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Adiabatic limits of η-invariants and the Meyer functions

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Abstract We construct a function on the orbifold fundamental group of the moduli space of smooth theta divisors, which we call the Meyer function for smooth theta divisors. In the construction, we use the adiabatic limits of the η-invariants of the mapping torus of theta divisors. We shall prove that the Meyer function for smooth theta divisors cobounds the signature cocycle, and we determine the values of the Meyer function for the Dehn twists. In particular, we give an analytic construction of the Meyer function of genus two.

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Contents

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1 Introduction

Let $\mathcal{M}_g$ be the mapping class group of a closed orientable surface Σ_g of genus g. In [30], Meyer introduced a 2-cocycle $\tau_g : \mathcal{M}_g \times \mathcal{M}_g \to \mathbb{Z}$, called the signature cocycle or the Meyer cocycle. By using the Meyer cocycle τ_g, he gave the formula for the signatures of surface bundles over surfaces. Since $\mathcal{M}_1 = SL_2(\mathbb{Z})$, $H^1(SL_2(\mathbb{Z}), \mathbb{Z}) = 0$ and $3[\tau_1] = 0$ in $H^2(\mathcal{M}_1, \mathbb{Z})$, there exists a unique function $\phi_1 : SL_2(\mathbb{Z}) \to \frac{1}{3}\mathbb{Z}$ that cobounds τ_1. The function ϕ_1 is called the Meyer function of genus one, which has the following property: Let $\pi : Z \to X$ be a Σ_1-bundle over a compact oriented surface with boundary $\partial X = c_1 \amalg \cdots \amalg c_k$. Let $A_1, \ldots, A_k$ be the monodromies around each component of the boundary. Since the Picard–Lefschetz transformation along c_i is an automorphism of $H^1(\Sigma_1, \mathbb{Z})$ preserving the intersection form, one has $A_i \in SL_2(\mathbb{Z})$ by fixing a symplectic basis of $H^1(\Sigma_1, \mathbb{Z})$. Then the signature of Z, which is defined as the signature of the cup-product pairing on $H^2(Z, \partial Z, \mathbb{R})$, satisfies

$$\mathrm{Sign}(Z) = -\sum_{i}^{k} \phi_1(A_i). \tag{1}$$

The explicit formula for ϕ_1 was obtained by Meyer [30].

In [2], Atiyah investigated the Meyer function ϕ_1 from several view points. For an odd dimensional closed oriented Riemannian manifold M, let $\eta(M)$ be the η-invariant of M with respect to the signature operator of M [3]. For $\sigma \in SL_2(\mathbb{Z})$, let $\pi : M_\sigma \to S^1$ be the mapping torus associated with σ, i.e., the Σ_1-bundle over S^1 with monodromy σ. Then Atiyah showed the following identity, when M_σ is equipped with a certain metric:

$$\phi_1(\sigma) = \eta(M_\sigma). \tag{2}$$

Moreover, he gave several interpretations of ϕ_1 in terms of the following quantities: (i) Hirzebruch's signature defect; (ii) the transformation low of the logarithm of the Dedekind η-function; (iii) the logarithm of the monodromy of the determinant line bundle; (iv) the value of the Shimizu L-function at the origin.

After Meyer and Atiyah, generalizations of their results to the cases of curves of higher genus or the case of higher dimensional complex tori were studied by many authors.

When $g = 2$ there exists a unique function $\phi_2 : \mathcal{M}_2 \to \frac{1}{5}\mathbb{Z}$ satisfying (1) for every Σ_2-bundles over compact oriented surfaces. The function ϕ_2 is called the Meyer function of genus two. While $[\tau_g] \in H^2(\mathcal{M}_g, \mathbb{Z})$ is not a torsion element for $g > 2$, the restriction of $[\tau_g]$ to the hyperelliptic mapping class group is known to be a torsion element. Therefore the *Meyer function for hyperelliptic curves* can be defined [20,32]. The relations between η-invariants and the Meyer function for hyperelliptic curves were studied in [32].

A natural extension of Eq. (2) to mapping torus of higher dimensional torus follows from the same idea as in Atiyah [2], which we give in Appendix A. The coincidence of the η-invariants of torus fibrations and the special values of the corresponding

L-functions was established by Bismut and Cheeger [11]. In their results, automorphic forms seem to play no role.

The purpose of this paper is to give a generalization of Eq. (2) in which an automorphic form of higher dimension plays a role similar to the role of Dedekind η-function in Atiyah's study. For this reason, we shall consider the signature cocycle of *smooth theta divisors* as a higher dimensional analogue of curves of genus two and we shall prove that the cohomology class of this cocycle vanishes rationally by constructing the *Meyer function for smooth theta divisors* explicitly. Let us explain our results in details.

Let $\mathfrak{S}_g$ be the Siegel upper half-space of degree g and let Γ_g be the Siegel modular group of degree g. Let $f : \mathbb{A}_g \to \mathfrak{S}_g$ be the universal family of principally polarized Abelian varieties. Then Γ_g acts on $\mathbb{A}_g$ and $\mathfrak{S}_g$, so that f is Γ_g-equivariant. Consider the universal family of theta divisors:

$$p : \Theta \to \mathfrak{S}_g, \quad \Theta \subset \mathbb{A}_g, \quad p = f|_{\Theta}.$$

Here the fiber $\Theta_\tau = p^{-1}(\tau)$ is the theta divisor of $\mathbb{A}_\tau := f^{-1}(\tau)$ for any $\tau \in \mathfrak{S}_g$, i.e., the zero divisor of the Riemann theta function. Let $\mathcal{N}_g := \{\tau \in \mathfrak{S}_g \mid \mathrm{Sing}\Theta_\tau \neq \emptyset\}$ be the Andreotti–Mayer locus. Then there is a Siegel modular form $\Delta_g(\tau)$ of weight $\frac{(g+3)\cdot g!}{2}$ with zero divisor $\mathcal{N}_g$ by [33,39]. We put $\mathfrak{S}_g^\circ = \mathfrak{S}_g - \mathcal{N}_g$, $\Theta^\circ = \Theta\big|_{\mathfrak{S}_g^\circ}$. After a slight modification of the Γ_g-action on $\mathbb{A}_g$, we construct a Γ_g-action on Θ° and a specific Γ_g-invariant Kähler metric g^{Θ° on Θ° such that $p : \Theta^\circ \to \mathfrak{S}_g^\circ$ is Γ_g-equivariant. (See Sects. 4 and 5 for the construction of g^{Θ°.) The quotient space $\Gamma_g\backslash\mathfrak{S}_g^\circ$ is regarded as the coarse moduli space of *smooth* theta divisors. Let us consider the orbifold fundamental group of $\Gamma_g\backslash\mathfrak{S}_g^\circ$, which will be one of the main objects in this paper:

$$\mathcal{S}_g := \pi_1^{orb}(\Gamma_g\backslash\mathfrak{S}_g^\circ).$$

Since $\mathcal{S}_1 = \mathcal{M}_1 = SL_2(\mathbb{Z})$ and $\mathcal{S}_2 = \mathcal{M}_2$, $\mathcal{S}_g$ is an analogue of the mapping class group.

Following Atiyah [2], we define a 2-cocycle $c_g \in Z^2(\mathcal{S}_g, \mathbb{Z})$ as follows. Let $\mathcal{B} := S^2 \setminus \amalg_{i=1}^3 D_i$ be a sphere with three holes and let $\amalg_{i=1}^3 \gamma_i = \partial\mathcal{B} \subset \mathcal{B}$ be the boundary. For given $\sigma_1, \sigma_2 \in \mathcal{S}_g$, let $\alpha : \mathcal{B} \to \Gamma_g\backslash\mathfrak{S}_g^\circ$ be a C^∞-map in the sense of orbifolds (i.e., $\pi_1(\mathcal{B})$-equivariant C^∞-map $\tilde{\mathcal{B}} \to \mathfrak{S}_g^\circ$ from the universal covering space $\tilde{\mathcal{B}}$ to $\mathfrak{S}_g^\circ$) such that its restrictions to γ_1 and γ_2 are representatives of σ_1 and σ_2, respectively. Let $X_{(\sigma_1,\sigma_2)} := \mathcal{B} \times_\alpha \Theta^\circ$ be the family of smooth theta divisors on $\mathcal{B}$ induced from $p : \Theta^\circ \to \mathfrak{S}_g^\circ$ via α. Then $X_{(\sigma_1,\sigma_2)}$ is a compact $2g$-dimensional oriented manifolds with non-empty boundary. Define the map $c_g : \mathcal{S}_g \times \mathcal{S}_g \to \mathbb{Z}$ by

$$c_g(\sigma_1, \sigma_2) := \mathrm{Sign}(X_{(\sigma_1,\sigma_2)}).$$

By the Novikov additivity for signature, c_g is a 2-cocycle of $\mathcal{S}_g$. We call c_g the signature cocycle of smooth theta divisors. By construction, $c_2 = \tau_2$. When g is odd, c_g is trivial, i.e., $c_g \equiv 0$.

For $\sigma \in \mathcal{S}_g$, we choose a map $\alpha : S^1 \to \Gamma_g \backslash \mathfrak{S}_g^\circ$ in the sense of orbifolds (i.e., equivariant path $\tilde{S^1} = \mathbb{R} \to \mathfrak{S}_g^\circ$), which is a representative of σ. Let $\pi : M_\sigma \to S^1$ be the mapping torus of a smooth theta divisor induced by α. Let g^{M_σ/S^1} be the metric on the relative tangent bundle TM_σ/S^1 induced from the metric g^{Θ°. Using the connection induced from the Levi–Civita connection on $T\mathbb{A}_g$, we define a family of metrics on M_σ by

$$g_\varepsilon^{M_\sigma} = g^{M_\sigma/S^1} \oplus \varepsilon^{-1} \pi^* dt^2, \quad \varepsilon \in \mathbb{R}_{>0}.$$

By Bismut–Cheeger [10], the limit $\eta^0(M_\sigma) := \lim_{\varepsilon \to 0} \eta(M_\sigma, g_\varepsilon^{M_\sigma})$ exists and is called the adiabatic limit of the η-invariants $\eta(M_\sigma, g_\varepsilon^{M_\sigma})$. Set

$$\Phi_g(\sigma) := \eta^0(M_\sigma) + \frac{(-1)^{g/2} 2^{g+3} (2^{g+2} - 1)}{(g+3)!} B_{\frac{g}{2}+1} \int_{S^1} \alpha^* d^c \log \|\Delta_g(\tau)\|^2, \quad (3)$$

where $d^c = \frac{1}{4\pi\sqrt{-1}}(\partial - \bar{\partial})$ and $\|\Delta_{2g}(\tau)\|^2 := (\det \mathrm{Im} \tau)^{\frac{(g+3)\cdot(g)!}{2}} |\Delta_g(\tau)|^2$ denotes the Petersson norm of the Siegel modular form $\Delta_g(\tau)$. Here B_k is the kth Bernoulli number when $k \in \mathbb{Z}$ and $B_k = 0$ when $k \in \frac{1}{2} + \mathbb{Z}$. The main results of this paper are stated as follows.

Theorem 1 *The value $\Phi_g(\sigma)$ is independent of the choice of α, and Φ_g descends to a real-valued function on $\mathcal{S}_g$ cobounding the signature cocycle $-c_g$, i.e.,*

$$-c_g(\sigma_1, \sigma_2) = \Phi_g(\sigma_1) + \Phi_g(\sigma_2) - \Phi_g(\sigma_1\sigma_2), \quad \sigma_1, \sigma_2 \in \mathcal{S}_g.$$

In particular, $[c_g] \otimes \mathbb{Q} = 0 \in H^2(\mathcal{S}_g, \mathbb{Q})$.

We call Φ_g the Meyer function for smooth theta divisors. When g is odd, Φ_g vanishes identically. When g is even, Φ_g is non-trivial by Theorem 3 below. From the uniqueness of the Meyer function of genus 2, it follows that $\phi_2 = \Phi_2$.

We next consider the uniqueness of a function on $\mathcal{S}_g$ cobounding c_g, which is equivalent to the vanishing of $H^1(\mathcal{S}_g, \mathbb{Z})$. In general, the uniqueness no longer holds.

Theorem 2 *The following equality holds*:

$$H^1(\mathcal{S}_g, \mathbb{Z}) = \begin{cases} 0 & \text{if} \quad 0 \le g \le 3, \\ \mathbb{Z} & \text{if} \quad g \ge 4. \end{cases}$$

To prove the non-triviality of Φ_g, we compute the value of Φ_g for the Dehn twists. The subgroup $\pi_1(\mathfrak{S}_g^\circ)$ of $\mathcal{S}_g$ is regarded as an analogue of the Torelli group by the exact sequence

$$1 \to \pi_1(\mathfrak{S}_g^\circ) \to \mathcal{S}_g \to \Gamma_g \to 1.$$

Then $\pi_1(\mathfrak{S}_g^\circ)$ is generated by lassoes surrounding the irreducible components of $\mathcal{N}_g$. By Debarre [19], $\mathcal{N}_g$ consists of two Γ_g-invariant components θ_g and $\mathcal{J}_g$ such that $\Gamma_g\backslash\theta_g$ and $\Gamma_g\backslash\mathcal{J}_g$ are irreducible divisors on the Siegel modular variety $\Gamma_g\backslash\mathfrak{S}_g$. Let $\sum_\lambda \theta_{g,\lambda}$ and $\sum_\mu \mathcal{J}_{g,\mu}$ be the irreducible decompositions of θ_g and $\mathcal{J}_g$, respectively. Consider lassoes surrounding $\theta_{g,\lambda}$ and $\mathcal{J}_{g,\mu}$, and denote their homotopy classes by Π_λ^1 and Π_μ^2, respectively. Then Π_λ^1 and Π_μ^2 are elements of $\pi_1(\mathfrak{S}_g^\circ) \subset \mathcal{S}_g$ such that $\{\Pi_\lambda^1, \Pi_\mu^2\}_{\lambda,\mu}$ generates $\pi_1(\mathfrak{S}_g^\circ)$.

Theorem 3 *The following equalities hold*:

$$\Phi_g(\Pi_\lambda^1) = \begin{cases} -\frac{4}{5} & \text{if } g = 2, \\ (-1)^{\frac{g}{2}+1} \frac{(g+1)2^{g+2}(2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1} & \text{if } g \geq 3. \end{cases}$$

$$\Phi_g(\Pi_\mu^2) = (-1)^{\frac{g}{2}+1} \frac{(g+1)2^{g+3}(2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1} \quad \text{if } g \geq 4.$$

When $g = 2$, the monodromy Π_λ^1 is the Dehn twist along a separating simple closed curve on a Riemann surface of genus two. In this case, the formula $\Phi_2(\Pi_\lambda^1) = \phi_2(\Pi_\lambda^1) = -\frac{4}{5}$ confirms a result of Matsumoto [29, Proposition 3.6]. We conjecture that the function Φ_g is a homomorphism on $\pi_1(\mathfrak{S}_g^\circ)$. If this conjecture is affirmative, then the value of Φ_g on $\pi_1(\mathfrak{S}_g^\circ)$ will be determined by Theorem 3. When $g = 2$, this conjecture is affirmative since the cocycle $\tau_2 = c_2$ is the pull-back of a cocycle of Γ_2.

We explain the strategy of the proof of Theorem 1 briefly.

(Step 1) For $\sigma_1, \sigma_2 \in \mathcal{S}_g$, consider the the family $\pi : X_{(\sigma_1,\sigma_2)} \to \mathcal{B}$ as defined above. For simplicity, set $X = X_{(\sigma_1,\sigma_2)}$. Endow X with the metric $g^{X/\mathcal{B}}$ on the relative tangent bundle $TX/\mathcal{B}$ induced by g^{Θ° via the classifying map $\alpha : \mathcal{B} \to \Gamma_g\backslash\mathfrak{S}_g^\circ$. Let $g^{\mathcal{B}}$ be a metric on $T\mathcal{B}$ that is a product metric on a color neighborhood of the boundary. By using the connection induced from the Levi–Civita connection on $T\mathbb{A}_g$, define a family of metrics by $g_\varepsilon^X := g^{X/\mathcal{B}} \oplus \varepsilon^{-1}\pi^* g^{\mathcal{B}}$, $\varepsilon \in \mathbb{R}_{>0}$. The Atiyah–Patodi–Singer index theorem applied to (X, g_ε^X) yields that

$$\mathrm{Sign}(X) = \int_{\mathcal{B}} \pi_* L(TX, g_\varepsilon^X) - \sum_{i=1}^{3} \eta(M_{\sigma_i}, g_\varepsilon^X|_{M_{\sigma_i}}), \quad \sigma_3 = (\sigma_1\sigma_2)^{-1}. \tag{4}$$

(Step 2) Let $\nabla^{X/\mathcal{B}}$ be the connection on the relative tangent bundle $TX/\mathcal{B}$ induced from the metric $g^{X/\mathcal{B}}$ and the connection on the fiber bundle $\pi : X \to \mathcal{B}$ (See Sect. 2). Since $\lim_{\varepsilon\to 0} L(TX, g_\varepsilon^X) = L(TX/\mathcal{B}, \nabla^{X/\mathcal{B}})$ and since the signature is independent of the choice of a metric, we take the limit $\varepsilon \to 0$ in (4) to get

$$c_g(\sigma_1, \sigma_2) = \int_{\mathcal{B}} \pi_* L(TX/\mathcal{B}, \nabla^{X/\mathcal{B}}) - \sum_{i=1}^{3} \eta^0(M_{\sigma_i}). \tag{5}$$

(Step 3) Let ∇^H be the holomorphic Hermitian connection on the holomorphic relative tangent bundle $T^{1,0}\Theta^\circ/\mathfrak{S}_g^\circ$. In Sect. 5, we shall prove that

$$\left(p_*\mathbf{L}(T^{1,0}\Theta^\circ/\mathfrak{S}_g^\circ, \nabla^H)\right)^{(2)} = k(g)dd^c \log \|\Delta_g(\tau)\|^2, \tag{6}$$

where $\mathbf{L}$ denotes the multiplicative genus of Chern forms corresponding to the power series $x/\tanh(x)$, $\omega^{(p)}$ denotes the p-form component of a differential form ω and $k(g)$ is a certain rational number containing the Bernoulli number $B_{\frac{g}{2}+1}$ (cf. Theorem 11). By the functoriality of the connection $\nabla^{X/\mathcal{B}}$ (Proposition 1) and by the Kählerness of the metric g^{Θ° (Theorem 8), we shall prove that (cf. Sects. 5 and 7)

$$\begin{aligned}&\left(\pi_* L(TX/\mathcal{B}, \nabla^{X/\mathcal{B}})\right)^{(2)} \\ &= \alpha^* \left(p_*\mathbf{L}(T^{1,0}\Theta^\circ/\mathfrak{S}_g^\circ, \nabla^H)\right)^{(2)} = d\left(k(g)\alpha^* d^c \log \|\Delta_g(\tau)\|^2\right).\end{aligned} \tag{7}$$

The assertion follows from (5), (6), (7) and the Stokes Theorem.

The remainder of this paper is organized as follows: In Sect. 2, we recall some results on the connection of the relative tangent bundle. In Sect. 3, we recall the definition of η-invariants. In Sect. 4, we recall some basic properties of theta divisors. In Sect. 5, we compute the Hirzebruch's L-form of the relative tangent bundle for the family of smooth theta divisors. In Sect. 6, we construct the signature cocycle c_g. In Sect. 7, we construct the Meyer function Φ_g and prove that Φ_g cobounds $-c_g$. In Sect. 8, we consider the uniqueness of a 1-cochain that cobounds c_g. In Sect. 9, we compute the value of Φ_g for the Dehn twists. In Sect. 10, we give another analytic expression of Φ_2 by using Dai's result concerning the η-forms [18].

Throughout this paper, we fix the following notation. For a complex manifold M, $T^{1,0}M$ (resp. $T^{0,1}M$) denotes the holomorphic (resp. anti-holomorphic) tangent bundle and TM denotes the real tangent bundle. We set $d^c := \frac{1}{4\pi\sqrt{-1}}(\partial - \bar{\partial})$. Hence $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$.

2 Preliminaries from Riemannian geometry

In this section, we recall some results of Riemannian geometry which will be used in the proof of the main theorem. Following [6], we define connections of fiber bundles and the connection of relative tangent bundles. Let M be a manifold and let $\pi : Z \to B$ be a fiber bundle with typical fiber M.

The *relative tangent bundle* $T(Z/B)$ is the subbundle of TZ defined by

$$T(Z/B) := \mathrm{Ker}\{\pi_* : TZ \to \pi^* TB\}.$$

A vector of $T(Z/B)$ is said to be *vertical*.

Definition 1 A subbundle $T_H Z \subset TZ$ with $TZ = T(Z/B) \oplus T_H Z$ is called a *connection* of the fiber bundle $\pi : Z \to B$.

For a connection, one has $T_H Z \cong \pi^* TB$ via the projection $\pi_* : TZ \to \pi^* TB$. A vector of $T_H Z$ is said to be *horizontal*.

When Z is trivial, i.e., $Z = M\times B$, TZ is naturally isomorphic to the direct sum $(\mathrm{pr}_1)^*TM \oplus (\mathrm{pr}_2)^*TB$. This connection is called the *trivial connection* of the trivial fiber bundle.

Given a connection, one can define the projection $P_Z : TZ\to T(Z/B)$ with kernel $T_H Z$. We often identify P_Z with the corresponding connection $T_H Z := \mathrm{Ker}(P_Z)$. In the rest of Sect. 2, we fix a connection $T_H Z$, or equivalently P_Z.

One can define the pull-back of a connection as follows: Let B' be a manifold and let $h : B'\to B$ be a C^∞-map. The fiber product $Z' := Z\times_B B' = \{(x, b)\in Z\times B' \mid \pi(x) = h(b)\}$ satisfies the following commutative diagram:

$$\begin{array}{ccc} Z' & \xrightarrow{\tilde{h}} & Z \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[h]{} & B \end{array} \qquad \tilde{h} = \mathrm{pr}_1,\ \pi' = \mathrm{pr}_2.$$

Lemma 1 *The map $P_Z\circ\tilde{h}_* : TZ'\to h^*T(Z/B)$ is surjective.*

Proof Since $\tilde{h}_*\big|_{T_{(x,b')}(Z'/B')} : T_{(x,b')}(Z'/B')\to T_x(Z/B)$ is an isomorphism for all $(x, b')\in Z'$ and since $P_Z\big|_{T(Z/B)} = \mathrm{id}_{T(Z/B)}$, $P_Z\circ\tilde{h}_*$ is surjective. □

Since $P_Z\circ\tilde{h}_*$ is surjective,

$$\begin{aligned} \dim\mathrm{Ker}(P_Z\circ\tilde{h}_*)_{(x,b')} &= \dim Z' - \mathrm{rank}T(Z/B) \\ &= \dim Z' - \mathrm{rank}T(Z'/B') = \dim T_b B'. \end{aligned}$$

Hence $\mathrm{Ker}(P_Z\circ\tilde{h}_*)$ is a subbundle of TZ'. Since $T(Z'/B')$ is canonically isomorphic to $\tilde{h}^*T(Z/B)$, the map $P_Z \circ \tilde{h}_*$ is identified with a projection from TZ' to $T(Z'/B')$.

Definition 2 The connection of $\pi' : Z'\to B'$ induced from $T_H Z$ by h is defined by

$$T_H Z' := \mathrm{Ker}\left(P_Z\circ\tilde{h}_* : TZ'\to T(Z/B)\right)$$

under the identification between $T(Z'/B')$ and $h^*T(Z/B)$ given by $(\tilde{h}_*)|_{T(Z'/B')}$. The projection corresponding to $T_H Z'$ is denoted by h^*P_Z.

Lemma 2 (a) *For any C^∞-map $h' : B''\to B'$,*

$$(h\circ h')^*P_Z = h'^*(h^*P_Z).$$

(b) *The following diagram is commutative*:

$$\begin{array}{ccc} TZ' & \xrightarrow{\tilde{h}_*} & TZ \\ {\scriptstyle P_{Z'}}\downarrow & & \downarrow{\scriptstyle P_Z} \\ T(Z'/B') & \xrightarrow[(\tilde{h}_*)|_{T(Z'/B')}]{} & T(Z/B). \end{array}$$

(c) *If h is a constant map, say $h(b') = b$ for all $b' \in B'$, then h^*P_Z is the trivial connection on the trivial fiber bundle $Z' = Z_b \times B'$, where $Z_b := \pi^{-1}(b)$.*

Proof (a) Set $Z'' := Z' \times_{B'} B''$. Let $\tilde{h}' : Z'' \to Z'$ be the lift of the map h'. Under the isomorphism $(h \circ h')^* T(Z/B) \cong h'^* T(Z'/B') \cong T(Z''/B'')$, we have

$$(h \circ h')^* P_Z = P_Z \circ (\tilde{h} \circ \tilde{h}')_* = (P_Z \circ \tilde{h}_*) \circ \tilde{h}'_* = h'^*(h^* P_Z).$$

(b) The assertion follows from Definition 2.

(c) Since

$$T_H Z' = \mathrm{Ker}\left(P_Z \circ \tilde{h}_* : TZ' \to T(Z/B)|_{Z_b}\right) = \mathrm{Ker}\left((\mathrm{pr}_1)_* : TZ' \to TZ_b\right),$$

$h^* P_Z$ is the trivial connection. □

Definition 3 Let Z be a manifold and let $\mathrm{Diff}(Z)$ be the group of C^∞-diffeomorphism of Z. For $\varphi \in \mathrm{Diff}(Z)$, the *mapping torus* $\pi : M_\varphi \to S^1 = \mathbb{R}/\mathbb{Z}$ is defined by

$$\pi : M_\varphi := (Z \times \mathbb{R})/\mathbb{Z}, \quad \pi := \mathrm{pr}_2,$$

where $\mathbb{Z}$ acts on $Z \times \mathbb{R}$ by

$$m \cdot (x, t) := (\varphi^m(x), t + m), \quad m \in \mathbb{Z}, \quad (x, t) \in Z \times \mathbb{R}.$$

If Z is oriented, let $\mathrm{Diff}^+(Z)$ be the group of orientation-preserving diffeomorphism of Z. For $\varphi \in \mathrm{Diff}^+(Z)$, M_φ is endowed with the orientation induced from the one on $M \times \mathbb{R}$. Notice that $M_\varphi = -M_{\varphi^{-1}}$, which is the same manifold equipped with the opposite orientation. Since the trivial connection $T_H(M \times \mathbb{R}) = \mathrm{pr}_2^* T\mathbb{R}$ is preserved by the $\mathbb{Z}$-action, it descends to a connection of M_φ. This connection is called the *canonical connection* of the mapping torus $\pi : M_\varphi \to S^1$.

We fix a metric $g^{Z/B}$ on the relative tangent bundle, a Riemannian metric g^B on B, and the connection $T_H Z$ and the corresponding projection P_Z. We define the Riemannian metric g^Z on the total space Z by

$$g^Z := g^{Z/B} \oplus \pi^* g^B$$

under the isomorphism $TZ \cong T(Z/B) \oplus T_H Z \cong T(Z/B) \oplus \pi^* TB$. Let ∇^Z be the Levi–Civita connection of (Z, g^Z). We define the connection $\nabla^{Z/B}$ on $T(Z/B)$ by

$$\nabla^{Z/B} := P_Z \circ \nabla^Z.$$

Then $\nabla^{Z/B}$ preserves the metric $g^{Z/B}$.

Lemma 3 *The connection $\nabla^{Z/B}$ is independent of the choice of g^B.*

Proof See [6, Proposition 10.2]. □

For $X \in TB$, let X_H be its horizontal lift, so that $\pi_* X_H = X$. If X is a smooth vector field on B, the Lie derivative operator $\mathcal{L}_X$ acts naturally on the tensor algebra of $T(Z/B)$. In particular, $(g^{Z/B})^{-1} \mathcal{L}_{X_H} g^{Z/B}$ is a self-adjoint endmorphism of $T(Z/B)$.

Theorem 4 *The connection $\nabla^{Z/B}$ is characterized by the following two properties*:

(1) *On each fiber of $\pi : Z \to B$ it coincides with the Levi–Civita connection associated with the metric $g^{Z/B}$.*
(2) *If $X \in TB$, then $\nabla^{Z/B}_{X_H} = \mathcal{L}_{X_H} + \frac{1}{2}(g^{Z/B})^{-1} \mathcal{L}_{X_H} g^{Z/B}$.*

Proof See [7, Theorem 1.1]. □

Proposition 1 *Let B' be a manifold and let $h : B \to B'$ be a C^∞-map. Set $Z := Z \times_{B'} B$ with $\tilde{h} : Z \to Z'$. Let $g^{Z/B} = h^* g^{Z'/B'}$ be the metric on $T(Z/B)$ induced from $g^{Z'/B'}$, and let $P_Z = h^* P_{Z'}$ be the connection of Z induced from $P_{Z'}$. Then $\nabla^{Z/B} = \tilde{h}^* \nabla^{Z'/B'}$.*

Proof If $X \in TZ$ is vertical, we have directly $\nabla^{Z/B}_X = (\tilde{h}^* \nabla^{Z'/B'})_X$ from Theorem 4 (1).

Let $X \in TB$ and set $Y := h_* X \in TB'$. Let X_H (resp. Y_H) be the horizontal lift of X (resp. Y) so that $\tilde{h}_* X_H = Y_H$. If $A \in T(Z'/B')$ be a smooth vertical vector field, or more generally a tensor algebra of $T(Z'/B')$, it follows that $\mathcal{L}_{X_H}(\tilde{h}^* A) = \tilde{h}^*(\mathcal{L}_{Y_H} A)$. Combined with Theorem 4 (2) and $g^{Z/B} = \tilde{h}^* g^{Z'/B'}$, this yields the equality

$$\nabla^{Z/B}_{X_H}\left(\tilde{h}^* A\right) = \tilde{h}^*(\nabla^{Z'/B'}_{Y_H} A), \quad A \in T(Z'/B'),$$

which completes the proof. □

With respect to the decomposition $TZ = T(Z/B) \oplus T_H Z$, we put for $\varepsilon \in \mathbb{R}^+$

$$g^{Z,\varepsilon} := g^{Z/B} \oplus \varepsilon^{-1} \pi^* g^B.$$

The Levi–Civita connections of $(Z, g^{Z,\varepsilon})$ and (B, g^B) are denoted by $\nabla^{Z,\varepsilon}$ and ∇^B, respectively. Let $R^{Z,\varepsilon}$ and R^B be the curvature of $\nabla^{Z,\varepsilon}$ and ∇^B, respectively. We define another connection ∇ on Z by

$$\nabla := \nabla^{Z/B} \oplus \pi^* \nabla^B,$$

and we put

$$S^{(\varepsilon)} := \nabla^{Z,\varepsilon} - \nabla \in \mathcal{A}^1(\mathrm{End}(TZ)), \quad S := S^{(1)}.$$

Then ∇ preserves the Riemannian metric $g^{Z,\varepsilon}$, and P_Z is parallel with respect to ∇, i.e., $\nabla \circ P_Z - P_Z \circ \nabla = 0$.

Let $\{e_1, \ldots, e_k\}$ be a local orthogonal framing for $(T(Z/B), g^{Z/B})$, and let $\{f_1, \ldots, f_l\}$ be a local orthogonal framing for $(T_H Z, \pi^* g^B)$.

Proposition 2 *With respect to the splitting $TZ = T(Z/B) \oplus T_H B$, the following identity holds*:

$$\lim_{\varepsilon \to 0} R^{Z,\varepsilon} = \begin{pmatrix} R^{Z/B} & P_Z(\nabla S) \\ 0 & \pi^* R^B \end{pmatrix}.$$

Proof See [12, Eq. (3.195)]. □

3 η-Invariants

In this section, we recall the definition and some properties of η-invariants. Let (M, g^M) be a closed oriented Riemannian manifold of dimension $(2l-1)$. Denote the space of C^∞ k-forms on M by $\mathcal{A}^k(M)$. Let $* : \mathcal{A}^k(M) \to \mathcal{A}^{2l-k-1}(M)$ be the Hodge star operator with respect to g^M. The signature operator $D : \oplus_{p\geq 0}\mathcal{A}^{2p}(M) \to \oplus_{p\geq 0} \mathcal{A}^{2p}(M)$ of M is defined by

$$D : \omega \longmapsto (\sqrt{-1})^l(-1)^{p+1}(*d - d*)\omega, \quad \omega \in \mathcal{A}^{2p}(M).$$

Then D is an elliptic self-adjoint differential operator of first order acting on $\oplus_{p\geq 0}\mathcal{A}^{2p}(M)$. Let $\sigma(D)$ be the spectrum of D. The η-function of M is defined by

$$\eta(s) := \sum_{\lambda \in \sigma(D)\setminus\{0\}} \frac{\mathrm{sign}\lambda}{|\lambda|^s}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$. Then $\eta(s)$ extends meromorphically to $\mathbb{C}$ and is holomorphic at $s = 0$ by [3,12].

Definition 4 The real number $\eta(0)$ is called the *η-invariant* of (M, g^M) and is denoted by $\eta(M, g^M)$.

Let (X, g^X) be a $4k$-dimensional, oriented, compact, Riemannian manifold with boundary Y. Put $g^Y := g^X\big|_Y$ and fix a color neighborhood $U \supset Y$ such that $U \cong Y \times [0, 1)$. Assume that $g^X\big|_U = g^Y \oplus dt^2$ under the above isomorphism. Let ∇^L be the Levi–Civita connection of (X, g^X) and let $R^L := (\nabla^L)^2$ be the curvature. Let $L(TX, \nabla^L)$ be the Hirzebruch L-form, i.e.,

$$L(TX, \nabla^L) := \det{}^{1/2}\left(\frac{-R^L/2\pi\sqrt{-1}}{\tanh(-R^L/2\pi\sqrt{-1})}\right). \tag{8}$$

Denote by $\mathrm{Sign}(X)$ the signature of X, i.e., the signature of the cup-product pairing on $H^{2k}(X, Y, \mathbb{Q})$, which is a homotopy invariant of the pair (X, Y). Note that one can also use the compact support cohomology $H^{2k}_c(X\backslash Y, \mathbb{Q}) \cong H^{2k}(X, Y, \mathbb{Q})$ to define $\mathrm{Sign}(X)$.

Theorem 5 (Atiyah–Patodi–Singer [3]) *The following equation holds*:

$$\mathrm{Sign}(X) = \int_X L(TX, \nabla^L) - \eta(Y, g^Y).$$

Let X, B and M be closed oriented manifolds. Let $\pi : X \to B$ be a C^∞-submersion, whose fibers are isomorphic to M. Assume that $\dim X = 4k$. Let $g^{X/B}$ be a metric on $T(X/B)$ and let g^B be a metric on TB. Let $T_H X \subset TX$ be a connection. We identify $T_H X$ with $\pi^* TB$ via π. With respect to the decomposition $TX = T(X/B) \oplus \pi^* TB$, we define the metric on X by $g^X := g^{X/B} \oplus \pi^* g^B$ and we consider the one parameter family of metrics on X defined by

$$g^X_\varepsilon := g^{X/B} \oplus \varepsilon^{-1} \pi^* g^B, \quad \varepsilon \in \mathbb{R}_+.$$

Theorem 6 ([10], [18, Corollary 4.1]) *The limit* $\lim_{\varepsilon\to 0} \eta(X, g^X_\varepsilon)$ *exists.*

The limit $\lim_{\varepsilon\to 0} \eta(X, g^X_\varepsilon)$ is called the *adiabatic limit of the η-invariants* and is denoted by $\eta^0(X)$.

4 Family of theta divisors

In this section we construct an action of the Siegel modular group on the universal family of theta divisors and we also construct a specific invariant Kähler metric on the total space of this family.

We first fix the notation. Let $\mathfrak{S}_g$ be the Siegel upper half-space of degree g and let Γ_g be the Siegel modular group, i.e.,

$$\mathfrak{S}_g := \{\tau \in M(g, \mathbb{C}) \mid {}^t\tau = \tau, \ \ \mathrm{Im}\tau > 0\}$$
$$\Gamma_g := \{\gamma \in GL(2g, \mathbb{Z}) \mid \gamma\, J_g\, {}^t\gamma = J_g\},$$

where $J_g = \left(\begin{smallmatrix} 0 & 1_g \\ -1_g & 0 \end{smallmatrix}\right)$ and 1_g denotes the $g \times g$ identity matrix. Γ_g acts on $\mathfrak{S}_g$ by

$$\gamma\cdot\tau := (A\tau + B)(C\tau + D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g, \quad \tau \in \mathfrak{S}_g.$$

For $\tau \in \mathfrak{S}_g$, write $\tau = ({}^t\tau_1, \ldots, {}^t\tau_g)$ and set

$$\Lambda_\tau := \mathbb{Z}\mathbf{e}_1 \oplus \cdots \oplus \mathbb{Z}\mathbf{e}_g \oplus \mathbb{Z}\tau_1 \oplus \cdots \oplus \mathbb{Z}\tau_g \ \subset \mathbb{C}^g$$

where $1_g = ({}^t\mathbf{e}_1, \ldots, {}^t\mathbf{e}_g)$ and $\tau = ({}^t\tau_1, \ldots, {}^t\tau_g) \in \mathfrak{S}_g$. Here all vectors denote row vectors. Define the $\mathbb{Z}^{2g}$-action on $\mathbb{C}^g \times \mathfrak{S}_g$ by

$$(m, n) \cdot (z, \tau) := (z + m\tau + n, \tau), \quad (z, \tau) \in \mathbb{C}^g \times \mathfrak{S}_g, \quad m, n \in \mathbb{Z}^{2g}.$$

Then

$$f : \mathbb{A}_g := (\mathbb{C}^g \times \mathfrak{S}_g)/\mathbb{Z}^{2g} \to \mathfrak{S}_g$$

is the universal family of principally polarized Abelian varieties over $\mathfrak{S}_g$, whose fiber over τ is $A_\tau := \mathbb{C}^g/\Lambda_\tau$. For $(a, b) \in \mathbb{R}^{2g}$, $z \in \mathbb{C}^g$ and $\tau \in \mathfrak{S}_g$ we define the theta function with characteristic by

$$\vartheta_{a,b}(z, \tau) := \sum_{n \in \mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}(n + a)\tau^t(n + a) + (n + a)^t(z + b)\right),$$

where $\mathbf{e}(t) = \exp(2\pi\sqrt{-1}t)$. Let

$$p : \Theta_{a,b} := \{(z, \tau) \in \mathbb{A}_g \mid \vartheta_{a,b}(z, \tau) = 0\} \to \mathfrak{S}_g.$$

be the *universal family of theta divisors*. For simplicity we write ϑ for $\vartheta_{0,0}$ and set $\Theta = \Theta_{0,0}$.

For any $(a, b) \in \mathbb{R}^{2g}$, we define an automorphism $t_{(a,b)} : \mathbb{A}_g \to \mathbb{A}_g$ by

$$t_{(a,b)} \cdot (z, \tau) := (z + a\tau + b, \tau).$$

Then $t_{(a,b)}$ has no fixed points when $(a, b) \in \mathbb{R}^{2g} \setminus \mathbb{Z}^{2g}$ and the subgroup $\mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$ acts trivially on $\mathbb{A}_g$. One has the Γ_g-action on $\mathbb{A}_g$ defined by

$$\gamma \cdot (z, \tau) := (z(C\tau + D)^{-1}, (A\tau + B)(C\tau + D)^{-1}),$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$, $z \in \mathbb{C}^g$, $\tau \in \mathfrak{S}_g$, so that f is Γ_g-equivariant. This action does not preserve the family $p : \Theta \to \mathfrak{S}_g$. However we can construct a Γ_g-action on Θ so that p is Γ_g-equivariant, after a slight modification of the definition of this Γ_g-action.

Theorem 7 ([22, Chap. II, Sec. 5, Theorem 6]) *For $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$, $\tau \in \mathfrak{S}_g$, (m, n), $(a, b) \in \mathbb{R}^{2g}$,*

$$\vartheta_{m,n}\left(t_{(a,b)} \cdot (z, \tau)\right) = \mathbf{e}\left(-\frac{1}{2}a\tau^t a - a^t(z + b + n)\right)\vartheta_{m+a,n+b}(z, \tau)$$

$$\vartheta_{m',n'}(\gamma \cdot (z, \tau)) = \mathbf{e}\left(\frac{1}{2}z(C\tau + D)^{-1}C^t z\right)\det(C\tau + D)^{\frac{1}{2}} \cdot u\vartheta_{m,n}(z, \tau).$$

Here

$$(m', n') = (m, n)\cdot\gamma^{-1} + \frac{1}{2}((C^t D)_0, (A^t B)_0),$$

where $M_0 = (m_{ij}\delta_{ij})$, $M = (m_{ij}) \in M(g, \mathbb{Z})$ *and* $u \in \mathbb{C}^*$ *is independent of* τ, z.

For $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, put

$$\tilde{\gamma} := t_{(a,b)}\circ\gamma \in \mathrm{Aut}(\mathbb{A}_g), \quad (a, b) := \frac{1}{2}((C^t D)_0, (A^t B)_0).$$

Proposition 3 (a) *The automorphism* $\tilde{\gamma}$ *preserves the family* $p : \Theta \to \mathfrak{S}_g$.
(b) *For any* $\gamma_1, \gamma_2 \in \Gamma_g$, *the following identity holds in* $Aut(\Theta)$:

$$\tilde{\gamma}_1\circ\tilde{\gamma}_2 = \widetilde{\gamma_1\gamma_2}$$

Proof (a) We set $(m, n) = (0, 0)$ in the second equality of Theorem 7 to get

$$\begin{aligned}
\vartheta_{0,0}(z, \tau) &= \mathbf{e}\left(-\frac{1}{2}z(C\tau + D)C^t z\right)\det(C\tau + D)^{-\frac{1}{2}}u^{-1}\vartheta_{a,b}(\gamma\cdot(z, \tau)) \\
&= \mathbf{e}\left(\frac{1}{2}a(\gamma\cdot\tau)^t a + a^t(z(C\tau + D)^{-1} + b + n)\right) \\
&\quad\times\mathbf{e}\left(-\frac{1}{2}z(C\tau + D)C^t z\right) \\
&\quad\times\det(C\tau + D)^{-\frac{1}{2}}u^{-1}\cdot\vartheta_{0,0}\left(t_{(a,b)}\circ\gamma\cdot(z, \tau)\right),
\end{aligned}$$

where the second equality follows from the first equality of Theorem 7. This implies that if $\vartheta(z, \tau) = 0$ then $\vartheta(\tilde{\gamma}\cdot(z, \tau)) = 0$.

(b) Since $\gamma\circ t_{(m,n)} = t_{(m,n)\cdot\gamma^{-1}}\circ\gamma$ for $\gamma \in \Gamma_g$ and $(m, n)\in\frac{1}{2}\mathbb{Z}^{2g}$, there exists $(m', n')\in\frac{1}{2}\mathbb{Z}^{2g}$ such that

$$(\widetilde{\gamma_1\gamma_2})^{-1}\tilde{\gamma}_1\circ\tilde{\gamma}_2 = t_{(m,n)}\circ(\gamma_1\gamma_2)^{-1}\circ t_{(m_1,n_1)}\circ\gamma_1\circ t_{(m_2,n_2)}\circ\gamma_2 = t_{(m',n')}.$$

Thus $(\widetilde{\gamma_1\gamma_2})^{-1}\tilde{\gamma}_1\circ\tilde{\gamma}_2$ is either the identity map or a holomorphic involution on $\Theta_{(\gamma_1\gamma_2\tau)}$ without fixed points. By Lemma 4 below, we get $\tilde{\gamma}_1\circ\tilde{\gamma}_2 = \widetilde{\gamma_1\gamma_2}$. □

Lemma 4 *If* Θ_τ *is smooth, then there is no holomorphic involution on* Θ_τ *without fixed points.*

Proof For a compact complex manifold X, let $\chi_{hol}(X)$ denote the arithmetic genus of X, i.e.,

$$\chi_{hol}(X) := \sum_{k\geq 0}(-1)^k h^k(X, \mathcal{O}_X).$$

Assume that ι is a holomorphic involution on Θ_τ without fixed points. Then

$$\chi_{hol}(\Theta_\tau) = 2\chi_{hol}(\Theta_\tau / < \iota >). \tag{9}$$

Let $\mathcal{I}_{\Theta_\tau}$ be the ideal sheaf of Θ_τ. From the exact sequence of sheaves $0 \to \mathcal{I}_{\Theta_\tau} \to \mathcal{O}_{A_\tau} \to \mathcal{O}_{\Theta_\tau} \to 0$ and the vanishing $\chi_{hol}(A_\tau) = 0$, we get

$$\chi_{hol}(\Theta_\tau) = \chi_{hol}(A_\tau) - \chi_{hol}(\mathcal{I}_{\Theta_\tau}) = -\chi_{hol}(\mathcal{I}_{\Theta_\tau}). \tag{10}$$

Let $[\Theta_\tau]$ be the line bundle on A_τ defined by the divisor Θ_τ. Then $[\Theta_\tau]$ is ample. Since $H^k(A_\tau, \mathcal{I}_{\Theta_\tau}) = H^k(A_\tau, [\Theta_\tau]^{-1})$, we get

$$\begin{aligned}\chi_{hol}(\mathcal{I}_{\Theta_\tau}) &= (-1)^g h^g(A_\tau, [\Theta_\tau]^{-1})\\ &= (-1)^g h^0(A_\tau, [\Theta_\tau] \otimes K_{A_\tau})\\ &= (-1)^g h^0(A_\tau, [\Theta_\tau]) = (-1)^g,\end{aligned}$$

where the first equality follows from the Kodaira vanishing theorem, the second equality follows from the Serre duality, and the third equality follows from the triviality of K_{A_τ}. Hence we get $\chi_{hol}(\Theta_\tau) = (-1)^{g+1}$, which contradicts (9). □

We set

$$g^{\mathbb{A}_g/\mathfrak{S}_g} := dz \cdot (\mathrm{Im}\tau)^{-1} \cdot {}^t d\bar{z}.$$

Then $g^{\mathbb{A}_g/\mathfrak{S}_g}$ is a Γ_g-invariant Hermitian metric on the relative tangent bundle $T(\mathbb{A}_g/\mathfrak{S}_g)$. The next purpose of this section is to construct a Γ_g-invariant Kähler metric on $T\mathbb{A}_g$ whose restriction to $T(\mathbb{A}_g/\mathfrak{S}_g)$ is $g^{\mathbb{A}_g/\mathfrak{S}_g}$.

Put $T^{2g} := \mathbb{R}^{2g}/\mathbb{Z}^{2g}$. Define a $\mathbb{Z}^{2g}$-action on $\mathbb{R}^{2g} \times \mathfrak{S}_g$ by $(m, n) \cdot (x, y, \tau) := (x + m, y + n, \tau)$ for $(m, n) \in \mathbb{Z}^{2g}$, $(x, y) \in \mathbb{R}^{2g}$, $\tau \in \mathfrak{S}_g$. Then $(\mathbb{R}^{2g} \times \mathfrak{S}_g)/\mathbb{Z}^{2g}$ is the trivial T^{2g}-bundle $T^{2g} \times \mathfrak{S}_g$. We define a C^∞-map $\tilde{\rho} : \mathbb{R}^{2g} \times \mathfrak{S}_g \to \mathbb{C}^g \times \mathfrak{S}_g$ by

$$\tilde{\rho}((x, y), \tau) := (x\tau + y, \tau), \quad x, y \in \mathbb{R}^g, \quad \tau \in \mathfrak{S}_g.$$

Since $\tilde{\rho}$ is a $\mathbb{Z}^{2g}$-equivariant map, $\tilde{\rho}$ induces a C^∞-isomorphism $\rho : T^{2g} \times \mathfrak{S}_g \to \mathbb{A}_g$ as T^{2g}-bundles over $\mathfrak{S}_g$. Define a Γ_g-action on $T^{2g} \times \mathfrak{S}_g$ by

$$\gamma \cdot ((x, y), \tau) := ((x, y)\gamma^{-1}, \gamma \cdot \tau), \quad \gamma \in \Gamma_g.$$

Lemma 5 *For all $\gamma \in \Gamma_g$, the following diagram is commutative.*

$$\begin{array}{ccc} T^{2g} \times \mathfrak{S}_g & \xrightarrow{\rho} & \mathbb{A}_g \\ \gamma \downarrow & & \downarrow \gamma \\ T^{2g} \times \mathfrak{S}_g & \xrightarrow[\rho]{} & \mathbb{A}_g \end{array}$$

Proof Let $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Since

$$\gamma^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}, \quad {}^tBD = {}^tDB, \quad {}^tAC = {}^tCA, \quad {}^tAD - {}^tCB = 1_g,$$

we get

$$\begin{aligned} \rho\gamma((x, y), \tau) &= \rho((x, y)\gamma^{-1}, \gamma\tau) \\ &= ((x^tD - y^tC)(A\tau + B)(C\tau + D)^{-1} \\ &\quad + (-x^tB + y^tA)(C\tau + D)(C\tau + D)^{-1}, (A\tau + B)(C\tau + D)^{-1}) \\ &= ((x\tau + y)(C\tau + D)^{-1}, (A\tau + B)(C\tau + D)^{-1}) \\ &= \gamma\rho((x, y), \tau). \end{aligned}$$

□

Since the trivial connection on $T^{2g} \times \mathfrak{S}_g$ is Γ_g-invariant, $\mathbb{A}_g$ has the induced Γ_g-invariant connection $T_H\mathbb{A}_g \subset T\mathbb{A}_g$ via the Γ_g-equivariant isomorphism ρ. We denote the Γ_g-equivariant projection corresponding to $T_H\mathbb{A}_g$ by P_ρ. Let $P_\rho^{\mathbb{C}} : T\mathbb{A}_g \otimes \mathbb{C} \to T(\mathbb{A}_g/\mathfrak{S}_g) \otimes \mathbb{C}$ be the complexification of P_ρ. Then $P_\rho^{\mathbb{C}}$ is also Γ_g-equivariant.

Let Z and B be complex manifolds and let $\pi : Z \to B$ be a holomorphic submersion. A connection P_Z on Z is said to be *compatible with the complex structure* if the horizontal lift of a $(1, 0)$ (resp. $(0, 1)$) vector is a $(1, 0)$ (resp. $(0, 1)$) vector, or equivalently, if $P : TZ \to T(Z/B)$ preserves the complex structure. Let $P_Z^{\mathbb{C}} : TZ \otimes \mathbb{C} \to T(Z/B) \otimes \mathbb{C}$ be the complexification. If P_Z is compatible with the complex structure, we get the decomposition $P_Z^{\mathbb{C}} = P_Z^{1,0} \oplus P_Z^{0,1}$ with respect to the decomposition

$$TZ \otimes \mathbb{C} = T^{1,0}Z \oplus T^{0,1}Z, \quad T(Z/B) \otimes \mathbb{C} = T^{1,0}(Z/B) \oplus T^{0,1}(Z/B),$$

such that $P_Z^{1,0}(T^{1,0}Z) = T^{1,0}(Z/B)$, $P^{0,1}Z(T^{0,1}Z) = T^{0,1}(Z/B)$. Hence $P_Z^{\mathbb{C}}$ induces the decomposition

$$T^{1,0}Z \cong T^{1,0}(Z/B) \oplus \pi^*T^{1,0}B.$$

Lemma 6 *The Γ_g-equivariant connection P_ρ is compatible with the complex structure. Hence $P_\rho^{\mathbb{C}}$ induces the Γ_g-equivariant C^∞-isomorphism*

$$T^{1,0}\mathbb{A}_g \cong T^{1,0}(\mathbb{A}_g/\mathfrak{S}_g) \oplus f^*T^{1,0}\mathfrak{S}_g.$$

Proof Since $\rho((x, y), \tau) = (x\tau + y, \tau)$ and $z_k = \sum_l x_l \tau_{lk} + y_k$, we get

$$
\begin{aligned}
\rho_*\left(\frac{\partial}{\partial \tau_{ij}}\right) &= \sum_{k=1}^{g} \frac{\partial z_k}{\partial \tau_{ij}} \frac{\partial}{\partial z_k} + \sum_{k=1}^{g} \frac{\partial \bar{z}_k}{\partial \tau_{ij}} \frac{\partial}{\partial \bar{z}_k} + \frac{\partial}{\partial \tau_{ij}} \\
&= \sum_{k,l=1}^{g} x_l \frac{\partial \tau_{lk}}{\partial \tau_{ij}} \frac{\partial}{\partial z_k} + \frac{\partial}{\partial \tau_{ij}} \\
&= x_i \frac{\partial}{\partial z_j} + x_j \frac{\partial}{\partial z_i} + \frac{\partial}{\partial \tau_{ij}}, \\
\rho_*\left(\frac{\partial}{\partial \bar{\tau}_{ij}}\right) &= x_i \frac{\partial}{\partial \bar{z}_j} + x_j \frac{\partial}{\partial \bar{z}_i} + \frac{\partial}{\partial \bar{\tau}_{ij}}.
\end{aligned} \tag{11}
$$

Notice that $\frac{\partial \tau_{lk}}{\partial \tau_{ij}} = \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}$, since τ is a symmetric matrix. From (11), the assertion follows. □

Let $g^{\mathfrak{S}_g}$ be the Bergman metric on $\mathfrak{S}_g$ with Kähler form

$$
\omega_{\mathfrak{S}_g} = -2\sqrt{-1}\partial\bar{\partial}\log\det\mathrm{Im}\tau. \tag{12}
$$

Then $g^{\mathfrak{S}_g}$ is Γ_g-invariant. With respect to the decomposition in Lemma 6, we define the Γ_g-invariant Hermitian metric $g^{\mathbb{A}_g}$ on $T^{1,0}\mathbb{A}_g$ by

$$
g^{\mathbb{A}_g} := g^{\mathbb{A}_g/\mathfrak{S}_g} \oplus f^* g^{\mathfrak{S}_g}.
$$

Then we have

$$
\begin{aligned}
g^{\mathbb{A}_g}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) &= g^{\mathbb{A}_g/\mathfrak{S}_g}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) \\
g^{\mathbb{A}_g}\left(\frac{\partial}{\partial z_i}, \rho_*\left(\frac{\partial}{\partial \tau_{kl}}\right)\right) &= 0 \\
g^{\mathbb{A}_g}\left(\rho_*\left(\frac{\partial}{\partial \tau_{ij}}\right), \rho_*\left(\frac{\partial}{\partial \tau_{kl}}\right)\right) &= g^{\mathfrak{S}_g}\left(\frac{\partial}{\partial \tau_{ij}}, \frac{\partial}{\partial \tau_{kl}}\right).
\end{aligned} \tag{13}
$$

Theorem 8 *The Hermitian metric $g^{\mathbb{A}_g}$ is Kähler.*

Proof Let L be the holomorphic line bundle over $\mathbb{A}_g$ defined by the divisor Θ, and let h_L be the Hermitian metric on L defined by

$$
\|\vartheta\|_L^2(z, \tau) := |\vartheta(z, \tau)|^2 \exp\left(-2\pi(\mathrm{Im}z)(\mathrm{Im}\tau)^{-1t}(\mathrm{Im}z)\right).
$$

Then

$$
c_1(L|_{A_\tau}, h_L) = \frac{\sqrt{-1}}{2} dz(\mathrm{Im}\tau)^{-1t}(d\bar{z}). \tag{14}
$$

Write

$$g^{\mathbb{A}_g/\mathfrak{S}_g} = \sum h_{ij} dz_i d\bar{z}_j, \quad g^{\mathfrak{S}_g} = \sum h'_{ijkl} d\tau_{ij} d\bar{\tau}_{kl}.$$

By (11) and (13), we get

$$\begin{aligned} 0 &= g^{\mathbb{A}_g}\left(\rho_*\left(\frac{\partial}{\partial \tau_{ij}}\right), \frac{\partial}{\partial z_k}\right) = x_i h_{jk} + x_j h_{ik} + g^{\mathbb{A}_g}\left(\frac{\partial}{\partial \tau_{ij}}, \frac{\partial}{\partial z_k}\right), \\ h'_{ijkl} &= g^{\mathbb{A}_g}\left(\rho_*(\frac{\partial}{\partial \tau_{ij}}), \rho_*\left(\frac{\partial}{\partial \tau_{kl}}\right)\right) \\ &= -x_i x_k h_{jl} - x_k x_j h_{il} - x_i x_l h_{jk} - x_j x_l h_{ik} + g^{\mathbb{A}_g}\left(\frac{\partial}{\partial \tau_{ij}}, \frac{\partial}{\partial \tau_{kl}}\right). \end{aligned}$$

Therefore

$$g^{\mathbb{A}_g}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = h_{ij} = (\mathrm{Im}\tau)^{-1}_{ij}, \tag{15}$$

$$g^{\mathbb{A}_g}\left(\frac{\partial}{\partial \tau_{ij}}, \frac{\partial}{\partial z_k}\right) = -x_i h_{jk} - x_j h_{ik}, \tag{16}$$

$$g^{\mathbb{A}_g}\left(\frac{\partial}{\partial \tau_{ij}}, \frac{\partial}{\partial \tau_{kl}}\right) = h'_{ijkl} + x_i x_k h_{jl} + x_j x_k h_{il} + x_i x_l h_{jk} + x_j x_l h_{ik}. \tag{17}$$

By (12) and (14),

$$h_{ij} = -\frac{1}{\pi}\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\|\vartheta\|^2(z,\tau) \tag{18}$$

$$h'_{ijkl} = -\frac{1}{\pi}\frac{\partial^2}{\partial \tau_{ij}\partial \bar{\tau}_{kl}} 4\pi \operatorname{logdetIm}\tau. \tag{19}$$

Since $z = x\tau + y$, we have $\mathrm{Im}z = x(\mathrm{Im}\tau)$, i.e., $x = \mathrm{Im}z(\mathrm{Im}\tau)^{-1}$. Set $E_{ij} := {}^t\mathbf{e}_i\mathbf{e}_j + {}^t\mathbf{e}_j\mathbf{e}_i$. Since $\mathrm{Im}z = \frac{1}{2\sqrt{-1}}(z - \bar{z})$ and $\mathrm{Im}\tau = \frac{1}{2\sqrt{-1}}(\tau - \bar{\tau})$, we get

$$\begin{aligned} &-\frac{1}{\pi}\frac{\partial^2}{\partial \tau_{ij}\partial \bar{z}_k}\log\|\vartheta\|^2(z,\tau) \\ &= 2\frac{\partial^2}{\partial \tau_{ij}\partial \bar{z}_k}\mathrm{Im}z(\mathrm{Im}\tau)^{-1t}(\mathrm{Im}z) \\ &= 2\left(\frac{-1}{2\sqrt{-1}}\right)\frac{\partial}{\partial \tau_{ij}}\{\mathbf{e}_k(\mathrm{Im}\tau)^{-1t}(\mathrm{Im}z) + \mathrm{Im}z(\mathrm{Im}\tau)^{-1t}\mathbf{e}_k\} \\ &= -2\left(\frac{-1}{2\sqrt{-1}}\right)\left(\frac{1}{2\sqrt{-1}}\right) \\ &\quad \times\{\mathbf{e}_k(\mathrm{Im}\tau)^{-1}E_{ij}(\mathrm{Im}\tau)^{-1t}(\mathrm{Im}z) + \mathrm{Im}z(\mathrm{Im}\tau)^{-1}E_{ij}(\mathrm{Im}\tau)^{-1t}\mathbf{e}_k\} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\{\mathbf{e}_k(\mathrm{Im}\tau)^{-1}E_{ij}{}^t x + xE_{ij}(\mathrm{Im}\tau)^{-1t}\mathbf{e}_k\} \\
&= -x_j h_{ik} - x_i h_{jk} \\
&= g^{\mathbb{A}_g}\left(\frac{\partial}{\partial \tau_{ij}}, \frac{\partial}{\partial z_k}\right),
\end{aligned} \tag{20}
$$

where the third equality follows from the identity

$$
\frac{\partial}{\partial \tau_{ij}}(\mathrm{Im}\tau)^{-1} = -\left(\frac{1}{2\sqrt{-1}}\right)(\mathrm{Im}\tau)^{-1}E_{ij}(\mathrm{Im}\tau)^{-1},
$$

the forth equality follows from the identity $x = (\mathrm{Im}z)(\mathrm{Im}\tau)^{-1}$ and the last equality follows from (16). Similarly, we get

$$
\begin{aligned}
&-\frac{1}{\pi}\frac{\partial^2}{\partial \tau_{ij}\partial \bar{\tau}_{kl}}\log\|\vartheta\|^2(z,\tau) \\
&= 2\frac{\partial^2}{\partial \tau_{ij}\partial \bar{\tau}_{kl}}\mathrm{Im}z(\mathrm{Im}\tau)^{-1t}\mathrm{Im}z \\
&= -2\left(\frac{-1}{2\sqrt{-1}}\right)\frac{\partial}{\partial \tau_{ij}}\mathrm{Im}z(\mathrm{Im}\tau)^{-1}E_{kl}(\mathrm{Im}\tau)^{-1t}\mathrm{Im}z \\
&= 2(-1)^2\left(\frac{-1}{2\sqrt{-1}}\right)\left(\frac{1}{2\sqrt{-1}}\right)\{\mathrm{Im}z(\mathrm{Im}\tau)^{-1}E_{ij}(\mathrm{Im}\tau)^{-1}E_{kl}(\mathrm{Im}\tau)^{-1t}\mathrm{Im}z \\
&\quad + \mathrm{Im}z(\mathrm{Im}\tau)^{-1}E_{kl}(\mathrm{Im}\tau)^{-1}E_{ij}(\mathrm{Im}\tau)^{-1t}\mathrm{Im}z\} \\
&= \frac{1}{2}\{xE_{ij}(\mathrm{Im}\tau)^{-1}E_{kl}{}^t x + xE_{kl}(\mathrm{Im}\tau)^{-1}E_{ij}{}^t x\} \\
&= x_i x_k h_{jl} + x_j x_k h_{il} + x_i x_l h_{jk} + x_j x_l h_{ik} \\
&= g^{\mathbb{A}_g}\left(\frac{\partial}{\partial \tau_{ij}}, \frac{\partial}{\partial \tau_{kl}}\right) - h'_{ijkl},
\end{aligned} \tag{21}
$$

where the last equality follows from (17).

Let Φ be the fundamental 2-form for $g^{\mathbb{A}_g}$. By (15), (18), (20) and (21), we get

$$
\Phi = -dd^c\log\|\vartheta\|_L^2(z,\tau) + f^*\omega_{\mathfrak{S}_g}.
$$

This completes the proof. □

Remark 1 By [21, Theorem 7.10], there exists a Γ_g-invariant Kähler metric $g^{\mathbb{A}_g}$ on $T\mathbb{A}_g$ such that $g^{\mathbb{A}_g}$ is a flat metric on each fiber and such that $p_* : T(\mathbb{A}_g/\mathfrak{S}_g)^\perp \to T\mathfrak{S}_g$ is an isometry. Here we gave an explicit construction of such a metric.

5 The L-form of the relative tangent bundle

Following [39, Proposition 5.1], we shall compute the Hirzebruch L-form of the relative tangent bundle of the family of smooth theta divisors, which will be used in Sects. 7 and 9.

A holomorphic function $f(\tau) \in \mathcal{O}(\mathfrak{S}_g)$ is a *Siegel modular form of weight k* if

$$f(\gamma \cdot \tau) = j(\tau, \gamma)^k \chi(\gamma) f(\tau), \quad \forall \gamma \in \Gamma_g,\ \forall \tau \in \mathfrak{S}_g,$$

where $j(\tau, \gamma) := \det(C\tau + D)$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\chi : \Gamma_g \to \mathbb{C}^*$ is a character. For a Siegel modular form $f(\tau)$ of weight k, define the *Petersson norm* by

$$\| f(\tau) \|^2 := (\det \mathrm{Im}\tau)^k |f(\tau)|^2. \tag{22}$$

By the automorphic property $\det \mathrm{Im}(\gamma \cdot \tau) = |j(\tau, \gamma)|^{-2} \det \mathrm{Im}(\tau)$ and the finiteness of $H_1(\Gamma_g, \mathbb{Z}) = \Gamma_g/[\Gamma_g, \Gamma_g]$, the norm $\| f(\tau)\|^2$ is a C^∞ Γ_g-invariant function on $\mathfrak{S}_g$. Set

$$\chi_g(\tau) := \prod_{a,b \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g,\ 4^t a \cdot b = 0 \in \mathbb{Z}/2\mathbb{Z}} \vartheta_{a,b}(0, \tau).$$

Then $\chi_g(\tau)$ is a Siegel modular form of weight $2^{g-2}(2^g + 1)$ and is called the *Igusa modular form*.

Let

$$\mathcal{N}_g := \{\tau \in \mathfrak{S}_g \mid \mathrm{Sing}\Theta_\tau \neq \emptyset\}$$

be the Andreotti–Mayer locus.

Theorem 9 ([19]) *The Andreotti–Mayer locus $\mathcal{N}_g$ is a divisor of $\mathfrak{S}_g$. There exist two Γ_g-invariant divisors θ_g and $\mathcal{J}_g$ on $\mathfrak{S}_g$ such that*

$$\mathcal{N}_g = \theta_g + 2\mathcal{J}_g,$$

where $\Gamma_g \backslash \theta_g$ and $\Gamma_g \backslash \mathcal{J}_g$ are irreducible divisors on $\Gamma_g \backslash \mathfrak{S}_g$. Here θ_g is the zero divisor of $\chi_g(\tau)$ and $\mathcal{J}_g = \emptyset$ if and only if $g = 2, 3$. There exist proper subvarieties $Z_1 \subset \theta_g$ and $Z_2 \subset \mathcal{J}_g$ with the following properties.

(1) *For any $\tau \in \theta_g^\circ := \theta_g \backslash Z_1$, $\mathrm{Sing}(\Theta_\tau)$ consists of one ordinary double point.*
(2) *For any $\tau \in \mathcal{J}_g^\circ := \mathcal{J}_g \backslash Z_2$, $\mathrm{Sing}(\Theta_\tau)$ consists of two ordinary double points which are mutually interchanged by the involution $z \to -z$.*

Theorem 10 ([39]) *There exists a Siegel cusp form $\Delta_g(\tau)$ of weight $\frac{(g+3)\cdot g!}{2}$ with zero divisor $\mathcal{N}_g$. In particular, there exists a Siegel modular form $J_g(\tau)$ of weight $\frac{(g+3)\cdot g!}{4} - 2^{g-3}(2^g + 1)$ with zero divisor $\mathcal{J}_g$ such that*

$$\Delta_g := \chi_g(\tau) J_g(\tau)^2.$$

We put

$$\mathfrak{S}_g^\circ := \mathfrak{S}_g - \mathcal{N}_g, \quad \Theta_g^\circ := \Theta|_{\mathfrak{S}_g^\circ}.$$

Then $p : \Theta^\circ \to \mathfrak{S}_g^\circ$ is a family of smooth theta divisors. Endow $T^{1,0}(\Theta^\circ/\mathfrak{S}_g^\circ)$ with the Hermitian metric $g^{\Theta^\circ/\mathfrak{S}_g^\circ} := g^{\mathbb{A}_g/\mathfrak{S}_g}\big|_{\Theta^\circ}$. Let $g^{\Theta^\circ} := g^{\mathbb{A}_g}|_{\Theta^\circ}$ be the Kähler metric on Θ° induced from $g^{\mathbb{A}_g}$. Regard $g^{\Theta^\circ/\mathfrak{S}_g^\circ}$ (resp. g^{Θ°) as a Riemannian metric on $T(\Theta^\circ/\mathfrak{S}_g^\circ)$ (resp. $T\Theta^\circ$). Let

$$T_H\Theta^\circ := T(\Theta^\circ/\mathfrak{S}_g^\circ)^\perp$$

be the orthogonal complement of $T(\Theta^\circ/\mathfrak{S}_g^\circ)$ in $T\Theta^\circ$ with respect to the metric g^{Θ°, which induces a connection $P_\Theta : T\Theta^\circ \to T\Theta^\circ/\mathfrak{S}_g^\circ$.

Lemma 7 *One has* $g^{\Theta^\circ} = g^{\Theta^\circ/\mathfrak{S}_g^\circ} \oplus p^*(g^{\mathfrak{S}_g}|_{\mathfrak{S}_g^\circ})$.

Proof Let N be the normal bundle of Θ° in $\mathbb{A}_g$. Endow N with the Hermitian metric induced from $g^{\mathbb{A}_g}$ via the C^∞-isomorphism $N \cong (T\Theta^\circ)^\perp$ in $T\mathbb{A}_g|_{\Theta^\circ}$. Then we have a C^∞ orthogonal decompositions $T\mathbb{A}_g|_{\Theta^\circ} \cong T\Theta^\circ \oplus N$ and $T(\mathbb{A}_g/\mathfrak{S}_g)|_{\Theta^\circ} = T(\Theta^\circ/\mathfrak{S}_g^\circ) \oplus N$. Hence we get the following equality of subvector bundles of $T\mathbb{A}_g|_{\Theta^\circ}$:

$$\begin{aligned} T_H\mathbb{A}_g|_{\Theta^\circ} &= T(\mathbb{A}_g/\mathfrak{S}_g)^\perp|_{\Theta^\circ} \quad (\text{in } T\mathbb{A}_g) \\ &= (T(\Theta^\circ/\mathfrak{S}_g^\circ) \oplus N)^\perp \quad (\text{in } T\Theta^\circ \oplus N) \\ &= T(\Theta^\circ/\mathfrak{S}_g^\circ)^\perp \quad (\text{in } T\Theta^\circ) \\ &= T_H\Theta^\circ. \end{aligned}$$

We thus have $p^*(g^{\mathfrak{S}_g}|_{\mathfrak{S}_g^\circ}) = f^*g^{\mathfrak{S}_g}|_{\Theta^\circ}$, which together with $g^{\Theta^\circ/\mathfrak{S}_g^\circ} = g^{\mathbb{A}_g/\mathfrak{S}_g}|_{\Theta^\circ}$, completes the proof. □

Lemma 8 *The connection P_Θ is compatible with the complex structure on Θ°.*

Proof Let $J \in \mathrm{End}(T\Theta^\circ)$ be the complex structure. Then the Riemannian metric g^{Θ° is invariant under the action of J. Therefore the orthogonal complement $T_H\Theta^\circ = T(\Theta^\circ/\mathfrak{S}_g^\circ)^\perp$ is also invariant under the action of J, which yields the assertion. □

We define the connection $\nabla^{\Theta^\circ/\mathfrak{S}_g^\circ}$ on $T(\Theta^\circ/\mathfrak{S}_g^\circ)$ by using $g^{\Theta^\circ/\mathfrak{S}_g^\circ}$ and P_Θ as in Sect. 2. Let ∇^h be the holomorphic Hermitian connection on $T^{1,0}(\Theta^\circ/\mathfrak{S}_g^\circ)$ with respect to the Hermitian metric $g^{\Theta^\circ/\mathfrak{S}_g^\circ}$.

Lemma 9 *Under the C^∞-isomorphism $T(\Theta^\circ/\mathfrak{S}_g^\circ) \otimes \mathbb{C} \cong T^{1,0}(\Theta^\circ/\mathfrak{S}_g^\circ) \oplus T^{0,1}(\Theta^\circ/\mathfrak{S}_g^\circ)$, the following equality of connections holds*:

$$\nabla^{\Theta^\circ/\mathfrak{S}_g^\circ} \otimes \mathbb{C} = \nabla^h \oplus \bar{\nabla}^h.$$

Proof Let ∇^L be the Levi–Civita connection on $(T\Theta^\circ, g^{\Theta^\circ})$ and let ∇^H be the holomorphic Hermitian connection on $T^{1,0}\Theta^\circ$. Let $P_\Theta^{\mathbb{C}}$ be the complexification of P_Θ. Since g^{Θ° is Kähler by Theorem 8, we get the decomposition by [25, Chap. I, Proposition 7.19]

$$\nabla^L \otimes \mathbb{C} = \nabla^H \oplus \bar{\nabla}^H$$

under the decomposition $T\Theta^\circ \otimes \mathbb{C} = T^{1,0}\Theta^\circ \oplus T^{0,1}\Theta^\circ$. By Lemma 8, we also get the decomposition $P_\Theta^{\mathbb{C}} = P_\Theta^{1,0} \oplus P_\Theta^{0,1}$. Then

$$\nabla^{\Theta^\circ/\mathfrak{S}_g^\circ} \otimes \mathbb{C} = (P_\Theta \nabla^L) \otimes \mathbb{C} = P_\Theta^{\mathbb{C}}(\nabla^L \otimes \mathbb{C}) = P_\Theta^{1,0}\nabla^H \oplus P_\Theta^{0,1}\bar{\nabla}^H.$$

Since $P_\Theta^{1,0}\nabla^H = \nabla^h$ by [25, Chap. I, Proposition 6.4], we get the result. □

Let B_k be the kth Bernoulli number when $k \in \mathbb{Z}$, i.e.,

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}.$$

We set $B_k = 0$ when $k \in \frac{1}{2} + \mathbb{Z}$.

Theorem 11 *Let g be even. The following equality holds*:

$$\begin{aligned}\left[p_* L(T(\Theta^\circ/\mathfrak{S}_g^\circ), \nabla^{\Theta^\circ/\mathfrak{S}_g^\circ})\right]^{(2)} &= \frac{(-1)^{g/2} 2^{g+1}(2^{g+2}-1)}{(g+1)(g/2+1)} B_{\frac{g}{2}+1} dd^c \log\det \operatorname{Im}\tau \\ &= \frac{(-1)^{g/2} 2^{g+3}(2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1} dd^c \log \|\Delta_g(\tau)\|^2,\end{aligned}$$

where f_* denotes the integration along the fibers and $\alpha^{(p)}$ denotes the p-form part of a from α.

Remark 2 When g is odd, say $2k+1$, since $\dim_{\mathbb{R}} \Theta_\tau = 4k$ and the L-form has only components of degree $4n$, the left-hand side of Theorem 11 is zero.

Proof The second equality follows from (22) and $\mathfrak{S}_g^\circ = \mathfrak{S}_g \backslash \mathrm{div}(\Delta_g)$. We prove the first equality. Let $R^h := (\nabla^h)^2$ be the curvature, which is a $(1,1)$-form with values in $\mathrm{End}(T^{1,0}(\Theta^\circ/\mathfrak{S}_g^\circ))$. Set

$$\mathbf{L}(x) := x/\tanh(x). \tag{23}$$

For a complex vector bundle E, let $\mathbf{L}(E)$ denote the multiplicative genus of Chern forms associated with $\mathbf{L}(x)$. By (8), we get

$$\begin{aligned}L\left(T(\Theta^\circ/\mathfrak{S}_g^\circ), \nabla^{\Theta/\mathfrak{S}_g^\circ}\right)^{(2g)} &= \det\left(\frac{-R^h/2\pi\sqrt{-1}}{\tanh(-R^h/2\pi\sqrt{-1})}\right)^{(g,g)} \\ &= \mathbf{L}\left(T^{1,0}(\Theta^\circ, \mathfrak{S}_g^\circ), \nabla^h\right)^{(g,g)}.\end{aligned} \tag{24}$$

Here the first equality follows from Lemma 9, the equality $\bar{R}^h = -{}^t R^h$ and the fact that $x/\tanh(x)$ is an even function.

Let G be a positive definite $g \times g$-Hermitian matrix and let $g_G := dz\, G\, {}^t\bar{d}z$ be a flat metric on $W := \mathbb{C}^g$ associated to G. Let $\mathbb{P}(W^\vee)$ be the projective space of hyperplanes of W and let E be the universal vector bundle of rank $(g-1)$ over $\mathbb{P}(W^\vee)$. Consider the following exact sequence of vector bundles over $\mathbb{P}(W^\vee)$:

$$0 \longrightarrow E \longrightarrow W^\vee = \mathbb{C}^g \longrightarrow N = W^\vee/E \longrightarrow 0. \tag{25}$$

Notice that $N = \mathcal{O}_{\mathbb{P}(W^\vee)}(1)$. Let $g_{E,G} := g_G|_E$ be the induced metric on E.

Let g_{1_g} be the restriction of the Hermitian metric $dz \cdot {}^t d\bar{z}$ on $T\mathbb{A}_g/\mathfrak{S}_g$ to the relative tangent bundle $T\Theta^\circ/\mathfrak{S}_g^\circ$. Let R be the curvature of the holomorphic Hermitian connection of $(T^{1,0}\Theta^\circ/\mathfrak{S}_g^\circ, g_{1_g})$. Set

$$\mathbf{L}(T^{1,0}\Theta^\circ/\mathfrak{S}_g^\circ, g_{1_g}) := \det \mathbf{L}\left(\frac{-R}{2\pi\sqrt{-1}}\right) \in \oplus_{p\geq 0} A^{p,p}(\Theta^\circ).$$

Let $\nu : \Theta_\tau \longrightarrow \mathbb{P}(W^\vee)$ be the Gauss map:

$$\nu : \Theta_\tau \ni z \longmapsto (T\Theta_\tau)_z \in \mathbb{P}(W^\vee),$$

which induces a finite covering with mapping degree $g!$. Then

$$(T\Theta_\tau, g^{\Theta_\tau}) = \nu^*(E, g_{E,(\mathrm{Im}\tau)^{-1}}). \tag{26}$$

By [38, Proposition 2.1], we have

$$\left[\mathbf{L}(T\Theta^\circ/\mathfrak{S}_g^\circ, g_{1_g})\right]^{(g,g)} \equiv 0.$$

Hence we obtain

$$\begin{aligned}\left[\mathbf{L}(T^{1,0}(\Theta^\circ/\mathfrak{S}_g^\circ), \nabla^h)\right]^{(g,g)} &= \left[\mathbf{L}(T^{1,0}(\Theta^\circ/\mathfrak{S}_g^\circ), \nabla^h)\right]^{(g,g)} \\ &\quad - \left[\mathbf{L}(T^{1,0}(\Theta^\circ/\mathfrak{S}_g^\circ), g_{1_g})\right]^{(g,g)} \\ &= -dd^c\left[\widetilde{\mathbf{L}}(T^{1,0}(\Theta^\circ/\mathfrak{S}_g^\circ); g_{1_g}, g^{\Theta^\circ/\mathfrak{S}_g^\circ})\right]^{(g-1,g-1)},\end{aligned} \tag{27}$$

where $\widetilde{\mathbf{L}}(T^{1,0}(\Theta^\circ/\mathfrak{S}_g^\circ); g_{1_g}, g^{\Theta^\circ/\mathfrak{S}_g^\circ})$ denotes the Bott–Chern secondary form [13,15] corresponding to $\mathbf{L}$. By (24), (26), and Proposition 7 below, we get

$$\begin{aligned}
p_*\left[\widetilde{\mathbf{L}}(T\Theta_\tau, g_{1_g}, g^{\Theta_\tau})\right]^{(g-1,g-1)} &= p_*\left[\nu^*\widetilde{\mathbf{L}}(E; g_{E,1_g}, g_{E,(\mathrm{Im}\tau)^{-1}})\right]^{(g-1,g-1)} \\
&= \deg\nu \int_{\mathbb{P}(W^\vee)} \widetilde{\mathbf{L}}(E; g_{E,1_g}, g_{E,(\mathrm{Im}\tau)^{-1}}) \\
&= -g!k(\mathbf{L}, g)\log\det\mathrm{Im}\tau, \qquad (28)
\end{aligned}$$

where $k(\mathbf{L}, g)$ is the constant defined in (72) below. By (27), (28) and the following Lemma 10, we complete the proof. □

Lemma 10 *The following equality holds*:

$$k(\mathbf{L}, 2k) = (-1)^g(2k+1)\frac{4^{k+1}(4^{k+1}-1)}{(2k+2)!}B_{k+1}.$$

Proof By (72) and the relation $\tanh'(x) = 1 - \tanh(x)^2$, we get

$$\begin{aligned}
k(\mathbf{L}, 2k) &= \left(\frac{\mathbf{L}'(0)}{\mathbf{L}(0)}\cdot\mathbf{L}^{-1}(x) - \frac{1}{2k}\mathbf{L}'(x)\cdot\mathbf{L}^{-2}(x)\right)\Big|_{x^{2k-1}} \\
&= -\frac{1}{2k}\left(\frac{\tanh(x)}{x^2} - \frac{\tanh'(x)}{x}\right)\Big|_{x^{2k-1}}, \qquad (29)
\end{aligned}$$

where $h(x)|_{x^g}$ is the coefficient of x^g for $h(x) \in \mathbb{C}[[x]]$. Combined with (29), the Taylor expansion

$$\tanh(x) = \sum_{n\geq 1}\frac{(-1)^{n+1}4^n(4^n-1)B_n}{(2n)!}x^{2n-1} \qquad (30)$$

yields the assertion. □

Remark 3 In Sect. 7, it will be crucial that $d^c\log\|\Delta_g(\tau)\|^2$ is Γ_g-invariant and that $dd^c\log\|\Delta_g(\tau)\|^2$ is an exact 2-form on $\Gamma_g\backslash\mathfrak{S}_g^\circ$.

6 The signature cocycle of smooth theta divisors

Since Γ_g acts on $\mathfrak{S}_g^\circ$ properly discontinuously, the quotient $\Gamma_g\backslash\mathfrak{S}_g^\circ$ has the structure of a complex orbifold and $\Gamma_g\backslash\mathfrak{S}_g^\circ$ is a coarse moduli space of smooth theta divisors. In this section, following [2], we construct a 2-cocycle of the orbifold fundamental group of $\Gamma_g\backslash\mathfrak{S}_g^\circ$, which is an analogue of the Meyer cocycle [2,37].

We fix a base point $* \in \mathfrak{S}_g^\circ$ such that $\{\gamma \in \Gamma_g \mid \gamma\cdot * = *\} = \{\pm 1_{2g}\}$. Let (B, b) be a topological space with base point b, and let $\pi : \widetilde{B}\to B$ be the universal covering. The fundamental group $\pi_1(B, b)$ acts on $\widetilde{B}$ as deck transformations. Fix a point $\tilde{b}\in\widetilde{B}$ with $\pi(\tilde{b}) = b$. We define the set $[B, \Gamma_g\backslash\mathfrak{S}_g^\circ]^{orb}$ by

$$\{(\alpha, \rho) \in C^0(\tilde{B}, \mathfrak{S}_g^\circ) \times \mathrm{Hom}(\pi_1(B, b), \Gamma_g) \mid \alpha(\tilde{b}) = *,\ \alpha(\gamma\cdot x) = \rho(\gamma)\cdot\alpha(x)\}/\sim.$$

Here $(\alpha_0, \rho_0) \sim (\alpha_1, \rho_1)$ if and only if $\rho_0 = \rho_1$ and there exists a homotopy $\tilde{p} : \widetilde{B} \times [0, 1] \to \mathfrak{S}_g^\circ$ connecting α_0 and α_1 such that $\tilde{\alpha}(*, 0) = \alpha_0$, $\tilde{\alpha}(*, 1) = \alpha_1$ and

$$\tilde{\alpha}(\gamma \cdot x, t) = \rho(\gamma) \cdot \tilde{\alpha}(x, t), \quad \gamma \in \Gamma_g, \ x \in \widetilde{B}, \ t \in [0, 1].$$

Definition 5 Define the *orbifold fundamental group* of $\Gamma_g \backslash \mathfrak{S}_g^\circ$ by

$$\begin{aligned} \mathcal{S}_g &:= [S^1, \Gamma_g \backslash \mathfrak{S}_g^\circ]^{orb} \\ &= \Big\{ (\alpha, \gamma) \in C^0(\mathbb{R}, \mathfrak{S}_g^\circ) \times \Gamma_g \mid \alpha(0) = *, \\ &\qquad \alpha(t+1) = \gamma \cdot \alpha(t), \ \ \forall t \in \mathbb{R} \Big\} / \sim . \end{aligned}$$

One has the following equivalent definition:

$$\mathcal{S}_g := \Big\{ (\alpha, \gamma) \in C^0([0, 1], \mathfrak{S}_g^\circ) \times \Gamma_g \mid \alpha(0) = \gamma^{-1} \cdot \alpha(1) = * \Big\} / \approx .$$

Here $(\alpha_0, \gamma_0) \approx (\alpha_1, \gamma_1)$ if and only if $\gamma_0 = \gamma_1$ and there exists a homotopy $\alpha(s, t) : [0, 1] \times [0, 1] \to \mathfrak{S}_g^\circ$ connecting α_0 and α_1 such that $\alpha(0, t) = \alpha_0(t)$, $\alpha(1, t) = \alpha_1(t)$, $\alpha(s, 0) = \gamma_0^{-1} \cdot \alpha(s, 1) = *$ for $s \in [0, 1]$.

The group law of $\mathcal{S}_g$ is defined as follows. Let $[(\alpha_1, \gamma_1)], [(\alpha_2, \gamma_2)] \in \mathcal{S}_g$. Then $\gamma_1 \cdot \alpha_2$ is a path connecting $\gamma_1 \cdot *$ and $(\gamma_1\gamma_2) \cdot *$. Define the new path $\alpha : [0, 1] \to \mathfrak{S}_g^\circ$ by

$$\alpha(t) := \begin{cases} \alpha_1(2t) & 0 \le t \le \frac{1}{2}, \\ \gamma_1 \cdot \alpha_2(2t - 1) & \frac{1}{2} \le t \le 1. \end{cases}$$

Then $[(\alpha_1, \gamma_1)] \cdot [(\alpha_2, \gamma_2)] := [(\alpha, \gamma_1\gamma_2)]$. For $\sigma = [(l, \gamma)] \in \mathcal{S}_g$, the inverse is given by

$$\sigma^{-1} = [(-(\gamma^{-1} \cdot l), \gamma^{-1})], \quad -l(t) := l(1 - t), \ t \in [0, 1]. \tag{31}$$

Let $p : \mathcal{S}_g \to \Gamma_g$ be the projection to the second factor. Since the kernel of p is isomorphic to $\pi_1(\mathfrak{S}_g^\circ, *)$, we have an exact sequence

$$1 \to \pi_1(\mathfrak{S}_g^\circ, *) \to \mathcal{S}_g \to \Gamma_g \to 1. \tag{32}$$

Remark 4 When $g = 1$, $\Gamma_1 \backslash \mathfrak{S}_1^\circ = SL_2(\mathbb{Z}) \backslash \mathfrak{S}_1$ is the moduli space of curves of genus 1 and $\mathcal{S}_1 = \mathcal{M}_1$. When $g = 2$, $\Gamma_2 \backslash \mathfrak{S}_2^\circ$ is the moduli space of curves of genus 2 by the Torelli theorem and $\mathcal{S}_2 = \mathcal{M}_2$. By (32), $\mathcal{S}_g$ is regarded as an analogue of the mapping class group.

Recall that a $\pi_1(B, b)$-equivariant map $(f, \rho) : (\tilde{B}, \tilde{b}) \to (\mathfrak{S}_g^\circ, *)$ is a pair (f, ρ), where $f \in C^0(\tilde{B}, \mathfrak{S}_g^\circ)$ and $\rho \in \mathrm{Hom}(\pi_1(B, b), \Gamma_g)$ satisfies the relations $f(\tilde{b}) = *$

and $f(\gamma \cdot x) = \rho(\gamma) \cdot f(x)$ for $\gamma \in \pi_1(B, b)$, $x \in \tilde{B}$. Given a $\pi_1(B, b)$-equivariant map (f, ρ), one obtains the homomorphism of groups $f_* : \pi_1(B, b) \to \mathcal{S}_g$ by $f_*([c]) = [(f \circ c, \rho([c]))]$ for $[c] \in \pi_1(B, b)$.

Let F be a compact oriented surface with non empty boundary. Fix a base point $b \in F$. Since F is homotopy equivalent to the n-bouquet $\mathbb{B}_n := S^1 \vee \cdots \vee S^1$ (n-times) for some $n \in \mathbb{Z}_{\geq 1}$, $\pi_1(F, b) \cong \pi_1(\mathbb{B}_n, *)$ is a free group of rank n. We have

$$\begin{aligned}[][\mathbb{B}_n, \Gamma_g \backslash \mathfrak{S}_g^\circ]^{orb} &\cong [S^1, \Gamma_g \backslash \mathfrak{S}_g^\circ]^{orb} \times \cdots \times [S^1, \Gamma_g \backslash \mathfrak{S}_g^\circ]^{orb} \quad (n \text{ times}) \\ &\cong \mathcal{S}_g \times \cdots \times \mathcal{S}_g \quad (n \text{ times}). \end{aligned} \tag{33}$$

Fix a set $\{g_1, \dots, g_n\}$ of generators of $\pi_1(F, b) \cong \pi_1(\mathbb{B}_n, *)$ as a free group of rank n. Since $[F, \Gamma_g \backslash \mathfrak{S}_g^\circ]^{orb} \equiv [\mathbb{B}_n, \Gamma_g \backslash \mathfrak{S}_g^\circ]^{orb}$ we obtain the bijection by (33)

$$[F, \Gamma_g \backslash \mathfrak{S}_g^\circ]^{orb} \;\cong\; \mathcal{S}_g \times \cdots \times \mathcal{S}_g \quad (n \text{ times}), \tag{34}$$

which is given by $[(f, \rho)] \longmapsto ([f_*(g_1), \rho(g_1)], \cdots, [f_*(g_n), \rho(g_n)])$.

From now, we denote by $\mathcal{B}$ a pants, i.e.,

$$\mathcal{B} = S^2 \backslash \amalg_{k=1}^3 D_k,$$

where D_1, D_2, D_3 are mutually disjoint open discs. Fix a base point $b \in \mathcal{B}$. Since $\mathcal{B}$ is homotopy equivalent to the 2-bouquet $\mathbb{B}_2$, $\pi_1(\mathcal{B}, b)$ is the free group of rank 2. Let g_1, g_2 be the generators of $\pi_1(\mathcal{B}, b)$ such that g_i is represented by a loop homotopy equivalent to ∂D_i. By (34) we have the bijection

$$[\mathcal{B}, \Gamma_g \backslash \mathfrak{S}_g^\circ]^{orb} \;\cong\; \mathcal{S}_g \times \mathcal{S}_g. \tag{35}$$

For $[(f, \rho)] \in [\mathcal{B}, \Gamma_g \backslash \mathfrak{S}_g^\circ]^{orb}$ the fiber product $\pi : \widetilde{\mathcal{B}} \times_f \Theta \to \widetilde{\mathcal{B}}$ is a $\pi_1(\mathcal{B}, b)$-equivariant fiber bundle because $f : \widetilde{\mathcal{B}} \to \mathfrak{S}_g^\circ$ is a $\pi_1(\mathcal{B}, b)$-equivariant map. Hence we get the fiber bundle $\pi : (\widetilde{\mathcal{B}} \times_f \Theta)/\pi_1(\mathcal{B}, b) \to \mathcal{B}$, which is uniquely determined by $[f] \in [\mathcal{B}, \Gamma_g \backslash \mathfrak{S}_g^\circ]^{orb}$ up to homotopy and which is a $2g$-dimensional compact oriented manifold with boundary. If $[(f, \rho)]$ corresponds to $(\sigma_1, \sigma_2) \in \mathcal{S}_g \times \mathcal{S}_g$ via the isomorphism (35), we set

$$X(\sigma_1, \sigma_2) := (\tilde{\mathcal{B}} \times_f \Theta)/\pi_1(\mathcal{B}, b).$$

Then $\pi : X(\sigma_1, \sigma_2) \to \mathcal{B}$ is a differentiable family of smooth theta divisors whose monodromy around ∂D_i is σ_i for $i = 1, 2$.

Recall that for $4k$-dimensional compact oriented manifold with boundary the signature $\mathrm{Sign}(X)$ is defined as the signature of the cup-product pairing on $H^{2k}(X, \partial X, \mathbb{Q})$.

Definition 6 Define the map $c_g : \mathcal{S}_g \times \mathcal{S}_g \to \mathbb{Z}$ by

$$c_g(\sigma_1, \sigma_2) := \mathrm{Sign}(X(\sigma_1, \sigma_2)), \qquad (\sigma_1, \sigma_2) \in \mathcal{S}_g \times \mathcal{S}_g.$$

We call c_g the *signature cocycle for smooth theta divisors*.

Remark 5 When g is odd, $c_g \equiv 0$ because $\mathrm{Sign}(X(\sigma_1, \sigma_2))$ always vanishes in this case.

Lemma 11 *The following equality holds*:

(a) $c_g(\sigma_1, \sigma_2) + c_g(\sigma_1\sigma_2, \sigma_3) = c_g(\sigma_2, \sigma_3) + c_g(\sigma_2\sigma_3, \sigma_1)$,
(b) *If* $\sigma_1\sigma_2\sigma_3 = I$, *then* $c_g(\sigma_1, \sigma_2) = c_g(\sigma_2, \sigma_3) = c_g(\sigma_3, \sigma_1)$,
(c) $c_g(\sigma_1, I) = c_g(I, \sigma_1) = 0$,
(d) $c_g(\sigma_1, \sigma_2) = c_g(\sigma_2, \sigma_1)$,
(e) $c_g(\sigma_1^{-1}, \sigma_2^{-1}) = -c_g(\sigma_1, \sigma_2)$,
(f) $c_g(\sigma_3\sigma_1\sigma_3^{-1}, \sigma_3\sigma_2\sigma_3^{-1}) = c_g(\sigma_1, \sigma_2)$,

where $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{S}_g$ *and* I *is the unit element. In particular,* c_g *is a* 2*-cocycle of the group* $\mathcal{S}_g$ *by* (*a*).

Proof By the same argument as in [2, p. 343], we obtain the assertion. □

Denote by $[c_g] \in H^2(\mathcal{S}_g, \mathbb{Z})$ the cohomology class of c_g. Then c_2 is the Meyer cocycle of genus two.

Remark 6 Let $\rho : \mathcal{S}_g \to \mathrm{Aut}(H^{g-1}(\Theta_*, \mathbb{Z}), <, >)$ be the monodromy representation, where $<\ ,\ >$ denotes the cup-product pairing. When g is even, $<\ ,\ >$ is skew-symmetric and $\mathrm{Aut}(H^{g-1}(\Theta_*, \mathbb{Z}), <, >) \cong \Gamma_{k_g}$, where $k_g = \frac{1}{2}\dim_{\mathbb{R}} H^{g-1}(\Theta_*, \mathbb{R})$. Hence we have the homomorphism $\rho : \mathcal{S}_g \to \Gamma_{k_g}$. In this case, c_g is the pull-back of the signature cocycle of Γ_{k_g} via the map ρ by [1, Sect. 4] and [2, Sect. 2]. When $g = 2$, ρ is equal to the homomorphism in (32). However this is not the case for general g, because $\dim_{\mathbb{R}} H^{g-1}(\Theta_*, \mathbb{R}) > g$ for $g > 2$.

7 Construction of the Meyer function for smooth theta divisors

As we explained in Sect. 1, the cohomology class of the Meyer cocycle τ_g is a torsion element of $H^2(\mathcal{M}_g, \mathbb{Z})$ for $g = 1, 2$ because $H^2(\mathcal{M}_g, \mathbb{Q}) = 0$. In this section we shall prove that the cohomology class of the signature cocycle c_g is a torsion element of $H^2(\mathcal{S}_g, \mathbb{Z})$ by constructing a 1-cochain that cobounds c_g explicitly. We don't know whether $H^2(\mathcal{S}_g, \mathbb{Q})$ vanishes or not when $g > 2$, while we will see that $H^2(\mathcal{S}_g, \mathbb{Z}) \neq 0$ for $g \geq 1$ in the next section.

Let $\sigma = [(\alpha, \gamma)] \in \mathcal{S}_g$. The fiber product $\mathbb{R} \times_\alpha \Theta^\circ$ is equipped with the $\pi_1(S^1)$-action such that $m \cdot (t, (z, \alpha(t))) = (t + m, \gamma^m \cdot (z, \alpha(t)))$. We define the *mapping torus* $M_{(\alpha,\gamma)}$ by

$$\pi : M_{(\alpha,\gamma)} := (\mathbb{R} \times_\alpha \Theta^\circ)/\pi_1(S^1) \to S^1, \qquad \pi = \mathrm{pr}_1.$$

Since the metric $g^{\Theta^\circ/\mathfrak{S}_g^\circ}$ on $T(\Theta^\circ/\mathfrak{S}_g^\circ)$ and the connection P_Θ on Θ° are Γ_g-invariant and since the map $\alpha : \widetilde{S^1} = \mathbb{R} \to \mathfrak{S}_g^\circ$ is $\pi_1(S^1)$-equivariant, the metric $g^{M_{(\alpha,\gamma)}/S^1}$ on $T(M_{(\alpha,\gamma)}/S^1)$ (resp. the connection $P_{(\alpha,\gamma)}$ on $M_{(\alpha,\gamma)}$) is induced from $g^{\Theta^\circ/\mathfrak{S}_g^\circ}$ (resp. P_Θ) via the map α. With respect to the decomposition $TM_{(\alpha,\gamma)} =$

$T(M_{(\alpha,\gamma)}/S^1) \oplus \pi^* T S^1$ associated with $P_{(\alpha,\gamma)}$, we define the one-parameter family of Riemannian metrics $g_\varepsilon^{M_{(\alpha,\gamma)}}$ on $M_{(\alpha,\gamma)}$ by

$$g_\varepsilon^{M_{(\alpha,\gamma)}} := g^{M_{(\alpha,\gamma)}/S^1} \oplus \varepsilon^{-1}\pi^* dt^2, \qquad \varepsilon \in \mathbb{R}_{>0}.$$

Here we regard S^1 as $\mathbb{R}/\mathbb{Z}$ and $t \in \mathbb{R}$ as a local coordinate of S^1. By Theorem 6, there exists the adiabatic limit

$$\eta^0(M_{(\alpha,\gamma)}) := \lim_{\varepsilon \to 0} \eta(M_{(\alpha,\gamma)}, g_\varepsilon^{M_{(\alpha,\gamma)}}).$$

Since the 1-form $d^c \log \| \Delta_g(\tau) \|^2$ is Γ_g-invariant, the pull-back $\alpha^* d^c \log \| \Delta_g(\tau) \|^2$ can be regarded as a 1-form on S^1.

Definition 7 For $\sigma \in \mathcal{S}_g$, let (α, γ) be a representative of σ, i.e., $\sigma = [(\alpha,\gamma)]$ and set

$$\Phi_g(\alpha,\gamma) := \eta^0(M_{(\alpha,\gamma)}) + \frac{(-1)^{g/2} 2^{g+3} (2^{g+2}-1) B_{\frac{g}{2}+1}}{(g+3)!} \int_{S^1} \alpha^* d^c \log \| \Delta_g(\tau) \|^2.$$

The following theorem is the main result of this paper.

Theorem 12 (a) *The value $\Phi_g(\alpha,\gamma)$ is independent of the choice of a representative (α,γ) of $\sigma \in \mathcal{S}_g$. In particular Φ_g is a function on $\mathcal{S}_g$.*

(b) *The function Φ_g satisfies*
- (b1) $c_g(\sigma_1,\sigma_2) = -\Phi_g(\sigma_1) - \Phi_g(\sigma_2) + \Phi_g(\sigma_1\sigma_2)$,
- (b2) $\Phi_g(I) = 0$,
- (b3) $\Phi_g(\sigma_1^{-1}) = -\Phi_g(\sigma_1)$,
- (b4) $\Phi_g(\sigma_2\sigma_1\sigma_2^{-1}) = \Phi_g(\sigma_1)$,

where $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{S}_g$. In particular, $[c_g] \otimes \mathbb{Q} = 0 \in H^2(\mathcal{S}_g, \mathbb{Q})$ by (b1).

Recall that the Meyer function ϕ_2 of genus two cobounds the Meyer cocycle τ_2 (cf. Introduction). As a consequence of Theorem 12, we get $\phi_2 = \Phi_2$ by the uniqueness of the Meyer function of genus 2. Since $\Delta_2(\tau)$ coincides with the Igusa modular form $\chi_2(\tau)$ up to a constant [39], we get the following analytic representation of the Meyer function ϕ_2.

Corollary 1 ([23]) *Let $\sigma = [(\alpha,\gamma)]$ be an element of $\mathcal{S}_2 = \mathcal{M}_2$. Then*

$$\phi_2(\sigma) = \eta^0(M_{(\alpha,\gamma)}) - \frac{2}{15} \int_{S^1} \alpha^* d^c \log \| \chi_2(\tau) \|^2.$$

Proof of Theorem 12 (*a*) Assume that (α_0, γ) and (α_1, γ) represent the same element $\sigma \in \mathcal{S}_g$. Put $I := [0,1]$. There exists a continuous map $\bar{\alpha} : I \times \mathbb{R} \to \mathfrak{S}_g^\circ$ satisfying

$$\bar{\alpha}(s,0) = *, \quad s \in I, \qquad \bar{\alpha}(s,t) = \gamma \cdot \bar{\alpha}(s,t+1), \quad (s,t) \in I \times \mathbb{R}$$

and

$$\bar{\alpha}(s,t)=\begin{cases}\alpha_0(t) & s\in[0,\frac{1}{3})\\ \alpha_1(t) & s\in(\frac{2}{3},1].\end{cases} \tag{36}$$

Since $\bar{\alpha}$ is $\pi_1(I\times S^1)$-equivariant, the fiber product $(I\times\mathbb{R})\times_{\bar{\alpha}}\Theta^\circ$ is endowed with the $\pi_1(I\times S^1)$-action, and we have the fiber bundle

$$\bar{\pi}: M_{(\bar{\alpha},\gamma)}:=(I\times\mathbb{R})\times_{\bar{\alpha}}\Theta^\circ/\pi_1(I\times S^1)\longrightarrow I\times S^1.$$

By the Γ_g-invariance of $g^{\Theta^\circ/\mathfrak{S}_g^\circ}$ and the $\pi_1(I\times S^1)$-equivariance of $\bar{\alpha}$, $g^{\Theta^\circ/\mathfrak{S}_g^\circ}$ induces a metric $g^{M_{(\bar{\alpha},\gamma)}/I\times S^1}$ on $T(M_{(\bar{\alpha},\gamma)}/I\times S^1)$, and the connection P_Θ induces a connection $P_{(\bar{\alpha},\gamma)}$ on $M_{(\bar{\alpha},\gamma)}$. With respect to the decomposition $TM_{(\bar{\alpha},\gamma)}=T(M_{(\bar{\alpha},\gamma)}/I\times S^1)\oplus\tilde{\pi}^*T(I\times S^1)$ associated with $P_{(\bar{\alpha},\gamma)}$, we set

$$g_\varepsilon^{M_{(\bar{\alpha},\gamma)}}:=g^{M_{(\bar{\alpha},\gamma)}/I\times S^1}\oplus\varepsilon^{-1}\pi^*(ds^2\oplus dt^2),\qquad \varepsilon\in\mathbb{R}_{>0}.$$

Let $\nabla^{M_{(\bar{\alpha},\gamma)}/(S^1\times I)}$ be the connection on $T(M_{(\bar{\alpha},\gamma)}/(S^1\times I))$ associated with $g^{M_{(\bar{\alpha},\gamma)}/(S^1\times I)}$ and $P_{(\bar{\alpha},\gamma)}$. By (36) and Lemma 2 ($c$), $g_\varepsilon^{M_{(\bar{\alpha},\gamma)}}$ is a product metric on a color neighborhood of the boundary $\partial M_{(\bar{\alpha},\gamma)}$, i.e.,

$$g_\varepsilon^{M_{(\bar{\alpha},\gamma)}}\big|_{[0,\frac{1}{3})\times S^1}=g_\varepsilon^{M_{(\alpha_0,\gamma)}}\oplus\varepsilon^{-1}dt^2,\quad g_\varepsilon^{M_{(\bar{\alpha},\gamma)}}\big|_{(\frac{2}{3},1]\times S^1}=g_\varepsilon^{M_{(\alpha_1,\gamma)}}\oplus\varepsilon^{-1}dt^2.$$

The Atiyah–Patodi–Singer index theorem applied to $(M_{(\bar{\alpha},\gamma)},g_\varepsilon^{M_{(\bar{\alpha},\gamma)}})$ yields that

$$\begin{aligned}\mathrm{Sign}(M_{(\bar{\alpha},\gamma)})=&\int_{I\times S^1}\tilde{\pi}_*L\left(TM_{(\bar{\alpha},\gamma)},g_\varepsilon^{M_{(\bar{\alpha},\gamma)}}\right)\\&-\left(\eta(M_{(\bar{\alpha}_0,\gamma)},g_\varepsilon^{M_{(\alpha_0,\gamma)}})-\eta(M_{(\bar{\alpha}_1,\gamma)},g_\varepsilon^{M_{(\alpha_1,\gamma)}})\right).\end{aligned} \tag{37}$$

Since I is contractible, $M_{(\bar{\alpha},\gamma)}$ is diffeomorphic to $M_{(\alpha_0,\gamma)}\times I$. Hence

$$\mathrm{Sign}(M_{(\bar{\alpha},\gamma)})=\mathrm{Sign}(M_{(\alpha_0,\gamma)})\times\mathrm{Sign}(I)=0. \tag{38}$$

Let $pr : M_{(\alpha,\gamma)} \to \Theta^\circ$ be the projection to the second factor. Then we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{I\times S^1} \bar{\pi}_* L\left(M_{(\bar{\alpha},\gamma)}, g_\varepsilon^{M_{(\bar{\alpha},\gamma)}}\right) \\
&= \int_{I\times S^1} \bar{\pi}_* \left(L\left(T(M_{(\bar{\alpha},\gamma)}/(I\times S^1))\right) \wedge \bar{\pi}^* L\left(T(I\times S^1)\right)\right) \\
&= \int_{I\times S^1} \left[\bar{\pi}_* L\left(T(M_{(\bar{\alpha},\gamma)}/(I\times S^1)), \nabla^{M_{(\bar{\alpha},\gamma)}/(I\times S^1)}\right)\right]^{(2)} \\
&= \int_{I\times S^1} \left[\bar{\pi}_* pr^* L\left(T(\Theta^\circ/\mathfrak{S}_{2g}^\circ), \nabla^{\Theta^\circ/\mathfrak{S}_g^\circ}\right)\right]^{(2)} \\
&= \int_{I\times S^1} \bar{\alpha}^* \left[p_* L\left(T(\Theta^\circ/\mathfrak{S}_g^\circ), \nabla^{\Theta^\circ/\mathfrak{S}_g^\circ}\right)\right]^{(2)}, \qquad (39)
\end{aligned}$$

where the first equality follows from Proposition 2, the third equality follows from Proposition 1 and we used the identity $\bar{\pi}_* p_2^* \omega = \bar{\alpha}^* p_* \omega$ for $\omega \in \mathcal{A}^k(\Theta^\circ)$ to get the last equality. By Theorem 11, we have

$$\begin{aligned}
&\int_{I\times S^1} \bar{\alpha}^* \left[p_* L\left(T(\Theta^\circ/\mathfrak{S}_g^\circ), \nabla^{\Theta^\circ/\mathfrak{S}_g^\circ}\right)\right]^{(2)} \\
&= \frac{(-1)^{g/2} 2^{g+3} (2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1} \int_{I\times S^1} \bar{\alpha}^* dd^c \log\|\varDelta_g(\tau)\|^2 \\
&= \frac{(-1)^{g/2} 2^{g+3} (2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1} \\
&\quad \times \left(\int_{\{1\}\times S^1} \alpha_1^* d^c \log\|\varDelta_g(\tau)\|^2 - \int_{\{0\}\times S^1} \alpha_0^* d^c \log\|\varDelta_g(\tau)\|^2\right), \qquad (40)
\end{aligned}$$

where we used the Γ_g-invariance of the 1-form $d^c \log\|\varDelta_g(\tau)\|^2$ to get the last equality. We obtain

$$\begin{aligned}
0 = &\lim_{\varepsilon\to 0} \int_{I\times S^1} \bar{\pi}_* L\left(TM_{(\bar{\alpha},\gamma)}, g_\varepsilon^{M_{(\bar{\alpha},\gamma)}}\right) \\
&- \left(\eta(M_{(\bar{\alpha}_0,\gamma)}, g_\varepsilon^{M_{(\alpha_0,\gamma)}}) - \eta(M_{(\bar{\alpha}_1,\gamma)}, g_\varepsilon^{M_{(\alpha_1,\gamma)}})\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{(-1)^{g/2} 2^{g+3}(2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1} \int_{S^1} \alpha_1^* d^c \log \| \Delta_g(\tau)\|^2 + \eta^0(M_{(\alpha_1,\gamma)}) \right) \\
&\quad - \left(\frac{(-1)^{g/2} 2^{g+3}(2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1} \int_{S^1} \alpha_0^* d^c \log \| \Delta_g(\tau)\|^2 + \eta^0(M_{(\alpha_0,\gamma)}) \right) \\
&= \Phi_g(\alpha_1,\gamma) - \Phi_g(\alpha_0,\gamma),
\end{aligned}
$$

where the first equality follows from (37) and (38), the second equality follows from (39), (40) and Theorem 6, and the last equality follows from Definition 7.

(b) Since $\eta(-M, g^M) = -\eta(M, g^M)$ for any odd dimensional Riemannian manifold (M, g^M) (cf. [3]), we have ($b3$). Let $\sigma_1 = [(\alpha_1, \gamma_1)]$, $\sigma_2 = [(\alpha_2, \gamma_2)]$, $\sigma_3 := (\sigma_1\sigma_2)^{-1} = [(\alpha_3, (\gamma_1\gamma_2)^{-1})] \in S_g$. Recall that $\mathcal{B} = S^2 \setminus \amalg_{k=1}^3 D_k$. By ($b3$), it suffices to show that

$$
\mathrm{Sign}(X(\sigma_1,\sigma_2)) = -\sum_{i=1}^{3} \Phi_g(\sigma_i) \tag{41}
$$

in order to prove ($b1$). Let U_i be a neighborhood of ∂D_i in $\mathcal{B}$ such that $U_i \cong [0,1) \times \partial D_i$. Let $\beta_i : \widetilde{U}_i \cong [0,1) \times \mathbb{R} \to \widetilde{\mathcal{B}}$ be the lift of the map $U_i \hookrightarrow \mathcal{B}$. As before, $g_1, g_2 \in \pi_1(\mathcal{B}, b)$ denote the generators represented by the loops ∂D_1, ∂D_2, respectively. Let $[(\alpha,\rho)] \in [\mathcal{B}, \Gamma_g \backslash \mathfrak{S}_g^\circ]^{orb}$ be the element corresponding to $(\sigma_1,\sigma_2) \in S_g \times S_g$ under the isomorphism (35). Since the loops ∂D_1, ∂D_2 and ∂D_3 represent g_1, g_2 and $(g_1g_2)^{-1} \in \pi_1(\mathcal{B}, b)$, we can assume that

$$
\alpha \circ \beta_i|_{\widetilde{U}_i}(s_i, t) = \alpha_i(t), \quad (s_i,t) \in \widetilde{U}_i \cong [0,1) \times \mathbb{R}, \qquad i = 1,2,3. \tag{42}
$$

Let $g^{X(\sigma_1,\sigma_2)/\mathcal{B}}$ (resp. $P_{X(\sigma_1,\sigma_2)}$) be the metric on $TX(\sigma_1,\sigma_2)$ (resp. the connection on $X(\sigma_1,\sigma_2)$) induced from the metric $g^{\Theta^\circ/\mathfrak{S}_g^\circ}$ (resp. the connection P_Θ) via the map α. Let $g^{\mathcal{B}}$ be a metric on $T\mathcal{B}$ such that $g^{\mathcal{B}}|_{U_i} = ds_i^2 \oplus dt^2$. With respect to the decomposition $TX(\sigma_1,\sigma_2) = T(X(\sigma_1,\sigma_2)/\mathcal{B}) \oplus \pi^* T\mathcal{B}$ associated with $P_{X(\sigma_1,\sigma_2)}$, we define the family of metrics on $TX(\sigma_1,\sigma_2)$ by

$$
g_\varepsilon^{X(\sigma_1,\sigma_2)} := g^{X(\sigma_1,\sigma_2)/\mathcal{B}} \oplus \varepsilon^{-1}\pi^* g^{\mathcal{B}}, \quad \varepsilon \in \mathbb{R}_{>0}.
$$

By (42) and Lemma 2 (c), we have

$$
g_\varepsilon^{X(\sigma_1,\sigma_2)}|_{U_i} = g_\varepsilon^{M_{(\alpha_i,\gamma)}} \oplus \varepsilon^{-1} ds_i^2, \qquad i = 1,2,3. \tag{43}
$$

Let $\nabla^{X(\sigma_1,\sigma_2)/\mathcal{B}}$ be the connection on $T(X(\sigma_1,\sigma_2))$ associated to the metric $g^{X(\sigma_1,\sigma_2)/\mathcal{B}}$ and the connection $P_{X(\sigma_1,\sigma_2)}$. Since the metric $g_\varepsilon^{X(\sigma_1,\sigma_2)}$ is a product metric on a color neighborhood of the boundary of $X(\sigma_1,\sigma_2)$ by (43), the Atiyah–Patodi–Singer index theorem applied to $(X(\sigma_1,\sigma_2), g_\varepsilon^{X(\sigma_1,\sigma_2)})$ yields that

$$\begin{aligned}
&\mathrm{Sign}\,(X(\sigma_1,\sigma_2))\\
&= \lim_{\varepsilon\to 0}\left(\int\limits_{X(\sigma_1,\sigma_2)} L\left(TX(\sigma_1,\sigma_2), g_\varepsilon^{X(\sigma_1,\sigma_2)}\right) - \sum_{i=1}^{3}\eta(M_{(\alpha_i,\gamma)}, g_\varepsilon^{M_{(\alpha_i,\gamma)}})\right)\\
&= \int\limits_{\mathcal{B}} \pi_* L\left(T(X(\sigma_1,\sigma_2)/\mathcal{B}), \nabla^{X(\sigma_1,\sigma_2)/\mathcal{B}}\right) - \sum_{i=1}^{3}\eta^0(M_{(\alpha_i,\gamma)})\\
&= \int\limits_{\mathcal{B}} \alpha^*\left[p_* L\left(T(\Theta^\circ/\mathfrak{S}_g^\circ), \nabla^{\Theta^\circ/\mathfrak{S}_g^\circ}\right)\right]^{(2)} - \sum_{i=1}^{3}\eta^0(M_{(\alpha_i,\gamma)})\\
&= \int\limits_{\mathcal{B}} \frac{(-1)^{g/2}2^{g+3}(2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1}\alpha^* dd^c \log\|\Delta_g(\tau)\|^2 - \sum_{i=1}^{3}\eta^0(M_{(\alpha_i,\gamma)})\\
&= \sum_{i=1}^{3}\int\limits_{\partial D_i} -\frac{(-1)^{g/2}2^{g+3}(2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1}\alpha_i^* d^c \log\|\Delta_g(\tau)\|^2\\
&\quad -\sum_{i=1}^{3}\eta^0(M_{(\alpha_i,\gamma)})\\
&\quad = -\sum_{i=1}^{3}\Phi_g(\sigma_i).
\end{aligned}$$

This completes the proof of $(b1)$. From $(b1)$ and Lemma 11 (c), $(b2)$ follows. By $(b1)$ and Lemma 11 (d), we have $\Phi_g(\sigma_1\sigma_2) = \Phi_g(\sigma_2\sigma_1)$ for any $\sigma_1, \sigma_2 \in \mathcal{S}_g$, from which $(b4)$ follows. □

8 The first cohomology of $\mathcal{S}_g$

The uniqueness of a 1-cochain that cobounds the 2-cocycle c_g is equivalent to the vanishing of $H^1(\mathcal{S}_g, \mathbb{Z})$. Indeed, if there is another 1-cochain $\Phi_g' : \mathcal{S}_g \to \mathbb{R}$ that cobounds c_g, the difference $\Phi_g - \Phi_g'$ is an element of $\mathrm{Hom}(\mathcal{S}_g, \mathbb{R}) \cong H^1(\mathcal{S}_g, \mathbb{R})$. (See [16] for generalities of cohomology of groups).

Let $k_1(g) = 2^{g-2}(2^g+1)$ and $k_2(g) = \frac{(g+3)\cdot g!}{4} - 2^{g-3}(2^g+1)$ denote the weights of the Siegel modular forms $\chi_g(\tau)$ and $J_g(\tau)$, respectively. Set $m_i(g) := \mathrm{L.C.D}(k_1(g), k_2(g))/k_i(g),\ i = 1, 2$. Then $\chi_g(\tau)^{m_1(g)} J_g(\tau)^{-m_2(g)}$ is a Γ_g-invariant holomorphic function on $\mathfrak{S}_g^\circ$.

While $H^1(\mathcal{S}_1, \mathbb{Z}) = H^1(\mathcal{S}_2, \mathbb{Z}) = 0$, the uniqueness is no longer valid for $g > 3$.

Theorem 13 *The following holds*:

$$H^1(\mathcal{S}_g, \mathbb{Z}) = \begin{cases} 0 & 1\le g\le 3,\\ \mathbb{Z} & g\ge 4.\end{cases}$$

For $g \geq 4$ *the generator of* $H^1(\mathcal{S}_g, \mathbb{Z})$ *is represented by a homomorphism* $\alpha \in \mathrm{Hom}(\mathcal{S}_g, \mathbb{Z})$ *defined by*

$$\sigma \longmapsto \frac{1}{2\pi\sqrt{-1}} \int_0^1 p^* d\log \chi_g(\tau)^{m_1(g)} J_g(\tau)^{-m_2(g)} \in \mathbb{Z}, \qquad \sigma = [(p, \gamma)] \in \mathcal{S}_g.$$

In particular, the cochain cobounding the signature cocycle c_g *is not unique when* $g \geq 2$.

The proof of Theorem 13 is divided into several lemmas below. By (32), we have the 5-term exact sequence (see [16, Chap. VII, Cororally 6.4])

$$\begin{aligned} 1 \to H^1(\Gamma_g, \mathbb{Z}) \to H^1(\mathcal{S}_g, \mathbb{Z}) \to H^1(\pi_1(\mathfrak{S}_g^\circ, *), \mathbb{Z})^{\Gamma_g} \\ \xrightarrow{\delta} H^2(\Gamma_g, \mathbb{Z}) \to H^2(\mathcal{S}_g, \mathbb{Z}), \end{aligned} \tag{44}$$

Here $H^1(\pi_1(\mathfrak{S}_g^\circ, *), \mathbb{Z})^{\Gamma_g}$ denotes the Γ_g-invariant subspace of $H^1(\pi_1(\mathfrak{S}_g^\circ, *), \mathbb{Z})$.

Lemma 12 *The following holds*:

$$H^1(\Gamma_g, \mathbb{Z}) = 0 \quad g \geq 1, \qquad H^2(\Gamma_g, \mathbb{Z}) = \begin{cases} \mathbb{Z}/12\mathbb{Z} & \text{if } g = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } g = 2 \\ \mathbb{Z} & \text{if } g \geq 3. \end{cases}$$

Proof See [14], [27, Corollary 5.2.3, Remark 5.2.4]. □

By the Hurwitz theorem [36, Chap. 7, Sect. 5, Proposition 2], we obtain

$$H^1(\pi_1(\mathfrak{S}_g^\circ, *), \mathbb{Z})^{\Gamma_g} \cong H^1(\mathfrak{S}_g^\circ, \mathbb{Z})^{\Gamma_g}. \tag{45}$$

Lemma 13 *Let* X *be a connected complex manifold of* $\dim_{\mathbb{C}} X \geq 2$. *Assume that*

$$H^1(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) = 0. \tag{46}$$

Let $D = \sum_{\lambda \in \Lambda} n_\lambda D_\lambda$ *be a divisor on* X *such that* $n_\lambda \neq 0$ *and* D_λ *is irreducible for all* $\lambda \in \Lambda$. *Then*

$$H^1(X - D, \mathbb{Z}) \cong \mathbb{Z}^\Lambda.$$

Here $\mathbb{Z}^\Lambda$ *denotes the direct product. The generator of the cohomology* $H^1(X - D, \mathbb{Z})$ *corresponding to* $\lambda \in \Lambda$ *is represented by the map* $l_\lambda \mapsto 1$ *and* $l_\mu \mapsto 0$ *for* $\mu \neq \lambda \in \Lambda$, *where* l_μ *denotes the loop around a small disk intersection* D_μ *transversally.*

Proof Since the real codimension of $\mathrm{Sing}D$ in X is greater than or equal to 4, we have $\pi_k(X, X - \mathrm{Sing}D, *) = 0$ for $1 \leq k \leq 3$. The relative Hurwitz theorem [36, Chap. 7, Sect. 5, Proposition 1] asserts that $H_k(X, X - \mathrm{Sing}D, \mathbb{Z}) = 0$ for $k \leq 3$. Hence $H^k(X, X - \mathrm{Sing}D, \mathbb{Z}) = 0$ for $k \leq 3$, which together with the cohomology exact sequence for the triple $(X, X - \mathrm{Sing}D, X - D)$, yields that

$$H^2(X, X - D, \mathbb{Z}) \cong H^2(X - \mathrm{Sing}D, X - D, \mathbb{Z}). \tag{47}$$

By the cohomology exact sequence for the pair $(X, X - D)$ and (46), we see that

$$H^1(X - D, \mathbb{Z}) \cong H^2(X, X - D, \mathbb{Z}) \cong H^2(X - \mathrm{Sing}D, X - D, \mathbb{Z}). \tag{48}$$

Since $D - \mathrm{Sing}D$ is a closed submanifold in $X - \mathrm{Sing}D$ and since $X - D = (X - \mathrm{Sing}D) - (D - \mathrm{Sing}D)$, the Thom isomorphism asserts that

$$H^2(X - \mathrm{Sing}D, X - D, \mathbb{Z}) \cong H^0(D - \mathrm{Sing}D, \mathbb{Z}). \tag{49}$$

By the irreducibility of D_λ, $D_\lambda - \mathrm{Sing}D_\lambda$ is path connected so that

$$H^0(D - \mathrm{Sing}D, \mathbb{Z}) \cong \mathbb{Z}^{\Lambda}. \tag{50}$$

The result follows from (48), (49) and (50). □

Lemma 14 *The following holds*:

$$H^1(\mathfrak{S}_g^\circ, \mathbb{Z})^{\Gamma_g} = \begin{cases} 0 & g = 1 \\ \mathbb{Z} & g = 2, 3 \\ \mathbb{Z}^{\oplus 2} & g \geq 4. \end{cases}$$

By regarding $H^1(\mathfrak{S}_g^\circ, \mathbb{C})$ as the de Rham cohomology group, the images of the generators under the natural map $H^1(\mathfrak{S}_g^\circ, \mathbb{Z}) \to H^1(\mathfrak{S}_g^\circ, \mathbb{C})$ are represented by the 1-forms $\frac{1}{2\pi\sqrt{-1}} d\log\chi_g(\tau)$ and $\frac{1}{2\pi\sqrt{-1}} d\log J_g(\tau)$. Here $J_g(\tau) \equiv 1$ and hence $d \log J_g(\tau) \equiv 0$ for $g \leq 3$.

Proof By Theorems 9 and 10, and Lemma 13, we get the assertion. □

Remark 7 Notice that the differential forms $\frac{1}{2\pi\sqrt{-1}} d\log\chi_g(\tau)$ and $\frac{1}{2\pi\sqrt{-1}} d\log J_g(\tau)$ are not Γ_g-invariant, but their cohomology classes are Γ_g-invariant.

Let $G := Sp(2g, \mathbb{R})$ be the symplectic group and let G^δ be the same group endowed with the discrete topology. Consider the universal covering

$$0 \to \mathbb{Z} \to \tilde{G} \to G \to 0, \tag{51}$$

which defines a central extension of G^δ by $\mathbb{Z}$. Let $e(G) \in H^2(G^\delta, \mathbb{Z})$ be the cohomology class corresponding to the central extension (51).

Recall that the automorphic factor $j(\tau, \gamma)$ is a nowhere vanishing holomorphic function on $\mathfrak{S}_g$. Since $\mathfrak{S}_g$ is simply connected, the logarithm of $j(\tau, \gamma)$ makes sense. Choose a branch of the logarithm of $j(\tau, \gamma)$ and denote it by $\log_\sigma j(\tau, \gamma)$ for $\gamma \in G^\delta$. Define the function $\lambda_\sigma : G^\delta \times G^\delta \to \mathbb{Z}$ by

$$(A, B) \longmapsto \frac{1}{2\pi\sqrt{-1}} \left(\log_\sigma j(B\cdot\tau, A) + \log_\sigma j(\tau, B) - \log_\sigma j(\tau, AB)\right) \tag{52}$$

for $(A, B) \in G^\delta \times G^\delta$.

Lemma 15 *The function λ_σ is a 2-cocycle of G^δ whose cohomology class is $e(G)$.*

Proof For $g = 1$, see [5, Lemma 2.1]. When $g \geq 1$, we closely follow [5]. Choose the branch $\log_\sigma j(\tau, \gamma)$ satisfying

$$\operatorname{Im} \log_\sigma j(\sqrt{-1}\cdot 1_{2g}, \gamma) \in [0, 2\pi). \tag{53}$$

Since the function λ_σ is measurable, the cohomology class $[\lambda_\sigma]$ is a constant multiple of $e(G)$ by [28, Theorem 2]. Therefore it suffices to determine the restriction of the cohomology class $[\lambda_\sigma]$ to the maximal compact subgroup of G. We identify the unitary group $U(g)$ with the maximal compact subgroup of G by the inclusion map defined by

$$\iota : U(g) \ni Z \longmapsto \begin{pmatrix} \operatorname{Re} Z & \operatorname{Im} Z \\ -\operatorname{Im} Z & \operatorname{Re} Z \end{pmatrix} \in G.$$

Since $j(\sqrt{-1}\cdot 1_{2g}, \iota(Z)) = \det(Z)^{-1}$ for $Z \in U(g)$ and the isotropy subgroup at $\sqrt{-1}\cdot 1_{2g} \in \mathfrak{S}_g$ is just $U(g)$, we have

$$2\pi\sqrt{-1}\lambda_\sigma(Z_1, Z_2) = \log_\sigma \det(Z_1 Z_2) - \log_\sigma \det(Z_1) - \log_\sigma \det(Z_2) \tag{54}$$

for $(Z_1, Z_2) \in U(g) \times U(g)$. By (54), the restriction of the cohomology class $[\lambda_\sigma]$ to $U(g)$ is the pull-back of the cohomology class corresponding to the universal covering

$$0 \to \mathbb{Z} \to \widetilde{U(1)} \cong \mathbb{R} \to U(1) \to 1,$$

via the map $\det : U(g) \to U(1)$. Since the induced map $(\det)_* : \pi_1(U(g)) \to \pi_1(U(1))$ is an isomorphism, we get $[\lambda_\sigma] = e(G)$. Since the cohomology class is independent of the choice of a branch of $\log_\sigma j(\tau, \gamma)$, we obtain the assertion. □

Lemma 16 *Let $\iota : \Gamma_g \to G^\delta$ be the natural inclusion. For $g \neq 2$ (resp. $g = 2$), the cohomology class $\iota^* e(G)$ is a generator of $H^2(\Gamma_g, \mathbb{Z})$ (resp. the free part of $H^2(\Gamma_2, \mathbb{Z})$).*

Proof Let $[\tau_g] \in H^2(G^\delta, \mathbb{Z})$ be the original signature cocycle of G (see [30] for definition). By [37, Theorem 1], we have $[\tau_g] = 4e(G)$. Let $\rho : \mathcal{M}_g \to \Gamma_g$ be the symplectic representation of the mapping class group obtained by the action on $H^1(\Sigma_g, \mathbb{Z})$. By

[30], $\rho^*\iota^*[\tau_g]$ is four times the generator of $H^2(\mathcal{M}_g,\mathbb{Z})$. Hence $4\iota^*e(G)$ is four times the generator of $H^2(\Gamma_g,\mathbb{Z})$, which yields the assertion. □

Lemma 17 *Let $g \geq 4$. The map $\delta : H^1(\pi_1(\mathfrak{S}_g^\circ,*),\mathbb{Z})^{\Gamma_g} \to H^2(\Gamma_g,\mathbb{Z})$ is given by*

$$(m,n) \longmapsto (k_1(g)m + k_2(g)n) \in H^2(\Gamma_g,\mathbb{Z}) \cong \mathbb{Z}$$

for $(m,n) \in H^1(\pi_1(\mathfrak{S}_g^\circ,*),\mathbb{Z})^{\Gamma_g} \cong \mathbb{Z}^{\oplus 2}$. For $g = 2,3$, the map δ is given by $m \mapsto k_1(g)m$.

Proof Let $\sigma : \Gamma_g \to \mathcal{S}_g$ be a section and write $\sigma(\gamma) = [(l_\gamma,\gamma)] \in \mathcal{S}_g$ for $\gamma \in \Gamma_g$. Let α be an element of $H^1(\pi_1(\mathfrak{S}_g^\circ,*),\mathbb{Z})^{\Gamma_g} \cong \mathrm{Hom}(\pi_1(\mathfrak{S}_g^\circ,*),\mathbb{Z})^{\Gamma_g}$. Then $\delta(\alpha) : \Gamma_g \times \Gamma_g \to \mathbb{Z}$ is given by

$$(A,B) \longmapsto \alpha\left(\sigma(A)\sigma(B)\sigma(AB)^{-1}\right) \in \mathbb{Z}, \quad (A,B) \in \Gamma_g \times \Gamma_g,$$

where we identify $\sigma(A)\sigma(B)\sigma(AB)^{-1}$ with the corresponding preimage under the inclusion $\pi_1(\mathfrak{S}_g^\circ,*) \to S_g$. Write

$$\sigma(A)\sigma(B)\sigma(AB)^{-1} = [(l_{(A,B)},1)] \in \pi_1(\mathfrak{S}_g^\circ,*),$$

where $l_{(A,B)}$ is a loop on $\mathfrak{S}_g^\circ$. By (31), $\sigma(AB)^{-1} = [(-(AB)^{-1} \cdot l_{(AB)}, (AB)^{-1})]$. Hence $l_{(A,B)}$ is the composition of the paths l_A, $A \cdot l_B$ and $-l_{(AB)}$. See Fig. 1.

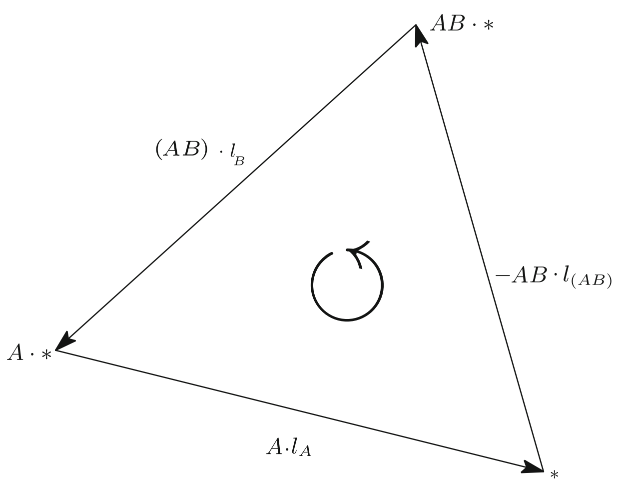

Fig. 1 Loop $l_{(A,B)}$

Under the identification in Lemma 14, $\delta(m,n)$ for $(m,n) \in H^1(\pi_1(\mathfrak{S}_g^\circ, *), \mathbb{Z})^{\Gamma_g} \cong \mathbb{Z}^{\oplus 2}$ is given by

$$\delta(m,n)(A,B) = \frac{1}{2\pi\sqrt{-1}} \int_{l_{(A,B)}} d\log\chi_g(\tau)^m J_g(\tau)^n \in \mathbb{Z}, \qquad (A,B) \in \Gamma_g \times \Gamma_g.$$

By using the path l_γ connecting $*$ and $\gamma \cdot *$, we define the branch $\log_\sigma j(\tau,\gamma)$ for $\gamma \in \Gamma_g$ satisfying

$$\log_\sigma j(*,\gamma) := \frac{1}{k_1(g)} \int_{l_\gamma} d\log\chi_g(\tau).$$

Then we get

$$\begin{aligned}
2\pi\sqrt{-1}\delta(1,0)(A,B) &= \int_{l_{(A,B)}} d\log\chi_g(\tau) \\
&= \int_{l_A} d\log\chi_g(\tau) + \int_{A\cdot l_B} d\log\chi_g(\tau) + \int_{-l_{(AB)}} d\log\chi_g(\tau) \\
&= k_1(g)\left[\log_\sigma j(*,A) - \log_\sigma j(*,AB)\right] + \int_{l_B} d\log\chi_g(A\cdot\tau) \\
&= k_1(g)\left[\log_\sigma j(*,A) - \log_\sigma j(*,AB)\right] \\
&\quad + \int_{l_B} \left[d\log\chi_g(\tau) + k_1(g) d\log_\sigma j(\tau,A)\right] \\
&= k_1(g)\left[\log_\sigma j(*,A) - \log_\sigma j(*,AB)\right] \\
&\quad + k_1(g)\left[\log_\sigma j(*,B) + \log_\sigma j(B\cdot *,A) - \log_\sigma j(*,A)\right] \\
&= k_1(g)\left[\log_\sigma j(B\cdot *,A) + \log_\sigma j(*,B) - \log_\sigma j(*,AB)\right].
\end{aligned}$$

By Lemmas 15 and 16, we see that $\delta(1,0) = k_1(g) \in H^2(\Gamma_g,\mathbb{Z}) \cong \mathbb{Z}$. Similarly we get $\delta(0,1) = k_2(g)$, which completes the proof. □

Proof of Theorem 13 Since $H^1(\Gamma_g,\mathbb{Z}) = 0$ in the exact sequence (44), we get $H^1(\mathcal{S}_g,\mathbb{Z}) = \mathrm{Ker}(\delta)$. By Lemma 17, we get $\mathrm{Ker}(\delta) = 0$, for $1 \le g \le 3$ and $\mathrm{Ker}(\delta) \cong \mathbb{Z}$ for $g \ge 4$. This completes the proof. □

In the proof of Theorem 13, we also obtain

Proposition 4 *One has* $H^2(\mathcal{S}_g,\mathbb{Z}) \neq 0$ *for* $g \ge 1$.

Proof Since $k_1(g) > 1$ for $g = 2, 3$ and G.C.D$(k_1(g), k_2(g)) > 1$ for $g \ge 4$, δ is not surjective by Lemma 17. By the exact sequence (44), we obtain the assertion. □

9 The value for the Dehn twists

In this section, we compute the value of Φ_g for the generators of the subgroup $\pi_1(\mathfrak{S}_g^\circ, *) \subset \mathcal{S}_g$ (cf. (32)) and we prove the rationality of Φ_g.

By Theorem 9, the Andreotti–Mayer locus $\mathcal{N}_g$ has two components θ_g and $\mathcal{J}_g$ such that $\Gamma_g\backslash\theta_g$ and $\Gamma_g\backslash\mathcal{J}_g$ are irreducible divisors on $\Gamma_g\backslash\mathfrak{S}_g$. Let $\sum_\lambda \theta_{g,\lambda}$ and $\sum_\mu \mathcal{J}_{g,\mu}$ be the irreducible decompositions of θ_g and $\mathcal{J}_g$, respectively. Consider a lasso in $\mathfrak{S}_g$ surrounding $\theta_{g,\lambda}$ (resp. $\mathcal{J}_{g,\mu}$) and denote its homotopy class by Π_λ^1 (resp. Π_μ^2). Then Π_λ^1 and Π_μ^2 define elements of $\pi_1(\mathfrak{S}_g^\circ, *) \subset \mathcal{S}_g$ up to conjugacy classes. After [24], we call Π_λ^1 and Π_μ^2 the *Dehn twists*.

Theorem 14 *The following equalities hold*:

$$\Phi_g(\Pi_\lambda^1) = \begin{cases} -\frac{4}{5} & \text{if } g = 2, \\ (-1)^{\frac{g}{2}+1} \frac{(g+1)2^{g+2}(2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1} & \text{if } g \geq 3. \end{cases}$$

$$\Phi_g(\Pi_\mu^2) = (-1)^{\frac{g}{2}+1} \frac{(g+1)2^{g+3}(2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1} \quad \text{if } g \geq 4.$$

Proof Let $\Delta := \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the unit disc and set $\Delta_r = \{z \in \Delta; |z| \leq r\}$ and $\Delta^* := \Delta\backslash\{0\}$. Let $\alpha_i : S^1 \to \mathfrak{S}_g^\circ$ be a representative of Π_λ^i. Recall that the Zariski open subset $\theta_g^\circ \subset \theta_g$ and $\mathcal{J}_g^\circ \subset \mathcal{J}_g$ were defined in Theorem 9. Let $\rho_i : \Delta \to \mathfrak{S}_g$ be a C^∞-map with the following properties:

(a) $\rho_i|_{\partial\Delta} = \alpha_i$ and $\rho_i(\Delta^*) \subset \mathfrak{S}_g^\circ$.
(b) $\rho_i|_{\Delta_{\frac{1}{3}}} : \Delta_{\frac{1}{3}} \to \rho_i(\Delta_{\frac{1}{3}}) \subset \mathfrak{S}_g$ is a holomorphic embedding with

$$\rho_i(re^{\sqrt{-1}\theta}) = \rho_i\left(\frac{2}{3}e^{\sqrt{-1}\theta}\right), \quad \frac{2}{3} \leq r \leq 1, \ \ 0 \leq \theta < 2\pi.$$

(c) $\rho_1(\Delta)$ intersects θ_g at $\rho_1(0) \in \theta_g^\circ$ transversally, and $\rho_2(\Delta)$ intersects $\mathcal{J}_g$ at $\rho_2(0) \in \mathcal{J}_g^\circ$ transversally.

Let

$$\varpi : X^i := \Delta \times_{\rho_i} \Theta \longrightarrow \Delta,$$

be the family of theta divisors over Δ induced from the universal family $p : \Theta \to \mathfrak{S}_g$ by ρ_i. Let $pr : X^i \to \Theta$ be the projection to the second factor. By Condition (c), X^i is a C^∞-manifold. By Conditions (a), (c) and Theorem 9, Sing $\left(\varpi^{-1}(0)\right)$ consists of one ordinary double point (resp. two ordinary double points) and $\varpi^{-1}(z)$ is a smooth theta divisor for $z \in \Delta^*$. Notice that ∂X^i endowed with the orientation induced from X^i is diffeomorphic to the mapping torus $M_{(\Pi_\lambda^i)^{-1}}$ endowed with the natural orientation (cf. Definition 3), i.e., $\partial X^i = -M_{\Pi_\lambda^i}$. For simplicity, set $M_i := M_{\Pi_\lambda^i}$.

Let g^{Δ} be a metric on $T\Delta$ such that

$$g^{\Delta} = \begin{cases} dr^2 + d\theta^2 & (|r| > \frac{2}{3}), \\ p^* g^{\mathfrak{S}_g} & (|r| < \frac{1}{3}). \end{cases} \tag{55}$$

Let $\mathcal{D}$ be the set of singular points of the central fiber $\varpi^{-1}(0)$. Let $g^{X^i/\Delta}$ be the metric on $T(X^i/\Delta)\big|_{X^i-\mathcal{D}}$ induced from the metric $g^{\Theta^\circ/\mathfrak{S}_g^\circ}$ via the map ρ_i. Let P_i be the connection induced from the connection P_Θ on Θ° via the map ρ_i. Using P_i, define the metric on $TX^i\big|_{X^i-\mathcal{D}}$ by $\tilde{g}^{X^i} := g^{X^i/\Delta} \oplus \varpi^* g^{\Delta}$. Since $pr\big|_{\varpi^{-1}(\Delta_{1/3})} : \varpi^{-1}(\Delta_{1/3}) \to \Theta$ is a holomorphic embedding and preserves the metric outside $\mathcal{D}$ by Lemma 7 and since the metric $g^{\Theta} := g^{\mathbb{A}_g}|_{\Theta}$ is defined on the total space Θ, the metric $\tilde{g}^{X^i}$ extends to a metric g^{X^i} on TX^i. Set

$$g_\varepsilon^{X^i} := g^{X^i} \oplus \varepsilon^{-1} \varpi^* g^{\Delta}, \quad \varepsilon \in \mathbb{R}_{>0}.$$

By Condition (b), ρ_i is constant in the radial direction when $\frac{2}{3} \leq r \leq 1$. Hence $g_\varepsilon^{X^i}$ is a product metric on a color neighborhood of the boundary ∂X^i by Lemma 2 (c) and (55). The Atiyah–Patodi–Singer index theorem applied to $(X^i, g_\varepsilon^{X^i})$ yields that

$$\mathrm{Sign}(X_g^i) = \int_{X^i} L(TX^i, g_\varepsilon^{X^i}) + \eta(M_i, g_\varepsilon^{M_i}). \tag{56}$$

Here, ∂X^i is identified with $-M_i$, and $g_\varepsilon^{M_i}$ is the restriction of $g_\varepsilon^{X^i}$ to the boundary $\partial X^i \cong -M_i$.

Lemma 18 *The following equality holds*:

$$\lim_{\varepsilon \to 0} L(TX^i, g_\varepsilon^{X^i})^{(2g)} = L(T(X^i/\Delta), \nabla^{X^i/\Delta})^{(2g)} + P(-t, \ldots, (-t)^g)|_{t^g} \cdot \sum_{p \in \mathcal{D}} \mu(p) \delta_p.$$

Here the differential form $L(T(X^i/\Delta), \nabla^{X^i/\Delta})$ on $X^i \backslash \mathcal{D}$ extends to a C^∞-differential form on X^i. The constant $\mu(p)$ is the Milnor number of the singular point $p \in \mathcal{D}$, δ_p is the Dirac delta current supported at p, and $P(x_1, \ldots, x_g) \in \mathbb{C}[[x_1, \ldots, x_g]]$ is defined by

$$\prod_{k=1}^{g} \mathbf{L}(x_k) = P(\sigma_1, \ldots, \sigma_g),$$

where $\sigma_1 = \sum_k x_k$, $\sigma_2 = \sum_{i>j} x_i x_j, \ldots, \sigma_g = \prod_k x_k$ are the elementary symmetric polynomials.

Proof On $X^i \setminus \mathcal{D}$, the assertion follows from Proposition 2. Let $U \subset X^i$ be an open neighborhood of $\mathcal{D}$ contained in $\varpi^{-1}(\Delta_{\frac{1}{3}})$. By Condition (*b*) and the equality (24), we have

$$L(TX^i, g_\varepsilon^{X^i})|_U = (pr|_U)^* L(T\Theta, g_\varepsilon^\Theta) = (pr|_U)^* \mathbf{L}(T^{1,0}\Theta, g_\varepsilon^\Theta), \tag{57}$$

where $g_\varepsilon^\Theta := g^\Theta \oplus \varepsilon^{-1} p^* g^{\mathfrak{S}_g}$. By [40, Main Theorem 2.2], we get

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathbf{L}(T^{1,0}\Theta, g_\varepsilon^\Theta)^{(2g)}|_{pr(U)} &= \mathbf{L}(T^{1,0}(\Theta/\mathfrak{S}_g), \nabla^h)^{(2g)}|_{pr(U)} \\ &\quad + P(-t, \dots, (-t)^g)|_{t^g} \cdot \sum_{p \in pr(\mathcal{D})} \mu(p)\delta_p,\end{aligned}$$

which together with (57), yields the assertion. □

Lemma 19 *The following equality holds*:

$$P(-t, \dots, (-t)^g)|_{t^g} = \mathbf{L}^{-1}(t)|_{t^g} = \frac{(-1)^{g/2} 2^{g+2}(2^{g+2}-1)}{(g+2)!} B_{\frac{g}{2}+1}$$

Proof Consider the exact sequence of vector bundles over $\mathbb{P}^g$:

$$0 \to \mathcal{O}(-1) \to \underline{\mathbb{C}}^{g+1} \to E := \underline{\mathbb{C}}^{g+1}/\mathcal{O}(-1) \to 0.$$

For a complex vector bundle F over $\mathbb{P}^g$, recall that $\mathbf{L}(F) \in H^*(\mathbb{P}^g, \mathbb{Q})$ denote the multiplicative genus of F associated with $\mathbf{L}(x)$ (cf. (24)) and let $c(F)$ denote the total Chern class of F. Set $t := c_1(\mathcal{O}(-1))$. Since $c(\mathcal{O}(-1)) \cdot c(E) = c(\underline{\mathbb{C}}^{g+1}) = 1$ and $c(\mathcal{O}(-1)) = 1 + t$, we have $c(E) = \sum_{k=0}^g (-t)^k$, which together with $\mathbf{L}(\mathcal{O}(-1)) = \mathbf{L}(t)$, $\mathbf{L}(E) = P(c_1(E), \dots, c_g(E))$ and $\mathbf{L}(\mathcal{O}(-1)) \cdot \mathbf{L}(E) = \mathbf{L}(\underline{\mathbb{C}}^{g+1}) = 1$, yields that

$$\begin{aligned}P((-t), \dots, (-t)^g) &= \mathbf{L}(E) = \mathbf{L}(\mathcal{O}(-1))^{-1} \\ &= \mathbf{L}^{-1}(t) \in H^*(\mathbb{P}^g, \mathbb{Q}) \cong \mathbb{Q}[t]/(t^{g+1}).\end{aligned}$$

This proves the first equality. Since $\mathbf{L}^{-1}(t) = \tanh(t)/t$ by (23), the second equality follows from (30). □

Since $p \in \mathcal{D}$ is a non-degenerate critical point of $\varpi : X^i \to \Delta$, we get $\mu(p) = 1$. Taking the limit $\varepsilon \to 0$ in (56), we get by Lemma 18, Theorem 11 and Lemma 19

$$\begin{aligned}\mathrm{Sign}(X^i) &= \int_{\Delta} \varpi_* pr^* L(T(\Theta^\circ/\mathfrak{S}_g), \nabla^{\Theta^\circ/\mathfrak{S}_g}) + \mathbf{L}^{-1}(t)|_{t^g} + \eta^0(M_i)\\ &= \int_{\Delta} \rho_i^* p_* L(T(\Theta^\circ/\mathfrak{S}_g), \nabla^{\Theta^\circ/\mathfrak{S}_g})\\ &\quad + i\frac{(-1)^{g/2}2^{g+2}(2^{g+2}-1)}{(g+2)!}B_{\frac{g}{2}+1} + \eta^0(M_i)\\ &= \frac{(-1)^{g/2}2^{g+1}(2^{g+2}-1)}{(g/2+1)(g+1)}B_{\frac{g}{2}+1}\int_{\Delta}\rho^* dd^c \mathrm{logdetIm}\tau\\ &\quad + i\frac{(-1)^{g/2}2^{g+2}(2^{g+2}-1)}{(g+2)!}B_{\frac{g}{2}+1} + \eta^0(M_i).\end{aligned} \tag{58}$$

By (58) and Definition 7, we get

$$\begin{aligned}\Phi_g(\Pi_\lambda^i) &= \frac{(-1)^{g/2}2^{g+3}(2^{g+2}-1)}{(g+3)!}B_{\frac{g}{2}+1}\\ &\quad \times \int_{\partial\Delta} \rho^* d^c\left(\log|\Delta_g(\tau)|^2(\mathrm{detIm}\tau)^{\frac{(g+3)\cdot(g)!}{2}}\right) + \eta^0(M_i)\\ &= -i\frac{(-1)^{g/2}2^{g+2}(2^{g+2}-1)}{(g+2)!}B_{\frac{g}{2}+1} + \mathrm{Sign}(X^i)\\ &\quad + \frac{(-1)^{g/2}2^{g+3}(2^{g+2}-1)}{(g+3)!}B_{\frac{g}{2}+1}\int_{\Delta}\rho^* dd^c \log|\Delta_g(\tau)|^2\\ &= i\frac{(-1)^{\frac{g}{2}+1}(g+1)2^{g+2}(2^{g+2}-1)}{(g+3)!}B_{\frac{g}{2}+1} + \mathrm{Sign}(X^i),\end{aligned} \tag{59}$$

where we used the Poincaré–Lelong formula and Theorem 10 to get the last equality.

When $g = 2$ and $i = 1$, since the singular fiber has two irreducible components and hence $\mathrm{Sign}(X^1) = -1$, the assertion follows. We complete the computation in the case $g \geq 3$ and $i = 1, 2$ by Lemma 20 below. □

Lemma 20 *Let $\pi : \mathfrak{X} \to \Delta$ be a proper surjective holomorphic map from a complex manifold $\mathfrak{X}$ of dimension $2n$ to the unit disk Δ. Assume that π has only finitely many critical points which are non-degenerate and lie in the central fiber $\mathfrak{X}_0$. If $n > 1$, then* $\mathrm{Sign}(\mathfrak{X}) = 0$.

Proof By the assumption, there are points $p_1, \ldots, p_l \in \mathfrak{X}_0$ and open neighborhoods U^k of p_k in $\mathfrak{X}$ such that

$$\pi(z_1^k, \ldots, z_{2n}^k) = (z_1^k)^2 + \cdots + (z_{2n}^k)^2, \quad (z_1^k, \ldots, z_{2n}^k) \in U^k,$$

and such that the induced map $\pi_* : T\mathfrak{X} \to T\Delta$ has maximal rank on $\mathfrak{X}\backslash\{p_1, \ldots, p_l\}$. Let $\varepsilon \in \mathbb{R}_{>0}$ be a small number. We may assume that each $V^k := \{(z_1^k, \ldots, z_{2n}^k) \in$

$\mathbb{C}^{2n} \mid |z_1^k|^2 + \cdots + |z_{2n}^k|^2 < \varepsilon^2\}$ is contained in U^k. Fix a $\rho \in \mathbb{R}_{>0}$ with $\rho < \varepsilon^2$. Set

$$D := \Delta_\rho, \quad X := \pi^{-1}(D), \quad X^\circ := X \setminus \cup_{k=1}^l V^k, \quad F := \pi^{-1}(0), \quad F^\circ := F \cap X^\circ.$$

Since $\mathfrak{X}$ is diffeomorphic to X, it suffices to show $\mathrm{Sign}(X) = 0$. Consider the following commutative diagram of the homologies induced from natural inclusions:

$$\begin{array}{ccccc} H_{2n}(X^\circ, \mathbb{Z}) & \xrightarrow{(f)} & H_{2n}(X \setminus \{p_1, \ldots, p_l\}, \mathbb{Z}) & \underset{\cong}{\xrightarrow{(a)}} & H_{2n}(X, \mathbb{Z}) \\ {\scriptstyle (e)}\uparrow{\scriptstyle \cong} & & & & {\scriptstyle \cong}\uparrow{\scriptstyle (b)} \\ H_{2n}(F^\circ, \mathbb{Z}) & \underset{(d)}{\xrightarrow{\cong}} & H_{2n}(F \setminus \{p_1, \ldots, p_l\}, \mathbb{Z}) & \underset{(c)}{\xrightarrow{\cong}} & H_{2n}(F, \mathbb{Z}). \end{array} \tag{60}$$

Here the isomorphisms (a) and (c) follow from the fact that the submanifold $\{p_1, \ldots, p_l\}$ of X (resp. F) has real codimension $4n$ (resp. $4n-2$), (b) and (d) follow from the fact that F (resp. F°) is a deformation retraction of X (resp. $F \setminus \{p_1, \ldots, p_l\}$). By Ehresmann's fibration theorem for manifolds with boundary [26, p. 23], there is an isomorphism of C^∞-fiber bundles $X^\circ \cong F^\circ \times \Delta$. Since Δ is contractible, we obtain the isomorphism (e).

By (60), the map (f) is an isomorphism. Hence we get the commutative diagram

$$\begin{array}{ccc} H_{2n}(X_t \cap X^\circ, \mathbb{Z}) & \xrightarrow{\cong} & H_{2n}(X^\circ, \mathbb{Z}) \\ \downarrow & & \downarrow{\scriptstyle \cong} \\ H_{2n}(X_t, \mathbb{Z}) & \longrightarrow & H_{2n}(X, \mathbb{Z}) \end{array} \tag{61}$$

for any $t \in \Delta$. By (61), the map $H_{2n}(X_t, \mathbb{Z}) \to H_{2n}(X, \mathbb{Z})$ is surjective for any $t \in \Delta$. Therefore every $c \in H_{2n}(X, \mathbb{Z})$ can be represented by a $2n$-cycle contained in the fiber X_t for any $t \in \Delta$. Since the intersection number of any two $2n$-cycles contained in different fibers is zero, the intersection matrix of the lattice $H_{2n}(X, \mathbb{Z})$ is the zero matrix, from which the assertion follows. □

Remark 8 When $g = 2$, $\sigma_2 \in \mathcal{M}_2$ is the Dehn twist along a separating simple closed curve on a Riemann surface of genus two. Since $\mathrm{Sign}(X_2) = -1$ and $B_2 = \frac{1}{30}$, we obtain $\phi_2(\sigma_2) = \Phi_2(\sigma_2) = -\frac{4}{5}$, which confirms [29, Proposition 3.6].

As a consequence, we obtained the following:

Theorem 15 *The value $\Phi_g(\sigma)$ is a rational number for any $\sigma \in \mathcal{S}_g$.*

Proof We keep the notation in Sect. 8. By Theorem 12, there exists a function $\Phi'_g : \mathcal{S}_g \to \mathbb{Q}$ which cobounds the signature cocycle c_g. By Theorem 13, there exits a constant $a \in \mathbb{R}$ such that

$$\Phi_g = \Phi'_g + aG,$$

where G is a generator of $\mathrm{Hom}(\mathcal{S}_g, \mathbb{Z}) \cong H^1(\mathcal{S}_g, \mathbb{Z})$. Let $\sigma := \Pi^1_\lambda$ be a generator of $\pi_1(\mathfrak{S}^\circ_g, *)$. Since $\Phi_g(\sigma) \in \mathbb{Q}$ by Theorem 14 and since $G(\sigma) = m_1(g) \neq 0$ by Theorem 13, we have $a \in \mathbb{Q}$, which yields the assertion. □

10 An interpretation of Φ_2 in terms of η-forms

In this final section, following Dai's results [18], we study the Meyer function Φ_2 of genus two from the view point of the Bismut–Cheeger *η-forms* and we give another analytic representation of Φ_2.

We first recall one of the main results in [18] briefly. Let $\pi : X \to B$ be a fiber bundle with typical compact fiber M such that $\dim_{\mathbb{R}} X = 4k - 1$ and $\dim_{\mathbb{R}} M = 2m$. Assume that X, B and M are oriented and the orientations are compatible. Give a metric g^B on TB, a metric $g^{X/B}$ on $T(X/B)$ and a connection P_X on X. Define the one parameter family of metrics on X by

$$g^X_\varepsilon := g^{X/B} \oplus \varepsilon^{-1}\pi^* g^B, \quad \varepsilon \in \mathbb{R}_{>0}.$$

Then one obtains the adiabatic limit $\eta^0(X)$ as in Sect. 3.

Let (E_r, d_r), $r \geq 2$ be the E_r-term of the Leray spectral sequence of the fiber bundle $\pi : X \to B$. The orientations of B and M give a natural basis ξ_2 of E_2^{4k-1}, which induces a basis ξ_r of E_r^{4k-1} for $r > 2$. (See [17, Sect. 4.3] for details.) Consider the symmetric bilinear product $E_r^{2k-1} \times E_r^{2k-1} \to \mathbb{R}$ defined by

$$(\omega_1, \omega_2) \mapsto (\omega_1 \cdot d_r\omega_2,\ \xi_r), \quad \omega_1, \omega_2 \in E_r^{2k-1},$$

and denote its signature by τ_r. Set $\tau := \sum_{r\geq 2} \tau_r$.

Let $R\pi_*\mathbb{C} := \oplus R^k\pi_*\mathbb{C}$ be the direct image sheaf, which is a locally constant sheaf. We identify $R\pi_*\mathbb{C}$ with the corresponding flat vector bundle on B. Since the fiber of $(R\pi_*\mathbb{C})_b$ is isomorphic to the space of harmonic forms on the fiber $X_b := \pi^{-1}(b)$, the vector bundle $R\pi_*\mathbb{C}$ carries the L^2-metric $g^{R\pi_*\mathbb{C}}$ and also carries the Hodge star operator $*_M \in C^\infty(B, \mathrm{End}(R\pi_*\mathbb{C}))$. Let $*_B$ be the Hodge star operator on the base space B. Define the involution τ acting on $\mathcal{A}^*(B, R\pi_*\mathbb{C})$ by $\tau := (-1)^{k+p(p-1)/2+q(q-1)/2} *_B \otimes *_M$ on $\mathcal{A}^p(B, R^q\pi_*\mathbb{C})$. Let $d^{R\pi_*\mathbb{C}}$ be the exterior differential acting on $\mathcal{A}^*(B, R\pi_*\mathbb{C})$. Set

$$D_B \otimes R\pi_*\mathbb{C} := \tau d^{R\pi_*\mathbb{C}} + d^{R\pi_*\mathbb{C}}\tau,$$

which is a differential operator acting on $\mathcal{A}^*(B, R\pi_*\mathbb{C})$.

Let $\hat{\eta}(X) \in \mathcal{A}^{\mathrm{odd}}(B)$ be the η-form of the family $\pi : X \to B$ associated with the metric $g^{X/B}$ and the connection P_X, introduced in [10].

Theorem 16 ([18, Theorem 0.3]) *The following equality holds*:

$$\eta^0(X) = 2\int_B L(TB, g^B) \wedge \hat{\eta} + \eta(D_B \otimes R\pi_*\mathbb{C}) + 2\tau,$$

where $\eta(D_B \otimes R\pi_*\mathbb{C})$ denotes the η-invariant of the differential operator $D_B \otimes R\pi_*\mathbb{C}$ (See [18, Sect. 4] *for the precise definition).*

We keep the notation in Sect. 7.

Theorem 17 *For $\sigma \in \mathcal{M}_2$, let (α, γ) be a representative of σ. Let $p : M_{(\alpha,\gamma)} \to S^1$ be the mapping torus associated with σ. Then*

$$\phi_2(\sigma) = \eta(D_{S^1} \otimes Rp_*\mathbb{C}) - \frac{4}{5}\int_{S^1} \alpha^* d^c \log \|\chi_2(\tau)\|^2.$$

Proof By Theorem 16, we have

$$\eta^0(M_{(\alpha,\gamma)}) = 2\int_{S^1} L(S^1, dt^2) \wedge \hat{\eta}(M_{(\alpha,\gamma)}) + \eta(D_{S^1} \otimes Rp_*\mathbb{C}) + 2\tau. \qquad (62)$$

Since $\dim_{\mathbb{R}} S^1 = 1$, all the differential d_r in the Leray spectral sequence (E_r, d_r) is the zero map and hence $\tau = 0$. Since $L(S^1, dt^2) = 1$, we get by Corollary 1 and (62),

$$\phi_2(\sigma) = 2\int_{S^1} \hat{\eta}(M_{(\alpha,\gamma)}) + \eta(D_{S^1} \otimes Rp_*\mathbb{C}) - \frac{2}{15}\int_{S^1} \alpha^* d^c \log \|\chi_2(\tau)\|^2. \qquad (63)$$

Let $f : \mathcal{C} := \Theta^\circ \to \mathfrak{S}_2^\circ$ be the universal family of curves of genus two. Recall that the Kähler metric $g^{\mathcal{C}} := g^{\Theta^\circ}$ and the connection $P_{\mathcal{C}} := P_\Theta$ were defined in Sect. 5. Denote by $\hat{\eta}_1(\mathcal{C})$ the 1-form component of the η-form of the family $f : \mathcal{C} \to \mathfrak{S}_2^\circ$ associated with $g^{\mathcal{C}}$ and $P_{\mathcal{C}}$. By the functorial property of the Bismut superconnection [6, Proposition 10.15] and the definition [10, Definition 4.33], the η-form has the functorial property $\hat{\eta}(M_{(\alpha,\gamma)}) = \alpha^*\hat{\eta}_1(\mathcal{C})$, which together with (62) and Theorem 18 below, yields the result. □

Theorem 18 *The following equality holds*:

$$\hat{\eta}_1(\mathcal{C}) = -\frac{1}{3} d^c \log \|\chi_2(\tau)\|^2.$$

Proof We recall the relation of the signature operator and the Dolbeault operator on Riemann surfaces. Let C be a compact Riemann surface. Let ι be the involution acting on $\mathcal{A}^*(C)$ defined by

$$\iota(\omega) := (\sqrt{-1})^{p(p-1)+1} * \omega, \quad \omega \in \mathcal{A}^p(C).$$

Denote by $\mathcal{A}^\pm(C)$ the ± 1 eigenspaces of the involution ι. Let D be the signature operator $d + d^* : \mathcal{A}^\pm(C) \to \mathcal{A}^\mp(C)$. Then the following diagram is commutative and the vertical arrows preserve the L^2-metrics.

$$\begin{array}{ccc} \mathcal{A}^{+}(C) & \xrightarrow{D} & \mathcal{A}^{-}(C) \\ {\scriptstyle f_+}\uparrow & & \uparrow{\scriptstyle f_-} \\ \mathcal{A}^{0,0}(C) \oplus \mathcal{A}^{1,0}(C) & \xrightarrow[\sqrt{2}\bar{\partial}]{} & \mathcal{A}^{0,1}(C) \oplus \mathcal{A}^{1,1}(C) \end{array} \tag{64}$$

Here, for $\omega^{i,j} \in \mathcal{A}^{i,j}(C)$,

$$f_{+}(\omega^{0,0}, \omega^{1,0}) := \frac{1}{\sqrt{2}} \left(\omega^{0,0} + \iota(\omega^{0,0}) \right) + \omega^{1,0},$$
$$f_{-}(\omega^{0,1}, \omega^{1,1}) := \omega^{0,1} + \frac{1}{\sqrt{2}} \left(\omega^{1,1} - \iota(\omega^{1,1}) \right).$$

The diagram (64), together with [8, p. 153], yields that

$$\hat{\eta}_1(\mathcal{C}_2) = -d^c \log \left(\det{}' \square_\tau^{0,1} \det{}' \square_\tau^{1,1} \right), \tag{65}$$

where $\det' \square_\tau^{i,j}$ is the regularized determinant of the $\bar{\partial}$-Laplacian $2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})$ acting on $\mathcal{A}^{i,j}(\mathcal{C}_\tau)$. By [39, Theorem 5.1], we have

$$\det{}' \square_\tau^{0,1} = \det{}' \square_\tau^{1,1} = \|\chi(\tau)\|^{\frac{1}{3}},$$

from which and (65) the assertion follows. □

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Appendix A: The Meyer function for tori

In this appendix, we investigate the signature cocycle for torus fibrations associated with $SL(4g-2, \mathbb{Z})$-vector bundles and relate it to η-invariants. We closely follow [2]. We refer to [11] for further studies of η-invariants of torus fibrations.

Recall that $\mathcal{B}$ is a sphere with three holes and let g_1 and g_2 be the generators of $\pi_1(\mathcal{B})$ as in Sect. 6. For $\sigma_1, \sigma_2 \in SL(4g-2, \mathbb{Z})$, we define the homomorphism $\rho : \pi_1(\mathcal{B}) \to SL(4g-2, \mathbb{Z})$ by

$$\rho(g_k) = \sigma_k, \quad k = 1, 2. \tag{66}$$

Let $p : E_\rho := \tilde{\mathcal{B}} \times_\rho \mathbb{R}^{4g-2} \to B$ be the flat real vector bundle of rank $2g-2$ associated with ρ and let $\Lambda_\rho := \tilde{\mathcal{B}} \times_\rho \mathbb{Z}^{4g-2} \subset E_\rho$ be the corresponding family of lattices.

Then the fiberwise quotient E_ρ/Λ_ρ is a torus fibration over $\mathcal{B}$, which is a compact oriented $4g$-dimensional manifold with boundary. We call E_ρ/Λ_ρ the torus fibration associated with E_ρ. We define

$$t_g : SL(4g-2,\mathbb{Z}) \times SL(4g-2,\mathbb{Z}) \longrightarrow \mathbb{Z}, \quad (\sigma_1,\sigma_2) \mapsto \operatorname{Sign}(E_\rho/\Lambda_\rho).$$

By the same argument as in [2, p. 343], t_g is a 2-cocycle of $SL(4g-2,\mathbb{Z})$. In particular, $t_1 \equiv \tau_1$. Since $H^1(SL(n,\mathbb{Z}),\mathbb{Z}) = 0$ for $n \geq 1$ and $H^2(SL(n,\mathbb{Z}),\mathbb{Z}) = 0$ for $n \geq 3$ by [31, Sect. 10], there exists a unique function $\psi_g : SL(4g-2,\mathbb{Z}) \to \mathbb{Z}$ for $g \geq 2$ which cobounds $-t_g$, i.e.,

$$t_g(\sigma_1,\sigma_2) = -\psi_g(\sigma_1) - \psi_g(\sigma_2) + \psi_g(\sigma_1\sigma_2), \quad \sigma_1,\sigma_2 \in SL(4g-2,\mathbb{Z}). \tag{67}$$

We call ψ_g the *Meyer function for tori*. The Novikov additivity for signatures yields

Proposition 5 *Let S be a compact oriented 2-dimensional manifold with boundary $\partial S = c_1 \amalg \cdots \amalg c_n$. Let E be a flat $SL(4g-2,\mathbb{Z})$ real vector bundle over S with monodromies $\sigma_k \in SL(4g-2,\mathbb{Z})$ on c_k, $1 \leq k \leq n$. Let $\pi : M \to S$ be the torus fibration associated with E. Assume that $g \geq 2$. Then*

$$\operatorname{Sign}(M) = -\sum_{k=1}^{n} \psi_g(\sigma_k).$$

Proof By the same argument as in [2, p. 357], we obtain the assertion. □

For $\sigma \in SL(4g-2,\mathbb{Z})$, let $p : E \to S^1$ be the flat real vector bundle over S^1 with monodromy σ. Let $p : M_\sigma \to S^1$ be the corresponding torus fibration. Fix a metric g^E and a connection ∇^E on E. Then g^E induces the metric g^{M_σ/S^1} on the relative tangent bundle $T(M_\sigma/S^1)$ and ∇^E induces the connection $TM_\sigma \cong T_H M_\sigma \oplus T(M_\sigma/S^1)$ of the torus fibration M_σ (see [6, Sect. 1.1]). Define the one parameter family of metrics on M_σ by

$$g_\varepsilon^{M_\sigma} := g^{M_\sigma/S^1} \oplus \varepsilon^{-1}\pi^* dt^2, \quad \varepsilon \in \mathbb{R}_{>0}.$$

Recall that $\eta^0(M_\sigma) := \lim_{\varepsilon\to 0} \eta(M_\sigma, g_\varepsilon^{M_\sigma})$ as in Sect. 3.

Proposition 6 *For any $\sigma \in SL(4g-2,\mathbb{Z})$, $\psi_g(\sigma) = \eta^0(M_\sigma)$.*

Proof By [11, Theorem 3.8], $\eta^0(M_\sigma)$ does not depend on g^E and ∇^E. Hence the map $\eta^0 : SL(4g-2,\mathbb{Z}) \to \mathbb{Z}$ defined by $\sigma \mapsto \eta^0(M_\sigma)$ is well-defined. By the uniqueness of the function that cobounds $-t_g$, it is enough to show that the function η^0 satisfies (67).

For $\sigma_1,\sigma_2 \in SL(4g-2,\mathbb{Z})$, let $\rho : \pi_1(\mathcal{B}) \to SL(4g-2,\mathbb{Z})$ be the homomorphism defined by (66). Let E_ρ be the flat vector bundle associated with ρ and denote the torus fibration associated with E_ρ by $p : X_\rho \to \mathcal{B}$. Notice that $\partial X_\rho = M_{\sigma_1} \amalg M_{\sigma_2} \amalg -M_{\sigma_1\sigma_2}$. Let ∇^{E_ρ} be a connection on E_ρ. Then we have the splitting (cf. [11, p. 353])

$$TX_\rho \cong p^* E_\rho \oplus p^* T\mathcal{B}. \tag{68}$$

Let g^{E_ρ} and $g^{\mathcal{B}}$ be metrics on the vector bundles E_ρ and $T\mathcal{B}$, which are product metrics on a color neighborhood of the boundary. Using the splitting (68), we define the one parameter family of metrics on TX_ρ by

$$g_\varepsilon^{X_\rho} := p^* g^{E_\rho} \oplus \varepsilon^{-1} p^* g^{\mathcal{B}}, \qquad \varepsilon \in \mathbb{R}_{>0}.$$

Since $g_\varepsilon^{X_\rho}$ is a product metric on a color neighborhood of the boundary, we get by the Atiyah–Patodi–Singer index theorem

$$\mathrm{Sign}(X_\rho) = \int_{X_\rho} L(TX_\rho, g_\varepsilon^{X_\rho}) - \eta(\partial X_\rho, g_\varepsilon^{X_\rho}|_{\partial X_\rho}). \tag{69}$$

By Proposition 2 and (68), we get

$$\lim_{\varepsilon\to 0} L(TX_\rho, g_\varepsilon^{X_\rho})^{(4g)} = \left(p^* L(E_\rho, g^{E_\rho}) p^* L(T\mathcal{B}, g^{\mathcal{B}})\right)^{(4g)} = 0, \tag{70}$$

because $\dim_{\mathbb{R}} \mathcal{B} = 2$ and $\mathrm{rank}\, E_\rho = 4g - 2$. Moreover,

$$\lim_{\varepsilon\to 0} \eta(\partial X_\rho, g_\varepsilon^{X_\rho}|_{\partial X_\rho}) = -\eta^0(M_{\sigma_1}) - \eta^0(M_{\sigma_1}) + \eta^0(M_{\sigma_1\sigma_2}). \tag{71}$$

Since $\mathrm{Sign}(X_\rho) = t_g(\sigma_1, \sigma_2)$, the assertion follows from (69), (70) and (71). □

Remark 9 By Proposition 6, we have $\eta^0(M_\sigma) \in \mathbb{Z}$, which confirms [11, Proposition 5.4]. By [34, Theorem 5.7], $\eta^0(M_\sigma) \neq 0$ for some torsion element $\sigma \in SL(4g-2, \mathbb{Z})$. Hence ψ is a non-trivial function on $SL(4g-2, \mathbb{Z})$.

Appendix B: An integration of the Bott–Chern secondary form

In this appendix, we prove the last equality in Eq. (28). We keep the notation in Sect. 5.

Proposition 7 *Let $F(x) \in \mathbb{C}[[x]]$ be a formal power series with $F(0) \neq 0$. For a complex vector bundle E, let $F(E)$ be the multiplicative genus associated with $F(x)$. Let $\widetilde{F}(E; g_{E,1_g}, g_{E,G})$ be the corresponding Bott–Chern secondary form. Then*

$$\int_{\mathbb{P}(W^\vee)} \widetilde{F}(E; g_{E,1_g}, g_{E,G}) = k(F, g) \mathrm{logdet} G.$$

Here $k(F, g)$ is the constant defined by

$$k(F, g) := \left(\frac{F'(0)}{F(0)} \cdot F^{-1}(x) - \frac{1}{g} F'(x) \cdot F^{-2}(x) \right) \Big|_{x^{g-1}}, \tag{72}$$

where $f(x)\big|_{x^k}$ denotes the coefficient of x^k in a formal power series $f(x)$.

Proof We follow [39, Proposition 5.1]. Put $H = \log G$ and $g_t := g_{\exp(tH)}$. Then $\{g_t\}_{0 \le t \le 1}$ is a one-parameter family of metrics connecting g_{1_g} and g_G. Its restriction to E is denoted by $g_{E,t}$. Let $W^\vee = E \oplus_t E_t^\perp$ be the orthogonal decomposition of $W^\vee$ relative to g_t. Let $g_{N,t}$ be the metric on N via the C^∞-identification $N \cong E_t^\perp$. With respect to this splitting, $H \in \mathrm{End}(W^\vee)$ can be written as follows:

$$H = \begin{pmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{pmatrix}, \quad H_{11}(t) \in \mathrm{End}(E). \tag{73}$$

Let $R_{E,t}$ be the curvature of $(E, g_{E,t})$, and put $c_1(E_t) := \frac{\sqrt{-1}}{2\pi} \mathrm{Tr}\, R_{E,t}$. Let $R_{N,t}$ be the curvature of $(N, g_{N,t})$ and put $c_1(N_t) := \frac{\sqrt{-1}}{2\pi} R_{N,t}$. Since $N_t = \mathcal{O}_{\mathbb{P}(W^\vee)}(1)$, the 2-form $c_1(N_t)$ represents $c_1\left(\mathcal{O}_{\mathbb{P}(W^\vee)}(1)\right)$. By [39, Eq. (5.12)], we have

$$\begin{aligned} &\left[\tilde{F}(E; g_{E,0}, g_{E,1})\right]^{(g-1,g-1)} \\ &= \frac{1}{g-1} \mathrm{Tr} H \int_0^1 \dot{F}(R_{E,t})^{(g-1,g-1)} dt - \frac{1}{g-1} \int_0^1 H_{22}(t) \dot{F}(R_{E,t})^{(g-1,g-1)} dt, \end{aligned} \tag{74}$$

where $\dot{F}(R_{E,t}) := \frac{d}{d\epsilon}|_{\epsilon=0} \det F(\epsilon 1_{g-1} + \frac{\sqrt{-1}}{2\pi} R_{E,t})$. By [38, Eq. (2.8)], we get

$$\det F\left(\frac{\sqrt{-1}}{2\pi} R_{E,t}\right) \cdot F(c_1(N_t)) = 1,$$

and

$$\begin{aligned} &\mathrm{Tr}\left[\left(F'\left(\frac{\sqrt{-1}}{2\pi} R_{E,t}\right)\right) F^{-1}\left(\frac{\sqrt{-1}}{2\pi} R_{E,t}\right)\right] + F'(c_1(N_t)) F^{-1}(c_1(N_t)) \\ &= \mathrm{Tr} F'(0_g) F^{-1}(0_g) = F'(0) F^{-1}(0) g, \end{aligned}$$

where 0_g is the $g \times g$ zero matrix. These, together with the definition of $k(F, g)$, yields that

$$\dot{F}(R_{E,t})^{(g-1,g-1)}$$
$$= \left[\det F\left(\frac{\sqrt{-1}}{2\pi}R_{E,t}\right)\mathrm{Tr}\left(F^{'}\left(\frac{\sqrt{-1}}{2\pi}R_{E,t}\right)F^{-1}\left(\frac{\sqrt{-1}}{2\pi}R_{E,t}\right)\right)\right]^{(g-1,g-1)}$$
$$= \left[F^{-1}(c_1(N_t))\{g\cdot F'(0)F^{-1}(0) - F'(c_1(N_t))F^{-1}(c_1(N_t))\}\right]^{(g-1,g-1)}$$
$$= g\cdot k(F,g)\,c_1(N_t)^{g-1}. \tag{75}$$

Comparing (74) and (75), we get

$$\int_{\mathbb{P}(W^\vee)} \widetilde{F}(E;g_{E,0},g_{E,1})$$
$$= \frac{g}{g-1}k(F,g)\times\left(\mathrm{Tr}\,H - \int_0^1 dt \int_{\mathbb{P}(W^\vee)} H_{22}(t)c_1(N_t)^{g-1}\right), \tag{76}$$

where we used the identity $\int_{\mathbb{P}(W^\vee)} c_1(N_t)^{g-1} = 1$. By [39, pp. 91 l.12–92 l.5], we have

$$\mathrm{Tr}H - \int_0^1 dt \int_{\mathbb{P}(W^\vee)} H_{22}(t)c_1(N_t)^{g-1} = \frac{g-1}{g}\mathrm{Tr}H,$$

which together with (76), yields that

$$\int_{\mathbb{P}(V^\vee)} \widetilde{F}(E;g_{E,0},g_{E,1}). = k(F,g)\mathrm{Tr}\,H.$$

This, combined with Tr $H = \log\det\ G$, yields the assertion. □

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