $\epsilon = h, s$.

In fact Ferry-Weinberger also prove a corresponding splitting result for the classical A-theory assembly map. In [171] Hu shows that a finite complex of non-positive curvature is a retract of a non-positively curved PL-manifold and concludes split injectivity of the classical L-theoretic assembly map for $R = \mathbb{Z}$.

The next result due to Bartels [26] is the algebraic K- and L-theory analogue of Theorem 3.14.

Theorem 3.31. (The K-and L-Theoretic Novikov Conjecture and Groups of Finite Asymptotic Dimension). *Let G be a group which admits a finite model for BG. Suppose that G has finite asymptotic dimension. Then*

(1) The assembly maps appearing in the Farrell-Jones Conjecture 1.28

$$A \colon H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

is injective for all $n \in \mathbb{Z}$;

(2) *If furthermore R carries an involution and $K_{-j}(R)$ vanishes for sufficiently large j, then the assembly maps appearing in the Farrell-Jones Conjecture 1.37*

$$A\colon H_n(BG;\mathbf{L}^{\langle -\infty\rangle}(R)) \to L_n^{\langle -\infty\rangle}(RG)$$

is injective for all $n \in \mathbb{Z}$;

We have already mentioned some further injectivity results about the Farrell-Jones Conjecture 2.5 in Theorem 3.23 (3) and (4)

Further results related to the Farrell-Jones Conjecture 2.5 can be found for instance in [10], [39]. **Comment 35** (By W.): `Complete and update list.`

3.4 List of Groups Satisfying the Conjectures

In the following table we list prominent classes of groups and state whether they are known to satisfy the Baum-Connes Conjecture 2.4 (with coefficients 2.74) or the Farrell-Jones Conjecture 2.5 (fibered 2.84). Some of the classes are redundant. A question mark means that the authors do not know about a corresponding result. The reader should keep in mind that there may exist results of which the authors are not aware.

type of group	Baum-Connes Conjecture 2.4 (with coefficients 2.74)	Farrell-Jones Conjecture 2.5 for K-theory for $R = \mathbb{Z}$ (fibered 2.84)	Farrell-Jones Conjecture 2.5 for L-theory for $R = \mathbb{Z}$ (fibered 2.84)
a-T-menable groups	true with coefficients (see Theorem 3.1)	?	injectivity is true after inverting 2 (see Propositions 2.21 and 2.55)
amenable groups	true with coefficients (see Theorem 3.1)	?	injectivity is true after inverting 2 (see Propositions 2.21 and 2.55)
elementary amenable groups	true with coefficients (see Theorem 3.1)	?	true fibered after inverting 2 (see Theorem 3.16)
virtually polycyclic	true with coefficients (see Theorem 3.1)	true rationally, true fibered in the range $n \leq 1$ (compare Remark 3.18)	true fibered after inverting 2 (see Theorem 3.16)
torsionfree virtually solvable subgroups of $GL(n, \mathbb{C})$	true with coefficients (see Theorem 3.1)	true in the range ≤ 1 [132, Theorem 1.1]	true fibered after inverting 2 [132, Corollary 5.3]

type of group	Baum-Connes Conjecture 2.4 (with coefficients 2.74)	Farrell-Jones Conjecture 2.5 for K-theory for $R = \mathbb{Z}$ (fibered 2.84)	Farrell-Jones Conjecture 2.5 for L-theory for $R = \mathbb{Z}$ (fibered 2.84)
discrete subgroups of Lie groups with finitely many path components	injectivity true (see Theorem 3.9 and Remark 3.11)	?	injectivity is true after inverting 2 (see Propositions 2.21 and 2.55)
subgroups of groups which are discrete cocompact subgroups of Lie groups with finitely many path components	injectivity true (see Theorem 3.9 and Remark 3.11)	true rationally, true fibered in the range $n \leq 1$ (see Theorem 3.16)	probably true fibered (see Remark 3.17). Injectivity is true after inverting 2 (see Propositions 2.21 and 2.55)
linear groups	injectivity is true (see Theorem 3.9 and Remark 3.10)	?	injectivity is true after inverting 2 (see Propositions 2.21 and 2.55)
arithmetic groups	injectivity is true (see Theorem 3.9 and Remark 3.10)	?	injectivity is true after inverting 2 (see Propositions 2.21 and 2.55)
torsionfree discrete subgroups of $GL(n, \mathbb{R})$	injectivity is true (see Theorem 3.9 and Remark 3.11)	true in the range $n \leq 1$ (see Theorem 3.20)	true (see Theorem 3.20)

type of group	Baum-Connes Conjecture 2.4 (with coefficients 2.74)	Farrell-Jones Conjecture 2.5 for K-theory for $R = \mathbb{Z}$ (fibered 2.84)	Farrell-Jones Conjecture 2.5 for L-theory for $R = \mathbb{Z}$ (fibered 2.84)
Groups with finite BG and finite asymptotic dimension	injectivity is true (see Theorem 3.14)	injectivity is true for all R (see Theorem 3.31)	injectivity is true for regular R (see Theorem 3.31)
G acts properly and isometrically on a complete Riemannian manifold M with non-positive sectional curvature	rational injectivity is true (see Theorem 3.8)	?	rational injectivity is true (see Propositions 2.21 and 2.55)
$\pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature	rationally injective (see Theorem 3.8)	?	injectivity true (see Theorem 3.30)
$\pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature which is A-regular	rationally injective (see Theorem 3.8)	true in the range $n \leq 1$, rationally surjective (see Theorem 3.20)	true (see Theorem 3.20)
$\pi_1(M)$ for a complete Riemannian manifold M with pinched negative sectional curvature	rational injectivity is true (see Theorem 3.9)	true in the range $n \leq 1$, rationally surjective (see Theorem 3.20)	true (see Theorem 3.20)
$\pi_1(M)$ for a closed Riemannian manifold M with non-positive sectional curvature	rationally injective (see Theorem 3.8)	true fibered in the range $n \leq 1$, true rationally (see Theorem 3.20)	true (see Theorem 3.20)
$\pi_1(M)$ for a closed Riemannian manifold M with negative sectional curvature	true for all subgroups (see Theorem 3.4)	true for all coefficients R (see Theorem 3.22)	true (see Theorem 3.20)
word hyperbolic groups	true for all subgroups (see Theorem 3.4). **Comment 36** (By W.): `Add reference to Kasparov-Yu for the version with coefficients`	injectivity is true for $A_{\mathcal{FIN}}$ and all R (see Theorem 3.29)	injectivity is true for $A_{\mathcal{FIN}}$ and all R with $K_n(R) = 0$ for sufficiently small n (see Theorem 3.29)

type of group	Baum-Connes Conjecture 2.4 (with coefficients 2.74)	Farrell-Jones Conjecture 2.5 for K-theory and $R = \mathbb{Z}$ (fibered 2.84)	Farrell-Jones Conjecture 2.5 for L-theory for $R = \mathbb{Z}$ (fibered 2.84)
one-relator groups	true with coefficients (see Theorem 3.2)	rational injectivity is true for $R = \mathbb{Z}$ and injectivity is true for regular R with $\mathbb{Q} \subseteq R$ (see Theorem 3.23)	true for all R after inverting 2 (see Theorem 3.23)
torsionfree one-relator groups	true with coefficients (see Theorem 3.2)	true for regular R (see Theorem 3.23)	true for all R after inverting 2 (see Theorem 3.23)
Haken 3-manifold groups (in particular knot groups)	true with coefficients (see Theorem 3.2)	true for regular R (see Theorem 3.23)	true for all R after inverting 2(see Theorem 3.23)
$\pi_1(M)$ for compact connected orientable 3-manifold	true with coefficients **Comment 37** (By W.): `I think that I have an argument using the Geometrization Conjecture`	true rationally, true fibered in the range $n \leq 1$ [297, Corollary 1.1.6] **Comment 38** (By W.): `Assuming the Geometrization Conjecture`	injectivity is true after inverting 2 (see Propositions 2.21 and 2.55) **Comment 39** (By W.): `Maybe we can say more.`
$SL(n,\mathbb{Z}), n \geq 3$	injectivity is true	compare Remark 3.21	injectivity is true after inverting 2 (see Propositions 2.21 and 2.55)
Artin's braid group B_n	true with coefficients [246, Theorem 5.25], [303]	true fibered in the range $n \leq 1$, true rationally [134]	injectivity is true after inverting 2 (see Propositions 2.21 and 2.55)
pure braid group C_n	true with coefficients	true in the range $n \leq 1$ (see Theorem 3.20)	injectivity is true after inverting 2 (see Propositions 2.21 and 2.55)
Thompson's group F	true with coefficients [109]	?	injectivity is true after inverting 2 (see Propositions 2.21 and 2.55)

Comment 40 (By W.): `Shall we mention [297], where virtually weak strongly poly-surface groups and the fibered Farrell-Jones Conjecture for pseudoisotopies are treated?`

Remark 3.32. The authors have no information about the status of these conjectures for mapping class groups of higher genus or the group of outer automorphisms of free groups. Since all of these spaces have finite models for $E_{\mathcal{FIN}}(G)$ Theorem 3.26 applies in these cases. **Comment 41** (By W.): `Emerson-Meyer will prove some results for the Novikov Conjecture.`

3.5 Inheritance Properties

In this section we list some inheritance properties of the various conjectures.

3.5.1 Directed Colimits

Let $\{G_i \mid i \in I\}$ be a directed system of groups. Let $G = \operatorname{colim}_{i \in I} G_i$ be the colimit. We do not require that the structure maps are injective. If the Fibered Farrell-Jones Conjecture 2.84 is true for each G_i, then it is true for G [132, Theorem 6.1]. **Comment 42** (By W.): `What is known for the Baum-Connes Conjecture with coefficients?`

Suppose that $\{G_i \mid i \in I\}$ is a system of subgroups of G directed by inclusion such that $G = \operatorname{colim}_{i \in I} G_i$. If each G_i satisfies the Farrell-Jones Conjecture 2.5, the Baum-Connes Conjecture 2.4 or the Baum-Connes Conjecture with Coefficients 2.74, then the same is true for G [37, Theorem 1.1], [246, Lemma 5.3]. We do not know a reference in Farrell-Jones case. An argument in that case uses Lemma 4.11, the fact that $K_n(RG) = \operatorname{colim}_{i \in I} K_n(RG_i)$ and that for suitable models we have $E_{\mathcal{F}}(G) = \bigcup_{i \in I} G \times_{G_i} E_{\mathcal{F} \cap G_i}(G_i)$.

3.5.2 Passing to Subgroups

The Baum-Connes Conjecture with Coefficients 2.74 and the Fibered Farrell-Jones Conjecture 2.84 pass to subgroups, i.e. if they hold for G, then also for any subgroup $H \subseteq G$. This claim for the Baum-Connes Conjecture with Coefficients 2.74 has been stated in [33], a proof can be found for instance in [65, Theorem 2.5]. For the Fibered Farrell-Jones Conjecture this is proven in [126, Theorem A.8 on page 289] for the special case $R = \mathbb{Z}$, but the proof also works for arbitrary rings R.

It is not known whether the Baum-Connes Conjecture 2.4 or the Farrell-Jones Conjecture 2.5 itself passes to subgroups.

3.5.3 Extensions of Groups

Let $p\colon G \to K$ be a surjective group homomorphism. Suppose that the Baum-Connes Conjecture with Coefficients 2.74 or the Fibered Farrell-Jones Conjecture 2.84 respectively holds for K and for $p^{-1}(H)$ for any subgroup $H \subset K$ which is finite or virtually cyclic respectively. Then the Baum-Connes Conjecture with Coefficients 2.74 or the Fibered Farrell-Jones Conjecture 2.84 respectively holds for G. This is proven in [254, Theorem 3.1] for the Baum-Connes Conjecture with Coefficients 2.74, and in [126, Proposition 2.2 on page 263] for the Fibered Farrell-Jones Conjecture 2.84 in the case $R = \mathbb{Z}$. The same proof works for arbitrary coefficient rings.

It is not known whether the corresponding statement holds for the Baum-Connes Conjecture 2.4 or the Farrell-Jones Conjecture 2.5 itself.

Let $H \subseteq G$ be a normal subgroup of G. Suppose that H is a-T-menable. Then G satisfies the Baum-Connes Conjecture with Coefficients 2.74 if and only if G/H does [65, Corollary 3.14]. The corresponding statement is not known for the Baum-Connes Conjecture 2.4.

3.5.4 Products of Groups

The group $G_1 \times G_2$ satisfies the Baum-Connes Conjecture with Coefficients 2.74 if and only if both G_1 and G_2 do [65, Theorem 3.17], [254, Corollary 7.12]. The corresponding statement is not known for the Baum-Connes Conjecture 2.4.

Let $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$ denote the infinite dihedral group. Whenever a version of the Fibered Farrell-Jones Conjecture 2.84 is known for $G = \mathbb{Z} \times \mathbb{Z}$, $G = \mathbb{Z} \times D_\infty$ and $D_\infty \times D_\infty$, then that version of the Fibered Farrell-Jones Conjecture is true for $G_1 \times G_2$ if and only if it is true for G_1 and G_2.

3.5.5 Subgroups of Finite Index

It is not known whether the Baum-Connes Conjecture 2.4, the Baum-Connes Conjecture with Coefficients 2.74, the Farrell-Jones Conjecture 2.5 or the Fibered Farrell-Jones Conjecture 2.84 is true for a group G if it is true for a subgroup $H \subseteq G$ of finite index.

3.5.6 Groups Acting on Trees

Comment 43 (By W.): `This subsection is not yet in a satisfactory form.`

Let G be a countable discrete group acting without inversion on a tree T. Then the Baum-Connes Conjecture with Coefficients 2.74 is true for G if and only if it holds for all stabilizers of the vertices of T. This is proven by Oyono-Oyono [255, Theorem 1.1]. This implies that Baum-Connes Conjecture with Coefficients 2.74 is stable under amalgamated products and HNN-extensions.

In Bartels-Lück [24] (see also Roushon [296]) certain inheritance properties, in particular for actions in trees, for the class of groups which satisfy the Fibered Version (see Fibered Meta Conjecture 2.95) of the Farrell-Jones Conjecture 2.90 for homotopy K-theory $KH_*(RG)$ and the Fibered Farrell-Jones Conjecture 2.84 are proven. For instance for a group G acting on a tree T the Farrell-Jones Conjecture 2.90 for homotopy K-theory $KH_*(RG)$ folds for G if it is true for all stabilizers [24, Theorem 0.5]. The same statement is true for the Fibered Version (see Fibered Meta Conjecture 2.95) of the Farrell-Jones Conjecture 2.90 for homotopy K-theory $KH_*(RG)$ (see [24, Theorem 0.5]) and for the L-theoretic version of the Farrell-Jones Conjecture 2.5 after inverting two (see [24, Theorem 0.5]).

A ring R is called *regular coherent* if every finitely presented R-module possesses a finite-dimensional resolution by finitely generated projective R-modules. A ring R is regular if and only if it is regular coherent and Noetherian. A group G is called *regular* or *regular coherent* respectively if for any regular ring R the group ring RG is regular respectively regular coherent. Every regular coherent group is torsionfree. The trivial group and free groups are regular coherent. Fundamental groups of closed 2-dimensional manifolds with the exception of $\mathbb{Z}/2$ are regular coherent. For more information about these notions we refer to [341, Theorem 19.1].

For a regular ring R and a group G acting on a tree T the K-theoretic Farrell-Jones Conjecture 2.5 for holds for G if it is true for the stabilizers of every vertex and every edge and the stabilizer of each edge is regular coherent [24, Theorem 0.11])

3.6 Miscellaneous

Comment 44 (By W.): `Add reference to Mathai's result that the Novikov Conjecture is true for groups of cohomological dimension` ≤ 2

Comment 45 (By W.): `Add remark that the counterexamples (groups with expanders) to the Baum-Connes Conjecture with coefficients are not counterexamples to the Novikov Conjectures. I think the argument is that they are limits of hyperbolic groups.`

Comment 46 (By W.): `Kasparov and Yu have announced a proof that hyperbolic groups satisfy the Baum-Connes Conjecture with coefficients. This involves the paper [369]. Later when this has appeared and is confirmed we have to make some changes.`

Exercises

3.1. Let G be group having Kazhdan's property (T). Show that every quotient group of G has property (T) and that G cannot be a free group or the fundamental group of a compact connected 2-manifold.

3.2. Let $1 \to H \to G \to Q \to 1$ be an extension of groups. Suppose that H is word-hyperbolic or a-T-menable and the same for Q. Show that G satisfies the Baum-Connes Conjecture with Coefficients 2.74.

3.3. The lamplighter group L is defined as the semi-direct product $\left(\bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2\right) \rtimes \mathbb{Z}$ with respect to the $\mathbb{Z}$-action on $\bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2$ given by shifting. Does it satisfy the Baum-Connes Conjecture 2.4 or the Farrell-Jones Conjecture 2.5 or some of their versions?

3.4. Suppose that there exists a group G for which the Baum-Connes Conjecture 2.4 or the Farrell-Jones Conjecture 2.5 is not true. Show that then G contains a finitely generated subgroup G' with the same property.

3.5. Show that the Fibered Farrell-Jones Conjecture 2.84 holds for all groups if and only if it holds for the fundamental groups of all closed connected orientable 4-manifolds M. **Comment 47** (By W.): `What can we say for Baum-Connes?`

3.6. Let M be a compact connected 4-manifold which can be written as a fiber bundle $F \to M \to B$ for compact connected orientable manifolds B and F which are different from $\{\bullet\}$. Show that $\pi_1(M)$ satisfies the Baum-Connes Conjecture with Coefficients 2.74. **Comment 48** (By W.): `This exercise works only if we know the Baum-Connes Conjecture with Coefficients 2.74 for closed connected 3-manifolds.`

3.7. Show that the Fibered Farrell-Jones Conjecture 2.84 holds for the group $\mathbb{Z}$ and every ring R. Suppose that the K-theoretic Fibered Farrell-Jones Conjecture 2.84 holds for the group $\mathbb{Z}$ and the ring R but with the family $\mathcal{VCY}$ replaced by the family $\mathcal{FIN}$. Show that this implies that $NK_n(RG) = 0$ holds for all groups G and $n \in \mathbb{Z}$. (This is for instance not true for $R = \mathbb{Z}$.)

3.8. Fix an equivariant homology theory. In the sequel we will consider the family of virtually cyclic subgroups. Suppose that the Fibered Meta Conjecture 2.95 holds for the groups $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times D_\infty$ and $D_\infty \times D_\infty$. Show for two groups G and H that the Fibered Meta Conjecture 2.95 holds for $G \times H$ if and only if it holds for both G and H.

3.9. Fix an equivariant homology theory. In the sequel we will consider the family of virtually cyclic subgroups. Suppose that the Fibered Meta Conjecture 2.95 holds for the groups $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times D_\infty$ and $D_\infty \times D_\infty$ and every virtually finitely generated free group. Show for two groups G and H that the Fibered Meta Conjecture 2.95 holds for $G * H$ if and only if it holds for both G and H using the fact that for any infinite cyclic subgroup $C \subseteq G \times H$ its preimage under the projection $p\colon G * H \to G \times H$ is free.

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4. Basic Technicalities

4.1 Introduction

4.2 Classifying Spaces for Families

4.2.1 *G*-CW-Complexes

A *G-CW-complex* X is a G-space X together with a filtration $X_{-1} = \emptyset \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X$ such that $X = \operatorname{colim}_{n\to\infty} X_n$ and for each n there is a G-pushout

$$\begin{array}{ccc} \coprod_{i\in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i\in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i\in I_n} Q_i^n} & X_n \end{array} .$$

This definition makes also sense for topological groups. The following alternative definition only applies to discrete groups. A G-CW-complex is a CW-complex with a G-action by cellular maps such that for each open cell e and each $g \in G$ with $ge \cap e \neq \emptyset$ we have $gx = x$ for all $x \in e$. There is an obvious notion of a G-CW-pair.

A G-CW-complex X is called *finite* if it is built out of finitely many G-cells $G/H_i \times D^n$. This is the case if and only if it is *cocompact*, i.e. the quotient space $G\backslash X$ is compact. More information about G-CW-complexes can be found for instance in [215, Sections 1 and 2], [332, Sections II.1 and II.2].

4.2.2 Families of Subgroups

A *family* $\mathcal{F}$ *of subgroups of* G is a set of subgroups of G closed under conjugation, i.e. $H \in \mathcal{F}, g \in G$ implies $g^{-1}Hg \in \mathcal{F}$, and finite intersections, i.e. $H, K \in \mathcal{F}$ implies $H \cap K \in \mathcal{F}$. Throughout the text we will use the notations

$$\mathcal{TR}, \quad \mathcal{FCY}, \quad \mathcal{FIN}, \quad \mathcal{CYC}, \quad \mathcal{VCY}_I, \quad \mathcal{VCY} \qquad \text{and} \qquad \mathcal{ALL}$$

for the families consisting of the trivial, all finite cyclic, all finite, all (possibly infinite) cyclic, all virtually cyclic of the first kind, all virtually cyclic, respectively all subgroups of a given group G. Recall that a group is called *virtually cyclic* if it is finite or contains an infinite cyclic subgroup of finite index. A group is *virtually cyclic of the first kind* if it admits a surjection onto an infinite cyclic group with finite kernel, compare Lemma 2.18.

4.2.3 The Definition of the Classifying Spaces for Families of Subgroups

Let $\mathcal{F}$ be a family of subgroups of G. A G-CW-complex, all whose isotropy groups belong to $\mathcal{F}$ and whose H-fixed point sets are contractible for all $H \in \mathcal{F}$, is called a *classifying space for the family* $\mathcal{F}$ and will be denoted $E_{\mathcal{F}}(G)$. Such a space is unique up to G-homotopy because it is characterized by the property that for any G-CW-complex X, all whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map from X to $E_{\mathcal{F}}(G)$. These spaces were introduced by tom Dieck [330], [332, I.6].

A functorial "bar-type" construction is given in [92, section 7].

If $\mathcal{F} \subset \mathcal{G}$ are families of subgroups for G, then by the universal property there is up to G-homotopy precisely one G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$.

The space $E_{\mathcal{TR}}(G)$ is the same as the space EG which is by definition the total space of the universal G-principal bundle $G \to EG \to BG$, or, equivalently, the universal covering of BG. A model for $E_{\mathcal{ALL}}(G)$ is given by the space $G/G = \{\bullet\}$ consisting of one point.

The space $E_{\mathcal{FIN}}(G)$ is also known as *the classifying space for proper G-actions* and denoted by $\underline{E}G$ in the literature. Recall that a G-CW-complex X is proper if and only if all its isotropy groups are finite (see for instance [215, Theorem 1.23 on page 18]). We often abbreviate $E_{\mathcal{FIN}}(G)$ by $\underline{\underline{E}}G$.$\underline{\underline{E}}G$

4.2.4 Specific Models for the Classifying Spaces for Families

There are often nice models for $E_{\mathcal{FIN}}(G)$. If G is word hyperbolic in the sense of Gromov, then the Rips-complex is a finite model [235], [236].

If G is a discrete subgroup of a Lie group L with finitely many path components, then for any maximal compact subgroup $K \subseteq L$ the space L/K with its left G-action is a model for $E_{\mathcal{FIN}}(G)$ [2, Corollary 4.14]. More information about $E_{\mathcal{FIN}}(G)$ can be found for instance in [33, section 2], [198], [217], [224], [225] and [308].

4.3 Spectra

4.3.1 *K*- and *L*-Theory Spectra over Groupoids

Let RINGS be the category of associative rings with unit. An *involution* on a R is a map $R \to R,\ r \mapsto \overline{r}$ satisfying $\overline{1} = 1$, $\overline{x+y} = \overline{x} + \overline{y}$ and $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$

for all $x, y \in R$. Let $\mathsf{RINGS}^{\mathrm{inv}}$ be the category of rings with involution. Let C^*-$\mathsf{ALGEBRAS}$ be the category of C^*-algebras. There are classical functors for $j \in -\infty \amalg \{j \in \mathbb{Z} \mid j \leq 2\}$

$$\mathbf{K}\colon \mathsf{RINGS} \to \mathsf{SPECTRA}; \tag{4.1}$$
$$\mathbf{L}^{\langle j \rangle}\colon \mathsf{RINGS}^{\mathrm{inv}} \to \mathsf{SPECTRA}; \tag{4.2}$$
$$\mathbf{K}^{\mathrm{top}}\colon C^*\text{-}\mathsf{ALGEBRAS} \to \mathsf{SPECTRA}. \tag{4.3}$$

The construction of such a non-connective algebraic K-theory functor goes back to Gersten [148] and Wagoner [340]. The spectrum for quadratic algebraic L-theory is constructed by Ranicki in [279]. In a more geometric formulation it goes back to Quinn [271]. In the topological K-theory case a construction using Bott periodicity for C^*-algebras can easily be derived from the Kuiper-Mingo Theorem (see [307, Section 2.2]). The homotopy groups of these spectra give the algebraic K-groups of Quillen (in high dimensions) and of Bass (in negative dimensions), the decorated quadratic L-theory groups, and the topological K-groups of C^*-algebras.

We emphasize again that in all three cases we need the non-connective versions of the spectra, i.e. the homotopy groups in negative dimensions are non-trivial in general. For example the version of the Farrell-Jones Conjecture where one uses connective K-theory spectra is definitely false in general, compare Remark 1.33.

Now let us fix a coefficient ring R (with involution). Then sending a group G to the group ring RG yields functors $R(-)\colon \mathsf{GROUPS} \to \mathsf{RINGS}$, respectively $R(-)\colon \mathsf{GROUPS} \to \mathsf{RINGS}^{\mathrm{inv}}$, where GROUPS denotes the category of groups. Let $\mathsf{GROUPS}^{\mathrm{inj}}$ be the category of groups with injective group homomorphisms as morphisms. Taking the reduced group C^*-algebra defines a functor $C^*_r\colon \mathsf{GROUPS}^{\mathrm{inj}} \to C^*$-$\mathsf{ALGEBRAS}$. The composition of these functors with the functors (4.1), (4.2) and (4.3) above yields functors

$$\mathbf{K}R(-)\colon \mathsf{GROUPS} \to \mathsf{SPECTRA}; \tag{4.4}$$
$$\mathbf{L}^{\langle j \rangle}R(-)\colon \mathsf{GROUPS} \to \mathsf{SPECTRA}; \tag{4.5}$$
$$\mathbf{K}^{\mathrm{top}}C^*_r(-)\colon \mathsf{GROUPS}^{\mathrm{inj}} \to \mathsf{SPECTRA}. \tag{4.6}$$

They satisfy

$$\begin{aligned} \pi_n(\mathbf{K}R(G)) &= K_n(RG); \\ \pi_n(\mathbf{L}^{\langle j \rangle}R(G)) &= L_n^{\langle j \rangle}(RG); \\ \pi_n(\mathbf{K}^{\mathrm{top}}C^*_r(G)) &= K_n(C^*_r(G)), \end{aligned}$$

for all groups G and $n \in \mathbb{Z}$. The next result essentially says that these functors can be extended to groupoids.

Theorem 4.7 (K- and L-Theory Spectra over Groupoids). *Let R be a ring (with involution). There exist covariant functors*

$$\mathbf{K}_R\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}; \tag{4.8}$$
$$\mathbf{L}_R^{\langle j\rangle}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}; \tag{4.9}$$
$$\mathbf{K}^{\mathrm{top}}\colon \mathsf{GROUPOIDS}^{\mathrm{inj}} \to \mathsf{SPECTRA} \tag{4.10}$$

with the following properties:

(1) If $F\colon \mathcal{G}_0 \to \mathcal{G}_1$ is an equivalence of (small) groupoids, then the induced maps $\mathbf{K}_R(F)$, $\mathbf{L}_R^{\langle j\rangle}(F)$ and $\mathbf{K}^{\mathrm{top}}(F)$ are weak equivalences of spectra.

(2) Let $I\colon \mathsf{GROUPS} \to \mathsf{GROUPOIDS}$ be the functor sending G to G considered as a groupoid, i.e. to $\mathcal{G}^G(G/G)$. This functor restricts to a functor $\mathsf{GROUPS}^{\mathrm{inj}} \to \mathsf{GROUPOIDS}^{\mathrm{inj}}$.
There are natural transformations from $\mathbf{K}R(-)$ to $\mathbf{K}_R \circ I$, from $\mathbf{L}^{\langle j\rangle}R(-)$ to $\mathbf{L}_R^{\langle j\rangle} \circ I$ and from $\mathbf{K}C_r^*(-)$ to $\mathbf{K}^{\mathrm{top}} \circ I$ such that the evaluation of each of these natural transformations at a given group is an equivalence of spectra.

(3) For every group G and all $n \in \mathbb{Z}$ we have

$$\pi_n(\mathbf{K}_R \circ I(G)) \cong K_n(RG);$$
$$\pi_n(\mathbf{L}_R^{\langle j\rangle} \circ I^{\mathrm{inv}}(G)) \cong L_n^{\langle j\rangle}(RG);$$
$$\pi_n(\mathbf{K}^{\mathrm{top}} \circ I(G)) \cong K_n(C_r^*(G)).$$

Proof. We only sketch the strategy of the proof. More details can be found in [92, Section 2].

Let $\mathcal{G}$ be a groupoid. Similar to the group ring RG one can define an R-linear category $R\mathcal{G}$ by taking the free R-modules over the morphism sets of $\mathcal{G}$. Composition of morphisms is extended R-linearly. By formally adding finite direct sums one obtains an additive category $R\mathcal{G}_\oplus$. Pedersen-Weibel [258] (compare also [57]) define a non-connective algebraic K-theory functor which digests additive categories and can hence be applied to $R\mathcal{G}_\oplus$. For the comparison result one uses that for every ring R (in particular for RG) the Pedersen-Weibel functor applied to $R_\oplus$ (a small model for the category of finitely generated free R-modules) yields the non-connective K-theory of the ring R and that it sends equivalences of additive categories to equivalences of spectra. In the L-theory case $R\mathcal{G}_\oplus$ inherits an involution and one applies the construction of [279, Example 13.6 on page 139] to obtain the $L^{\langle 1\rangle} = L^h$-version. The versions for $j \leq 1$ can be obtained by a construction which is analogous to the Pedersen-Weibel construction for K-theory, compare [61, Section 4]. In the C^*-case one obtains from $\mathcal{G}$ a C^*-category $C_r^*(\mathcal{G})$ and assigns to it its topological K-theory spectrum. There is a construction of the topological K-theory spectrum of a C^*-category in [92, Section 2]. However, the construction given there depends on two statements, which appeared in [145, Proposition 1 and Proposition 3], and those statements are incorrect, as already pointed out by Thomason in [329]. The construction in [92, Section

2] can easily be fixed but instead we recommend the reader to look at the more recent construction of Joachim [175].

4.4 Equivariant Homology Theories

A *G-homology theory* $\mathcal{H}_*^G$ is the "obvious" G-equivariant generalization of the concept of a homology theory. It assigns to every G-CW-complex X a $\mathbb{Z}$-graded abelian group $\mathcal{H}_*^G(X)$. It is natural in X, satisfies G-homotopy invariance and possesses long exact Mayer-Vietoris sequences. An equivariant homology theory $\mathcal{H}_*^?$ assigns to every discrete group G a G-homology theory $\mathcal{H}_*^G$. These are linked by an induction structure. We explain how a functor from the orbit category $\mathsf{Or}(G)$ to the category of spectra leads to a G-homology theory (see Proposition 4.20) and how more generally a functor from the category of groupoids leads to an equivariant homology theory (see Proposition 4.22). We then describe the main examples of such spectra valued functors which were already used in order to formulate the Farrell-Jones and the Baum-Connes Conjectures in Section 2.2.

4.4.1 *G*-Homology Theories

Fix a group G and an associative commutative ring Λ with unit. A *G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules* is a collection of covariant functors $\mathcal{H}_n^G$ from the category of G-CW-pairs to the category of Λ-modules indexed by $n \in \mathbb{Z}$ together with natural transformations

$$\partial_n^G(X,A)\colon \mathcal{H}_n^G(X,A) \to \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A,\emptyset)$$

for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

(1) G-homotopy invariance

If f_0 and f_1 are G-homotopic maps $(X,A) \to (Y,B)$ of G-CW-pairs, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for $n \in \mathbb{Z}$.

(2) Long exact sequence of a pair

Given a pair (X,A) of G-CW-complexes, there is a long exact sequence

$$\cdots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X,A) \xrightarrow{\partial_{n+1}^G} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \\ \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X,A) \xrightarrow{\partial_n^G} \mathcal{H}_{n-1}^G(A) \xrightarrow{\mathcal{H}_{n-1}^G(i)} \cdots,$$

where $i\colon A \to X$ and $j\colon X \to (X,A)$ are the inclusions.

(3) Excision

Let (X,A) be a G-CW-pair and let $f\colon A \to B$ be a cellular G-map of

G-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G-CW-pair. Then the canonical map $(F, f)\colon (X, A) \to (X \cup_f B, B)$ induces for each $n \in \mathbb{Z}$ an isomorphism

$$\mathcal{H}_n^G(F, f)\colon \mathcal{H}_n^G(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(X \cup_f B, B).$$

(4) Disjoint union axiom

Let $\{X_i \mid i \in I\}$ be a family of G-CW-complexes. Denote by $j_i\colon X_i \to \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

$$\bigoplus_{i \in I} \mathcal{H}_n^G(j_i)\colon \bigoplus_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\left(\coprod_{i \in I} X_i\right)$$

is bijective for each $n \in \mathbb{Z}$.

Of course a G-homology theory for the trivial group $G = \{1\}$ is a homology theory (satisfying the disjoint union axiom) in the classical non-equivariant sense.

The disjoint union axiom ensures that we can pass from finite G-CW-complexes to arbitrary ones using the following lemma.

Lemma 4.11. *Let $\mathcal{H}_*^G$ be a G-homology theory. Let X be a G-CW-complex and $\{X_i \mid i \in I\}$ be a directed system of G-CW-subcomplexes directed by inclusion such that $X = \cup_{i \in I} X_i$. Then for all $n \in \mathbb{Z}$ the natural map*

$$\operatorname{colim}_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

is bijective.

Proof. Compare for example with [328, Proposition 7.53 on page 121], where the non-equivariant case for $I = \mathbb{N}$ is treated.

Example 4.12 (Bredon Homology). The most basic G-homology theory is *Bredon homology*. The *orbit category* $\mathsf{Or}(G)$ has as objects the homogeneous spaces G/H and as morphisms G-maps. Let X be a G-CW-complex. It defines a contravariant functor from the orbit category $\mathsf{Or}(G)$ to the category of CW-complexes by sending G/H to $\operatorname{map}_G(G/H, X) = X^H$. Composing it with the functor cellular chain complex yields a contravariant functor

$$C_*^c(X)\colon \mathsf{Or}(G) \to \mathbb{Z}\text{-}\mathsf{CHCOM}$$

into the category of $\mathbb{Z}$-chain complexes. Let Λ be a commutative ring and let

$$M\colon \mathsf{Or}(G) \to \Lambda\text{-}\mathsf{MODULES}$$

be a covariant functor. Then one can form the tensor product over the orbit category (see for instance [215, 9.12 on page 166]) and obtains the Λ-chain complex $C_*^c(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} M$. Its homology is the Bredon homology of X with coefficients in M

$$H_*^G(X; M) = H_*(C_*^c(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} M).$$

Thus we get a G-homology theory H_*^G with values in Λ-modules. For a trivial group G this reduces to the cellular homology of X with coefficients in the Λ-module M.

4.4.2 The Axioms of an Equivariant Homology Theory

The notion of a G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules for a commutative ring Λ was defined in Subsection 4.4.1. We now recall the axioms of an equivariant homology theory from [219, Section 1]. We will see in Section 4.3.1 that the G-homology theories we used in the formulation of the Baum-Connes and the Farrell-Jones Conjectures in Subsection 2.2.2 are in fact the values at G of suitable equivariant homology theories.

Let $\alpha\colon H \to G$ be a group homomorphism. Given a H-space X, define the *induction of X with α* to be the G-space $\operatorname{ind}_\alpha X$ which is the quotient of $G \times X$ by the right H-action $(g, x) \cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g, x) \in G \times X$. If $\alpha\colon H \to G$ is an inclusion, we also write ind_H^G instead of $\operatorname{ind}_\alpha$.

An *equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules* consists of a G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules for each group G together with the following so called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X, A) such that $\ker(\alpha)$ acts freely on X, there are for each $n \in \mathbb{Z}$ natural isomorphisms

$$\operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(\operatorname{ind}_\alpha(X, A))$$

satisfying the following conditions.

(1) Compatibility with the boundary homomorphisms
$\partial_n^G \circ \operatorname{ind}_\alpha = \operatorname{ind}_\alpha \circ \partial_n^H$.
(2) Functoriality
Let $\beta\colon G \to K$ be another group homomorphism such that $\ker(\beta \circ \alpha)$ acts freely on X. Then we have for $n \in \mathbb{Z}$

$$\operatorname{ind}_{\beta\circ\alpha} \;=\; \mathcal{H}_n^K(f_1) \circ \operatorname{ind}_\beta \circ \operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X, A) \to \mathcal{H}_n^K(\operatorname{ind}_{\beta\circ\alpha}(X, A)),$$

where $f_1\colon \operatorname{ind}_\beta \operatorname{ind}_\alpha(X, A) \xrightarrow{\cong} \operatorname{ind}_{\beta\circ\alpha}(X, A)$, $(k, g, x) \mapsto (k\beta(g), x)$ is the natural K-homeomorphism.
(3) Compatibility with conjugation
For $n \in \mathbb{Z}$, $g \in G$ and a G-CW-pair (X, A) the homomorphism

$$\operatorname{ind}_{c(g)\colon G \to G}\colon \mathcal{H}_n^G(X, A) \to \mathcal{H}_n^G(\operatorname{ind}_{c(g)\colon G\to G}(X, A))$$

agrees with $\mathcal{H}_n^G(f_2)$, where the G-homeomorphism

$$f_2\colon (X,A) \to \mathrm{ind}_{c(g)\colon G\to G}(X,A)$$

sends x to $(1, g^{-1}x)$ and $c(g)\colon G \to G$ sends g' to $gg'g^{-1}$.

If the G-homology theory $\mathcal{H}_*^G$ is defined or considered only for proper G-CW-pairs (X,A), we call it a *proper G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules.*

4.4.3 Basic Properties and Examples of Equivariant Homology Theories

Example 4.13. Let $\mathcal{K}_*$ be a homology theory for (non-equivariant) CW-pairs with values in Λ-modules. Examples are singular homology, oriented bordism theory or topological K-homology. Then we obtain two equivariant homology theories with values in Λ-modules, whose underlying G-homology theories for a group G are given by the following constructions

$$\mathcal{H}_n^G(X,A) = \mathcal{K}_n(G\backslash X, G\backslash A);$$
$$\mathcal{H}_n^G(X,A) = \mathcal{K}_n(EG \times_G (X,A)).$$

Example 4.14. Given a proper G-CW-pair (X,A), one can define the G-bordism group $\Omega_n^G(X,A)$ as the abelian group of G-bordism classes of maps $f\colon (M,\partial M) \to (X,A)$ whose sources are oriented smooth manifolds with cocompact orientation preserving proper smooth G-actions. The definition is analogous to the one in the non-equivariant case. This is also true for the proof that this defines a proper G-homology theory. There is an obvious induction structure coming from induction of equivariant spaces. Thus we obtain an equivariant proper homology theory $\Omega_*^?$.

Example 4.15. Let Λ be a commutative ring and let

$$M\colon \mathsf{GROUPOIDS} \to \Lambda\text{-}\mathsf{MODULES}$$

be a contravariant functor. For a group G we obtain a covariant functor

$$M^G\colon \mathsf{Or}(G) \to \Lambda\text{-}\mathsf{MODULES}$$

by its composition with the transport groupoid functor $\mathcal{G}^G$ defined in (4.21). Let $H_*^G(-;M)$ be the G-homology theory given by the Bredon homology with coefficients in M^G as defined in Example 4.12. There is an induction structure such that the collection of the $H^G(-;M)$ defines an equivariant homology theory $H_*^?(-;M)$. This can be interpreted as the special case of Proposition 4.22, where the covariant functor $\mathsf{GROUPOIDS} \to \Omega\text{-}\mathsf{SPECTRA}$ is the composition of M with the functor sending a Λ-module to the associated Eilenberg-MacLane spectrum. But there is also a purely algebraic construction.

The next lemma was used in the proof of the Transitivity Principle 2.11.

Lemma 4.16. *Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in Λ-modules. Let G be a group and let $\mathcal{F}$ a family of subgroups of G. Let Z be a G-CW-complex. Consider $N \in \mathbb{Z} \cup \{\infty\}$. For $H \subseteq G$ let $\mathcal{F} \cap H$ be the family of subgroups of H given by $\{K \cap H \mid K \in \mathcal{F}\}$. Suppose for each $H \subset G$, which occurs as isotropy group in Z, that the map induced by the projection* $\mathrm{pr}\colon E_{\mathcal{F}\cap H}(H) \to \{\bullet\}$

$$\mathcal{H}_n^H(\mathrm{pr})\colon \mathcal{H}_n^H(E_{\mathcal{F}\cap H}(H)) \to \mathcal{H}_n^H(\{\bullet\})$$

is bijective for all $n \in \mathbb{Z}, n \leq N$.

Then the map induced by the projection $\mathrm{pr}_2\colon E_{\mathcal{F}}(G) \times Z \to Z$

$$\mathcal{H}_n^G(\mathrm{pr}_2)\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z) \to \mathcal{H}_n^G(Z)$$

is bijective for $n \in \mathbb{Z}, n \leq N$.

Proof. We first prove the claim for finite-dimensional G-CW-complexes by induction over $d = \dim(Z)$. The induction beginning $\dim(Z) = -1$, i.e. $Z = \emptyset$, is trivial. In the induction step from $(d-1)$ to d we choose a G-pushout

$$\begin{array}{ccc}
\coprod_{i\in I_d} G/H_i \times S^{d-1} & \longrightarrow & Z_{d-1} \\
\downarrow & & \downarrow \\
\coprod_{i\in I_d} G/H_i \times D^d & \longrightarrow & Z_d
\end{array}$$

If we cross it with $E_{\mathcal{F}}(G)$, we obtain another G-pushout of G-CW-complexes. The various projections induce a map from the Mayer-Vietoris sequence of the latter G-pushout to the Mayer-Vietoris sequence of the first G-pushout. By the Five-Lemma it suffices to prove that the following maps

$$\mathcal{H}_n^G(\mathrm{pr}_2)\colon \mathcal{H}_n^G\left(E_{\mathcal{F}}(G) \times \coprod_{i\in I_d} G/H_i \times S^{d-1}\right) \to \mathcal{H}_n^G\left(\coprod_{i\in I_d} G/H_i \times S^{d-1}\right);$$
$$\mathcal{H}_n^G(\mathrm{pr}_2)\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z_{d-1}) \to \mathcal{H}_n^G(Z_{d-1});$$
$$\mathcal{H}_n^G(\mathrm{pr}_2)\colon \mathcal{H}_n^G\left(E_{\mathcal{F}}(G) \times \coprod_{i\in I_d} G/H_i \times D^{d}\right) \to \mathcal{H}_n^G\left(\coprod_{i\in I_d} G/H_i \times D^{d}\right)$$

are bijective for $n \in \mathbb{Z}, n \leq N$. This follows from the induction hypothesis for the first two maps. Because of the disjoint union axiom and G-homotopy invariance of $\mathcal{H}_*^?$ the claim follows for the third map if we can show for any $H \subseteq G$ which occurs as isotropy group in Z that the map

$$\mathcal{H}_n^G(\mathrm{pr}_2)\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times G/H) \to \mathcal{H}^G(G/H) \tag{4.17}$$

is bijective for $n \in \mathbb{Z}, n \leq N$. The G-map

$$G \times_H \operatorname{res}_G^H E_{\mathcal{F}}(G) \to G/H \times E_{\mathcal{F}}(G) \qquad (g,x) \mapsto (gH, gx)$$

is a G-homeomorphism where res_G^H denotes the restriction of the G-action to an H-action. Obviously $\operatorname{res}_G^H E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}\cap H}(H)$. We conclude from the induction structure that the map (4.17) can be identified with the map

$$\mathcal{H}_n^G(\mathrm{pr})\colon \mathcal{H}_n^H(E_{\mathcal{F}\cap H}(H)) \to \mathcal{H}^H(\{\bullet\})$$

which is bijective for all $n \in \mathbb{Z}, n \leq N$ by assumption. This finishes the proof in the case that Z is finite-dimensional. The general case follows by a colimit argument using Lemma 4.11.

4.4.4 Constructing Equivariant Homology Theories

Recall that a (non-equivariant) spectrum yields an associated (non-equivariant) homology theory. In this section we explain how a spectrum over the orbit category of a group G defines a G-homology theory. We would like to stress that our approach using spectra over the orbit category should be distinguished from approaches to equivariant homology theories using spectra with G-action or the more complicated notion of G-equivariant spectra in the sense of [208], see for example [59] for a survey. The latter approach leads to a much richer structure but only works for compact Lie groups.

We briefly fix some conventions concerning spectra. We always work in the very convenient *category* SPACES *of compactly generated spaces* (see [322], [360, I.4]). In that category the adjunction homeomorphism $\operatorname{map}(X \times Y, Z) \xrightarrow{\cong} \operatorname{map}(X, \operatorname{map}(Y,Z))$ holds without any further assumptions such as local compactness and the product of two CW-complexes is again a CW-complex. Let SPACES^+ be the category of pointed compactly generated spaces. Here the objects are (compactly generated) spaces X with base points for which the inclusion of the base point is a cofibration. Morphisms are pointed maps. If X is a space, denote by X_+ the pointed space obtained from X by adding a disjoint base point. For two pointed spaces $X = (X,x)$ and $Y = (Y,y)$ define their *smash product* as the pointed space

$$X \wedge Y = X \times Y/(\{x\} \times Y \cup X \times \{y\}),$$

and the *reduced cone* as

$$\operatorname{cone}(X) = X \times [0,1]/(X \times \{1\} \cup \{x\} \times [0,1]).$$

A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps* $\sigma(n)\colon E(n) \wedge S^1 \longrightarrow E(n+1)$. A *map* of spectra $\mathbf{f}\colon \mathbf{E} \to \mathbf{E}'$ is a sequence of maps $f(n)\colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e. we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ for all $n \in \mathbb{Z}$.

Maps of spectra are sometimes called functions in the literature, they should not be confused with the notion of a map of spectra in the stable category (see [4, III.2.]). The category of spectra and maps will be denoted $\mathsf{SPECTRA}$. Recall that the homotopy groups of a spectrum are defined by

$$\pi_i(\mathbf{E}) \;=\; \operatorname{colim}_{k\to\infty} \pi_{i+k}(E(k)),$$

where the system $\pi_{i+k}(E(k))$ is given by the composition

$$\pi_{i+k}(E(k)) \xrightarrow{S} \pi_{i+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma(k)_*} \pi_{i+k+1}(E(k+1))$$

of the suspension homomorphism S and the homomorphism induced by the structure map. A *weak equivalence* of spectra is a map $\mathbf{f}\colon \mathbf{E} \to \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups.

Given a spectrum $\mathbf{E}$ and a pointed space X, we can define their smash product $X \wedge \mathbf{E}$ by $(X \wedge \mathbf{E})(n) := X \wedge E(n)$ with the obvious structure maps. It is a classical result that a spectrum $\mathbf{E}$ defines a homology theory by setting

$$H_n(X, A; \mathbf{E}) = \pi_n\left((X_+ \cup_{A_+} \operatorname{cone}(A_+)) \wedge \mathbf{E}\right).$$

We want to extend this to G-homology theories. This requires the consideration of spaces and spectra over the orbit category. Our presentation follows [92], where more details can be found.

In the sequel $\mathcal{C}$ is a small category. Our main example is the orbit category $\mathsf{Or}(G)$, whose objects are homogeneous G-spaces G/H and whose morphisms are G-maps.

Definition 4.18. *A* covariant (contravariant) $\mathcal{C}$-space X *is a covariant (contravariant) functor*

$$X\colon \mathcal{C} \;\to\; \mathsf{SPACES}.$$

A map between $\mathcal{C}$-spaces is a natural transformation of such functors. Analogously a pointed $\mathcal{C}$-space *is a functor from $\mathcal{C}$ to SPACES^+ and a $\mathcal{C}$-spectrum a functor to $\mathsf{SPECTRA}$.*

Example 4.19. *Let Y be a left G-space. Define the associated* contravariant $\mathsf{Or}(G)$-space $\operatorname{map}_G(-, Y)$ *by*

$$\operatorname{map}_G(-,Y)\colon\ \mathsf{Or}(G) \to \mathsf{SPACES}, \qquad G/H \;\mapsto \operatorname{map}_G(G/H, Y) = Y^H.$$

If Y is pointed then $\operatorname{map}_G(-,Y)$ *takes values in pointed spaces.*

Let X be a contravariant and Y be a covariant $\mathcal{C}$-space. Define their *balanced product* to be the space

$$X \times_{\mathcal{C}} Y \;:=\; \coprod_{c\in \operatorname{ob}(\mathcal{C})} X(c) \times Y(c)/\sim$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for all morphisms $\phi\colon c \to d$ in $\mathcal{C}$ and points $x \in X(d)$ and $y \in Y(c)$. Here $x\phi$ stands for $X(\phi)(x)$ and ϕy for $Y(\phi)(y)$. If X and Y are pointed, then one defines analogously their *balanced smash product* to be the pointed space

$$X \wedge_{\mathcal{C}} Y \;=\; \bigvee_{c \in \mathrm{ob}(\mathcal{C})} X(c) \wedge Y(c)/\sim .$$

In [92] the notation $X \otimes_{\mathcal{C}} Y$ was used for this space. Doing the same construction level-wise one defines the *balanced smash product* $X \wedge_{\mathcal{C}} \mathbf{E}$ of a contravariant pointed $\mathcal{C}$-space and a covariant $\mathcal{C}$-spectrum $\mathbf{E}$.

The proof of the next result is analogous to the non-equivariant case. Details can be found in [92, Lemma 4.4], where also cohomology theories are treated.

Proposition 4.20 (Constructing G-Homology Theories). *Let* $\mathbf{E}$ *be a covariant* $\mathsf{Or}(G)$*-spectrum. It defines a* G*-homology theory* $H_*^G(-;\mathbf{E})$ *by*

$$H_n^G(X,A;\mathbf{E}) \;=\; \pi_n\left(\mathrm{map}_G\left(-,(X_+ \cup_{A_+} \mathrm{cone}(A_+))\right) \wedge_{\mathsf{Or}(G)} \mathbf{E}\right).$$

In particular we have

$$H_n^G(G/H;\mathbf{E}) \;=\; \pi_n(\mathbf{E}(G/H)).$$

Recall that we seek an equivariant homology theory and not only a G-homology theory. If the $\mathsf{Or}(G)$-spectrum in Proposition 4.20 is obtained from a $\mathsf{GROUPOIDS}$-spectrum in a way we will now describe, then automatically we obtain the desired induction structure.

Let $\mathsf{GROUPOIDS}$ be the category of small groupoids with covariant functors as morphisms. Recall that a groupoid is a category in which all morphisms are isomorphisms. A covariant functor $f\colon \mathcal{G}_0 \to \mathcal{G}_1$ of groupoids is called *injective*, if for any two objects x, y in $\mathcal{G}_0$ the induced map $\mathrm{mor}_{\mathcal{G}_0}(x,y) \to \mathrm{mor}_{\mathcal{G}_1}(f(x), f(y))$ is injective. Let $\mathsf{GROUPOIDS}^{\mathrm{inj}}$ be the subcategory of $\mathsf{GROUPOIDS}$ with the same objects and injective functors as morphisms. For a G-set S we denote by $\mathcal{G}^G(S)$ its associated *transport groupoid*. Its objects are the elements of S. The set of morphisms from s_0 to s_1 consists of those elements $g \in G$ which satisfy $gs_0 = s_1$. Composition in $\mathcal{G}^G(S)$ comes from the multiplication in G. Thus we obtain for a group G a covariant functor

$$\mathcal{G}^G\colon \mathsf{Or}(G) \to \mathsf{GROUPOIDS}^{\mathrm{inj}}, \qquad G/H \mapsto \mathcal{G}^G(G/H). \tag{4.21}$$

A functor of small categories $F\colon \mathcal{C} \to \mathcal{D}$ is called an *equivalence* if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that both $F \circ G$ and $G \circ F$ are naturally equivalent to the identity functor. This is equivalent to the condition that F induces a bijection on the set of isomorphisms classes of objects and for any objects $x, y \in \mathcal{C}$ the map $\mathrm{mor}_{\mathcal{C}}(x,y) \to \mathrm{mor}_{\mathcal{D}}(F(x), F(y))$ induced by F is bijective.

Proposition 4.22 (Constructing Equivariant Homology Theories). *Consider a covariant* $\mathsf{GROUPOIDS}^{\mathrm{inj}}$*-spectrum*

$$\mathbf{E}\colon \mathsf{GROUPOIDS}^{\mathrm{inj}} \to \mathsf{SPECTRA}.$$

Suppose that $\mathbf{E}$ *respects equivalences, i.e. it sends an equivalence of groupoids to a weak equivalence of spectra. Then* $\mathbf{E}$ *defines an equivariant homology theory* $H_*^?(-;\mathbf{E})$*, whose underlying* G*-homology theory for a group* G *is the* G*-homology theory associated to the covariant* $\mathsf{Or}(G)$*-spectrum* $\mathbf{E} \circ \mathcal{G}^G\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ *in the previous Proposition 4.20, i.e.*

$$H_*^G(X,A;\mathbf{E}) \;=\; H_*^G(X,A;\mathbf{E}\circ\mathcal{G}^G).$$

In particular we have

$$H_n^G(G/H;\mathbf{E}) \;\cong\; H_n^H(\{\bullet\};\mathbf{E}) \;\cong\; \pi_n(\mathbf{E}(I(H))),$$

where $I(H)$ *denotes* H *considered as a groupoid with one object. The whole construction is natural in* $\mathbf{E}$.

Proof. We have to specify the induction structure for a homomorphism $\alpha\colon H \to G$. We only sketch the construction in the special case where α is injective and $A = \emptyset$. The details of the full proof can be found in [301, Theorem 2.10 on page 21].

The functor induced by α on the orbit categories is denoted in the same way

$$\alpha\colon \mathsf{Or}(H) \to \mathsf{Or}(G), \qquad H/L \mapsto \operatorname{ind}_\alpha(H/L) = G/\alpha(L).$$

There is an obvious natural equivalence of functors $\mathsf{Or}(H) \to \mathsf{GROUPOIDS}^{\mathrm{inj}}$

$$T\colon \mathcal{G}^H \to \mathcal{G}^G \circ \alpha.$$

Its evaluation at H/L is the equivalence of groupoids $\mathcal{G}^H(H/L) \to \mathcal{G}^G(G/\alpha(L))$ which sends an object hL to the object $\alpha(h)\alpha(L)$ and a morphism given by $h \in H$ to the morphism $\alpha(h) \in G$. The desired isomorphism

$$\operatorname{ind}_\alpha\colon H_n^H(X;\mathbf{E}\circ\mathcal{G}^H) \to H_n^G(\operatorname{ind}_\alpha X;\mathbf{E}\circ\mathcal{G}^G)$$

is induced by the following map of spectra

$$\begin{aligned} &\operatorname{map}_H(-,X_+) \wedge_{\mathsf{Or}(H)} \mathbf{E}\circ\mathcal{G}^H \xrightarrow{\operatorname{id}\wedge\mathbf{E}(T)} \operatorname{map}_H(-,X_+) \wedge_{\mathsf{Or}(H)} \mathbf{E}\circ\mathcal{G}^G\circ\alpha \\ &\xleftarrow{\cong} (\alpha_*\operatorname{map}_H(-,X_+)) \wedge_{\mathsf{Or}(G)} \mathbf{E}\circ\mathcal{G}^G \xleftarrow{\cong} \operatorname{map}_G(-,\operatorname{ind}_\alpha X_+) \wedge_{\mathsf{Or}(G)} \mathbf{E}\circ\mathcal{G}^G. \end{aligned}$$

Here $\alpha_*\operatorname{map}_H(-,X_+)$ is the pointed $\mathsf{Or}(G)$-space which is obtained from the pointed $\mathsf{Or}(H)$-space $\operatorname{map}_H(-,X_+)$ by induction, i.e. by taking the balanced product over $\mathsf{Or}(H)$ with the $\mathsf{Or}(H)$-$\mathsf{Or}(G)$ bimodule $\operatorname{mor}_{\mathsf{Or}(G)}(??,\alpha(?))$ [92, Definition 1.8]. Notice that $\mathbf{E}\circ\mathcal{G}^G\circ\alpha$ is the same as the restriction of the

$\mathsf{Or}(G)$-spectrum $\mathbf{E} \circ \mathcal{G}^G$ along α which is often denoted by $\alpha^*(\mathbf{E} \circ \mathcal{G}^G)$ in the literature [92, Definition 1.8]. The second map is given by the adjunction homeomorphism of induction α_* and restriction α^* (see [92, Lemma 1.9]). The third map is the homeomorphism of $\mathsf{Or}(G)$-spaces which is the adjoint of the obvious map of $\mathsf{Or}(H)$-spaces $\operatorname{map}_H(-, X_+) \to \alpha^* \operatorname{map}_G(-, \operatorname{ind}_\alpha X_+)$ whose evaluation at H/L is given by $\operatorname{ind}_\alpha$.

4.4.5 Assembly Maps in Terms of Homotopy Colimits

In this section we describe a homotopy-theoretic formulation of the Baum-Connes and Farrell-Jones Conjectures. For the classical assembly maps which in our set-up correspond to the trivial family such formulations were described in [358].

For a group G and a family $\mathcal{F}$ of subgroups we denote by $\mathsf{Or}_{\mathcal{F}}(G)$ the *restricted orbit category*. Its objects are homogeneous spaces G/H with $H \in \mathcal{F}$. Morphisms are G-maps. If $\mathcal{F} = \mathcal{ALL}$ we get back the (full) orbit category, i.e. $\mathsf{Or}(G) = \mathsf{Or}_{\mathcal{ALL}}(G)$.

Metaconjecture 4.23 (Homotopy-Theoretic Isomorphism Conjecture). *Let G be a group and $\mathcal{F}$ a family of subgroups. Let $\mathbf{E}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ be a covariant functor. Then*

$$A_{\mathcal{F}}\colon \operatorname{hocolim}_{\mathsf{Or}_{\mathcal{F}}(G)} \mathbf{E}|_{\mathsf{Or}_{\mathcal{F}}(G)} \;\to\; \operatorname{hocolim}_{\mathsf{Or}(G)} \mathbf{E} \simeq \mathbf{E}(G/G)$$

is a weak equivalence of spectra.

Here hocolim is the homotopy colimit of a covariant functor to spectra, which is itself a spectrum. The map $A_{\mathcal{F}}$ is induced by the obvious functor $\mathsf{Or}_{\mathcal{F}}(G) \to \mathsf{Or}(G)$. The equivalence $\operatorname{hocolim}_{\mathsf{Or}(G)} \mathbf{E} \simeq \mathbf{E}(G/G)$ comes from the fact that G/G is a final object in $\mathsf{Or}(G)$. For information about homotopy-colimits we refer to [46], [92, Section 3] and [102].

Remark 4.24. If we consider the map on homotopy groups that is induced by the map $A_{\mathcal{F}}$ which appears in the Homotopy-Theoretic Isomorphism Conjecture above, then we obtain precisely the map with the same name in Meta-Conjecture 2.1 for the homology theory $H_*^G(-;\mathbf{E})$ associated with $\mathbf{E}$ in Proposition 4.20, compare [92, Section 5]. In particular the Baum-Connes Conjecture 2.4 and the Farrell-Jones Conjecture 2.5 can be seen as special cases of Meta-Conjecture 4.23.

4.4.6 Universal Property of the Homotopy-Theoretic Assembly Map

The Homotopy-Theoretic Isomorphism Conjecture 4.23 is in some sense the most conceptual formulation of an Isomorphism Conjecture because it has a universal property as the universal approximation from the left by a (weakly) excisive $\mathcal{F}$-homotopy invariant functor. This is explained in detail in [92, Section 6]. This universal property is important if one wants to identify different models for the assembly map, compare e.g. [22, Section 6] and [157].

4.4.7 Naturality under Induction

Consider a covariant functor $\mathbf{E}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ which respects equivalences. Let $H_*^?(-;\mathbf{E})$ be the associated equivariant homology theory (see Proposition 4.22). Then for a group homomorphism $\alpha\colon H \to G$ and H-CW-pair (X,A) we obtain a homomorphism

$$\operatorname{ind}_\alpha\colon H_n^H(X,A;\mathbf{E}) \to H_n^G(\operatorname{ind}_\alpha(X,A);\mathbf{E})$$

which is natural in (X,A). Note that we did not assume that $\ker(\alpha)$ acts freely on X. In fact the construction sketched in the proof of Proposition 4.22 still works even though $\operatorname{ind}_\alpha$ may not be an isomorphism as it is the case if $\ker(\alpha)$ acts freely. We still have functoriality as described in 2 in Subection 4.4.2.

Now suppose that $\mathcal{H}$ and $\mathcal{G}$ are families of subgroups for H and G such that $\alpha(K) \in \mathcal{G}$ holds for all $K \in \mathcal{H}$. Then we obtain a G-map $f\colon \operatorname{ind}_\alpha E_{\mathcal{H}}(H) \to E_{\mathcal{G}}(G)$ from the universal property of $E_{\mathcal{G}}(G)$. Let $p\colon \operatorname{ind}_\alpha\{\bullet\} = G/\alpha(H) \to \{\bullet\}$ be the projection. Let $I\colon \mathsf{GROUPS} \to \mathsf{GROUPOIDS}$ be the functor sending G to $\mathcal{G}^G(G/G)$. Then the following diagram, where the horizontal arrows are induced from the projections to the one point space, commutes for all $n \in \mathbb{Z}$.

$$\begin{array}{ccc}
H_n^H(E_{\mathcal{H}}(H);\mathbf{E}) & \xrightarrow{A_{\mathcal{H}}} & H_n^H(\{\bullet\};\mathbf{E}) = \pi_n(\mathbf{E}(I(H))) \\
{\scriptstyle H_n^G(f)\circ\operatorname{ind}_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle H_n^G(p)\circ\operatorname{ind}_\alpha = \pi_n(\mathbf{E}(I(\alpha)))} \\
H_n^G(E_{\mathcal{G}}(G);\mathbf{E}) & \xrightarrow{A_{\mathcal{G}}} & H_n^G(\{\bullet\};\mathbf{E}) = \pi_n(\mathbf{E}(I(G))).
\end{array}$$

If we take the special case $\mathbf{E} = \mathbf{K}_R$ and $\mathcal{H} = \mathcal{G} = \mathcal{VCY}$, we get the following commutative diagram, where the horizontal maps are the assembly maps appearing in the Farrell-Jones Conjecture 2.5 and α_* is the change of rings homomorphism (induction) associated to α.

$$\begin{array}{ccc}
H_n^H(E_{\mathcal{VCY}}(H);\mathbf{K}_R) & \xrightarrow{A_{\mathcal{VCY}}} & K_n(RH) \\
{\scriptstyle H_n^G(f)\circ\operatorname{ind}_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_*} \\
H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) & \xrightarrow{A_{\mathcal{VCY}}} & K_n(RG).
\end{array}$$

We see that we can define a kind of induction homomorphism on the source of the assembly maps which is compatible with the induction structure given on their target. We get analogous diagrams for the L-theoretic version of the Farrell-Jones-Isomorphism Conjecture 2.5, for the Bost Conjecture 2.72 and for the Baum-Connes Conjecture for maximal group C^*-algebras (see (2.71) in Subsection 2.12.2).

Remark 4.25. The situation for the Baum-Connes Conjecture 2.4 itself, where one has to work with reduced C^*-algebras, is more complicated. Recall that not every group homomorphism $\alpha\colon H \to G$ induces a homomorphisms of C^*-algebras $C_r^*(H) \to C_r^*(G)$. (It does if $\ker(\alpha)$ is finite.) But it turns out that the source $H_n^H(E_{\mathcal{FIN}}(H); \mathbf{K}^{\mathrm{top}})$ always admits such a homomorphism. The point is that the isotropy groups of $E_{\mathcal{FIN}}(H)$ are all finite and the spectra-valued functor $\mathbf{K}^{\mathrm{top}}$ extends from $\mathsf{GROUPOIDS}^{\mathrm{inj}}$ to the category $\mathsf{GROUPOIDS}^{\mathrm{finker}}$, which has small groupoids as objects but as morphisms only those functors $f\colon \mathcal{G}_0 \to \mathcal{G}_1$ with finite kernels (in the sense that for each object $x \in \mathcal{G}_0$ the group homomorphism $\mathrm{aut}_{\mathcal{G}_0}(x) \to \mathrm{aut}_{\mathcal{G}_1}(f(x))$ has finite kernel). This is enough to get for any group homomorphism $\alpha\colon H \to G$ an induced map $\mathrm{ind}_\alpha\colon H_n^H(X, A; \mathbf{K}^{\mathrm{top}}) \to H_n^G(\mathrm{ind}_\alpha(X, A); \mathbf{K}^{\mathrm{top}})$ provided that X is proper. Hence one can define an induction homomorphism for the source of the assembly map as above.

In particular the Baum-Connes Conjecture 2.4 predicts that for any group homomorphism $\alpha\colon H \to G$ there is an induced induction homomorphism $\alpha_*\colon K_n(C_r^*(H)) \to K_n(C_r^*(G))$ on the targets of the assembly maps although there is no induced homomorphism of C^*-algebras $C_r^*(H) \to C_r^*(G)$ in general.

4.5 Miscellaneous

Exercises

4.1. Test

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5. Analytic Methods

5.1 Introduction

In Section 2.2.2 we formulated the Baum-Connes Conjecture 2.5 and the Farrell-Jones Conjecture 2.4 in abstract homological terms. We have seen that this formulation was very useful in order to understand formal properties of assembly maps. But in order to actually prove cases of the conjectures one needs to interpret the assembly maps in a way that is more directly related to geometry or analysis. In this section we wish to explain such approaches to the assembly maps. We briefly survey some of the methods of proof that are used to attack the Baum-Connes Conjecture 2.4 and the Farrell-Jones Conjecture 2.5.

5.2 Analytic Equivariant K-Homology

Recall that the covariant functor $\mathbf{K}^{\mathrm{top}}\colon \mathsf{GROUPOIDS}^{\mathrm{inj}} \to \mathsf{SPECTRA}$ introduced in (4.10) defines an equivariant homology theory $H_*^?(-;\mathbf{K}^{\mathrm{top}})$ in the sense of Subection 4.4.2 such that

$$H_n^G(G/H;\mathbf{K}^{\mathrm{top}}) = H_n^H(\{\bullet\};\mathbf{K}^{\mathrm{top}}) = \begin{cases} R(H) & \text{for even n;} \\ 0 & \text{for odd n,} \end{cases}$$

holds for all groups G and subgroups $H \subseteq G$ (see Proposition 4.22). Next we want to give for a proper G-CW-complex X an analytic definition of $H_n^G(X;\mathbf{K}^{\mathrm{top}})$.

Consider a locally compact proper G-space X. Recall that a G-space X is called *proper* if for each pair of points x and y in X there are open neighborhoods V_x of x and W_y of y in X such that the subset $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ of G is finite. A G-CW-complex X is proper if and only if all its isotropy groups are finite [215, Theorem 1.23]. Let $C_0(X)$ be the C^*-algebra of continuous functions $f\colon X \to \mathbb{C}$ which vanish at infinity. The C^*-norm is the supremum norm. A *generalized elliptic G-operator* is a triple (U,ρ,F), which consists of a unitary representation $U\colon G \to \mathcal{B}(H)$ of G on a Hilbert space H, a $*$-representation $\rho\colon C_0(X) \to \mathcal{B}(H)$ such that $\rho(f \circ l_{g^{-1}}) = U(g) \circ \rho(f) \circ U(g)^{-1}$ holds for $g \in G$, and a bounded selfadjoint

G-operator $F\colon H \to H$ such that the operators $\rho(f)(F^2-1)$ and $[\rho(f), F]$ are compact for all $f \in C_0(X)$. Here $\mathcal{B}(H)$ is the C^*-algebra of bounded operators $H \to H$, $l_{g^{-1}}\colon H \to H$ is given by multiplication with g^{-1}, and $[\rho(f), F] = \rho(f) \circ F - F \circ \rho(f)$. We also call such a triple (U, ρ, F) an *odd cycle*. If we additionally assume that H comes with a $\mathbb{Z}/2$-grading such that ρ preserves the grading if we equip $C_0(X)$ with the trivial grading, and F reverses it, then we call (U, ρ, F) an *even cycle*. This means that we have an orthogonal decomposition $H = H_0 \oplus H_1$ such that U, ρ and F look like

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix} \qquad \rho = \begin{pmatrix} \rho_0 & 0 \\ 0 & \rho_1 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}. \tag{5.1}$$

An important example of an even cocycle is described in Section 5.6. A cycle (U, ρ, f) is called *degenerate*, if for each $f \in C_0(X)$ we have $[\rho(f), F] = \rho(f)(F^2-1) = 0$. Two cycles (U_0, ρ_0, F_0) and (U_1, ρ_1, F_1) of the same parity are called *homotopic*, if $U_0 = U_1$, $\rho_0 = \rho_1$ and there exists a norm continuous path $F_t, t \in [0,1]$ in $\mathcal{B}(H)$ such that for each $t \in [0,1]$ the triple (U_0, ρ_0, F_t) is again a cycle of the same parity. Two cycles (U_0, ρ_0, F_0) and (U_1, ρ_1, F_1) are called *equivalent*, if they become homotopic after taking the direct sum with degenerate cycles of the same parity. Let $K_n^G(C_0(X))$ for even n be the set of equivalence classes of even cycles and $K_n^G(C_0(X))$ for odd n be the set of equivalence classes of odd cycles. These become abelian groups by the direct sum. The neutral element is represented by any degenerate cycle. The inverse of an even cycle is represented by the cycle obtained by reversing the grading of H. The inverse of an odd cycle (U, ρ, F) is represented by $(U, \rho, -F)$.

A proper G-map $f\colon X \to Y$ induces a map of C^*-algebras $C_0(f)\colon C_0(Y) \to C_0(X)$ by composition and thus in the obvious way a homomorphism of abelian groups $K_0^G(f)\colon K_0^G(C_0(X)) \to K_0^G(C_0(Y))$. It depends only on the proper G-homotopy class of f. One can show that this construction defines a G-homology theory on the category of finite proper G-CW-complexes. It extends to a G-homology theory K_*^G for all proper G-CW-complexes by

$$K_n^G(X) = \operatorname{colim}_{Y \in I(X)} K_n^G(C_0(Y)) \tag{5.2}$$

where $I(X)$ is the set of proper finite G-CW-subcomplexes $Y \subseteq X$ directed by inclusion. This definition is forced upon us by Lemma 4.11. The groups $K_n^G(X)$ and $K_n^G(C_0(X))$ agree for finite proper G-CW-complexes, in general they are different.

The cycles were introduced by Atiyah [12]. The equivalence relation, the group structure and the homological properties of $K_n^G(X)$ were established by Kasparov [186]. More information about analytic K-homology can be found in Higson-Roe [170].

5.3 Analytic Assembly Map

For for every G-CW-complex X the projection $\mathrm{pr}\colon X \to \{\bullet\}$ induces a map

$$H_n^G(X;\mathbf{K}^{\mathrm{top}}) \to H_n^G(\{\bullet\};\mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G)). \tag{5.3}$$

In the case where X is the proper G-space $E_{\mathcal{FIN}}(G)$ we obtain the assembly map appearing in the Baum-Connes Conjecture 2.4. We explain its analytic analogue

$$\mathrm{ind}_G\colon K_n^G(X) \to K_n(C_r^*(G)). \tag{5.4}$$

Note that we need to assume that X is a proper G-space since $K_n^G(X)$ was only defined for such spaces. It suffices to define the map for a finite proper G-CW-complex X. In this case it assigns to the class in $K_n^G(X) = K_n^G(C_0(X))$ represented by a cycle (U,ρ,F) its G-index in $K_n(C_r^*(G))$ in the sense of Mishencko-Fomenko [244]. At least in the simple case, where G is finite, we can give its precise definition. The odd K-groups vanish in this case and $K_0(C_r^*(G))$ reduces to the complex representation ring $R(G)$. If we write F in matrix form as in (5.1) then $P\colon H \to H$ is a G-equivariant Fredholm operator. Hence its kernel and cokernel are G-representations and the G-index of F is defined as $[\ker(P)]-[\mathrm{coker}(P)] \in R(G)$. In the case of an infinite group the kernel and cokernel are a priori not finitely generated projective modules over $C_r^*(G)$, but they are after a certain pertubation. Moreover the choice of the pertubation does not affect $[\ker(P)] - [\mathrm{coker}(P)] \in K_0(C_r^*(G))$.

The identification of the two assembly maps (5.3) and (5.4) has been carried out in Hambleton-Pedersen [157] using the universal characterization of the assembly map explained in [92, Section 6]. In particular for a proper G-CW-complex X we have an identification $H_n^G(X;\mathbf{K}^{\mathrm{top}}) \cong K_n^G(X)$. Notice that $H_n^G(X;\mathbf{K}^{\mathrm{top}})$ is defined for all G-CW-complexes, whereas $K_n^G(X)$ has only been introduced for proper G-CW-complexes.

Thus the Baum-Connes Conjecture 2.4 gives an index-theoretic interpretations of elements in $K_0(C_r^*(G))$ as generalized elliptic operators or cycles (U,ρ,F). We have explained already in Subsection 1.3.1 an application of this interpretation to the Trace Conjecture for Torsionfree Groups 1.10 and in Subsection 2.11.2 to the Stable Gromov-Lawson-Rosenberg Conjecture 2.62.

5.4 Equivariant KK-Theory

Kasparov [188] developed *equivariant KK-theory*, which we will briefly explain next. It is one of the basic tools in the proofs of theorems about the Baum-Connes Conjecture 2.4.

A *G-C^*-algebra A* is a C^*-algebra with a G-action by $*$-automorphisms. To any pair of separable G-C^*-algebras (A,B) Kasparov assigns abelian groups $KK_n^G(A,B)$. If G is trivial, we write briefly $KK_n(A,B)$. We do not give the rather complicated definition but state the main properties.

If we equip $\mathbb{C}$ with the trivial G-action, then $KK_n^G(C_0(X),\mathbb{C})$ reduces to the abelian group $K_n^G(C_0(X))$ introduced in Section 5.2. The topological

K-theory $K_n(A)$ of a C^*-algebra coincides with $KK_n(\mathbb{C}, A)$. The equivariant KK-groups are covariant in the second and contravariant in the first variable under homomorphism of C^*-algebras. One of the main features is the bilinear Kasparov product

$$KK_i^G(A,B) \times KK_j^G(B,C) \to KK_{i+j}(A,C),\ (\alpha,\beta) \mapsto \alpha \otimes_B \beta. \quad (5.5)$$

It is associative and natural. A homomorphism $\alpha\colon A \to B$ defines an element in $KK_0(A,B)$. There are natural *descent homomorphisms*

$$j_G\colon KK_n^G(A,B) \to KK_n(A \rtimes_r G, B \rtimes_r G), \quad (5.6)$$

where $A \rtimes_r G$ and $B \rtimes_r G$ denote the reduced crossed product C^*-algebras.

5.5 Dirac-Dual Dirac Method

A G-C^*-algebra A is called *proper* if there exists a locally compact proper G-space X and a G-homomorphism $\sigma\colon C_0(X) \to \mathcal{B}(A)$, $f \mapsto \sigma_f$ satisfying $\sigma_f(ab) = a\sigma_f(b) = \sigma_f(a)b$ for $f \in C_0(X)$, $a,b \in A$ and for every net $\{f_i \mid i \in I\}$, which converges to 1 uniformly on compact subsets of X, we have $\lim_{i\in I} \parallel \sigma_{f_i}(a) - a \parallel \ = \ 0$ for all $a \in A$. A locally compact G-space X is proper if and only if $C_0(X)$ is proper as a G-C^*-algebra.

Given a proper G-CW-complex X and a G-C^*-algebra A, we put

$$KK_n^G(X;A) = \operatorname{colim}_{Y \in I(X)} KK_n^G(C_0(Y), A), \quad (5.7)$$

where $I(Y)$ is the set of proper finite G-CW-subcomplexes $Y \subseteq X$ directed by inclusion. We have $KK_n^G(X;\mathbb{C}) = K_n^G(X)$. There is an analytic index map

$$\operatorname{ind}_G^A\colon KK_n^G(X;A) \to K_n(A \rtimes_r G), \quad (5.8)$$

which can be identified with the assembly map appearing in the Baum-Connes Conjecture with Coefficients 2.74 The following result is proved in Tu [333] extending results of Kasparov-Skandalis [187], [190].

Theorem 5.9. *The Baum-Connes Conjecture with coefficients 2.74 holds for a proper G-C^*-algebra A, i.e.* $\operatorname{ind}_G^A\colon KK_n^G(E_{\mathcal{FIN}}(G);A) \to K_n(A \rtimes G)$ *is bijective.*

Now we are ready to state the *Dirac-dual Dirac method* which is the key strategy in many of the proofs of the Baum-Connes Conjecture 2.4 or the Baum-Connes Conjecture with coefficients 2.74.

Theorem 5.10 (Dirac-Dual Dirac Method). *Let G be a countable (discrete) group. Suppose that there exist a proper G-C^*-algebra A, elements*

$\alpha \in KK_i^G(A,\mathbb{C})$, called the Dirac element*, and $\beta \in KK_i^G(\mathbb{C},A)$, called the* dual Dirac element*, satisfying*

$$\beta \otimes_A \alpha \;=\; 1 \quad \in KK_0^G(\mathbb{C},\mathbb{C}).$$

Then the Baum-Connes Conjecture 2.4 is true, or, equivalently, the analytic index map

$$\mathrm{ind}_G\colon K_n^G(X) \to K_n(C_r^*(G))$$

of 5.4 is bijective.

Proof. The index map ind_G is a retract of the bijective index map ind_G^A from Theorem 5.9. This follows from the following commutative diagram

$$\begin{tikzcd}
K_n^G(E_{\mathcal{FIN}}(G)) \arrow[r, "-\otimes_{\mathbb{C}}\beta"] \arrow[d, "\mathrm{ind}_G"] & KK_n^G(E_{\mathcal{FIN}}(G);A) \arrow[r, "-\otimes_A\alpha"] \arrow[d, "\mathrm{ind}_G^A"] & K_n^G(E_{\mathcal{FIN}}(G)) \arrow[d, "\mathrm{ind}_G"] \\
K_n(C_r^*(G)) \arrow[r, "-\otimes_{C_r^*(G)} j_G(\beta)"] & K_n(A \rtimes_r G) \arrow[r, "-\otimes_{A\rtimes_r} j_G(\alpha)"] & K_n(C_r^*(G))
\end{tikzcd}$$

and the fact that the composition of both the top upper horizontal arrows and lower upper horizontal arrows are bijective.

5.6 An Example of a Dirac Element

In order to give a glimpse of the basic ideas from operator theory we briefly describe how to define the Dirac element α in the case where G acts by isometries on a complete Riemannian manifold M. Let $T_{\mathbb{C}}M$ be the complexified tangent bundle and let $\mathrm{Cliff}(T_{\mathbb{C}}M)$ be the associated Clifford bundle. Let A be the proper G-C^*-algebra given by the sections of $\mathrm{Cliff}(T_{\mathbb{C}}M)$ which vanish at infinity. Let H be the Hilbert space $L^2(\wedge T_{\mathbb{C}}^*M)$ of L^2-integrable differential forms on $T_{\mathbb{C}}M$ with the obvious $\mathbb{Z}/2$-grading coming from even and odd forms. Let U be the obvious G-representation on H coming from the G-action on M. For a 1-form ω on M and $u \in H$ define a $*$-homomorphism $\rho\colon A \to \mathcal{B}(H)$ by

$$\rho_\omega(u) \;:=\; \omega \wedge u + i_\omega(u).$$

Now $D = (d + d^*)$ is a symmetric densely defined operator $H \to H$ and defines a bounded selfadjoint operator $F\colon H \to H$ by putting $F = \frac{D}{\sqrt{1+D^2}}$. Then (U,ρ,F) is an even cocycle and defines an element $\alpha \in K_0^G(M) = KK_0^G(C_0(M),\mathbb{C})$. More details of this construction and the construction of the dual Dirac element β under the assumption that M has non-positive curvature and is simply connected, can be found for instance in [335, Chapter 9].

5.7 Banach KK-Theory

Skandalis showed that the Dirac-dual Dirac method cannot work for all groups [314] as long as one works with KK-theory in the unitary setting. The problem is that for a group with property (T) the trivial and the regular unitary representation cannot be connected by a continuous path in the space of unitary representations, compare also the discussion in [180]. This problem can be circumvented if one drops the condition unitary and works with a variant of KK-theory for Banach algebras as worked out by Lafforgue [201], [203], [204].

5.8 Comparing to Other Theories

Every natural transformation of G-homology theories leads to a comparison between the associated assembly maps. For example one can compare topological K-theory to periodic cyclic homology [80], i.e. for every Banach algebra completion $\mathcal{A}(G)$ of $\mathbb{C}G$ inside $C_r^*(G)$ there exists a commutative diagram

$$\begin{array}{ccc} K_*(BG) & \longrightarrow & K_*(\mathcal{A}(G)) \\ \downarrow & & \downarrow \\ H_*(BG; HP_*(\mathbb{C})) & \longrightarrow & HP_*(\mathcal{A}(G)). \end{array}$$

This is used in [80] to prove injectivity results for word hyperbolic groups. Similar diagrams exist for other cyclic theories (compare for example [267]).

A suitable model for the cyclotomic trace $trc\colon K_n(RG) \to TC_n(RG)$ from (connective) algebraic K-theory to topological cyclic homology [44] leads for every family $\mathcal{F}$ to a commutative diagram

$$\begin{array}{ccc} H_n(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{Z}}^{con}) & \longrightarrow & K_n^{con}(\mathbb{Z}G) \\ \downarrow & & \downarrow \\ H_n(E_{\mathcal{F}}(G); \mathbf{TC}_{\mathbb{Z}}) & \longrightarrow & TC_n(\mathbb{Z}G). \end{array}$$

Injectivity results about the left hand and the lower horizontal map lead to injectivity results about the upper horizontal map. This is the principle behind Theorem 3.24 and 3.26.

5.9 Miscellaneous

Exercises

5.1. Test

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6. Controlled Methods

6.1 Introduction

6.2 Controlled Topology and Algebra

To a topological problem one can often associate a notion of "size". We describe a prototypical example. Let M be a Riemannian manifold. Recall that an h-cobordism W over $M = \partial^- W$ admits retractions $r^\pm \colon W \times I \to W$, $(x,t) \mapsto r^\pm_t(x,t)$ which retract W to $\partial^\pm W$, i.e. which satisfy $r^\pm_0 = \mathrm{id}_W$ and $r^\pm_1(W) \subset \partial^\pm W$. Given $\epsilon > 0$ we say that W is ϵ-controlled if the retractions can be chosen in such a way that for every $x \in W$ the paths (called tracks of the h-cobordism) $p^\pm_x \colon I \to M$, $t \mapsto r^-_1 \circ r^\pm_t(x)$ both lie within an ϵ-neighbourhood of their starting point. The usefulness of this concept is illustrated by the following theorem [139].

Theorem 6.1. *Let M be a compact Riemannian manifold of dimension ≥ 5. Then there exists an $\epsilon = \epsilon_M > 0$, such that every ϵ-controlled h-cobordism over M is trivial.*

If one studies the s-Cobordism Theorem 1.21 and its proof one is naturally lead to algebraic analogues of the notions above. A (geometric) R-module over the space X is by definition a family $M = (M_x)_{x \in X}$ of free R-modules indexed by points of X with the property that for every compact subset $K \subset X$ the module $\oplus_{x \in K} M_x$ is a finitely generated R-module. A morphism ϕ from M to N is an R-linear map $\phi = (\phi_{y,x}) \colon \oplus_{x \in X} M_x \to \oplus_{y \in X} N_y$. Instead of specifying fundamental group data by paths (analogues of the tracks of the h-cobordism) one can work with modules and morphisms over the universal covering $\widetilde{X}$, which are invariant under the operation of the fundamental group $G = \pi_1(X)$ via deck transformations, i.e. we require that $M_{gx} = M_x$ and $\phi_{gy,gx} = \phi_{y,x}$. Such modules and morphisms form an additive category which we denote by $\mathcal{C}^G(\widetilde{X}; R)$. In particular one can apply to it the non-connective K-theory functor $\mathbf{K}$ (compare [258]). In the case where X is compact the category is equivalent to the category of finitely generated free RG-modules and hence $\pi_* \mathbf{K} \mathcal{C}^G(\widetilde{X}; R) \cong K_*(RG)$. Now suppose $\widetilde{X}$ is equipped with a G-invariant metric, then we will say that a morphism $\phi = (\phi_{y,x})$ is ϵ-controlled if $\phi_{y,x} = 0$, whenever x and y are further than ϵ apart. (Note that ϵ-controlled morphisms do not form a category because the composition of two such morphisms will in general be 2ϵ-controlled.)

Theorem 6.1 has the following algebraic analogue [272] (see also Section 4 in [261]).

Theorem 6.2. *Let M be a compact Riemannian manifold with fundamental group G. There exists an $\epsilon = \epsilon_M > 0$ with the following property. The K_1-class of every G-invariant automorphism of modules over $\widetilde{M}$ which together with its inverse is ϵ-controlled lies in the image of the classical assembly map*

$$H_1(BG; \mathbf{K}R) \to K_1(RG) \cong K_1(\mathcal{C}^G(\widetilde{M}; R)).$$

To understand the relation to Theorem 6.1 note that for $R = \mathbb{Z}$ such an ϵ-controlled automorphism represents the trivial element in the Whitehead group which is in bijection with the h-cobordisms over M, compare Theorem 1.21.

There are many variants to the simple concept of "metric ϵ-control" we used above. In particular it is very useful to not measure size directly in M but instead use a map $p\colon M \to X$ to measure size in some auxiliary space X. (For example we have seen in Subsection 1.5.3 and Subsection 1.7.2 that "bounded" control over $\mathbb{R}^k$ may be used in order to define or describe negative K-groups.)

Before we proceed we would like to mention that there are analogous control-notions for pseudoisotopies and homotopy equivalences. The tracks of a pseudoisotopy $f\colon M \times I \to M \times I$ are defined as the paths in M which are given by the composition

$$p_x\colon I = \{x\} \times I \subset M \times I \xrightarrow{f} M \times I \xrightarrow{p} M$$

for each $x \in M$, where the last map is the projection onto the M-factor. Suppose $f\colon N \to M$ is a homotopy equivalence, $g\colon M \to N$ its inverse and h_t and h'_t are homotopies from $f \circ g$ to id_M respectively from $g \circ f$ to id_N then the tracks are defined to be the paths in M that are given by $t \mapsto h_t(x)$ for $x \in M$ and $t \mapsto f \circ h'_t(y)$ for $y \in N$. In both cases, for pseudoisotopies and for homotopy equivalences, the tracks can be used to define ϵ-control.

6.3 Assembly as Forget Control

If instead of a single problem over M one defines a family of problems over $M \times [1, \infty)$ and requires the control to tend to zero for $t \to \infty$ in a suitable sense, then one obtains something which is a homology theory in M. Relaxing the control to just bounded families yields the classical assembly map. This idea appears in [273] in the context of pseudoisotopies and in a more categorical fashion suitable for higher algebraic K-theory in [61] and [262].

We spell out some details in the case of algebraic K-theory, i.e. for geometric modules.

Let M be a Riemannian manifold with fundamental group G and let $\mathcal{S}(1/t)$ be the space of all functions $[1,\infty) \to [0,\infty)$, $t \mapsto \delta_t$ such that $t \mapsto t \cdot \delta_t$ is bounded. Similarly let $\mathcal{S}(1)$ be the space of all functions $t \mapsto \delta_t$ which are bounded. Note that $\mathcal{S}(1/t) \subset \mathcal{S}(1)$. A G-invariant morphism ϕ over $\widetilde{M} \times [1,\infty)$ is $\mathcal{S}$-controlled for $\mathcal{S} = \mathcal{S}(1)$ or $\mathcal{S}(1/t)$ if there exists an $\alpha > 0$ and a $\delta_t \in \mathcal{S}$ (both depending on the morphism) such that $\phi_{(x,t),(x',t')} \neq 0$ implies that $|t - t'| \leq \alpha$ and $d_{\widetilde{M}}(x,x') \leq \delta_{\min\{t,t'\}}$. We denote by $\mathcal{C}^G(\widetilde{M} \times [1,\infty), \mathcal{S}; R)$ the category of all $\mathcal{S}$-controlled morphisms. Furthermore $\mathcal{C}^G(\widetilde{M} \times [1,\infty), \mathcal{S}; R)^\infty$ denotes the quotient category which has the same objects, but where two morphisms are identified, if their difference factorizes over an object which lives over $\widetilde{M} \times [1,N]$ for some large but finite number N. This passage to the quotient category is called "taking germs at infinity". It is a special case of a Karoubi quotient, compare [57].

Theorem 6.3 (Classical Assembly as Forget Control). *Suppose M is aspherical, i.e. M is a model for BG, then for all $n \in \mathbb{Z}$ the map*

$$\pi_n(\mathbf{K}\mathcal{C}^G(\widetilde{M} \times [1,\infty), \mathcal{S}(1/t); R)^\infty) \to \pi_n(\mathbf{K}\mathcal{C}^G(\widetilde{M} \times [1,\infty), \mathcal{S}(1); R)^\infty)$$

can be identified up to an index shift with the classical assembly map that appears in Conjecture 1.28, i.e. with

$$H_{n-1}(BG; \mathbf{K}(R)) \to K_{n-1}(RG).$$

Note that the only difference between the left and the right hand side is that on the left morphism are required to become smaller in a $1/t$-fashion, whereas on the right hand side they are only required to stay bounded in the $[1,\infty)$-direction.

Using so called equivariant continuous control (see [8] and [22, Section 2] for the equivariant version) one can define an equivariant homology theory which applies to arbitrary G-CW-complexes. This leads to a "forget-control description" for the generalized assembly maps that appear in the Farrell-Jones Conjecture 2.5. Alternatively one can use stratified spaces and stratified Riemannian manifolds in order to describe generalized assembly maps in terms of metric control. Compare [126, 3.6 on p.270] and [273, Appendix].

6.4 Methods to Improve Control

From the above description of assembly maps we learn that the problem of proving surjectivity results translates into the problem of improving control. A combination of many different techniques is used in order to achieve such control-improvements. We discuss some prototypical arguments which go back to [113] and [117] and again restrict attention to K-theory. Of course this can only give a very incomplete impression of the whole program which is mainly due to Farrell-Hsiang and Farrell-Jones. The reader should consult [135] and [177] for a more detailed survey.

We restrict to the basic case, where M is a compact Riemannian manifold with negative sectional curvature. In order to explain a contracting property of the geodesic flow $\Phi\colon \mathbb{R} \times S\widetilde{M} \to S\widetilde{M}$ on the unit sphere bundle $S\widetilde{M}$, we introduce the notion of foliated control. We think of $S\widetilde{M}$ as a manifold equipped with the one-dimensional foliation by the flow lines of $\widetilde{\Phi}$ and equip it with its natural Riemannian metric. Two vectors v and w in $S\widetilde{M}$ are called foliated (α, δ)-controlled if there exists a path of length α inside one flow line such that v lies within distance $\delta/2$ of the starting point of that path and w lies within distance $\delta/2$ of its endpoint.

Two vectors v and $w \in S\widetilde{M}$ are called asymptotic if the distance between their associated geodesic rays is bounded. These rays will then determine the same point on the sphere at infinity which can be introduced to compactify $\widetilde{M}$ to a disk. Recall that the universal covering of a negatively curved manifold is diffeomorphic to $\mathbb{R}^n$. Suppose v and w are α-controlled asymptotic vectors, i.e. their distance is smaller than $\alpha > 0$. As a consequence of negative sectional curvature the vectors $\Phi_t(v)$ and $\Phi_t(w)$ are foliated $(C\alpha, \delta_t)$-controlled, where $C > 1$ is a constant and $\delta_t > 0$ tends to zero when t tends to ∞. So roughly speaking the flow contracts the directions transverse to the flow lines and leaves the flow direction as it is, at least if we only apply it to asymptotic vectors.

This property can be used in order to find foliated (α, δ)-controlled representatives of K-theory classes with arbitrary small δ if one is able to define a suitable transfer from M to $S\widetilde{M}$, which yields representatives whose support is in an asymptotic starting position for the flow. Here one needs to take care of the additional problem that in general such a transfer may not induce an isomorphism in K-theory.

Finally one is left with the problem of improving foliated control to ordinary control. Corresponding statements are called "Foliated Control Theorems". Compare [21], [116], [118], [119] and [123].

If such an improvement were possible without further hypothesis, we could prove that the classical assembly map, i.e. the assembly map with respect to the trivial family is surjective. We know however that this is not true in general. It fails for example in the case of topological pseudoisotopies or for algebraic K-theory with arbitrary coefficients. In fact the geometric arguments that are involved in a "Foliated Control Theorem" need to exclude flow lines in $S\widetilde{M}$ which correspond to "short" closed geodesic loops in SM. But the techniques mentioned above can be used in order to achieve ϵ-control for arbitrary small $\epsilon > 0$ outside of a suitably chosen neighbourhood of "short" closed geodesics. This is the right kind of control for the source of the assembly map which involves the family of cyclic subgroups. (Note that a closed a loop in M determines the conjugacy class of a maximal infinite cyclic subgroup inside $G = \pi_1(M)$.) We see that even in the torsionfree case the class of cyclic subgroups of G naturally appears during the proof of a surjectivity result.

Another source for processes which improve control are expanding self-maps. Think for example of an n-torus $\mathbb{R}^n/\mathbb{Z}^n$ and the self-map f_s which is induced by $m_s \colon \mathbb{R}^n \to \mathbb{R}^n$, $x \to sx$ for a large positive integer s. If one pulls an automorphism back along such a map one can improve control, but unfortunately the new automorphism describes a different K-theory class. Additional algebraic arguments nevertheless make this technique very successful. Compare for example [113]. Sometimes a clever mixture between flows and expanding self-maps is needed in order to achieve the goal, compare [120]. Recent work of Farrell-Jones (see [129], [130], [131] and [176]) makes use of a variant of the Cheeger-Fukaya-Gromov collapsing theory.

Remark 6.4 (Algebraicizing the Farrell-Jones Approach). In this Subsection we sketched some of the geometric ideas which are used in order to obtain control over an h-cobordism, a pseudisotopy or an automorphism of a geometric module representing a single class in K_1. In Subsection 6.3 we used families over the cone $M \times [1, \infty)$ in order to described the whole algebraic K-theory assembly map at once in categorical terms without ever referring to a single K-theory element. The recent work [25] shows that the geometric ideas can be adapted to this more categorical set-up, at least in the case where the group is the fundamental group of a Riemannian manifold with strictly negative curvature. However serious difficulties had to be overcome in order to achieve this. One needs to formulate and prove a Foliated Control Theorem in this context and also construct a transfer map to the sphere bundle for higher K-theory which is in a suitable sense compatible with the control structures.

6.5 The Descent Principle

In Theorem 6.3 we described the classical assembly map as a forget control map using G-invariant geometric modules over $\widetilde{M} \times [1, \infty)$. If in that context one does not require the modules and morphisms to be invariant under the G-action one nevertheless obtains a forget control functor between additive categories for which we introduce the notation

$$\mathcal{D}(1/t) = \mathcal{C}(\widetilde{M} \times [1, \infty), \mathcal{S}(1/t); R)^{\infty} \to \mathcal{D}(1) = \mathcal{C}(\widetilde{M} \times [1, \infty), \mathcal{S}(1); R)^{\infty}.$$

Applying K-theory yields a version of a "coarse" assembly map which is the algebraic K-theory analogue of the map described in Section 2.12.5. A crucial feature of such a construction is that the left hand side can be interpreted as a locally finite homology theory evaluated on $\widetilde{M}$. It is hence an invariant of the proper homotopy type of $\widetilde{M}$. Compare [8] and [356]. It is usually a lot easier to prove that this coarse assembly map is an equivalence. Suppose for example that M has non-positive curvature, choose a point $x_0 \in M$ (this will destroy the G-invariance) and with increasing $t \in [1, \infty)$ move the modules along geodesics towards x_0. In this way one can show that the coarse assembly map is an isomorphism. Such coarse assembly maps exist also in the context of algebraic L-theory and topological K-theory, compare [167], [284].

Results about these maps (compare e.g. [26], [61], [366], [368]) lead to injectivity results for the classical assembly map by the "descent principle" (compare [58], [61], [284]) which we will now briefly describe in the context of algebraic K-theory. (We already stated an analytic version in Section 2.12.5.) For a spectrum $\mathbf{E}$ with G-action we denote by $\mathbf{E}^{hG}$ the homotopy fixed points. Since there is a natural map from fixed points to homotopy fixed points we obtain a commutative diagram

$$\begin{tikzcd}
\mathbf{K}(\mathcal{D}(1/t))^G \arrow[r] \arrow[d] & \mathbf{K}(\mathcal{D}(1))^G \arrow[d] \\
\mathbf{K}(\mathcal{D}(1/t))^{hG} \arrow[r] & \mathbf{K}(\mathcal{D}(1))^{hG}.
\end{tikzcd}$$

If one uses a suitable model K-theory commutes with taking fixed points and hence the upper horizontal map can be identified with the classical assembly map by Theorem 6.3. Using that K-theory commutes with infinite products [60], one can show by an induction over equivariant cells, that the vertical map on the left is an equivalence. Since we assume that the map $\mathbf{K}(\mathcal{D}(1/t)) \to \mathbf{K}(\mathcal{D}(1))$ is an equivalence, a standard property of the homotopy fixed point construction implies that the lower horizontal map is an equivalence. It follows that the upper horizontal map and hence the classical assembly map is split injective. A version of this argument which involves the assembly map for the family of finite subgroups can be found in [293].

6.6 Miscellaneous

Exercises

6.1. Test

last edited on 29.12.05
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7. Cyclic Methods

7.1 Introduction

7.2 Miscellaneous

Exercises

7.1. Test

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8. Guide for Computations

8.1 Introduction

Our ultimate goal is to compute K- and L-groups such as $K_n(RG)$, $L_n^{\langle -\infty \rangle}(RG)$ and $K_n(C_r^*(G))$. Assuming that the Baum-Connes Conjecture 2.4 or the Farrell-Jones Conjecture 2.5 is true, this reduces to the computation of the left hand side of the corresponding assembly map, i.e. to $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\mathrm{top}})$, $H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$ and $H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle})$. This is much easier since here we can use standard methods from algebraic topology such as spectral sequences, Mayer-Vietoris sequences and Chern characters. Nevertheless such computations can be pretty hard. Roughly speaking, one can obtain a general reasonable answer after rationalization, but integral computations have only been done case by case and no general pattern is known.

8.2 *K*- and *L*- Groups for Finite Groups

In all these computations the answer is given in terms of the values of $K_n(RG)$, $L_n^{\langle -\infty \rangle}(RG)$ and $K_n(C_r^*(G))$ for finite groups G. Therefore we briefly recall some of the results known for finite groups focusing on the case $R = \mathbb{Z}$

8.2.1 Topological *K*-Theory for Finite Groups

Let G be a finite group. By $r_F(G)$, we denote the number of isomorphism classes of irreducible representations of G over the field F. By $r_{\mathbb{R}}(G; \mathbb{R})$, $r_{\mathbb{R}}(G; \mathbb{C})$, respectively $r_{\mathbb{R}}(G; \mathbb{H})$ we denote the number of isomorphism classes of irreducible real G-representations V, which are of real, complex respectively of quaternionic type, i.e. $\mathrm{aut}_{\mathbb{R}G}(V)$ is isomorphic to the field of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$. Let $RO(G)$ respectively $R(G)$ be the *real* respectively the *complex representation ring.*

Notice that $\mathbb{C}G = l^1(G) = C_r^*(G) = C^*_{\max}(G)$ holds for a finite group, and analogous for the real versions.

Proposition 8.1. *Let G be a finite group.*

(1) *We have*

$$K_n(C_r^*(G)) \;\cong\; \begin{cases} R(G) \cong \mathbb{Z}^{r_{\mathbb{C}}(G)} & \textit{for } n \textit{ even;} \\ 0 & \textit{for } n \textit{ odd;} \end{cases}$$

(2) *There is an isomorphism of topological K-groups*

$$K_n(C_r^*(G;\mathbb{R})) \;\cong\; K_n(\mathbb{R})^{r_{\mathbb{R}}(G;\mathbb{R})} \times K_n(\mathbb{C})^{r_{\mathbb{R}}(G;\mathbb{C})} \times K_n(\mathbb{H})^{r_{\mathbb{R}}(G;\mathbb{H})}.$$

Moreover $K_n(\mathbb{C})$ is 2-periodic with values $\mathbb{Z}$, 0 for $n = 0, 1$, $K_n(\mathbb{R})$ is 8-periodic with values $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0 for $n = 0, 1, \ldots, 7$ and $K_n(\mathbb{H}) = K_{n+4}(\mathbb{R})$ for $n \in \mathbb{Z}$;

Proof. One gets isomorphisms of C^*-algebras

$$C_r^*(G) \;\cong\; \prod_{j=1}^{r_{\mathbb{C}}(G)} M_{n_i}(\mathbb{C})$$

and

$$C_r^*(G;\mathbb{R}) \;\cong\; \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{R})} M_{m_i}(\mathbb{R}) \times \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{C})} M_{n_i}(\mathbb{C}) \times \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{H})} M_{p_i}(\mathbb{H})$$

from [309, Theorem 7 on page 19, Corollary 2 on page 96, page 102, page 106]. Now the claim follows from Morita invariance and the well-known values for $K_n(\mathbb{R})$, $K_n(\mathbb{C})$ and $K_n(\mathbb{H})$ (see for instance [328, page 216]).

To summarize, the values of $K_n(C_r^*(G))$ and $K_n(C_r^*(G;\mathbb{R}))$ are explicitly known for finite groups G and are in the complex case in contrast to the real case always torsion free.

8.2.2 Algebraic K-Theory for Finite Groups

Here are some facts about the algebraic K-theory of integral group rings of finite groups.

Proposition 8.2. *Let G be a finite group.*

(1) *$K_n(\mathbb{Z}G) = 0$ for $n \leq -2$;*
(2) *We have*

$$K_{-1}(\mathbb{Z}G) \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2)^s,$$

where

$$r = 1 - r_{\mathbb{Q}}(G) + \sum_{p \mid |G|} r_{\mathbb{Q}_p}(G) - r_{\mathbb{F}_p}(G)$$

and the sum runs over all primes dividing the order of G. (Recall that $r_F(G)$ denotes the number of isomorphism classes of irreducible representations of G over the field F.) There is an explicit description of the integer s in terms of global and local Schur indices [64]. If G contains a normal abelian subgroup of odd index, then $s = 0$;

(3) The group $\widetilde{K}_0(\mathbb{Z}G)$ *is finite;*
(4) The group $\mathrm{Wh}(G)$ *is a finitely generated abelian group and its rank is* $r_{\mathbb{R}}(G) - r_{\mathbb{Q}}(G)$*;*
(5) The groups $K_n(\mathbb{Z}G)$ *are finitely generated for all* $n \in \mathbb{Z}$*;*
(6) We have $K_{-1}(\mathbb{Z}G) = 0$, $\widetilde{K}_0(\mathbb{Z}G) = 0$ *and* $\mathrm{Wh}(G) = 0$ *for the following finite groups* $G = \{1\}$, $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, D_6, D_8, *where* D_m *is the dihedral group of order* m.
If p *is a prime, then* $K_{-1}(\mathbb{Z}[\mathbb{Z}/p]) = K_{-1}(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}/p]) = 0$.
We have

$$K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}, \quad \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/6]) = 0, \quad \mathrm{Wh}(\mathbb{Z}/6) = 0$$
$$K_{-1}(\mathbb{Z}[D_{12}]) \cong \mathbb{Z}, \quad \widetilde{K}_0(\mathbb{Z}[D_{12}]) = 0, \quad \mathrm{Wh}(D_{12}) = 0.$$

(7) Let $\mathrm{Wh}_2(G)$ *denote the cokernel of the assembly map*

$$H_2(BG; \mathbf{K}(\mathbb{Z})) \to K_2(\mathbb{Z}G).$$

We have $\mathrm{Wh}_2(G) = 0$ *for* $G = \{1\}$, $\mathbb{Z}/2$, $\mathbb{Z}/3$ *and* $\mathbb{Z}/4$. *Moreover* $|\,\mathrm{Wh}_2(\mathbb{Z}/6)| \le 2$, $|\,\mathrm{Wh}_2(\mathbb{Z}/2 \times \mathbb{Z}/2)| \ge 2$ *and* $\mathrm{Wh}_2(D_6) = \mathbb{Z}/2$.

Proof. 1 and 2 are proved in [64].

3 is proved in [325, Proposition 9.1 on page 573].

4 This is proved for instance in [253].

5 See [199], [269].

6 and 7 The computation $K_{-1}(\mathbb{Z}G) = 0$ for $G = \mathbb{Z}/p$ or $\mathbb{Z}/p \times \mathbb{Z}/p$ can be found in [27, Theorem 10.6, p. 695] and is a special case of [64].

The vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ is proven for $G = D_6$ in [283, Theorem 8.2] and for $G = D_8$ in [283, Theorem 6.4]. The cases $G = \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/6$, and $(\mathbb{Z}/2)^2$ are treated in [88, Corollary 5.17]. Finally, $\widetilde{K}_0(\mathbb{Z}D_{12}) = 0$ follows from [88, Theorem 50.29 on page 266] and the fact that $\mathbb{Q}D_{12}$ as a $\mathbb{Q}$-algebra splits into copies of $\mathbb{Q}$ and matrix algebras over $\mathbb{Q}$, so its maximal order has vanishing class group by Morita equivalence.

The claims about $\mathrm{Wh}_2(\mathbb{Z}/n)$ for $n = 2, 3, 4, 6$ and for $\mathrm{Wh}_2((\mathbb{Z}/2)^2)$ are taken from [95, Proposition 5], [103, p.482] and [323, p. 218 and 221].

We get $K_2(\mathbb{Z}D_6) \cong (\mathbb{Z}/2)^3$ from [323, Theorem 3.1]. The assembly map $H_2(B\mathbb{Z}/2; \mathbf{K}(\mathbb{Z})) \to K_2(\mathbb{Z}[\mathbb{Z}/2])$ is an isomorphism by [103, Theorem on p. 482]. Now construct a commutative diagram

$$\begin{tikzcd}
H_2(B\mathbb{Z}/2; \mathbf{K}(\mathbb{Z})) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H_2(BD_6; \mathbf{K}(\mathbb{Z})) \arrow[d] \\
K_2(\mathbb{Z}[\mathbb{Z}/2]) \arrow[r] & K_2(\mathbb{Z}D_6)
\end{tikzcd}$$

whose lower horizontal arrow is split injective and whose upper horizontal arrow is an isomorphism by the Atiyah-Hirzebruch spectral sequence. Hence the right vertical arrow is split injective and $\mathrm{Wh}_2(D_6) = \mathbb{Z}/2$.

Let us summarize. We already mentioned that a complete computation of $K_n(\mathbb{Z})$ is not known. Also a complete computation of $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$ for arbitrary primes p is out of reach (see [241, page 29,30]). There is a complete formula for $K_{-1}(\mathbb{Z}G)$ and $K_n(\mathbb{Z}G) = 0$ for $n \leq -2$ and one has a good understanding of $\mathrm{Wh}(G)$ (see [253]). We have already mentioned Borel's formula for $K_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $n \in \mathbb{Z}$ (see Remark 1.31). For more rational information see also 8.11.

8.2.3 Algebraic *L*-Theory for Finite Groups

Here are some facts about L-groups of finite groups.

Proposition 8.3. *Let G be a finite group. Then*

(1) For each $j \leq 1$ the groups $L_n^{\langle j \rangle}(\mathbb{Z}G)$ are finitely generated as abelian groups and contain no p-torsion for odd primes p. Moreover, they are finite for odd n.
(2) For every decoration $\langle j \rangle$ we have

$$L_n^{\langle j \rangle}(\mathbb{Z}G)[1/2] \;\cong\; L_n^{\langle j \rangle}(\mathbb{R}G)[1/2] \cong \begin{cases} \mathbb{Z}[1/2]^{r_{\mathbb{R}}(G)} & n \equiv 0 \ \ (4); \\ \mathbb{Z}[1/2]^{r_{\mathbb{C}}(G)} & n \equiv 2 \ \ (4); \\ 0 & n \equiv 1,3 \ \ (4); \end{cases}$$

(3) If G has odd order and n is odd, then $L_n^{\epsilon}(\mathbb{Z}G) = 0$ for $\epsilon = p, h, s$.

Proof. 1 See for instance [158].

2 See [279, Proposition 22.34 on page 253].

3 See [14] or [158].

The L-groups of $\mathbb{Z}G$ are pretty well understood for finite groups G. More information about them can be found in [158].

8.3 Rational Computations for Infinite Groups

Next we state what is known rationally about the K- and L-groups of an infinite (discrete) group, provided the Baum-Connes Conjecture 2.4 or the relevant version of the Farrell-Jones Conjecture 2.5 is known.

In the sequel let $(\mathcal{FCY})$ be the set of conjugacy classes (C) for finite cyclic subgroups $C \subseteq G$. For $H \subseteq G$ let $N_GH = \{g \in G \mid gHg^{-1} = H\}$ be its *normalizer*, let $Z_GH = \{g \in G \mid ghg^{-1} = h \text{ for } h \in H\}$ be its *centralizer*, and put

$$W_GH := N_GH/(H \cdot Z_GH),$$

where $H \cdot Z_GH$ is the normal subgroup of N_GH consisting of elements of the form hu for $h \in H$ and $u \in Z_GH$. Notice that W_GH is finite if H is finite.

Recall that the *Burnside ring* $A(G)$ of a finite group is the Grothendieck group associated to the abelian monoid of isomorphism classes of finite G-sets with respect to the disjoint union. The ring multiplication comes from the cartesian product. The zero element is represented by the empty set, the unit is represented by $G/G = \{\bullet\}$. For a finite group G the abelian groups $K_q(C_r^*(G))$, $K_q(RG)$ and $L^{\langle -\infty \rangle}(RG)$ become modules over $A(G)$ because these functors come with a Mackey structure and $[G/H]$ acts by $\operatorname{ind}_H^G \circ \operatorname{res}_G^H$.

We obtain a ring homomorphism

$$\chi^G \colon A(G) \to \prod_{(H) \in \mathcal{FIN}} \mathbb{Z}, \qquad [S] \mapsto (|S^H|)_{(H) \in \mathcal{FIN}}$$

which sends the class of a finite G-set S to the element given by the cardinalities of the H-fixed point sets. This is an injection with finite cokernel. This leads to an isomorphism of $\mathbb{Q}$-algebras

$$\chi_{\mathbb{Q}}^G := \chi^G \otimes_{\mathbb{Z}} \operatorname{id}_{\mathbb{Q}} \colon A(G) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{(H) \in (\mathcal{FIN})} \mathbb{Q}. \tag{8.4}$$

For a finite cyclic group C let

$$\theta_C \ \in \ A(C) \otimes_{\mathbb{Z}} \mathbb{Z}[1/|C|] \tag{8.5}$$

be the element which is sent under the isomorphism $\chi_{\mathbb{Q}}^C \colon A(C) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{(H) \in \mathcal{FIN}} \mathbb{Q}$ of (8.4) to the element, whose entry is one if $(H) = (C)$ and is zero if $(H) \neq (C)$. Notice that θ_C is an idempotent. In particular we get a direct summand $\theta_C \cdot K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \mathbb{Q}$ in $K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \mathbb{Q}$ and analogously for $K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L^{\langle -\infty \rangle}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$.

8.3.1 Rationalized Topological *K*-Theory for Infinite Groups

The next result is taken from [221, Theorem 0.4 and page 127]. Recall that Λ^G is the ring $\mathbb{Z} \subseteq \Lambda^G \subseteq \mathbb{Q}$ which is obtained from $\mathbb{Z}$ by inverting the orders of the finite subgroups of G.

Theorem 8.6 (Rational Computation of Topological K-Theory for Infinite Groups). *Suppose that the group G satisfies the Baum-Connes Conjecture 2.4. Then there is an isomorphism*

$$\bigoplus_{p+q=n} \ \bigoplus_{(C) \in (\mathcal{FCY})} K_p(BZ_GC) \otimes_{\mathbb{Z}[W_GC]} \theta_C \cdot K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \Lambda^G$$
$$\xrightarrow{\cong} K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G.$$

If we tensor with $\mathbb{Q}$, we get an isomorphism

$$\bigoplus_{p+q=n} \ \bigoplus_{(C) \in (\mathcal{FCY})} H_p(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$
$$\xrightarrow{\cong} K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

8.3.2 Rationalized Algebraic K-Theory for Infinite Groups

Recall that for algebraic K-theory of the integral group ring we know because of Proposition 2.20 that in the Farrell-Jones Conjecture we can reduce to the family of finite subgroups. A reduction to the family of finite subgroups also works if the coefficient ring is a regular $\mathbb{Q}$-algebra, compare 2.17. The next result is a variation of [219, Theorem 0.4].

Theorem 8.7 (Rational Computation of Algebraic K-Theory). *Suppose that the group G satisfies the Farrell-Jones Conjecture 2.5 for the algebraic K-theory of RG, where either $R = \mathbb{Z}$ or R is a regular ring with $\mathbb{Q} \subset R$. Then we get an isomorphism*

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{FCY})} H_p(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 8.8. If in Theorem 8.7 we assume the Farrell-Jones Conjecture for the algebraic K-theory of RG but make no assumption on the coefficient ring R, then we still obtain that the map appearing there is split injective.

Example 8.9 (The Comparison Map from Algebraic to Topological K-theory). If we consider $R = \mathbb{C}$ as coefficient ring and apply $- \otimes_{\mathbb{Z}} \mathbb{C}$ instead of $- \otimes_{\mathbb{Z}} \mathbb{Q}$, the formulas simplify. Suppose that G satisfies the Baum-Connes Conjecture 2.4 and the Farrell-Jones Conjecture 2.5 for algebraic K-theory with $\mathbb{C}$ as coefficient ring. Recall that $\operatorname{con}(G)_f$ is the set of conjugacy classes (g) of elements $g \in G$ of finite order. We denote for $g \in G$ by $\langle g \rangle$ the cyclic subgroup generated by g.

Then we get the following commutative square, whose horizontal maps are isomorphisms and whose vertical maps are induced by the obvious change of theory homomorphism (see [219, Theorem 0.5])

$$\begin{array}{ccc} \bigoplus_{p+q=n} \bigoplus_{(g)\in\operatorname{con}(G)_f} H_p(Z_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \xrightarrow{\cong} & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\ \downarrow & & \downarrow \\ \bigoplus_{p+q=n} \bigoplus_{(g)\in\operatorname{con}(G)_f} H_p(Z_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \xrightarrow{\cong} & K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

The Chern character appearing in the lower row of the commutative square above has already been constructed by different methods in [31]. The construction in [219] works also for $\mathbb{Q}$ (and even smaller rings) and other theories like algebraic K- and L-theory. This is important for the proof of Theorem 2.59 and to get the commutative square above.

Example 8.10 (A Formula for $K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$). Suppose that the Farrell-Jones Conjecture is true rationally for $K_0(\mathbb{Z}G)$, i.e. the assembly map

$$A_{\mathcal{VCY}}\colon H_0^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Then we obtain

$$K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong$$
$$K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \bigoplus\nolimits_{(C)\in(\mathcal{FCY})} H_1(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot K_{-1}(RC) \otimes_{\mathbb{Z}} \mathbb{Q}\ .$$

Notice that $\widetilde{K}_0(\mathbb{Z}G)\otimes_{\mathbb{Z}}\mathbb{Q}$ contains only contributions from $K_{-1}(\mathbb{Z}C)\otimes_{\mathbb{Z}}\mathbb{Q}$ for finite cyclic subgroups $C \subseteq G$.

Remark 8.11. Note that these statements are interesting already for finite groups. For instance Theorem 8.6 yields for a finite group G and $R = \mathbb{C}$ an isomorphism

$$\bigoplus_{(C)\in(\mathcal{FCY})} \Lambda_G \otimes_{\Lambda_G[W_GC]} \theta_C \cdot R(C) \otimes_{\mathbb{Z}} \Lambda_G \ \cong\ R(G) \otimes_{\mathbb{Z}} \Lambda_G.$$

which in turn implies Artin's Theorem 2.24.

8.3.3 Rationalized Algebraic *L*-Theory for Infinite Groups

Here is the L-theory analogue of the results above. Compare [219, Theorem 0.4].

Theorem 8.12 (Rational Computation of Algebraic L-Theory for Infinite Groups). *Suppose that the group G satisfies the Farrell-Jones Conjecture 2.5 for L-theory. Then we get for all $j \in \mathbb{Z}, j \leq 1$ an isomorphism*

$$\bigoplus_{p+q=n} \ \bigoplus_{(C)\in(\mathcal{FCY})} H_p(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot L_q^{\langle j\rangle}(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$\xrightarrow{\cong} L_n^{\langle j\rangle}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 8.13 (Separation of Variables). Notice that in Theorem 8.6, 8.7 and 8.12 we see again the principle we called *separation of variables* in Remark 1.30. There is a group homology part which is independent of the coefficient ring R and the K- or L-theory under consideration and a part depending only on the values of the theory under consideration on RC or $C_r^*(C)$ for all finite cyclic subgroups $C \subseteq G$.

8.4 Integral Computations for Infinite Groups

As mentioned above, no general pattern for integral calculations is known or expected. We mention at least one situation where a certain class of groups can be treated simultaneously. Let $\mathcal{MFI}$ be the subset of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ which are maximal in $\mathcal{FIN}$. Consider the following assertions on the group G.

(M) $M_1, M_1 \in \mathcal{MFI}, M_1 \cap M_2 \neq 1 \;\Rightarrow\; M_1 = M_2$;
(NM) $M \in \mathcal{MFI} \;\Rightarrow\; N_G M = M$;
(VCL_I) If V is an infinite virtually cyclic subgroup of G, then V is of type I (see Lemma 2.18);
(FJK_N) The Isomorphism Conjecture of Farrell-Jones for algebraic K-theory 2.5 is true for $\mathbb{Z}G$ in the range $n \le N$ for a fixed element $N \in \mathbb{Z} \amalg \{\infty\}$, i.e. the assembly map $A\colon \mathcal{H}_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \xrightarrow{\cong} K_n(RG)$ is bijective for $n \in \mathbb{Z}$ with $n \le N$.

Let $\widetilde{K}_n(C_r^*(H))$ be the cokernel of the map $K_n(C_r^*(\{1\})) \to K_n(C_r^*(H))$ and $\overline{L}_n^{\langle j\rangle}(RG)$ be the cokernel of the map $L_n^{\langle j\rangle}(R) \to L_n^{\langle j\rangle}(RG)$. This coincides with $\widetilde{L}_n^{\langle j\rangle}(R)$, which is the cokernel of the map $L_n^{\langle j\rangle}(\mathbb{Z}) \to L_n^{\langle j\rangle}(R)$ if $R = \mathbb{Z}$ but not in general. Denote by $\mathrm{Wh}_n^R(G)$ the n-th Whitehead group of RG which is the $(n-1)$-th homotopy group of the homotopy fiber of the assembly map $BG_+ \wedge \mathbf{K}(R) \to \mathbf{K}(RG)$. It agrees with the previous defined notions if $R = \mathbb{Z}$. The next result is taken from [93, Theorem 4.1].

Theorem 8.14. *Let $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ be a ring such that the order of any finite subgroup of G is invertible in Λ. Let $(\mathcal{MFI})$ be the set of conjugacy classes (H) of subgroups of G such that H belongs to $\mathcal{MFI}$. Then:*

(1) If G satisfies (M), (NM) and the Baum-Connes Conjecture 2.4, then for $n \in \mathbb{Z}$ there is an exact sequence of topological K-groups

$$0 \to \bigoplus_{(H)\in(\mathcal{MFI})} \widetilde{K}_n(C_r^*(H)) \to K_n(C_r^*(G)) \to K_n(G\backslash E_{\mathcal{FIN}}(G)) \to 0,$$

which splits after applying $-\otimes_{\mathbb{Z}} \Lambda$;

(2) If G satisfies (M), (NM), (VCL_I) and the L-theory part of the Farrell-Jones Conjecture 2.5, then for all $n \in \mathbb{Z}$ there is an exact sequence

$$\cdots \to H_{n+1}(G\backslash E_{\mathcal{FIN}}(G); \mathbf{L}^{\langle -\infty\rangle}(R)) \to \bigoplus_{(H)\in(\mathcal{MFI})} \overline{L}_n^{\langle -\infty\rangle}(RH)$$
$$\to L_n^{\langle -\infty\rangle}(RG) \to H_n(G\backslash E_{\mathcal{FIN}}(G); \mathbf{L}^{\langle -\infty\rangle}(R)) \to \cdots$$

It splits after applying $-\otimes_{\mathbb{Z}} \Lambda$, more precisely

$$L_n^{\langle -\infty\rangle}(RG) \otimes_{\mathbb{Z}} \Lambda \to H_n(G\backslash E_{\mathcal{FIN}}(G); \mathbf{L}^{\langle -\infty\rangle}(R)) \otimes_{\mathbb{Z}} \Lambda$$

is a split-surjective map of Λ-modules;

(3) If G satisfies (M), (NM) and the Farrell-Jones Conjecture 2.5 for $L_n(RG)[1/2]$, then the conclusion of assertion 2 still holds if we invert 2 everywhere. Moreover, in the case $R = \mathbb{Z}$ the sequence reduces to a short exact sequence

$$0 \to \bigoplus_{(H)\in(\mathcal{MFI})} \widetilde{L}_n^{\langle j\rangle}(\mathbb{Z}H)[1/2] \to L_n^{\langle j\rangle}(\mathbb{Z}G)[1/2]$$
$$\to H_n(G\backslash E_{\mathcal{FIN}}(G);\mathbf{L}(\mathbb{Z})[1/2] \to 0,$$

which splits after applying $-\otimes_{\mathbb{Z}[1/2]} \Lambda[1/2]$.

(4) If G satisfies (M), (NM), and (FJK$_N$), then there is for $n\in\mathbb{Z}, n\le N$ an isomorphism

$$H_n(E_{\mathcal{VCY}}(G),E_{\mathcal{FIN}}(G);\mathbf{K}_R)\oplus \bigoplus_{(H)\in(\mathcal{MFI})} \mathrm{Wh}_n^R(H) \xrightarrow{\cong} \mathrm{Wh}_n^R(G),$$

where $\mathrm{Wh}_n^R(H)\to\mathrm{Wh}_n^R(G)$ *is induced by the inclusion* $H\to G$.

Remark 8.15 (Role of $G\backslash E_{\mathcal{FIN}}(G)$). Theorem 8.14 illustrates that for such computations a good understanding of the geometry of the orbit space $G\backslash E_{\mathcal{FIN}}(G)$ is necessary.

Remark 8.16. In [93] it is explained that the following classes of groups do satisfy the assumption appearing in Theorem 8.14 and what the conclusions are in the case $R=\mathbb{Z}$. Some of these cases have been treated earlier in [40], [231].

- Extensions $1\to\mathbb{Z}^n\to G\to F\to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0\in\mathbb{Z}^n$;
- Fuchsian groups F;
- One-relator groups G.

Theorem 8.14 is generalized in [222] in order to treat for instance the semidirect product of the discrete three-dimensional Heisenberg group by $\mathbb{Z}/4$. For this group $G\backslash E_{\mathcal{FIN}}(G)$ is S^3.

A calculation for 2-dimensional crystallographic groups and more general cocompact NEC-groups is presented in [231] (see also [257]). For these groups the orbit spaces $G\backslash E_{\mathcal{FIN}}(G)$ are compact surfaces possibly with boundary.

Example 8.17. Let F be a cocompact Fuchsian group with presentation

$$F=\langle a_1,b_1,\dots,a_g,b_g,c_1,\dots,c_t \mid$$
$$c_1^{\gamma_1}=\dots=c_t^{\gamma_t}=c_1^{-1}\cdots c_t^{-1}[a_1,b_1]\cdots[a_g,b_g]=1\rangle$$

for integers $g,t\ge 0$ and $\gamma_i>1$. Then $G\backslash E_{\mathcal{FIN}}(G)$ is a closed orientable surface of genus g. The following is a consequence of Theorem 8.14 (see [231] for more details).

- There are isomorphisms

$$K_n(C_r^*(F))\cong\begin{cases}\left(2+\sum_{i=1}^t(\gamma_i-1)\right)\cdot\mathbb{Z} & n=0;\\ (2g)\cdot\mathbb{Z} & n=1:\end{cases}$$

- The inclusions of the maximal subgroups $\mathbb{Z}/\gamma_i = \langle c_i \rangle$ induce an isomorphism
$$\bigoplus_{i=1}^{t} \mathrm{Wh}_n(\mathbb{Z}/\gamma_i) \xrightarrow{\cong} \mathrm{Wh}_n(F)$$
for $n \leq 1$;
- There are isomorphisms
$$L_n(\mathbb{Z}F)[1/2] \cong \begin{cases} \left(1 + \sum_{i=1}^t \left[\frac{\gamma_i}{2}\right]\right) \cdot \mathbb{Z}[1/2] & n \equiv 0 \ (4); \\ (2g) \cdot \mathbb{Z}[1/2] & n \equiv 1 \ (4); \\ \left(1 + \sum_{i=1}^t \left[\frac{\gamma_i - 1}{2}\right]\right) \cdot \mathbb{Z}[1/2] & n \equiv 2 \ (4); \\ 0 & n \equiv 3 \ (4), \end{cases}$$
where $[r]$ for $r \in \mathbb{R}$ denotes the largest integer less than or equal to r.
From now on suppose that each γ_i is odd. Then the number m above is odd and we get for $\epsilon = p$ and s
$$L_n^\epsilon(\mathbb{Z}F) \cong \begin{cases} \mathbb{Z}/2 \bigoplus \left(1 + \sum_{i=1}^t \frac{\gamma_i - 1}{2}\right) \cdot \mathbb{Z} & n \equiv 0 \ (4); \\ (2g) \cdot \mathbb{Z} & n \equiv 1 \ (4); \\ \mathbb{Z}/2 \bigoplus \left(1 + \sum_{i=1}^t \frac{\gamma_i - 1}{2}\right) \cdot \mathbb{Z} & q \equiv 2 \ (4); \\ (2g) \cdot \mathbb{Z}/2 & n \equiv 3 \ (4). \end{cases}$$
For $\epsilon = h$ we do not know an explicit formula. The problem is that no general formula is known for the 2-torsion contained in $\widetilde{L}^h_{2q}(\mathbb{Z}[\mathbb{Z}/m])$, for m odd, since it is given by the term $\widehat{H}^2(\mathbb{Z}/2; \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/m]))$, see [15, Theorem 2].

Information about the left hand side of the Farrell-Jones assembly map for algebraic K-theory in the case where G is $SL_3(\mathbb{Z})$ can be found in [334].

8.5 Techniques for Computations

We briefly outline some methods that are fundamental for computations and for the proofs of some of the theorems above.

8.5.1 The Equivariant Atiyah-Hirzebruch Spectral Sequence

Let $\mathcal{H}^G_*$ be a G-homology theory with values in Λ-modules. Then there are two spectral sequences which can be used to compute it. The first one is the rather obvious equivariant version of the *Atiyah-Hirzebuch spectral sequence.* It converges to $\mathcal{H}^G_n(X)$ and its E^2-term is given in terms of Bredon homology
$$E^2_{p,q} \;=\; H^G_p(X; \mathcal{H}^G_q(G/H))$$
of X with respect to the coefficient system, which is given by the covariant functor $\mathsf{Or}(G) \to \Lambda\text{-}\mathsf{MODULES}$, $G/H \mapsto \mathcal{H}^G_q(G/H)$. More details can be found for instance in [92, Theorem 4.7].

8.5.2 The p-Chain Spectral Sequence

There is another spectral sequence, the *p-chain spectral sequence* [93]. Consider a covariant functor $\mathbf{E}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$. It defines a G-homology theory $\mathcal{H}_*^G(-;\mathbf{E})$ (see Proposition 4.20). The p-chain spectral sequence converges to $\mathcal{H}_n^G(X)$ but has a different setup and in particular a different E^2-term than the equivariant Atiyah-Hirzebruch spectral sequence. We describe the E^1-term for simplicity only for a proper G-CW-complex.

A *p-chain* is a sequence of conjugacy classes of finite subgroups

$$(H_0) < \ldots < (H_p)$$

where $(H_{i-1}) < (H_i)$ means that H_{i-1} is subconjugate, but not conjugate to (H_i). Notice for the sequel that the group of automorphism of G/H in $\mathsf{Or}(G)$ is isomorphic to NH/H. To such a p-chain there is associated the NH_p/H_p-NH_0/H_0-set

$$\begin{aligned} S((H_0) < \ldots < (H_p)) \;&=\; \operatorname{map}(G/H_{p-1}, G/H_p)^G \times_{NH_{p-1}/H_{p-1}} \\ &\qquad \ldots \times_{NH_1/H_1} \operatorname{map}(G/H_0, G/H_1)^G. \end{aligned}$$

The E^1-term $E^1_{p,q}$ of the p-chain spectral sequence is

$$\bigoplus_{(H_0)<\ldots<(H_p)} \pi_q\left(\left(X^{H_p} \times_{NH_p/H_p} S((H_0) < \ldots < (H_p))\right)_+ \wedge_{NH_0/H_0} \mathbf{E}(G/H_0)\right)$$

where Y_+ means the pointed space obtained from Y by adjoining an extra base point. There are many situations where the p-chain spectral sequence is much more useful than the equivariant Atiyah-Hirzebruch spectral sequence. Sometimes a combination of both is necessary to carry through the desired calculation.

8.5.3 Equivariant Chern Characters

Equivariant Chern characters have been studied in [219] and [221] and allow to compute equivariant homology theories for proper G-CW-complexes. The existence of the equivariant Chern character says that under certain conditions the Atiyah-Hirzebruch spectral sequence collapses and, indeed, the source of the equivariant Chern character is canonically isomorphic to $\bigoplus_{p+q} E^2_{p,q}$, where $E^2_{p,q}$ is the E^2-term of the equivariant Atiyah-Hirzebruch spectral sequence.

The results of Section 8.3 are essentially proved by applying the equivariant Chern character to the source of the assembly map for the family of finite subgroups.

8.6 Miscellaneous

Exercises

8.1. Test

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9. Solutions of the Exercises

Chapter 1

1.1. By the Baum-Connes Conjecture 1.1 and the the version 1.17 of the Farrell-Jones Conjecture we have to compute $K_n(BG)$ for $n \in \mathbb{Z}$ and $H_1(BG;\mathbb{Z})$ and use the facts $K_0(\mathbb{Z}) \cong \mathbb{Z}$ and $K_1(\mathbb{Z}) = \{\pm 1\}$. Recall that $K_q(\{\bullet\})$ is $\mathbb{Z}$ for even q and 0 for odd q and that $K_n(\mathbb{Z})$ is 0 for $n \leq -1$, $\mathbb{Z}$ for $n = 0$ and $\mathbb{Z}/2$ for $n = 1$. Since the homology of BG is concentrated in dimensions ≤ 2 and the composition of the inclusion $\{\bullet\} \to BG$ with the projection $BG \to \{\bullet\}$ is the identity, the Atiyah-Hirzebruch spectral sequence collapses completely. So we get

$$K_n(C_r^*(G)) = \begin{cases} \mathbb{Z}^{2g} & \text{if } n \text{ is odd;} \\ \mathbb{Z}^2 & \text{if } n \text{ is even and } g \geq 1; \\ \mathbb{Z} & \text{if } n \text{ is even and } g = 0; \end{cases}$$

$$K_n(\mathbb{Z}G) = \begin{cases} \{0\} & \text{if } n \leq -1; \\ \mathbb{Z} & \text{if } n = 0; \\ \mathbb{Z}^{2g} \times \{\pm 1\} & \text{if } n = 1. \end{cases}$$

1.2. The Fourier transform yields an isometric isomorphism of Hilbert spaces $l^2(\mathbb{Z}) \to L^2(S^1)$. We can view $C(S^1)$ as a closed $*$-subalgebra of $\mathcal{B}(L^2(S^1))$ by sending $f \in C(S^1)$ to the operator $L^2(S^1) \to L^2(S^1)$ which sends $g \in L^2(S^1)$ to the element $f \cdot g \in L^2(S^1)$ given by $f \cdot g(z) = f(z) \cdot g(z)$. Since the functions $S^1 \to \mathbb{C}$ given by finite Laurent series $\sum_{n=a}^{b} \lambda_n \cdot z^n$ build a dense $*$-subalgebra of $C(S^1)$, we conclude $C_r^*(\mathbb{Z}) = C(S^1)$.

Since $\int_{S^1} z^n d\mu$ is the same as the integral $\frac{1}{2\pi i} \cdot \int_\gamma z^{n-1} dz$ over the curve $\gamma\colon [0,1] \to \mathbb{C},\ t \mapsto \exp(2\pi i t)$, we conclude from the Residue Theorem that $\int_{S^1} z^n d\mu$ is 1 for $n = 0$ and is 0 for $n \neq 0$. This implies by the continuity of the trace that the standard trace sends $f \in C(S^1)$ to $\int_{S^1} f d\mu$.

Let $A \in M(n,n,C(S^1))$ be an (n,n)-matrix over $C(S^1)$. It can also be viewed as a continuous function $\overline{A}\colon S^1 \to M(n,n,\mathbb{C})$. Then $\operatorname{tr}_{C_r^*(\mathbb{Z})}(A)$ agrees with $\int_{S^1} \operatorname{tr}_{\mathbb{C}}(\overline{A}(z))\, d\mu$. Now suppose that $A^2 = A$. Then $\overline{A}(z)^2 = \overline{A}(z)$ for all $z \in S^1$. This implies that $\operatorname{tr}_{\mathbb{C}}(\overline{A}(z)) = \dim_{\mathbb{C}}(\operatorname{im}(A(z))$ lies in $\mathbb{Z}$. Since $\overline{A}$ and hence $\operatorname{tr}_{\mathbb{C}}(\overline{A}(z))$ are continuous, the function $\operatorname{tr}_{\mathbb{C}}(\overline{A}(z))$ is constant with an integer as value. Therefore $\operatorname{tr}_{C_r^*(\mathbb{Z})}(A)$ is an integer.

1.3. Let $g \in G$ be an element of finite order $n \geq 2$. Then $n^{-1} \cdot \sum_{i=1}^{n} g^i$ is a non-trivial idempotent in RG.

1.4. The inverse is $1 - t^2 - t^3$. The map $\mathbb{Z}[\mathbb{Z}/5] \to \mathbb{C}$ sending t to $\exp(2\pi i/5)$, the determinant over $\mathbb{C}$ and the map $\mathbb{C}^{\mathrm{inv}} \to (0,\infty), z \mapsto |z|$ together induce a map $\mathrm{Wh}(\mathbb{Z}/5) \to (0,\infty)$ of abelian groups, where we equip the target with the group structure given by multiplying positive real numbers. It sends the element given by the unit above to $|1-\cos(2\pi/5)|$, which is different from 1.

1.5. (1) If $f\colon P_* \to Q_*$ is a R-chain homotopy equivalence, then its algebraic mapping cone $\mathrm{cone}(f_*)$ is contractible. This implies $\mathrm{cone}(f)_{\mathrm{odd}} \cong \mathrm{cone}(f)_{\mathrm{ev}}$, where $\mathrm{cone}(f)_{\mathrm{odd}}$ and $\mathrm{cone}(f)_{\mathrm{ev}}$ respectively is the direct sum over all odd-dimensional and even-dimensional chain modules respectively. Hence $o(P_*) = o(Q_*)$.

(2) is obvious.

(3) Suppose that $\widetilde{o(P_*)} = 0$. By adding elementary finite projective R-chain complexes $\ldots 0 \to 0 \to Q \xrightarrow{\mathrm{id}} Q \to 0 \to 0 \to \ldots$ to P_* one can change P_* within its R-chain homotopy class such that all R-chain modules are finitely generated free except the top-dimensional one. But this top dimensional one must be stably free because of $\widetilde{o(P_*)} = 0$. Now add another appropriate elementary finite free R-chain complex to turn also this top-dimensional module into a finitely generated free one.

If P_* is up to homotopy finite free, $\widetilde{o(P_*)} = 0$ follows from assertion (1).

1.6. If $n \le 2$, every closed oriented manifold is aspherical or a sphere. Suppose that $n \ge 3$. The collapse map $M\sharp N \to M \vee N$ is $(n-1)$-connected since up to homotopy the one-point union $M \vee N$ is obtained from $M\sharp N$ by attaching a n-cell D^n. Hence $\pi_1(M\sharp N)$ is isomorphic to the free amalgamated product $\pi_1(M) * \pi_1(N)$. Since all higher homotopy groups of $M\sharp N$ vanish, $\pi_k(M \vee N) = 0$ for $2 \le k \le n-1$. Since $M\sharp N$ is aspherical, $\pi_1(M\sharp N)$ is torsionfree. Hence $\pi_k(M)$ and $\pi_k(N)$ vanish for $2 \le k \le n-1$ and both $\pi_1(M)$ and $\pi_1(N)$ are torsionfree. If $\pi_1(M)$ is finite, then it must be trivial and hence M is homotopy equivalent to S^n. By the Hurewicz Theorem a CW-complex X is aspherical if and only if $\widetilde{H}_n(\widetilde{X};\mathbb{Z})$ vanishes for all $n \in \mathbb{Z}$. If $\pi_1(M)$ is infinite, the universal covering $\widetilde{M}$ is a non-compact manifold and hence satisfies $H_n(\widetilde{M}) = 0$. Hence M is aspherical. It remains to rule out the case that both M and N are aspherical. Suppose they are aspherical. By elementary covering theory and Mayer-Vietoris arguments, one checks that $\widetilde{H}_n(\widetilde{M \vee N};\mathbb{Z})$ vanishes for all $n \in \mathbb{Z}$. Hence $M \vee N$ is aspherical. This implies that the collapse map $M\sharp N \to M \vee N$ is a homotopy equivalence. This leads to a contradiction since $H_n(M) \cong H_n(M) \cong H_n(M\sharp N) \cong \mathbb{Z}$ and $H_n(M \vee N) \cong H_n(M) \oplus H_n(N)$ holds.

1.7. We conclude from Corollary 1.34 and Corollary 1.35 that $\mathcal{P}^{\mathrm{Diff}}(M) \to \mathcal{P}(M)$ induces a rational isomorphism on π_n if and only if

$$0 \ = \ \pi_n(\mathcal{P}^{\mathrm{Diff}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} \ \cong \ \bigoplus_{k=1}^{\infty} H_{n-4k+1}(M;\mathbb{Q})$$

holds.

1.8. The group G is the knot groups associated to the trefoil knot. Now the same argument as in Example 1.4 shows that G is torsionfree and

$$\begin{aligned} L_n^{\langle -j\rangle}(\mathbb{Z}G) \ &\cong \ H_n(S^1;\mathbf{L}^{\langle j\rangle}(\mathbb{Z})) \\ &\cong \ L_n^{\langle j\rangle}(\mathbb{Z}) \oplus L_{n-1}^{\langle j\rangle}(\mathbb{Z}) \ \cong \ \begin{cases} \mathbb{Z} & n = 0,1 \mod 4; \\ \mathbb{Z}/2 & n = 2,3 \mod 4. \end{cases} \end{aligned}$$

1.9. The implication (1) $\Rightarrow$ (2) follows from elementary covering theory. Suppose (2) holds. By elementary covering theory there is a covering $p\colon \overline{N} \to N$ and a map $\overline{f}\colon M \to \overline{N}$ such that the image of $\pi_1(p)$ and $\pi_1(f)$ agree and $p \circ \overline{f} = f$. The map $\pi_1(\overline{f})$ is bijective. Hence $\overline{f}$ is homotopic to a homeomorphism by the Borel Conjecture 1.49.

1.10. Let $f\colon M \to S^n$ be a representative for an element in $\mathcal{S}^{\mathrm{top}}(S^n)$. Since the Poincaré Conjecture 1.23 is known in dimensions $n \neq 3$, **Comment 49** (By W.): `Later drop the condition` $n \neq 3$ `depending on what the status of the proof by Perelman is.` there exists a homeomorphism $g\colon S^n \to M$. We can assume without loss of generality that $f \circ g\colon S^n \to S^n$ has degree one, otherwise compose g with a homeomorphism $S^n \to S^n$ of degree -1. This implies that $f \circ g$ is homotopic to $\mathrm{id}\colon S^n \to S^n$. Hence f and id_{S^n} define the same class in $\mathcal{S}(S^n)$.

1.11. From the discussion in Section 1.11 applied to the obvious group homomorphisms $G \to G \times H$ and $G \times H \to G$ it follows that the assembly map for G is a retract of the one for $G \times H$.

Chapter 2

2.1. We can write $SL_2(\mathbb{Z})$ as the free amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. Now apply the sequence (0.1) and the fact $K_n(C_r^*(H))$ is $R_{\mathbb{C}}(H)$ for even n and is trivial for odd n if H is a finite group. The result is that $K_n(C_r^*(SDL_2(\mathbb{Z}))$ is $\mathbb{Z}^8$ for even n and $\{0\}$ for odd n.

2.2. Because of the Baum-Connes Conjecture 2.4 it suffices to prove that $K_n^G(E_{\mathcal{FIN}}(G))$ is finitely generated for all $n \in \mathbb{Z}$. Since $G\backslash E_{\mathcal{FIN}}(G)$ is compact, $E_{\mathcal{FIN}}(G)$ is built by finitely many equivariant cells $G/H \times D^n$. We show that for every G-CW-complex X which is built of finitely many equivariant cells $G/H_i \times D^{n_i}$ for finite $H_i \subseteq G$ that $K_n^G(X)$ is finitely generated for all $n \in \mathbb{Z}$. We do this by induction over the number of equivariant cells. In the induction step we consider X which is obtained from Y by attaching the cell $G/H \times D$. Then the associated long exact Mayer-Vietoris sequence together with the induction hypothesis that $K_n^G(Y)$ is finitely generated for all $n \in \mathbb{Z}$ and the fact that $K_n^G(G/H)$ is zero or $R_{\mathbb{C}}(H)$ yield the induction step.

2.3. Let G be of the first kind. Then there is an epimorphism $f \colon G \to \mathbb{Z}$ which induces an epimorphism $H_1(G;\mathbb{Z}) \to H_1(\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}$. Hence $H_1(G;\mathbb{Z})$ is infinite. Suppose that $H_1(G;\mathbb{Z})$ is infinite. Then the obvious composition $G \to H_1(G;\mathbb{Z}) \to H_1(G;\mathbb{Z})/\operatorname{tors}(H_1(G;\mathbb{Z}))$ is an epimorphism onto an infinite cyclic group.

2.4. Because of Proposition 2.22 it suffices to show that any virtually cyclic subgroup V of G is of the first kind. The intersection $V \cap T$ is a torsionfree virtually cyclic group and hence isomorphic to $\mathbb{Z}$. Since $V/(V \cap T)$ is a subgroup of G/T, the finite group $T/(V \cap T)$ has odd order. Since $\operatorname{aut}(\mathbb{Z}) = \{\pm 1\}$ has order 2, $V \cap T$ is central in V. Since D_∞ has a trivial center, V cannot map surjectively to D_∞.

2.5. Up to conjugacy S_3 has two non-trivial cyclic subgroups, one of order two and one of order three. Denote in the sequel by F the trivial representation. Let F^- be the $\mathbb{Z}/2$-representation given by $-\operatorname{id} \colon F \to F$. Using the canonical epimorphism $S_3 \to \mathbb{Z}_2$, we consider F^- also as S_3-representation. Let V be the irreducible 2-dimensional S_3-representation which is obtained from F^3 with the S_3-permutation action by dividing out the diagonal $\{(x,x,x) \mid x \in F\}$. Denote by W the analogously defined 2-dimensional $\mathbb{Z}/3$-representation. Then $R_F(\mathbb{Z}/2)$ is the free abelian group with $\{F, F^-\}$ as bases and $R_F(S_3)$ is the free abelian group with $\{F, F^-, V\}$ as bases. The induction homomorphism induced by the inclusion $\mathbb{Z}/2 \to S_3$ is

$$\begin{aligned} i_2 \colon & R_F(\mathbb{Z}/2) \to R_F(S_3), \\ & \mu_F \cdot [F] + \mu_{F^-} \cdot [F^-] \ \mapsto \ \mu_F \cdot [F] + \mu_{F^-} \cdot [F^-] + (\mu_F + \mu_{F^-}) \cdot [V]. \end{aligned}$$

Suppose F does not contain a non-trivial third root of unity. Then W is irreducible and $R_F(\mathbb{Z}/3)$ is a free abelian group with $\{F, W\}$ as basis. The induction homomorphism induced by the inclusion $\mathbb{Z}/3 \to S_3$ is

$$\begin{aligned} i_3 \colon & R_F(\mathbb{Z}/3) \to R_F(S_3) \\ & \qquad \mu_F \cdot [F] + \mu_W \cdot [W] \ \mapsto \ \mu_F \cdot [F] + \mu_F \cdot [F^-] + 2\mu_W \cdot [V]. \end{aligned}$$

The following sequence is exact

$$R_F(\mathbb{Z}/2) \oplus R_F(\mathbb{Z}/3) \xrightarrow{i_2 \oplus i_3} R_F(S_3) \xrightarrow{\delta} \mathbb{Z}/2 \to 0,$$

if we define δ by

$$\delta\left(\lambda_F \cdot [F] + \lambda_{F^-} \cdot [F^-] + \lambda_V \cdot [V]\right) \;=\; \overline{\lambda_F} + \overline{\lambda_{F^-}} + \overline{\lambda_V}.$$

Suppose F contains a third root of unity. Then W is reducible and decomposes into $W = W_1 \oplus W_2$ for two non-isomorphic 1-dimensional $\mathbb{Z}/3$-representations and $R_F(\mathbb{Z}/3)$ is the free abelian group with $\{F, W_1, W_2\}$ as basis. The induction homomorphism induced by the inclusion $\mathbb{Z}/3 \to S_3$ is

$$\begin{aligned} i_3 \colon R_F(\mathbb{Z}/3) \to R_F(S_3)\mu_F \cdot [F] + \mu_{W_1} \cdot [W_1] + \mu_{W_2} \cdot [W_2] \\ \mapsto \; \mu_F \cdot [F] + \mu_F \cdot [F^-] + (\mu_{W_1} + \mu_{W_2}) \cdot [V]. \end{aligned}$$

Hence $i_2 \oplus i_3 \colon R_F(\mathbb{Z}/2) \oplus R_F(\mathbb{Z}/3) \to R_F(S_3)$ is surjective.

2.6. Since by assumption Conjecture 2.29 holds for G and $R = \mathbb{Q}$, it suffices to treat the case, where G is finite. By Artin's Theorem 2.24 the proof can be reduced further to finite cyclic groups G.

2.7. Consider an element x in $K_0(\mathbb{C}G)$. It can be written as $x = [P] - [Q]$ for finitely generated projective $\mathbb{C}G$-modules P and Q. The assumptions on P and Q imply that there are positive integers $m(P)$,$n(p)$,$m(Q)$, and $m(Q)$ such that

$$\begin{aligned} m(P) \cdot [P] &= n(P) \cdot [\mathbb{C}G]; \\ m(P) \cdot [Q] &= n(Q) \cdot [\mathbb{C}G], \end{aligned}$$

holds in $K_0(\mathbb{C}G)$. This implies that the induction map

$$K_0(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{Q} \to K_0(\mathbb{C}G) \otimes_{\mathbb{C}} \mathbb{Q}$$

is surjective. Proposition 2.38 (1) implies that $\operatorname{class}_0(G)_f$ is isomorphic to $\mathbb{C}$. Hence G is torsionfree.

2.8. Since G is finite, $\mathbb{Q}G$ is semisimple and hence any finitely generated $\mathbb{Q}G$-module is finitely generated projective. The functor sending a $\mathbb{Z}G$-module M to the $\mathbb{Q}G$-module $\mathbb{Q}G \otimes_{\mathbb{Z}G} M \;\cong_{\mathbb{Q}G}\; \mathbb{Q} \otimes_{\mathbb{Z}} M$ is exact. Hence j is well-defined.

Consider a finitely generated projective $\mathbb{Q}G$-module P. Choose an idempotent matrix $A \in M_n(\mathbb{Q}G)$ such that the image of the map $r_A \colon \mathbb{Q}G^n \to \mathbb{Q}G^n$ given by right multiplication with A is $\mathbb{Q}G$-isomorphic to P. Choose an integer $l \geq 1$ such that $B := l \cdot A$ is a matrix over $\mathbb{Z}G$. Let M be the cokernel of the map $r_B \colon \mathbb{Z}G^n \to \mathbb{Z}G^n$. Then $G_0(\mathbb{Z}G) \to K_0(\mathbb{Q}G)$ maps $[M]$ to $[P]$. Hence $j \colon G_0(\mathbb{Z}G) \to K_0(\mathbb{Q}G)$ is surjective.

We have already mentioned that $K_0(\mathbb{Z}) \to K_0(\mathbb{Z}G)$ is rationally an isomorphism. The map $K_0(\mathbb{Q}) \to K_0(\mathbb{Q}G)$ is rationally an isomorphism if and only if G is trivial.

2.9. (1) This follows from Theorem 2.47

(2) The statement without applying $- \otimes_{\mathbb{Z}} \Lambda^G$ together with the version 2.29 of the Farrell-Jones Conjecture for $R = \mathbb{C}$ would imply the original Trace Conjecture due to Baum and Connes for which Roy constructed a counterexample.

2.10. Let $V \subseteq G$ be a virtually cyclic subgroup. The order of any finite subgroup of V is invertible in R. Using Lemma 2.18 one can show that V contains an infinite cyclic subgroup C such that $[V : C]$ is invertible in R. Since R is regular, $R[\mathbb{Z}]$ is regular and $[V : C]$ is invertible in $R[\mathbb{Z}]$. Hence RV is regular and $NK_n(RV)$ is trivial for all $n \in \mathbb{Z}$ and all virtually cyclic subgroups of G. The equivariant Atyiah-Hirzebruch spectral sequence implies that $H_n(E_{\mathcal{VCY}}(G); \mathbf{NK}_R) = 0$ for all $n \in \mathbb{Z}$.

2.11. We have to show for any group homomorphism $\phi\colon K \to G$ that the assembly map

$$A_{\phi^*\mathcal{G}} \colon \mathcal{H}_n^K(E_{\phi^*\mathcal{G}}(K)) \to \mathcal{H}_n^K(\{\bullet\})$$

is bijective for all $n \in \mathbb{Z}$. Since by assumption about $\mathcal{F}$ this is true for $A_{\phi^*\mathcal{F}}$, it suffices to show the bijectivity of the relative assembly map

$$A_{\phi^*\mathcal{F} \to \phi^*\mathcal{G}} \colon \mathcal{H}_n^K(E_{\phi^*\mathcal{F}}(K)) \to \mathcal{H}_n^K(E_{\phi^*\mathcal{G}}(K)).$$

By the Transitivity Principle 2.11 it remains to show for any $L \in \phi^*\mathcal{G}$ that the assembly map

$$A_{L \cap \phi^*\mathcal{F}} \colon \mathcal{H}_n^L(E_{L \cap \phi^*\mathcal{F}}(L)) \to \mathcal{H}_n^L(\{\bullet\})$$

is bijectivive. This follows from the assumption about $\mathcal{F}$ because of $L \cap \phi^*\mathcal{F} = (\phi|_L)^*\mathcal{F}$.

Chapter 3

3.1. If a quotient group of G acts isometrically on some affine Hilbert space without a fixed point, the same is true for G. Obviously $\mathbb{Z}$ acts freely on $\mathbb{R}$ and hence does not have property (T). A free group maps surjectively onto $\mathbb{Z}$. The fundamental group of a compact connected surface maps surjectively onto $\mathbb{Z}$ or is finite.

3.2. The class of word hyperbolic groups and the class of a-T-menable groups is closed under extensions with finite quotients. Hence for any finite subgroup $K \subseteq Q$ the group $p^{-1}(K)$ for $p\colon G \to Q$ the projection is again word hyperbolic or a-T-menable. Any word hyperbolic and any a-T-menable group satisfies the Baum-Connes Conjecture with Coefficients 2.74 (see Theorem 3.1). **Comment 50** (By W.): `Add reference for word hyperbolic groups.` Now apply the results of Subsection 3.5.3.

3.3. The groups $\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}/2$ and $\mathbb{Z}$ are abelian. Hence L is elementary-amenable and in particular amenable. Hence L satisfies the Baum-Connes Conjecture with Coefficients 2.74 and in particular the Baum-Connes Conjecture 2.4 by Theorem 3.1.

The lamplighter group L satisfies the L-theoretic version of the Fibered Farrell-Jones Conjecture 2.84 with $R = \mathbb{Z}$ and in particular the L-theoretic version Farrell-Jones Conjecture 2.5 after inverting two by Theorem 3.16.

Since L belongs to the class $\mathcal{C}_0$, the Fibered Version of the Farrell-Jones Conjecture 2.90 for $KH_*(RL)$ for all rings R, the injectivity part of Farrell-Jones Conjecture 0.8 for $K_*(RL)$ for all regular rings R with $\mathbb{Q} \subset R$ is true and for $R = \mathbb{Z}$ the injectivity part of the Farrell-Jones Conjecture 2.5 for $K_*(\mathbb{Z}G)$ is true rationally (see Theorem 3.23).

Let R is a regular ring such that 2 is invertible in R. For a finite group H the ring RH and $R[H \times \mathbb{Z}]$ are regular coherent. Any finitely presented $R[\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}/2]$-module or $R[(\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}/2) \times \mathbb{Z}]$-module respectively is the induction of a finitely presented RH-module or $R[H \times \mathbb{Z}]$-module respectively for some finite subgroup $H \subset \bigoplus_{n\in\mathbb{Z}}\mathbb{Z}/2$. Hence the rings $R[\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}/2]$ and $R[(\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}/2) \times \mathbb{Z}]$ are regular coherent.

This is enough to using the Mayer-Vietoris sequences for HNN-extensions that the Farrell-Jones Conjecture 0.8 for $K_*(RL)$ for all regular rings R with $1/2 \in R$ is true. **Comment 51** (By W.): `This should be checked in detail.`

3.4. Obviously G is the directed union of its finitely generated subgroups. Now apply the results mentioned in Subsection 3.5.1.

3.5. A group G is finitely presented if and only if it is the fundamental group of a closed connected orientable 4-manifold.

Every group G is the directed union of its finitely generated subgroups.

Any finitely generated group G can be written as the colimit of a system of finitely presented groups. Namely fix a presentation with a finite set of generators S and some set of relations R. Let I be the set of finite subsets of R directed by inclusion. Now consider the system $\{G_i \mid J \in I\}$, where G_J has is the finitely presented group with S as set of generators and J as set of relations. The structure maps are given by the obvious projections. The colimit of this system is G.

Now apply the results mentioned in Subsection 3.5.1.

3.6. Obviously $\dim(B)$ and $\dim(F)$ are 1, 2 or 3. Hence they satisfy the Baum-Connes Conjecture with Coefficients 2.74. **Comment 52** (By W.): `Add reference.`

Suppose $\dim(B)$ is 3. Then F is S^1 or D^1 and we get an exact sequence $1 \to C \to \pi_1(E) \to \pi_1(B) \to 1$ for some cyclic group C. Since the Baum-Connes Conjecture with Coefficients 2.74 is true for any virtually cylic group, the claim follows from the results of Subsection 3.5.3.

Suppose $\dim(B)$ is 2. Then $\dim(F)$ is also 2. If B is aspherical, then we have the exact sequence $1 \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to 1$ and all finite subgroups of $\pi_1(B)$ are trivial, and the claim follows from the results of Subsection 3.5.3. If B is not aspherical, B must be S^2 or D^2. In this case we get an extension $1 \to C \to \pi_1(F) \to \pi_1(E) \to 1$ for a cyclic group C. Since F is aspherical or S^2, C must be trivial or infinite cyclic. If C is trivial, the claim is true. If C is infinite cylic, then $\pi_1(F)$ contains a normal cyclic subgroup and hence F must be T^2. Therefore $\pi_1(E)$ is virtually cylic and satisfies Baum-Connes Conjecture with Coefficients 2.74.

Suppose $\dim(B) = 1$. Then $B = D^1$ or $B = S^1$. If $B = D^1$, then $\pi_1(E) \cong \pi_1(F)$ and the claim follows. Suppose $B = S^1$. Then we obtain an exact sequence $1 \to \pi_1(F) \to \pi_1(E) \to \mathbb{Z} \to 1$. Since very finite subgroup of $\mathbb{Z}$ is trivial, the claim follows from the results of Subsection 3.5.3.

3.7. Since $\mathbb{Z}$ is virtually cyclic, the Fibered Meta Conjecture 2.95 is always true for $G = \mathbb{Z}$. Suppose that the K-theoretic Fibered Farrell-Jones Conjecture 2.84 holds for the group $\mathbb{Z}$ and the ring R but with the family $\mathcal{VCY}$ replaced by the family $\mathcal{FIN}$. Obviously $\mathcal{FIN} = \mathcal{TR}$ since $\mathbb{Z}$ is torsionfree. Let $\phi\colon G \times \mathbb{Z} \to \mathbb{Z}$ be the projection. By assumption the assembly map

$$H_n^{G\times\mathbb{Z}}(E_{p^*\mathcal{TR}}(G \times \mathbb{Z}); \mathbf{K}_R) \xrightarrow{\cong} H_n^{G\times\mathbb{Z}}(\{\bullet\}; \mathbf{K}_R) = K_n(R[G \times \mathbb{Z}])$$

is bijective for all $n \in \mathbb{Z}$. Obviously a model for $E_{p^*\mathcal{TR}}(G \times \mathbb{Z}); \mathbf{K}_R)$ is $\mathbb{R}$ with the $G \times \mathbb{Z}$-action for which G acts trivially and $\mathbb{Z}$ by translation. Then the source of the assembly map can be identified with $K_n(RG) \oplus K_{n-1}(RG)$ and the assembly map above is the restriction to $K_n(RG) \oplus K_{n-1}(RG)$ of the isomorphism appearing in the Bass-Heller-Swan decomposition

$$K_n(RG) \oplus K_{n-1}(RG) \oplus NK_n(RG) \oplus NK_n(RG) \xrightarrow{\cong} K_n(R[G \times \mathbb{Z}]).$$

This implies $NK_n(RG) = 0$.

3.8. Since the Fibered Meta Conjecture 2.95 passes to subgroups, it holds for both G and H if it is true for $G \times H$.

Suppose that the Fibered Meta Conjecture 2.95 holds for G, H, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times D_\infty$ and $D_\infty \times D_\infty$. Recall from Section 2.14 that the Fibered Meta Conjecture 2.95 satisfies the obvious version of the Transitivity Principle 2.11. This implies for a group extensions $1 \to G_0 \to G_1 \xrightarrow{p} G_2 \to 1$ that the Fibered Meta Conjecture 2.95 holds for G_1 if it holds for G_2 and the group $p^{-1}(V)$ for every virtually cyclic subgroup $V \subseteq G$. Hence the Fibered Meta Conjecture 2.95 holds for $V \times W$ for any two virtually cyclic groups $V \times W$ since $V \times W$ maps surjectively with finite kernel to one of the groups $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times D_\infty$ and $D_\infty \times D_\infty$. Now use the projection $G \times V \to V$ to show that $G \times V$ satisfies the Fibered Meta Conjecture 2.95 for every virtually cyclic group V. The Fibered Meta Conjecture 2.95 follows for $G \times H$ using the projection $G \times H \to H$.

3.9. Since the Fibered Meta Conjecture 2.95 passes to subgroups, it holds for both G and H if it is true for $G * H$.

Suppose that the Fibered Meta Conjecture 2.95 holds for G and H. By an exercise above we can assume that the Fibered Meta Conjecture 2.95 holds for $G \times H$. Let $p\colon G * H \to G \times H$ be the obvious group homomorphism. We have already explained in the solution of an exercise above that the Fibered Meta Conjecture 2.95 holds for $G * H$ if it holds for $p^{-1}(V)$ for every virtually cyclic subgroup $V \subseteq G$.

Since every virtually free group is a colimit of virtually finitely generated free subgroups and the Fibered Meta Conjecture 2.95 holds for every virtually finitely generated free subgroups it holds for every virtually free group.

Let $C \subseteq G \times H$ be infinite cyclic. Then $p^{-1}(C)$ is free. Hence $p^{-1}(V)$ is virtually free for any virtually cyclic group $V \subseteq G$. This implies by assumption that the Fibered Meta Conjecture 2.95 holds for $p^{-1}(V)$ for every virtually cyclic subgroup $V \subseteq G$.

Chapter 4

4.1. Test

Chapter 5

5.1. Test

Chapter 6

6.1. Test

Chapter 7

7.1. Test

Chapter 8

8.1. Test

Chapter 9

9.1. Test

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References

[1] H. Abels. Parallelizability of proper actions, global K-slices and maximal compact subgroups. *Math. Ann.*, 212:1–19, 1974/75.

[2] H. Abels. A universal proper G-space. *Math. Z.*, 159(2):143–158, 1978.

[3] U. Abresch. Über das Glätten Riemannscher Metriken. Habilitationsschrift, Bonn, 1988.

[4] J. F. Adams. *Stable homotopy and generalised homology.* University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.

[5] A. Alves and P. Ontaneda. A formula for the Whitehead group of a three-dimensional crystallographic group. preprint, arXiv:math.KT/0310156, to appear in Topology, 2003.

[6] C. Anantharaman and J. Renault. Amenable groupoids. In *Groupoids in analysis, geometry, and physics (Boulder, CO, 1999)*, volume 282 of *Contemp. Math.*, pages 35–46. Amer. Math. Soc., Providence, RI, 2001.

[7] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*, volume 36 of *Monographies de L'Enseignement Mathématique.* Geneva, 2000. With a foreword by Georges Skandalis and Appendix B by E. Germain.

[8] D. R. Anderson, F. X. Connolly, S. C. Ferry, and E. K. Pedersen. Algebraic K-theory with continuous control at infinity. *J. Pure Appl. Algebra*, 94(1):25–47, 1994.

[9] D. R. Anderson and W. C. Hsiang. The functors K_{-i} and pseudo-isotopies of polyhedra. *Ann. of Math. (2)*, 105(2):201–223, 1977.

[10] C. S. Aravinda, F. T. Farrell, and S. K. Roushon. Surgery groups of knot and link complements. *Bull. London Math. Soc.*, 29(4):400–406, 1997.

[11] C. S. Aravinda, F. T. Farrell, and S. K. Roushon. Algebraic K-theory of pure braid groups. *Asian J. Math.*, 4(2):337–343, 2000.

[12] M. F. Atiyah. Global theory of elliptic operators. In *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)*, pages 21–30. Univ. of Tokyo Press, Tokyo, 1970.

[13] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. *Astérisque*, 32-33:43–72, 1976.

[14] A. Bak. Odd dimension surgery groups of odd torsion groups vanish. *Topology*, 14(4):367–374, 1975.

[15] A. Bak. The computation of surgery groups of finite groups with abelian 2-hyperelementary subgroups. In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 384–409. Lecture Notes in Math., Vol. 551. Springer-Verlag, Berlin, 1976.

[16] P. Balmer and M. Matthey. Codescent theory ii: Cofibrant approximations. arXiv:math.KT/0308056, 2003.

[17] P. Balmer and M. Matthey. Codescent theory. I. Foundations. *Topology Appl.*, 145(1-3):11–59, 2004.

[18] P. Balmer and M. Matthey. Model theoretic reformulation of the Baum-Connes and Farrell-Jones conjectures. *Adv. Math.*, 189(2):495–500, 2004.

[19] M. Banagl and A. Ranicki. Generalized Arf invariants in algebraic L-theory. arXiv: math.AT/0304362, 2003.

[20] A. Bartels. On the domain of the assembly map in algebraic K-theory. *Algebr. Geom. Topol.(electronic)*, 3:1037–1050, 2003.

[21] A. Bartels, F. T. Farrell, L. E. Jones, and H. Reich. A foliated squeezing theorem for geometric modules. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory.* 2003. Proceedings of the summer school "High dimensional manifold theory" in Trieste May/June 2001.

[22] A. Bartels, T. Farrell, L. Jones, and H. Reich. On the isomorphism conjecture in algebraic K-theory. *Topology*, 43(1):157–213, 2004.

[23] A. Bartels and W. Lück. Induction theorems and isomorphism conjectures for K- and L-theory. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 331, Münster, arXiv:math.KT/0404486, 2004.

[24] A. Bartels and W. Lück. Isomorphism conjecture for homotopy K-theory and groups acting on trees. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 342, Münster, arXiv:math.KT/0407489, 2004.

[25] A. Bartels and H. Reich. On the Farrell-Jones conjecture for higher algebraic K-theory. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 282, Münster, to appear in JAMS, 2003.

[26] A. C. Bartels. Squeezing and higher algebraic K-theory. *K-Theory*, 28(1):19–37, 2003.

[27] H. Bass. *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[28] H. Bass. Some problems in "classical" algebraic K-theory. In *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 3–73. Lecture Notes in Math., Vol. 342. Springer, Berlin, 1973.

[29] H. Bass. Euler characteristics and characters of discrete groups. *Invent. Math.*, 35:155–196, 1976.

[30] H. Bass, A. Heller, and R. G. Swan. The Whitehead group of a polynomial extension. *Inst. Hautes Études Sci. Publ. Math.*, (22):61–79, 1964.

[31] P. Baum and A. Connes. Chern character for discrete groups. In *A fête of topology*, pages 163–232. Academic Press, Boston, MA, 1988.

[32] P. Baum and A. Connes. Geometric K-theory for Lie groups and foliations. *Enseign. Math. (2)*, 46(1-2):3–42, 2000.

[33] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-algebras: 1943–1993*

(San Antonio, TX, 1993), pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[34] P. Baum and R. G. Douglas. K-homology and index theory. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, pages 117–173. Amer. Math. Soc., Providence, R.I., 1982.

[35] P. Baum, N. Higson, and R. Plymen. Representation theory of p-adic groups: a view from operator algebras. In *The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998)*, pages 111–149. Amer. Math. Soc., Providence, RI, 2000.

[36] P. Baum, M. Karoubi, and J. Roe. On the Baum-Connes conjecture in the real case. Preprint, 2002.

[37] P. Baum, S. Millington, and R. Plymen. Local-global principle for the Baum-Connes conjecture with coefficients. Preprint, 2002.

[38] G. Baumslag, E. Dyer, and A. Heller. The topology of discrete groups. *J. Pure Appl. Algebra*, 16(1):1–47, 1980.

[39] E. Berkove, F. T. Farrell, D. Juan-Pineda, and K. Pearson. The Farrell-Jones isomorphism conjecture for finite covolume hyperbolic actions and the algebraic K-theory of Bianchi groups. *Trans. Amer. Math. Soc.*, 352(12):5689–5702, 2000.

[40] E. Berkove, D. Juan-Pineda, and K. Pearson. The lower algebraic K-theory of Fuchsian groups. *Comment. Math. Helv.*, 76(2):339–352, 2001.

[41] A. J. Berrick. *An approach to algebraic K-theory.* Pitman (Advanced Publishing Program), Boston, Mass., 1982.

[42] A. J. Berrick, I. Chatterji, and G. Mislin. From acyclic groups to the Bass conjecture for amenable groups. *Math. Ann.*, 329(4):597–621, 2004.

[43] B. Blackadar. *K-theory for operator algebras.* Springer-Verlag, New York, 1986.

[44] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, 111(3):465–539, 1993.

[45] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[46] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.

[47] W. Browder. *Surgery on simply-connected manifolds.* Springer-Verlag, New York, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65.

[48] M. Burger and A. Valette. Idempotents in complex group rings: theorems of Zalesskii and Bass revisited. *J. Lie Theory*, 8(2):219–228, 1998.

[49] D. Burghelea and R. Lashof. Stability of concordances and the suspension homomorphism. *Ann. of Math. (2)*, 105(3):449–472, 1977.

[50] S. Cappell, A. Ranicki, and J. Rosenberg, editors. *Surveys on surgery theory. Vol. 1.* Princeton University Press, Princeton, NJ, 2000. Papers dedicated to C. T. C. Wall.

[51] S. Cappell, A. Ranicki, and J. Rosenberg, editors. *Surveys on surgery theory. Vol. 2.* Princeton University Press, Princeton, NJ, 2001. Papers dedicated to C. T. C. Wall.

[52] S. E. Cappell. Mayer-Vietoris sequences in hermitian K-theory. In *Algebraic K-theory, III: Hermitian K-theory and geometric applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 478–512. Lecture Notes in Math., Vol. 343. Springer, Berlin, 1973.

[53] S. E. Cappell. Splitting obstructions for Hermitian forms and manifolds with $Z_2 \subset \pi_1$. *Bull. Amer. Math. Soc.*, 79:909–913, 1973.

[54] S. E. Cappell. Manifolds with fundamental group a generalized free product. I. *Bull. Amer. Math. Soc.*, 80:1193–1198, 1974.

[55] S. E. Cappell. Unitary nilpotent groups and Hermitian K-theory. I. *Bull. Amer. Math. Soc.*, 80:1117–1122, 1974.

[56] S. E. Cappell and J. L. Shaneson. On 4-dimensional s-cobordisms. *J. Differential Geom.*, 22(1):97–115, 1985.

[57] M. Cárdenas and E. K. Pedersen. On the Karoubi filtration of a category. *K-Theory*, 12(2):165–191, 1997.

[58] G. Carlsson. Deloopings in algebraic K-theory. In this Handbook.

[59] G. Carlsson. A survey of equivariant stable homotopy theory. *Topology*, 31(1):1–27, 1992.

[60] G. Carlsson. On the algebraic K-theory of infinite product categories. *K-theory*, 9:305–322, 1995.

[61] G. Carlsson and E. K. Pedersen. Controlled algebra and the Novikov conjectures for K- and L-theory. *Topology*, 34(3):731–758, 1995.

[62] G. Carlsson, E. K. Pedersen, and W. Vogell. Continuously controlled algebraic K-theory of spaces and the Novikov conjecture. *Math. Ann.*, 310(1):169–182, 1998.

[63] D. W. Carter. Localization in lower algebraic K-theory. *Comm. Algebra*, 8(7):603–622, 1980.

[64] D. W. Carter. Lower K-theory of finite groups. *Comm. Algebra*, 8(20):1927–1937, 1980.

[65] J. Chabert and S. Echterhoff. Permanence properties of the Baum-Connes conjecture. *Doc. Math.*, 6:127–183 (electronic), 2001.

[66] J. Chabert, S. Echterhoff, and R. Meyer. Deux remarques sur l'application de Baum-Connes. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(7):607–610, 2001.

[67] J. Chabert, S. Echterhoff, and R. Nest. The Connes-Kasparov conjecture for almost connected groups and for linear p-adic groups. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 251, 2003.

[68] S. Chang. A higher Hirzebruch-type invariant and the Borel conjecture. preprint, to appear in Geometriae Dedicata, 2002.

[69] S. Chang and S. Weinberger. On invariants of Hirzebruch and Cheeger-Gromov. *Geom. Topol.*, 7:311–319 (electronic), 2003.

[70] I. Chatterji and G. Mislin. Traces and reduced group c^*-algebra. preprint, 2004.

[71] I. Chatterji and K. Ruane. Some geometric groups with rapid decay. preprint, arXiv:math.GT/0310356, 2004.

[72] J. Cheeger and M. Gromov. Bounds on the von Neumann dimension of L^2-cohomology and the Gauss-Bonnet theorem for open manifolds. *J. Differential Geom.*, 21(1):1–34, 1985.

[73] J. Cheeger and M. Gromov. On the characteristic numbers of complete manifolds of bounded curvature and finite volume. In *Differential geometry and complex analysis*, pages 115–154. Springer-Verlag, Berlin, 1985.

[74] P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg, and A. Valette. *Groups with the Haagerup property*, volume 197 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001.

[75] G. Cliff and A. Weiss. Moody's induction theorem. *Illinois J. Math.*, 32(3):489–500, 1988.

[76] T. Cochran. Non-commutative knot theory. Preprint, 2001.

[77] T. D. Cochran, K. E. Orr, and P. Teichner. Knot concordance, Whitney towers and L^2-signatures. *Ann. of Math. (2)*, 157(2):433–519, 2003.

[78] M. M. Cohen. *A course in simple-homotopy theory.* Springer-Verlag, New York, 1973. Graduate Texts in Mathematics, Vol. 10.

[79] A. Connes. *Noncommutative geometry.* Academic Press Inc., San Diego, CA, 1994.

[80] A. Connes and H. Moscovici. Cyclic cohomology, the Novikov conjecture and hyperbolic groups. *Topology*, 29(3):345–388, 1990.

[81] F. X. Connolly and M. O. M. da Silva. The groups $N^rK_0(\mathbf{Z}\pi)$ are finitely generated $\mathbf{Z}[\mathbf{N}^r]$-modules if π is a finite group. *K-Theory*, 9(1):1–11, 1995.

[82] F. X. Connolly and J. F. Davis. The surgery obstruction groups of the infinite dihedral group. *Geom. Topol.*, 8:1043–1078 (electronic), 2004.

[83] F. X. Connolly and T. Koźniewski. Nil groups in K-theory and surgery theory. *Forum Math.*, 7(1):45–76, 1995.

[84] F. X. Connolly and S. Prassidis. On the exponent of the cokernel of the forget-control map on K_0-groups. *Fund. Math.*, 172(3):201–216, 2002.

[85] F. X. Connolly and S. Prassidis. On the exponent of the NK_0-groups of virtually infinite cyclic groups. *Canad. Math. Bull.*, 45(2):180–195, 2002.

[86] F. X. Connolly and A. Ranicki. On the calculation of UNIL_*. arXiv:math.AT/0304016v1, 2003.

[87] J. Cuntz. K-theoretic amenability for discrete groups. *J. Reine Angew. Math.*, 344:180–195, 1983.

[88] C. W. Curtis and I. Reiner. *Methods of representation theory. Vol. II.* John Wiley & Sons Inc., New York, 1987. With applications to finite groups and orders, A Wiley-Interscience Publication.

[89] C. W. Curtis and I. Reiner. *Methods of representation theory. Vol. I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1990. With applications to finite groups and orders, Reprint of the 1981 original, A Wiley-Interscience Publication.

[90] K. R. Davidson. *C^*-algebras by example*, volume 6 of *Fields Institute Monographs.* American Mathematical Society, Providence, RI, 1996.

[91] J. F. Davis. Manifold aspects of the Novikov conjecture. In *Surveys on surgery theory, Vol. 1*, pages 195–224. Princeton Univ. Press, Princeton, NJ, 2000.

[92] J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, 15(3):201–252, 1998.

[93] J. F. Davis and W. Lück. The p-chain spectral sequence. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 257, Münster. To appear in a special issue of *K-Theory* dedicated to H. Bass, 2002.

[94] P. de la Harpe. Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint. *C. R. Acad. Sci. Paris Sér. I Math.*, 307(14):771–774, 1988.

[95] R. K. Dennis, M. E. Keating, and M. R. Stein. Lower bounds for the order of $K_2(FG)$ and $Wh_2(G)$. *Math. Ann.*, 223(2):97–103, 1976.

[96] A. Dold. Relations between ordinary and extraordinary homology. Colloq. alg. topology, Aarhus 1962, 2-9, 1962.

[97] A. Dold. *Lectures on algebraic topology.* Springer-Verlag, Berlin, second edition, 1980.

[98] S. K. Donaldson. Irrationality and the h-cobordism conjecture. *J. Differential Geom.*, 26(1):141–168, 1987.

[99] A. N. Dranishnikov, G. Gong, V. Lafforgue, and G. Yu. Uniform embeddings into Hilbert space and a question of Gromov. *Canad. Math. Bull.*, 45(1):60–70, 2002.

[100] A. W. M. Dress. Contributions to the theory of induced representations. In *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 183–240. Lecture Notes in Math., Vol. 342. Springer, Berlin, 1973.

[101] A. W. M. Dress. Induction and structure theorems for orthogonal representations of finite groups. *Ann. of Math. (2)*, 102(2):291–325, 1975.

[102] E. Dror Farjoun. Homotopy and homology of diagrams of spaces. In *Algebraic topology (Seattle, Wash., 1985)*, pages 93–134. Springer-Verlag, Berlin, 1987.

[103] M. J. Dunwoody. $K_2(\mathbb{Z}\pi)$ for π a group of order two or three. *J. London Math. Soc. (2)*, 11(4):481–490, 1975.

[104] W. Dwyer, B. Williams, and M. Weiss. A parametrized index theorem for algebraic k-theory euler class. preprint, revised version, 2001.

[105] W. Dywer, T. Schick, and S. Stolz. Remarks on a conjecture of Gromov and Lawson. arXiv:math.GT/0208011, 2002.

[106] B. Eckmann. Cyclic homology of groups and the Bass conjecture. *Comment. Math. Helv.*, 61(2):193–202, 1986.

[107] B. Eckmann. Projective and Hilbert modules over group algebras, and finitely dominated spaces. *Comment. Math. Helv.*, 71(3):453–462, 1996.

[108] H. Emerson and R. Meyer. A descent principle for the Dirac dual Dirac method. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 332, arXiv:math.GT/0405388, 2004.

[109] D. S. Farley. A proper isometric action of Thompson's group V on Hilbert space. preprint, 2001.

[110] F. T. Farrell. The nonfiniteness of Nil. *Proc. Amer. Math. Soc.*, 65(2):215–216, 1977.

[111] F. T. Farrell. The exponent of UNil. *Topology*, 18(4):305–312, 1979.

[112] F. T. Farrell and W. C. Hsiang. On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, Proc. Sympos. Pure Math., XXXII, pages 325–337. Amer. Math. Soc., Providence, R.I., 1978.

[113] F. T. Farrell and W. C. Hsiang. The topological-Euclidean space form problem. *Invent. Math.*, 45(2):181–192, 1978.

[114] F. T. Farrell and W. C. Hsiang. On Novikov's conjecture for nonpositively curved manifolds. I. *Ann. of Math. (2)*, 113(1):199–209, 1981.

[115] F. T. Farrell and W. C. Hsiang. The Whitehead group of poly-(finite or cyclic) groups. *J. London Math. Soc. (2)*, 24(2):308–324, 1981.

[116] F. T. Farrell and L. E. Jones. h-cobordisms with foliated control. *Bull. Amer. Math. Soc. (N.S.)*, 15(1):69–72, 1986.

[117] F. T. Farrell and L. E. Jones. K-theory and dynamics. I. *Ann. of Math. (2)*, 124(3):531–569, 1986.

[118] F. T. Farrell and L. E. Jones. Foliated control with hyperbolic leaves. *K-Theory*, 1(4):337–359, 1987.

[119] F. T. Farrell and L. E. Jones. Foliated control theory. I, II. *K-Theory*, 2(3):357–430, 1988.

[120] F. T. Farrell and L. E. Jones. The surgery L-groups of poly-(finite or cyclic) groups. *Invent. Math.*, 91(3):559–586, 1988.

[121] F. T. Farrell and L. E. Jones. Negatively curved manifolds with exotic smooth structures. *J. Amer. Math. Soc.*, 2(4):899–908, 1989.

[122] F. T. Farrell and L. E. Jones. A topological analogue of Mostow's rigidity theorem. *J. Amer. Math. Soc.*, 2(2):257–370, 1989.

[123] F. T. Farrell and L. E. Jones. Foliated control without radius of injectivity restrictions. *Topology*, 30(2):117–142, 1991.

[124] F. T. Farrell and L. E. Jones. Rigidity in geometry and topology. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 653–663, Tokyo, 1991. Math. Soc. Japan.

[125] F. T. Farrell and L. E. Jones. Stable pseudoisotopy spaces of compact non-positively curved manifolds. *J. Differential Geom.*, 34(3):769–834, 1991.

[126] F. T. Farrell and L. E. Jones. Isomorphism conjectures in algebraic K-theory. *J. Amer. Math. Soc.*, 6(2):249–297, 1993.

[127] F. T. Farrell and L. E. Jones. Topological rigidity for compact non-positively curved manifolds. In *Differential geometry: Riemannian geometry (Los Angeles, CA, 1990)*, pages 229–274. Amer. Math. Soc., Providence, RI, 1993.

[128] F. T. Farrell and L. E. Jones. The lower algebraic K-theory of virtually infinite cyclic groups. *K-Theory*, 9(1):13–30, 1995.

[129] F. T. Farrell and L. E. Jones. Collapsing foliated Riemannian manifolds. *Asian J. Math.*, 2(3):443–494, 1998.

[130] F. T. Farrell and L. E. Jones. Rigidity for aspherical manifolds with $\pi_1 \subset GL_m(\mathbb{R})$. *Asian J. Math.*, 2(2):215–262, 1998.

[131] F. T. Farrell and L. E. Jones. Local-collapsing theory. to appear in Pacific Journal, 2003.

[132] F. T. Farrell and P. A. Linnell. K-theory of solvable groups. *Proc. London Math. Soc. (3)*, 87(2):309–336, 2003.

[133] F. T. Farrell and P. A. Linnell. Whitehead groups and the Bass conjecture. *Math. Ann.*, 326(4):723–757, 2003.

[134] F. T. Farrell and S. K. Roushon. The Whitehead groups of braid groups vanish. *Internat. Math. Res. Notices*, (10):515–526, 2000.

[135] T. Farrell. The Borel conjecture. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 225–298. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school "High dimensional manifold theory" in Trieste May/June 2001, Number 1. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.

[136] T. Farrell, L. Göttsche, and W. Lück, editors. *High dimensional manifold theory*. Number 9 in ICTP Lecture Notes. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school "High dimensional manifold theory" in Trieste May/June 2001, Number 1. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.

[137] T. Farrell, L. Göttsche, and W. Lück, editors. *High dimensional manifold theory*. Number 9 in ICTP Lecture Notes.

Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school "High dimensional manifold theory" in Trieste May/June 2001, Number 2. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.

[138] T. Farrell, L. Jones, and W. Lück. A caveat on the isomorphism conjecture in L-theory. *Forum Math.*, 14(3):413–418, 2002.

[139] S. Ferry. The homeomorphism group of a compact Hilbert cube manifold is an anr. *Ann. Math. (2)*, 106(1):101–119, 1977.

[140] S. Ferry. A simple-homotopy approach to the finiteness obstruction. In *Shape theory and geometric topology (Dubrovnik, 1981)*, pages 73–81. Springer-Verlag, Berlin, 1981.

[141] S. Ferry and A. Ranicki. A survey of Wall's finiteness obstruction. In *Surveys on surgery theory, Vol. 2*, volume 149 of *Ann. of Math. Stud.*, pages 63–79. Princeton Univ. Press, Princeton, NJ, 2001.

[142] S. C. Ferry, A. A. Ranicki, and J. Rosenberg. A history and survey of the Novikov conjecture. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, pages 7–66. Cambridge Univ. Press, Cambridge, 1995.

[143] S. C. Ferry, A. A. Ranicki, and J. Rosenberg, editors. *Novikov conjectures, index theorems and rigidity. Vol. 1.* Cambridge University Press, Cambridge, 1995. Including papers from the conference held at the Mathematisches Forschungsinstitut Oberwolfach, Oberwolfach, September 6–10, 1993.

[144] S. C. Ferry and S. Weinberger. Curvature, tangentiality, and controlled topology. *Invent. Math.*, 105(2):401–414, 1991.

[145] Z. Fiedorowicz. The Quillen-Grothendieck construction and extension of pairings. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), I*, pages 163–169. Springer, Berlin, 1978.

[146] M. H. Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.

[147] M. H. Freedman. The disk theorem for four-dimensional manifolds. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 647–663, Warsaw, 1984. PWN.

[148] S. M. Gersten. On the spectrum of algebraic K-theory. *Bull. Amer. Math. Soc.*, 78:216–219, 1972.

[149] E. Ghys. Groupes aléatoires. preprint, Lyon, to appear in Séminaire Bourbaki, 55ème année, 2002-2003, no. 916, 2003.

[150] D. Grayson. Higher algebraic K-theory. II (after Daniel Quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 217–240. Lecture Notes in Math., Vol. 551. Springer-Verlag, Berlin, 1976.

[151] M. Gromov. Large Riemannian manifolds. In *Curvature and topology of Riemannian manifolds (Katata, 1985)*, pages 108–121. Springer-Verlag, Berlin, 1986.

[152] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[153] M. Gromov. Spaces and questions. *Geom. Funct. Anal.*, (Special Volume, Part I):118–161, 2000. GAFA 2000 (Tel Aviv, 1999).

[154] M. Gromov. Random walk in random groups. *Geom. Funct. Anal.*, 13(1):73–146, 2003.

[155] E. Guentner, N. Higson, , and S. Weinberger. The Novikov conjecture for linear groups. in preparation, 2003.

[156] I. Hambleton. Algebraic K- and L-theory and applications to the topology of manifolds. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 299–369. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school "High dimensional manifold theory" in Trieste May/June 2001, Number 1. http://www.ictp.trieste.it/˜pub_off/lectures/vol9.html.

[157] I. Hambleton and E. K. Pedersen. Identifying assembly maps in K- and L-theory. *Math. Ann.*, 328(1-2):27–57, 2004.

[158] I. Hambleton and L. R. Taylor. A guide to the calculation of the surgery obstruction groups for finite groups. In *Surveys on surgery theory, Vol. 1*, pages 225–274. Princeton Univ. Press, Princeton, NJ, 2000.

[159] A. E. Hatcher. Concordance spaces, higher simple-homotopy theory, and applications. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, pages 3–21. Amer. Math. Soc., Providence, R.I., 1978.

[160] J.-C. Hausmann. On the homotopy of nonnilpotent spaces. *Math. Z.*, 178(1):115–123, 1981.

[161] J. Hempel. 3-*Manifolds.* Princeton University Press, Princeton, N. J., 1976. Ann. of Math. Studies, No. 86.

[162] L. Hesselholt and I. Madsen. On the K-theory of nilpotent endomorphisms. In *Homotopy methods in algebraic topology (Boulder, CO, 1999)*, volume 271 of *Contemp. Math.*, pages 127–140. Amer. Math. Soc., Providence, RI, 2001.

[163] N. Higson. The Baum-Connes conjecture. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, pages 637–646 (electronic), 1998.

[164] N. Higson. Bivariant K-theory and the Novikov conjecture. *Geom. Funct. Anal.*, 10(3):563–581, 2000.

[165] N. Higson and G. Kasparov. E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.*, 144(1):23–74, 2001.

[166] N. Higson, V. Lafforgue, and G. Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002.

[167] N. Higson, E. K. Pedersen, and J. Roe. C^*-algebras and controlled topology. *K-Theory*, 11(3):209–239, 1997.

[168] N. Higson and J. Roe. On the coarse Baum-Connes conjecture. In *Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993)*, pages 227–254. Cambridge Univ. Press, Cambridge, 1995.

[169] N. Higson and J. Roe. Amenable group actions and the Novikov conjecture. *J. Reine Angew. Math.*, 519:143–153, 2000.

[170] N. Higson and J. Roe. *Analytic K-homology*. Oxford University Press, Oxford, 2000. Oxford Science Publications.

[171] B. Hu. Retractions of closed manifolds with nonpositive curvature. In *Geometric group theory (Columbus, OH, 1992)*, volume 3 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 135–147. de Gruyter, Berlin, 1995.

[172] B. Z. Hu. Whitehead groups of finite polyhedra with nonpositive curvature. *J. Differential Geom.*, 38(3):501–517, 1993.

[173] K. Igusa. The stability theorem for smooth pseudoisotopies. *K-Theory*, 2(1-2):vi+355, 1988.

[174] H. Inassaridze. *Algebraic K-theory*, volume 311 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1995.

[175] M. Joachim. K-homology of C^*-categories and symmetric spectra representing K-homology. *Math. Ann.*, 327(4):641–670, 2003.

[176] L. Jones. A paper for F.T. Farrell on his 60-th birthday. preprint, to appear in the Proceedings of the school/conference on "High-dimensional Manifold Topology" in Trieste, May/June 2001.

[177] L. Jones. Foliated control theory and its applications. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 405– 460. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school "High dimensional manifold theory" in Trieste May/June 2001, Number 2. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.

[178] L. E. Jones. A paper for F. T. Farrell on his 60th birthday. In *High-dimensional manifold topology*, pages 200–260. World Sci. Publishing, River Edge, NJ, 2003.

[179] D. Juan-Pineda and S. Prassidis. A controlled approach to the isomorphism conjecture. preprint, arXiv:math.KT/0404513, 2004.

[180] P. Julg. Remarks on the Baum-Connes conjecture and Kazhdan's property T. In *Operator algebras and their applications (Waterloo, ON, 1994/1995)*, volume 13 of *Fields Inst. Commun.*, pages 145–153. Amer. Math. Soc., Providence, RI, 1997.

[181] P. Julg. Travaux de N. Higson et G. Kasparov sur la conjecture de Baum-Connes. *Astérisque*, (252):Exp. No. 841, 4, 151–183, 1998. Séminaire Bourbaki. Vol. 1997/98.

[182] P. Julg. La conjecture de Baum-Connes à coefficients pour le groupe $\mathrm{Sp}(n,1)$. *C. R. Math. Acad. Sci. Paris*, 334(7):533–538, 2002.

[183] P. Julg and G. Kasparov. Operator K-theory for the group $SU(n,1)$. *J. Reine Angew. Math.*, 463:99–152, 1995.

[184] D. M. Kan and W. P. Thurston. Every connected space has the homology of a $K(\pi,1)$. *Topology*, 15(3):253–258, 1976.

[185] M. Karoubi. Bott periodicity, generalizations and variants. In this Handbook.

[186] G. Kasparov. Novikov's conjecture on higher signatures: the operator K-theory approach. In *Representation theory of groups and algebras*, pages 79–99. Amer. Math. Soc., Providence, RI, 1993.

[187] G. Kasparov and G. Skandalis. Groupes "boliques" et conjecture de Novikov. *C. R. Acad. Sci. Paris Sér. I Math.*, 319(8):815–820, 1994.

[188] G. G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Invent. Math.*, 91(1):147–201, 1988.

[189] G. G. Kasparov. K-theory, group C^*-algebras, and higher signatures (conspectus). In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, pages 101–146. Cambridge Univ. Press, Cambridge, 1995.

[190] G. G. Kasparov and G. Skandalis. Groups acting on buildings, operator K-theory, and Novikov's conjecture. *K-Theory*, 4(4):303–337, 1991.

[191] M. A. Kervaire. Le théorème de Barden-Mazur-Stallings. *Comment. Math. Helv.*, 40:31–42, 1965.

[192] N. Keswani. Homotopy invariance of relative eta-invariants and C^*-algebra K-theory. *Electron. Res. Announc. Amer. Math. Soc.*, 4:18–26 (electronic), 1998.

[193] N. Keswani. Von Neumann eta-invariants and C^*-algebra K-theory. *J. London Math. Soc. (2)*, 62(3):771–783, 2000.

[194] R. C. Kirby and L. C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations.* Princeton University Press, Princeton, N.J., 1977. With notes by J. Milnor and M. F. Atiyah, Annals of Mathematics Studies, No. 88.

[195] M. Kolster. K-theory and arithmetics. Lectures - ICTP Trieste 2002.

[196] M. Kreck. Surgery and duality. *Ann. of Math. (2)*, 149(3):707–754, 1999.

[197] M. Kreck and W. Lück. *The Novikov Conjecture: Geometry and Algebra*, volume 33 of *Oberwolfach Seminars.* Birkhäuser, 2004.

[198] P. H. Kropholler and G. Mislin. Groups acting on finite-dimensional spaces with finite stabilizers. *Comment. Math. Helv.*, 73(1):122–136, 1998.

[199] A. O. Kuku. K_n, SK_n of integral group-rings and orders. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, pages 333–338. Amer. Math. Soc., Providence, RI, 1986.

[200] A. O. Kuku and G. Tang. Higher K-theory of group-rings of virtually infinite cyclic groups. *Math. Ann.*, 325(4):711–726, 2003.

[201] V. Lafforgue. Une démonstration de la conjecture de Baum-Connes pour les groupes réductifs sur un corps p-adique et pour certains groupes discrets possédant la propriété (T). *C. R. Acad. Sci. Paris Sér. I Math.*, 327(5):439–444, 1998.

[202] V. Lafforgue. A proof of property (RD) for cocompact lattices of $\mathrm{SL}(3, \mathbf{R})$ and $\mathrm{SL}(3, \mathbf{C})$. *J. Lie Theory*, 10(2):255–267, 2000.

[203] V. Lafforgue. Banach KK-theory and the Baum-Connes conjecture. In *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, volume 202 of *Progr. Math.*, pages 31–46. Birkhäuser, Basel, 2001.

[204] V. Lafforgue. K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes. *Invent. Math.*, 149(1):1–95, 2002.

[205] T. Y. Lam. *A first course in noncommutative rings.* Springer-Verlag, New York, 1991.

[206] E. C. Lance. *Hilbert C^*-modules.* Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists.

[207] R. Lee and R. H. Szczarba. The group $K_3(Z)$ is cyclic of order forty-eight. *Ann. of Math. (2)*, 104(1):31–60, 1976.

[208] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory.* Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[209] P. A. Linnell. Decomposition of augmentation ideals and relation modules. *Proc. London Math. Soc. (3)*, 47(1):83–127, 1983.

[210] P. A. Linnell. Division rings and group von Neumann algebras. *Forum Math.*, 5(6):561–576, 1993.

[211] E. Lluis-Puebla, J.-L. Loday, H. Gillet, C. Soulé, and V. Snaith. *Higher algebraic K-theory: an overview*, volume 1491 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992.

[212] J.-L. Loday. K-théorie algébrique et représentations de groupes. *Ann. Sci. École Norm. Sup. (4)*, 9(3):309–377, 1976.

[213] J. Lott. The zero-in-the-spectrum question. *Enseign. Math. (2)*, 42(3-4):341–376, 1996.

[214] W. Lück. The geometric finiteness obstruction. *Proc. London Math. Soc. (3)*, 54(2):367–384, 1987.

[215] W. Lück. *Transformation groups and algebraic K-theory.* Springer-Verlag, Berlin, 1989. Mathematica Gottingensis.

[216] W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. II. Applications to Grothendieck groups, L^2-Euler characteristics and Burnside groups. *J. Reine Angew. Math.*, 496:213–236, 1998.

[217] W. Lück. The type of the classifying space for a family of subgroups. *J. Pure Appl. Algebra*, 149(2):177–203, 2000.

[218] W. Lück. A basic introduction to surgery theory. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 1–224.

Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school "High dimensional manifold theory" in Trieste May/June 2001, Number 1. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.

[219] W. Lück. Chern characters for proper equivariant homology theories and applications to K- and L-theory. *J. Reine Angew. Math.*, 543:193–234, 2002.

[220] W. Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.

[221] W. Lück. The relation between the Baum-Connes conjecture and the trace conjecture. *Invent. Math.*, 149(1):123–152, 2002.

[222] W. Lück. K-and L-theory of the semi-direct product discrete three-dimensional Heisenberg group by $\mathbb{Z}/4$. in preparation, 2003.

[223] W. Lück. $L2$-invariants from the algebraic point of view. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 295, Münster, arXiv:math.GT/0310489 v1, 2003.

[224] W. Lück. Survey on classifying spaces for families of subgroups. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 308, Münster, arXiv:math.GT/0312378 v1, 2004.

[225] W. Lück and D. Meintrup. On the universal space for group actions with compact isotropy. In *Geometry and topology: Aarhus (1998)*, pages 293–305. Amer. Math. Soc., Providence, RI, 2000.

[226] W. Lück and H. Reich. The Baum-Connes and the Farrell-Jones Conjectures in K- and L-Theory. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 324, Münster, arXiv:math.GT/0402405, accepted for the handbook of K-theory, Springer, 2003.

[227] W. Lück, H. Reich, J. Rognes, and M. Varisco. *In preparation*, 2003.

[228] W. Lück and M. Rørdam. Algebraic K-theory of von Neumann algebras. *K-Theory*, 7(6):517–536, 1993.

[229] W. Lück and T. Schick. Approximating L^2-signatures by their finite-dimensional analogues. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 190, Münster, to appear in Forum, 2001.

[230] W. Lück and T. Schick. Various L^2-signatures and a topological L^2-signature theorem. In *High-dimensional manifold topology*, pages 362–399. World Sci. Publishing, River Edge, NJ, 2003.

[231] W. Lück and R. Stamm. Computations of K- and L-theory of cocompact planar groups. *K-Theory*, 21(3):249–292, 2000.

[232] I. Madsen and M. Rothenberg. Equivariant pseudo-isotopies and K_{-1}. In *Transformation groups (Osaka, 1987)*, volume 1375 of *Lecture Notes in Math.*, pages 216–230. Springer, Berlin, 1989.

[233] V. Mathai. Spectral flow, eta invariants, and von Neumann algebras. *J. Funct. Anal.*, 109(2):442–456, 1992.

[234] M. Matthey and G. Mislin. Equivariant K-homology and restriction to finite cyclic subgroups. Preprint, 2003.

[235] D. Meintrup. *On the Type of the Universal Space for a Family of Subgroups.* PhD thesis, Westfälische Wilhelms-Universität Münster, 2000.

[236] D. Meintrup and T. Schick. A model for the universal space for proper actions of a hyperbolic group. *New York J. Math.*, 8:1–7 (electronic), 2002.

[237] R. Meyer and R. Nest. The Baum-Connes conjecture via localization of categories. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 306, arXiv:math.GT/0312292, to appear in Topology, 2003.

[238] J. Milnor. On manifolds homeomorphic to the 7-sphere. *Ann. of Math. (2)*, 64:399–405, 1956.

[239] J. Milnor. *Lectures on the h-cobordism theorem.* Princeton University Press, Princeton, N.J., 1965.

[240] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.*, 72:358–426, 1966.

[241] J. Milnor. *Introduction to algebraic K-theory.* Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 72.

[242] J. Milnor. Towards the Poincaré conjecture and the classification of 3-manifolds. *Notices Amer. Math. Soc.*, 50(10):1226–1233, 2003.

[243] I. Mineyev and G. Yu. The Baum-Connes conjecture for hyperbolic groups. *Invent. Math.*, 149(1):97–122, 2002.

[244] A. S. Miščenko and A. T. Fomenko. The index of elliptic operators over C^*-algebras. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(4):831–859, 967, 1979. English translation in *Math. USSR-Izv.* 15 (1980), no. 1, 87–112.

[245] G. Mislin. Wall's finiteness obstruction. In *Handbook of algebraic topology*, pages 1259–1291. North-Holland, Amsterdam, 1995.

[246] G. Mislin and A. Valette. *Proper group actions and the Baum-Connes conjecture.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003.

[247] S. A. Mitchell. On the Lichtenbaum-Quillen conjectures from a stable homotopy-theoretic viewpoint. In *Algebraic topology and its applications*, volume 27 of *Math. Sci. Res. Inst. Publ.*, pages 163–240. Springer, New York, 1994.

[248] J. A. Moody. Brauer induction for G_0 of certain infinite groups. *J. Algebra*, 122(1):1–14, 1989.

[249] G. J. Murphy. *C^*-algebras and operator theory.* Academic Press Inc., Boston, MA, 1990.

[250] J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of number fields*. Springer-Verlag, Berlin, 2000.

[251] A. J. Nicas. On the higher Whitehead groups of a Bieberbach group. *Trans. Amer. Math. Soc.*, 287(2):853–859, 1985.

[252] R. Oliver. Projective class groups of integral group rings: a survey. In *Orders and their applications (Oberwolfach, 1984)*, volume 1142 of *Lecture Notes in Math.*, pages 211–232. Springer, Berlin, 1985.

[253] R. Oliver. *Whitehead groups of finite groups*. Cambridge University Press, Cambridge, 1988.

[254] H. Oyono-Oyono. Baum-Connes Conjecture and extensions. *J. Reine Angew. Math.*, 532:133–149, 2001.

[255] H. Oyono-Oyono. Baum-Connes conjecture and group actions on trees. *K-Theory*, 24(2):115–134, 2001.

[256] D. S. Passman. *Infinite crossed products*. Academic Press Inc., Boston, MA, 1989.

[257] K. Pearson. Algebraic K-theory of two-dimensional crystallographic groups. *K-Theory*, 14(3):265–280, 1998.

[258] E. Pedersen and C. Weibel. A non-connective delooping of algebraic K-theory. In *Algebraic and Geometric Topology; proc. conf. Rutgers Uni., New Brunswick 1983*, volume 1126 of *Lecture notes in mathematics*, pages 166–181. Springer, 1985.

[259] E. K. Pedersen. On the K_{-i}-functors. *J. Algebra*, 90(2):461–475, 1984.

[260] E. K. Pedersen. On the bounded and thin h-cobordism theorem parameterized by $\mathbf{R}^k$. In *Transformation groups, Poznań 1985*, volume 1217 of *Lecture Notes in Math.*, pages 306–320. Springer, Berlin, 1986.

[261] E. K. Pedersen. Controlled methods in equivariant topology, a survey. In *Current trends in transformation groups*, volume 7 of *K-Monogr. Math.*, pages 231–245. Kluwer Acad. Publ., Dordrecht, 2002.

[262] E. K. Pedersen and C. A. Weibel. K-theory homology of spaces. In *Algebraic topology (Arcata, CA, 1986)*, pages 346–361. Springer-Verlag, Berlin, 1989.

[263] G. K. Pedersen. *C^*-algebras and their automorphism groups*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1979.

[264] M. Pimsner and D. Voiculescu. K-groups of reduced crossed products by free groups. *J. Operator Theory*, 8(1):131–156, 1982.

[265] M. V. Pimsner. KK-groups of crossed products by groups acting on trees. *Invent. Math.*, 86(3):603–634, 1986.

[266] R. T. Powers. Simplicity of the C^*-algebra associated with the free group on two generators. *Duke Math. J.*, 42:151–156, 1975.

[267] M. Puschnigg. The Kadison-Kaplansky conjecture for word-hyperbolic groups. *Invent. Math.*, 149(1):153–194, 2002.

[268] D. Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972.

[269] D. Quillen. Finite generation of the groups K_i of rings of algebraic integers. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 179–198. Lecture Notes in Math., Vol. 341. Springer-Verlag, Berlin, 1973.

[270] D. Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer-Verlag, Berlin, 1973.

[271] F. Quinn. A geometric formulation of surgery. In *Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969)*, pages 500–511. Markham, Chicago, Ill., 1970.

[272] F. Quinn. Ends of maps. I. *Ann. of Math. (2)*, 110(2):275–331, 1979.

[273] F. Quinn. Ends of maps. II. *Invent. Math.*, 68(3):353–424, 1982.

[274] A. Ranicki. Foundations of algebraic surgery. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 491–514. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school "High dimensional manifold theory" in Trieste May/June 2001, Number 2. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.

[275] A. A. Ranicki. Algebraic L-theory. II. Laurent extensions. *Proc. London Math. Soc. (3)*, 27:126–158, 1973.

[276] A. A. Ranicki. Algebraic L-theory. III. Twisted Laurent extensions. In *Algebraic K-theory, III: Hermitian K-theory and geometric application (Proc. Conf. Seattle Res. Center, Battelle Memorial Inst., 1972)*, pages 412–463. Lecture Notes in Mathematics, Vol. 343. Springer, Berlin, 1973.

[277] A. A. Ranicki. *Exact sequences in the algebraic theory of surgery.* Princeton University Press, Princeton, N.J., 1981.

[278] A. A. Ranicki. The algebraic theory of finiteness obstruction. *Math. Scand.*, 57(1):105–126, 1985.

[279] A. A. Ranicki. *Algebraic L-theory and topological manifolds.* Cambridge University Press, Cambridge, 1992.

[280] A. A. Ranicki. *Lower K- and L-theory.* Cambridge University Press, Cambridge, 1992.

[281] A. A. Ranicki. On the Novikov conjecture. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, pages 272–337. Cambridge Univ. Press, Cambridge, 1995.

[282] H. Reich. $L2$-Betti numbers, isomorphism conjectures and noncommutative localization. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 250, Münster, to appear in the Proceedings of the Workshop on Noncommutative Localization - ICMS Edinburgh, April 2002, 2003.

[283] I. Reiner. Class groups and Picard groups of group rings and orders. pages iv+44, Providence, R. I., 1976. American Mathematical Society.

Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics, No. 26.
[284] J. Roe. *Index theory, coarse geometry, and topology of manifolds.* Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996.
[285] J. Rognes. $K_4(\mathbb{Z})$ is the trivial group. *Topology*, 39(2):267–281, 2000.
[286] J. Rognes and C. Weibel. Two-primary algebraic K-theory of rings of integers in number fields. *J. Amer. Math. Soc.*, 13(1):1–54, 2000. Appendix A by Manfred Kolster.
[287] J. Rosenberg. K-theory and geometric topology. In this Handbook.
[288] J. Rosenberg. C^*-algebras, positive scalar curvature and the Novikov conjecture. III. *Topology*, 25:319–336, 1986.
[289] J. Rosenberg. *Algebraic K-theory and its applications.* Springer-Verlag, New York, 1994.
[290] J. Rosenberg. Analytic Novikov for topologists. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, pages 338–372. Cambridge Univ. Press, Cambridge, 1995.
[291] J. Rosenberg and S. Stolz. A "stable" version of the Gromov-Lawson conjecture. In *The Čech centennial (Boston, MA, 1993)*, pages 405–418. Amer. Math. Soc., Providence, RI, 1995.
[292] D. Rosenthal. Continuous control and splitting the l-theoretic assembly map. preprint, arXiv:math.AT/03120446, to appear in Forum, 2003.
[293] D. Rosenthal. Splitting with continuous control in algebraic K-theory. *K-Theory*, 32(2):139–166, 2004.
[294] D. Rosenthal and D. Schütz. On the algebraic k- and l-theory of word hyperbolic groups. preprint, Preprintreihe des SFB 478 – Geometrische Strukturen in der Mathematik des Mathematischen Instituts der Westfälischen Wilhelms-Universität Münster, Heft 343, 2004.
[295] C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology.* Springer-Verlag, Berlin, 1982. Reprint.
[296] S. K. Roushon. The Farrell-Jones isomorphism conjecture for 3-manifold groups. preprint, arXiv:math.KT/0405211, 2004.
[297] S. K. Roushon. The isomorphism conjecture for 3-manifold groups and k-theory of poly-virtually-surface-groups. preprint, arXiv:math.KT/0408243, 2004.
[298] L. H. Rowen. *Ring theory. Vol. II.* Academic Press Inc., Boston, MA, 1988.
[299] R. Roy. The trace conjecture—a counterexample. *K-Theory*, 17(3):209–213, 1999.
[300] R. Roy. A counterexample to questions on the integrality property of virtual signature. *Topology Appl.*, 100(2-3):177–185, 2000.
[301] J. Sauer. K-theory for proper smooth actions of totally disconnected groups. Ph.D. thesis, 2002.

[302] J. A. Schafer. The Bass conjecture and group von Neumann algebras. *K-Theory*, 19(3):211–217, 2000.

[303] T. Schick. Finite group extensions and the Baum-Connes conjecture. in preparation.

[304] T. Schick. A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. *Topology*, 37(6):1165–1168, 1998.

[305] T. Schick. Operator algebras and topology. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 571–660. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school "High dimensional manifold theory" in Trieste May/June 2001, Number 2. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.

[306] P. Schneider. Über gewisse Galoiscohomologiegruppen. *Math. Z.*, 168(2):181–205, 1979.

[307] H. Schröder. *K-theory for real C^*-algebras and applications.* Longman Scientific & Technical, Harlow, 1993.

[308] J.-P. Serre. Cohomologie des groupes discrets. In *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*, pages 77–169. Ann. of Math. Studies, No. 70. Princeton Univ. Press, Princeton, N.J., 1971.

[309] J.-P. Serre. *Linear representations of finite groups.* Springer-Verlag, New York, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[310] J.-P. Serre. *Galois cohomology.* Springer-Verlag, Berlin, 1997. Translated from the French by Patrick Ion and revised by the author.

[311] J. L. Shaneson. Wall's surgery obstruction groups for $G \times Z$. *Ann. of Math. (2)*, 90:296–334, 1969.

[312] W.-X. Shi. Deforming the metric on complete Riemannian manifolds. *J. Differential Geom.*, 30(1):223–301, 1989.

[313] J. R. Silvester. *Introduction to algebraic K-theory.* Chapman & Hall, London, 1981. Chapman and Hall Mathematics Series.

[314] G. Skandalis. Une notion de nucléarité en K-théorie (d'après J. Cuntz). *K-Theory*, 1(6):549–573, 1988.

[315] G. Skandalis. Progrès récents sur la conjecture de Baum-Connes. Contribution de Vincent Lafforgue. *Astérisque*, (276):105–135, 2002. Séminaire Bourbaki, Vol. 1999/2000.

[316] G. Skandalis, J. L. Tu, and G. Yu. The coarse Baum-Connes conjecture and groupoids. *Topology*, 41(4):807–834, 2002.

[317] C. Soulé. On higher p-adic regulators. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 372–401. Springer, Berlin, 1981.

[318] C. Soulé. Éléments cyclotomiques en K-théorie. *Astérisque*, (147-148):225–257, 344, 1987. Journées arithmétiques de Besançon (Besançon, 1985).

[319] V. Srinivas. *Algebraic K-theory.* Birkhäuser Boston Inc., Boston, MA, 1991.

[320] C. Stark. Topological rigidity theorems. In R. Daverman and R. Sher, editors, *Handbook of Geometric Topology*, chapter 20. Elsevier, 2002.

[321] C. W. Stark. Surgery theory and infinite fundamental groups. In *Surveys on surgery theory, Vol. 1*, volume 145 of *Ann. of Math. Stud.*, pages 275–305. Princeton Univ. Press, Princeton, NJ, 2000.

[322] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.

[323] M. R. Stein. Excision and K_2 of group rings. *J. Pure Appl. Algebra*, 18(2):213–224, 1980.

[324] S. Stolz. Manifolds of positive scalar curvature. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 661–708. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school "High dimensional manifold theory" in Trieste May/June 2001, Number 2. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.

[325] R. G. Swan. Induced representations and projective modules. *Ann. of Math. (2)*, 71:552–578, 1960.

[326] R. G. Swan. *K-theory of finite groups and orders.* Springer-Verlag, Berlin, 1970. Lecture Notes in Mathematics, Vol. 149.

[327] R. G. Swan. Higher algebraic K-theory. In *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, volume 58 of *Proc. Sympos. Pure Math.*, pages 247–293. Amer. Math. Soc., Providence, RI, 1995.

[328] R. M. Switzer. *Algebraic topology—homotopy and homology.* Springer-Verlag, New York, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212.

[329] R. W. Thomason. Beware the phony multiplication on Quillen's $\mathcal{A}^{-1}\mathcal{A}$. *Proc. Amer. Math. Soc.*, 80(4):569–573, 1980.

[330] T. tom Dieck. Orbittypen und äquivariante Homologie. I. *Arch. Math. (Basel)*, 23:307–317, 1972.

[331] T. tom Dieck. *Transformation groups and representation theory.* Springer-Verlag, Berlin, 1979.

[332] T. tom Dieck. *Transformation groups.* Walter de Gruyter & Co., Berlin, 1987.

[333] J.-L. Tu. The Baum-Connes conjecture and discrete group actions on trees. *K-Theory*, 17(4):303–318, 1999.

[334] S. Upadhyay. Controlled algebraic K-theory of integral group ring of $\mathrm{SL}(3, \mathbf{Z})$. *$K$-Theory*, 10(4):413–418, 1996.

[335] A. Valette. *Introduction to the Baum-Connes conjecture.* Birkhäuser Verlag, Basel, 2002. From notes taken by Indira Chatterji, With an appendix by Guido Mislin.

[336] K. Varadarajan. *The finiteness obstruction of C. T. C. Wall*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1989. A Wiley-Interscience Publication.

[337] W. Vogell. Algebraic K-theory of spaces, with bounded control. *Acta Math.*, 165(3-4):161–187, 1990.

[338] W. Vogell. Boundedly controlled algebraic K-theory of spaces and its linear counterparts. *J. Pure Appl. Algebra*, 76(2):193–224, 1991.

[339] W. Vogell. Continuously controlled A-theory. *K-Theory*, 9(6):567–576, 1995.

[340] J. B. Wagoner. Delooping classifying spaces in algebraic K-theory. *Topology*, 11:349–370, 1972.

[341] F. Waldhausen. Algebraic K-theory of generalized free products. I, II. *Ann. of Math. (2)*, 108(1):135–204, 1978.

[342] F. Waldhausen. Algebraic K-theory of topological spaces. I. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, pages 35–60. Amer. Math. Soc., Providence, R.I., 1978.

[343] F. Waldhausen. Algebraic K-theory of topological spaces. I. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, Proc. Sympos. Pure Math., XXXII, pages 35–60. Amer. Math. Soc., Providence, R.I., 1978.

[344] F. Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, pages 318–419. Springer-Verlag, Berlin, 1985.

[345] F. Waldhausen. Algebraic K-theory of spaces, concordance, and stable homotopy theory. In *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*, pages 392–417. Princeton Univ. Press, Princeton, NJ, 1987.

[346] C. T. C. Wall. Finiteness conditions for CW-complexes. *Ann. of Math. (2)*, 81:56–69, 1965.

[347] C. T. C. Wall. Finiteness conditions for CW complexes. II. *Proc. Roy. Soc. Ser. A*, 295:129–139, 1966.

[348] C. T. C. Wall. Poincaré complexes. I. *Ann. of Math. (2)*, 86:213–245, 1967.

[349] C. T. C. Wall. *Surgery on compact manifolds*. American Mathematical Society, Providence, RI, second edition, 1999. Edited and with a foreword by A. A. Ranicki.

[350] N. E. Wegge-Olsen. *K-theory and C^*-algebras*. The Clarendon Press Oxford University Press, New York, 1993. A friendly approach.

[351] C. Weibel. Algebraic K-theory of rings of integers in local and global fields. In this Handbook.

[352] C. Weibel. An introduction to algebraic K-theory. In preparation, see http://math.rutgers.edu/~weibel/Kbook.html.

[353] C. A. Weibel. K-theory and analytic isomorphisms. *Invent. Math.*, 61(2):177–197, 1980.

[354] C. A. Weibel. Mayer-Vietoris sequences and module structures on NK_*. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 466–493. Springer, Berlin, 1981.

[355] C. A. Weibel. Homotopy algebraic K-theory. In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, volume 83 of *Contemp. Math.*, pages 461–488. Amer. Math. Soc., Providence, RI, 1989.

[356] M. Weiss. Excision and restriction in controlled K-theory. *Forum Math.*, 14(1):85–119, 2002.

[357] M. Weiss and B. Williams. Automorphisms of manifolds and algebraic K-theory. I. *K-Theory*, 1(6):575–626, 1988.

[358] M. Weiss and B. Williams. Assembly. In *Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993)*, pages 332–352. Cambridge Univ. Press, Cambridge, 1995.

[359] M. Weiss and B. Williams. Automorphisms of manifolds. In *Surveys on surgery theory, Vol. 2*, volume 149 of *Ann. of Math. Stud.*, pages 165–220. Princeton Univ. Press, Princeton, NJ, 2001.

[360] G. W. Whitehead. *Elements of homotopy theory*. Springer-Verlag, New York, 1978.

[361] B. Wilking. Rigidity of group actions on solvable Lie groups. *Math. Ann.*, 317(2):195–237, 2000.

[362] B. Williams. K-theory and surgery. In this Handbook.

[363] M. Yamasaki. L-groups of crystallographic groups. *Invent. Math.*, 88(3):571–602, 1987.

[364] G. Yu. On the coarse baum-connes conjecture. *K-theory*, 9:199–221, 1995.

[365] G. Yu. Localization algebras and the coarse Baum-Connes conjecture. *K-Theory*, 11(4):307–318, 1997.

[366] G. Yu. Asymptotic Fredholm modules, vector bundles with small variation and a nonvanishing theorem for K-theoretic indices. *K-Theory*, 13(4):331–346, 1998.

[367] G. Yu. The Novikov conjecture for groups with finite asymptotic dimension. *Ann. of Math. (2)*, 147(2):325–355, 1998.

[368] G. Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000.

[369] G. Yu. Hyperbolic groups admit proper affine isometric actions on l-spaces. preprint, arXiv:math.GR/0411234, to apear in GAFA, 2004.

[370] G. L. Yu. Baum-Connes conjecture and coarse geometry. *K-Theory*, 9(3):223–231, 1995.

[371] G. L. Yu. Coarse Baum-Connes conjecture. *K-Theory*, 9(3):199–221, 1995.

Notation

last edited on 9.1.05
last compiled on March 29, 2005

Comment 53 (By W.): `These entries by Holger have to be carried over to other ones by Wolfgang. In the final version this file icnotationreich.tex has to be taken out.`

last edited on 29.12.05
last compiled on March 29, 2005

Index

10. Comments by Lück (temporary chapter)

Comment 54 (By W.): `This chapter has to be taken out in the final version)`

10.1 Mathematical Comments and Problems

10.2 Mathematical Items which have to be Added

10.3 Comments about Latex

(1) 22.12.04: In labels never let blanks or $-signs appear. Moreover, use sec: the:, pro:, lem:, def:, con: or rem: and so on to indicate whether it is a Theorem, Proposition, Lemma, Definition, Conjecture or Remark and so on. For instance
\ label{con:BCC_ torsionfree_ intro};

10.4 Comments about the Layout

(1) 22.12.204: Except in the main Introduction (Chapter 0) or in environments like Definition I emphasize a notion if and only if it appears in the index. For instance

    ... \emph{Whitehead group}%
    \index{Whitehead group}
    $Wh(G)$%
    \indexnotation{Wh(G)} ... ;

(2) 22.12.04: In itemize or enumerate descriptions I use semicolons at the end of an item except for the last which ends with a point. I have done this also in enumerate envirinments appearing in Theorems or Lemmas.;
(3) 22.12.04: After commutative squares or other diagrams do not insert a point or comma in contrast to equations or other displayments;
(4) 22.12.04: Theorems, Conjectures, Questions and Problems are cited in the index. For instance
\index{Theorem!Dirac-Dual Dirac Method}

This is not done for Definitions as new notions regardless whether they appear in the text or in an definitions have their own entry in the index. This is also not done for Lemmas, Corollaries and Remarks;

(5) 27.12.04: I sometimes have used $\underline{E}G$ and $\underline{\underline{E}}G$ for $E_{\mathcal{FIN}}(G)$ and$E_{\mathcal{VCY}}(G)$. There are macros \eub{# 1} and \edub{# 1} to generate $\underline{E}G$ and $\underline{\underline{E}}G$;
(6) 29.12.04: There two commands for inserting comments, namely \commenth and \commentw depending on whether Holger or Wolfgang are inserting it;
(7) 31.12.04 Denote the complex or real representation ring by $R_{\mathbb{C}}(G)$ and $R_{\mathbb{R}}(G)$;
(8) 31.12.04 Use always the proof environment. However, it does not seem to produce the qed-sign. This must be fixed;
(9) 3.1.05 Assembly maps are denote by the letter A with possible subscripts indicating the family, e.g. $A_{\mathcal{FIN}}$ or $A_{\mathcal{FIN}\to\mathcal{VCY}}$. We do not write the group G or the dimension n in connection with the assembly map. For instance, we write $A_{\mathcal{FIN}\to\mathcal{VCY}}\colon K_n^G(E_{\mathcal{FIN}}(G)) \to K_n^G(E_{\mathcal{VCY}}(G))$;
(10) 4.1.05 Notation for matrix algebras: $Sl_n(R)$, $GL_n(R)$, $M_n(R)$, $M_{m,n}(R)$;
(11) 8.1.05 I use $\ldots$ except for the beginning or ends or long exact sequence, where I use $\cdots$. For instance

$$\cdots \xrightarrow{\partial_{n+1}} K_n^{G_0}(\underline{E}G_0) \to K_n^{G_1}(\underline{E}G_1) \oplus K_n^{G_2}(\underline{E}G_2) \to K_n^{G}(\underline{E}G) \xrightarrow{\partial_n}$$
$$K_{n-1}^{G_0}(\underline{E}G_0) \to K_{n-1}^{G_1}(\underline{E}G_1) \oplus K_{n-1}^{G_2}(\underline{E}G_2) \to K_{n-1}^{G_0}(\underline{E}G) \to \cdots$$

and $i = 1, 2, \ldots, n$;
(12) 9.1.05: Use everywhere the macro \pt for the one point space$\{\bullet\}$. Then we can change it later if we want to;
(13) 9.1.05: I have used smooth everywhere and replaced differentiable by smooth;

10.5 Comments about the Spelling

(1) 22.12.04: I use analogue (British) instead of analog (American);
(2) 27.12.04: torsionfree (One word);
(3) 27.12.04: semisimple (One word);
(4) 28.12.04: pseudoisotopy (One word);
(5) 29.12.04: prove, proved proven;
(6) 29.12.04: choose, chosed, chosen;
(7) 9.1.05: We should agree on a unified use of the words any, each and every. I am not certain about the rules;

10.6 Further Reminders and Comments

(1) 9.1.05: We have to make a principle decision what we will present in the Chapter 4 about Basic Technicalities. In particular we have to decide

whether we present the definitions of the middle, lower and higher algebraic K-groups at one place and do not give partial but incomplete explanations at several places in the text. Of course some repetition is okay. The same question arises for algebraic L-theory and topological K-theory although there it seems to be less significant because the definition is easier in these cases;

(2) 16.1.05: At some place we should explain where the name assembly map commes from;

10.7 Additional References

These references may have not yet appeared in the text but we should later decide whether they should be incoporated:

[5], [16], [17], [18], [66], [179], [108], [237], [369],

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11. Comments by Reich (temporary chapter)

Comment 55 (By W.): `This chapter has to be taken out in the final version)`

11.1 Mathematical Comments and Problems

11.2 Mathematical Items which have to be Added

11.3 Comments about Latex

11.4 Comments about the Layout

11.5 Comments about the Spelling

11.6 Further Reminders and Comments

11.7 Additional References

These references may have not yet appeared in the text but we should later decide whether they should be incoporated:

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