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ON THE IMMERSION OF MANIFOLDS IN EUCLIDEAN SPACE

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By an *immersion* $f: M^k \to E^{k+l}$ of a k dimensional manifold in $k+l$ dimensional euclidean space, we mean a differentiable map (for convenience all manifolds and differentiable maps will be assumed C^∞), of $M = M^k$ into E^{k+l} which is regular; i.e., the induced map on the tangent space at each point of M is one-one. We will assume M is oriented and connected. We let B_τ be the tangent sphere bundle of the closed manifold M, and B_ν be the normal sphere bundle of M induced by the immersion f. We let W_i (resp. $\overline{W}_i$) be the integral Stiefel-Whitney characteristic classes of B_τ (resp. B_ν) of dimension i. f induces a map of B_ν into the unit sphere S^{k+l-1} in E^{k+l} by translating the unit normal vectors to M in E^{k+l} to the origin. This map is called the normal map (Chern [3]), and since $\dim B_\nu = k + l - 1$, we can define the *normal degree* of f as the degree of this map. Similarly, we can define a map by translating unit tangent vectors to the origin, and if $f: M^k \to E^{2k}$ and hence $\dim B_\tau = 2k - 1$, we can define a trangential degree.

In Section 1 we study the relations between the Gysin homology sequence of the Whitney sum of two sphere bundles and Gysin sequences of the components. In Section 2, we apply this result to the Whitney sum of B_τ and B_ν to show that:

(a) If $f: M^k \to E^{2k}$ is an immersion of a closed manifold with orientation $M \in H_k(M^k)$ then the trangential degree of f is $\overline{W}_k \cdot M$ (i.e., the Kronecker index $\langle M, \overline{W}_k \rangle$).

(b) If $f: M^k \to E^{k+l}$ with $l > 1$ is an immersion of a closed manifold with orientation M then the normal degree of f is $-W_k \cdot M$.

Further, if $f: M \to M'$ is an immersion of M in any connected oriented manifold M', not necessarily closed, of dimension $k+l$, $l > 1$, then the concept of normal degree may be generalized to be an integer mod $W'_{k+l} \cdot M'$. Here W'_{k+l} is the Stiefel-Whitney class of the tangent bundle of M', M' represents the basic class if M' is closed, and if M' is not closed $W_{k+l} \cdot M'$ is defined to be zero. Then we obtain

(b′) The normal degree of an immersion $f: M^k \to M'^{k+l}$, $l > 1$, is $-W_k \cdot M \bmod (W'_{k+l} \cdot M')$. Since $-W_k \cdot M =$ Euler characteristic of M, (a) is the known result that the normal degree is the Euler characteristic (Chern [3]).

In Section 3 we show that for completely regular immersions (see Section 3 for definition), $f: M^k \to E^{2k}$, k even, the tangential degree is twice the algebraic intersection number of Whitney [13]. Using results of Whitney, this enables us to prove :

Let M^k be a closed oriented manifold of $\dim k$, *k even. For any immersion $f: M^k \to E^{2k}$, $\overline{W}_k \cdot M$ is even ; and for every even integer n there exists an immersion $f: M^k \to E^{2k}$ such that $\overline{W}_k \cdot M = n$.*

As a corollary we obtain a theorem of Milnor [8] that there exists an immersion of real projective 3-space P^3 in E^4.

In Section 4 and 5 we study the tangential map $t: M^k \to G(k, l)$ where $G(k, l)$ is the Grassmann manifold of oriented k-planes in E^{k+l}, associated to the immersion $f: M^k \to E^{k+l}$ by assigning to each point of M^k the tangent plane at that point translated to the origin. If $l > k$, then it is well known that $t^*: H^*(G(k, l)) \to H^*(M)$ is determined by the characteristic classes of M. For the case $l \leqq k$, we get the following results :

1. Let f and g be immersions of M^k in E^{k+l} with $l > 1$ if k is odd, and with the same normal Stiefel-Whitney class $\overline{W}_l$ with integer coefficients. Then the induced tangential map $t^*: H^*(G(k, l)) \to H^*(M)$ of f and g are the same. Furthermore, if l is odd or f is an imbedding the condition that the classes $\overline{W}_l$ are the same is unnecessary. This theorem is true if coefficients are the integers, Z_2 or the rationals.

2. If M^k may be imbedded in E^{k+1} or immersed in E^{k+3}, where k is of the form $4(2^r - 1)$, then in $H^k(M^k)$ we have $P_k = W_k \bmod 2$. (For $k = 4$, this is essentially a theorem of Pontrjagin.) The same result holds, say, if M^k may be immersed in E^{2k} with a $(k-3)$-normal frame.

In general we give a complete review of the results on the characteristic classes of M^k obtainable from the cohomology of $G(k, l)$ and the fact that M^k may be immersed or imbedded in E^{k+l}. We obtain a number of known results, for example, a result of Kervaire (Theorem 5.5).

Unless we say otherwise, the coefficient group for homology and cohomology will be the integers. All the Stiefel-Whitney classes in the first four sections will have integral coefficients. All manifolds will be connected and oriented.

1. Gysin sequence of a Whitney sum

Let (B_i, S_i, M), $p_i: B_i \to M$, $i = 1, 2$, be two sphere bundles, S_i a sphere of $\dim d_i - 1$, with structure group R_i, the rotation group on the euclidean space E_i of dim d_i. We consider $R_1 \times R_2 \subset R$, where R is the rotation group on the euclidean space E of dim $d_1 + d_2$. The *Whitney sum* (B, S, M), $\dim S = d_1 + d_2 - 1$ of the two bundles is the sphere bundle with group R defined by taking as coordinate functions the direct sum of the coordinate function of the two given bundles, considered as having values in R, Then we have natural fibre preserving inclusions $f_i\colon B_i \to B$, $i = 1, 2$. f_i may be defined by considering the associated vector bundles with fibres E_i and E respectively, then f_i is induced by the natural inclusion $E_i \to E_1 + E_2 = E$ of the fibres, restricted to the unit sphere. This gives a global map since by the definition of Whitney sum, the coordinate functions of the sum bundle act on each factor of the fibre separately in the fashion given by the component bundles.

THEOREM 1.1. *f_i induces a map of the Gysin cohomology sequence of (B, S, M) into that of (B_i, S_i, M):*

$$\begin{array}{ccccccccc}
\longrightarrow & H^{r-d_1-d_2}(M) & \longrightarrow & H^r(M) & \longrightarrow & H^r(B) & \longrightarrow & H^{r+1-d_1-d_2}(M) & \longrightarrow \\
& \downarrow G_i^* & & \downarrow I & & \downarrow f_i^* & & \downarrow G_i^* & \\
\longrightarrow & H^{r-d_i}(M) & \longrightarrow & H^r(M) & \longrightarrow & H^r(B_i) & \longrightarrow & H^{r+1-d_i}(M) & \longrightarrow
\end{array}$$

where I is the identity, and letting W_i be the Stiefel-Whitney class of $B_i, S_i, M)$ *of* $\dim d_i$,

$$G_1^*(x) = (-1)^{d_1 d_2} x \cup W_2,\ G_2^*(x) = x \cup W_1 \,.$$

REMARK. W_i, $i = 1, 2$, are integral Stiefel-Whitney classes, but we may use any coefficient group for the terms in the Gysin sequences, then the cup product is under the natural pairing of the integers with the group. In particular, we may use real numbers mod 1 and topologize our cohomology groups. Then under Pontrjagin duality we get a map of the homology sequence of B_i into that of B, both with integer coefficients; and cup product goes over into cap product with the class W_i under duality.

In proving this theorem we use a number of results from Thom's thesis [11]. Following Thom we let A be the mapping cylinder of $B \to M$ and let $A' = A - B$, then we have the maps

$$\begin{aligned}
j\ &: H^r(M) \to H^r(A) \\
\beta\ &: H^r(A') \to H^r(A) \\
\varphi^*\ &: H^r(M) \to H^{r-d_1-d_2}(A')
\end{aligned}$$

where j is an isomorphism induced by the projection $A \to M$, β by the inclusion $A \to (A, B)$, and φ^* is the isomorphism obtained by Thom by considering a carapace on A' as a carapace on M by redefining supports. We have corresponding maps of $B_i \to M$. Then for example, $f_1\colon (A_1, B_1) \to$

(A, B) induces (where we have again used f_i for all maps defined by $f_i : B_i \to B$)

$$\begin{array}{ccccccccc}
 & H^{r-d_1-d_2}(M) & & H^r(M) & & & & H^{r+1-d_1-d_2}(M) & \\
 & \downarrow \varphi^* & & \downarrow j & & & & \downarrow \varphi^* & \\
\longrightarrow & H^r(A') & \longrightarrow & H^r(A) & \longrightarrow & H^r(B) & \longrightarrow & H^{r+1}(A') & \longrightarrow \\
 & \downarrow f_1^* & & \downarrow f_1^* & & \downarrow f_1^* & & \downarrow f_1^* & \\
\longrightarrow & H^r(A_1') & \longrightarrow & H^r(A_1) & \longrightarrow & H^r(B_1) & \longrightarrow & H^{r+1}(A_1') & \longrightarrow \\
 & \uparrow \varphi_1^* & & \uparrow j_1 & & & & \uparrow \varphi_1^* & \\
 & H^{r-d_1}(M) & & H^r(M) & & & & H^{r+1-d_1}(M) &
\end{array}$$

This gives a map of the Gysin sequence of (B, S, M) into that of $(B_i, S_i\, M)$; it is only necessary to identify the maps. Since

$$\begin{array}{ccc}
H^r(A) & \xrightarrow{f_1^*} & H^r(A_1) \\
\uparrow j & & \uparrow j_1 \\
H^r(M) & \xrightarrow{I} & H^r(M)
\end{array}$$

commutes, $j_1^{-1} f_1^* j = I : H^r(M) \to H^r(M)$. It remains to identify the map

$$G_1^* = \varphi_1^{*-1} f_1^* \varphi^*$$

As shown by Thom [11], if w is a fixed generator of $H^0(M)$ and if we let

$$\varphi^*(w) = U \in H^{d_1+d_2}(A'), \quad \varphi_1^*(w) = U_i \in H^{d_i}(A_i'),$$

we have :

$$\varphi^*(x) = j^*(x) \cup U, \qquad\qquad W = j^{-1}\beta U,$$

W the Stiefel-Whitney class of dim $d_1 + d_2$ in (B, S, M) and similarly for (B_i, S_i, M).

Now $f_1^* \varphi^*(x) = f_1^* j(x) \cup f_1^* U = j_1(x) \cup f_1^* U$. Hence if we can prove

$$f_1^* U = (-1)^{d_1 d_2} j_1(W_2) \cup U_1 \,, \tag{1.2}$$

we have

$$f_1^* \varphi^*(x) = (-1)^{d_1 d_2} j_1(x) \cup j_1(W_2) \cup U_1 = (-1)^{d_1 d_2} j_1(x \cup W_2) \cup U_1$$

and

$$G_1^*(x) = (-1)^{d_1 d_2} x \cup W_2 \,.$$

Remark 1. The proof for G_2^* is identical except for order of terms in the cup product in (1.2) (and hence the difference in sign) and will not be repeated.

Remark 2. If we use compact coefficient groups for the cohomology groups of the Gysin sequences, U and U_i are taken with *integer* coefficients and the cup products $j_1(x) \cup f_1^* U$ are under the natural pairing of the integers with the compact coefficient groups. In (1.2) on the other hand, everything is with integer coefficients.

It remains to prove (1.2):

Let

$$\mu: H^{p+d_1}(A_1, B_1) \times H^{q+d_2}(A_2, B_2) \to H^{p+q+d_1+d_2}((A_1, B_1 \times (A_2, B_2))$$

be the natural pairing. Note that

$$H^r((A_2, B_1) \times (A_2, B_2)) = H^r(A_1 \times A_2, A_1 \times B_2 \cup B_1 \times A_2) = H^r(A_1' \times A_2') .$$

Let $\overline{U}$ be the class $\overline{\varphi}^*(\overline{w})$ in $H^{d_1+d_2}(A_1' \times A_2')$ of the sphere bundle $(A_1 \times B_2 \cup B_1 \times A_2) \to M \times M$ corresponding to the generator $\overline{w} \in H^0(M \times M)$ where $\overline{w} = \mu_1(w_1 \otimes w_2)$, $\mu_1: H^{d_1}(M) \otimes H^{d_2}(M) \to H^{d_1+d_2}(M \times M)$. Then Thom [11] shows that $\mu(U_1 \otimes U_2) = \overline{U}$. But the Whitney sum bundle $A' \to M$ is induced by the diagonal map $d: M \to M \times M$; i.e.,

$$\begin{array}{ccc} A' & \xrightarrow{d'} & A_1' \times A_2' \\ \downarrow & & \downarrow \\ M & \xrightarrow{d} & M \times M \end{array}$$

commutes, where d' is the induced map, and hence $d'^*\overline{U} = U$.

Consider the sequence of maps

$$(A_1, B_1) \xrightarrow{d_1} (A_1, B_1) \times A_1 \xrightarrow{p_1} (A_1, B_1) \times M \xrightarrow{i_2} (A_1, B_1) \times (A_2, B_2)$$

where

i_2 is induced by the inclusion of M in A_2 (identity on first factor)

p_1 is induced by the projection of A_1 on M (identity on first factor)

d_1 is induced by the diagonal map $A_1 \to A_1 \times A_1$

Then this sequence of maps is the same as the following:

$$(A_1, B_1) \xrightarrow{f_1} (A, B) \xrightarrow{d'} (A_1, B_1) \times (A_2, B_2)$$

since they are both fibre preserving and correspond to the diagonal map $M \to M \times M$ in the base, it is sufficient to check them on each fibre. Let S_1 be a fibre in B_1 over $x \in M$ and Y_1 the mapping cylinder of $S_1 \to x$, then both fibre maps are induced by:

$$Y_1 \longrightarrow Y_1 \times x \longrightarrow Y_1 \times Y_2 .$$

Hence

$$d_1^* p_1^* i_2^* \mu(U_1 \otimes U_2) = f_1^* d'^* \overline{U} = f_1^* U .$$

To compute the left hand expression, we compute on each component of the above products separately, and we have by "abuse of notation":

$$i_2^*(U_1 \otimes U_2) = U_1 \otimes j_2^{-1}\beta_2 U_2 = U_1 \otimes W_2 ,$$

since $j_2^{-1}\beta_2: H(A_2, B_2) \to H(M)$ is the same as that given by the inclusions

$M \to A_2 \to (A_2, B_2)$. Further

$$p_1^*(U_1 \otimes W_2) = U_1 \otimes j_1(W_2)$$
$$d_1^*(U_1 \otimes j_1(W_2)) = U_1 \cup j_1(W_2)$$

i.e., this last map actually is

$$H^{d_1}(A_1, B_1) \otimes H^{d_2}(A_1) \longrightarrow H^{d_1 d_2}((A_1, B_1) \times A_1) \xrightarrow{d_1^*} H^{d_1+d_2}((A_1, B_1) .$$

Hence

$$f_1^* U = U_1 \cup j_1(W_2) = (-1)^{d_1 d_2} j_1(W_2) \cup U_1. \qquad \text{q.e.d.}$$

2. Applications to immersed manifolds

APPLICATION TO NORMAL DEGREE.[1] Let M be a compact oriented n-dimensional manifold and $f: M \to M'$ be an immersion of M in an oriented manifold of dim $n + N$, $N \geqq 2$. Let B_ν be the normal bundle of M in M' and B_τ be the tangent bundle of M; then $\dim B_\nu = n + N - 1$ and $\dim B_\tau = 2n - 1$. The map f induces a map $f_\nu: B_\nu \to T$, where T is the tangent bundle of M'. Then $f_{\nu *}: H_{n+N-1}(B_\nu) \to H_{n+N-1}(T)$. Now consider the Gysin sequence of T. Note that the right square is commutative.

$$\begin{array}{ccccccc} H_{n+N}(M') & \xrightarrow{W'_{n+N} \frown} & H_0(M') & \longrightarrow & H_{n+N-1}(T) & \longrightarrow & H_{n+N-1}(M') \longrightarrow \\ & & & & \uparrow & & \uparrow \\ & & & & H_{n+N-1}(B_\nu) & \longrightarrow & H_{n+N-1}(M) . \end{array}$$

Since $H_{n+N-1}(M)$ is zero, the image $f_{\nu *}(B_\nu)$ of the basic class B_ν of $H_{n+N-1}(B_\nu)$ is contained in the kernel of $H_{n+N-1}(T) \to H_{n+N-1}(M')$ and hence in the image of $H_0(M')$. Let $W'_{n+N} \cdot M'$ be the value of W'_{n+N} on the basic class M' of $H_{n+N}(M')$.[2] The image of $H_0(M')$ is isomorphic to the integers mod $(W'_{n+N} \cdot M')$; if M' is compact $W'_{n+N} \cdot M' = -\Omega_{M'}$, where $\Omega_{M'}$ is the Euler characteristic of M'. Hence the immersion defines a *normal degree* mod $(W'_{n+N} \cdot M')$. We use the homology version of Theorem 1.1 (see remark following theorem) to compute this degree. Consider

$$\begin{array}{ccccc} 0 & & 0 & & \\ \uparrow & & \uparrow & & \\ H_{n+N-1}(B_\nu) & \longrightarrow & H_{n+N-1}(B_\tau \otimes B_\nu) & \longrightarrow & H_{n+N-1}(T) \\ \uparrow \approx & & \uparrow \approx & & \uparrow \\ H_n(M) & \xrightarrow{W_n \frown} & H_0(M) & \longrightarrow & H_0(M') \\ \uparrow & & \uparrow & & \\ 0 & & 0 & & \end{array}$$

[1] The definition of normal degree given below is due to S.S. Chern who suggested the problem solved here.

[2] If M is not closed let $W'_{n+N} \cdot M' = 0$.

The vertical maps are part of the respective Gysin homology sequences. For the tangent bundle on M' we use *singular* homology theory and spectral sequences to define the Gysin homology sequence. (The Gysin sequences for non-triangulable space using compact supports are in general distinct from those using singular theory; e.g., if M' is a euclidean space. For a sphere bundle over the compact manifold M, the two Gysin sequences coincide.) The top line of horizontal maps are induced by inclusions. The left hand square is commutative by Theorem 1.1. Since $B_\tau \oplus B_\nu \to M$ and $T \to M'$ are sphere bundles of the same dimension, we have a map of their spectral sequences and hence of their Gysin sequences using *singular* homology theory. Hence we see that if we choose the orientation of B properly, we have the degree of the normal map is $-W_n \cdot M = \Omega_M$ modulo $W'_{n+N} \cdot M'$, where Ω_M is the Euler characteristic of M. Hence we have proved:

THEOREM 2.1. *The degree of the normal map induced by the immersion of a compact orientable manifold M into an orientable manifold M',* $\dim M = n$, $\dim M' = n + N$, $N \geqq 2$, *is* $\Omega_M \bmod W'_{n+N} \cdot M'$. *If M' is compact, the degree of the normal map is* $\Omega_M \bmod \Omega_{M'}$.

REMARK. If M' is euclidean space, the degree of the normal map is simply the integer Ω_M (since $W'_{n+N} = 0$), which agrees with previous known results.

APPLICATION TO TANGENTIAL DEGREE. Let $f: M \to E^{2k}$ be an immersion of a *oriented* manifold of dim k into $2k$-dim euclidean space. Then f induces a map $\varphi: B_\tau \to S^{2k-1}$ of the tangent bundle of M into the unit sphere of E^{2k}, by translating the unit tangent vectors of M in E^{2k} to the origin of E^{2k}. Since B_τ is of dim $2k-1$ we can talk about the degree of φ. This degree is called the *tangential degree* of the immersion f.

THEOREM 2.2. *If $\overline{W}_k \in H^k(M)$ is the normal characteristic class of* dim k *of the immersion f, then the tangential degree of f is $\overline{W}_k \cdot M$.*

PROOF. Let B_ν be the normal bundle of M induced by f, and let $B_\tau \oplus B_\nu$ be the Whitney sum. Then $B_\tau \oplus B_\nu$ is the bundle induced by f over M by the tangent bundle T of E^{2k}. The natural injection $B_\tau \to B_\tau \oplus B_\nu$ (see Section 1) and the induced map $B_\tau \oplus B_\nu \to T$ are bundle maps and induce maps of the corresponding Gysin sequences. We again use *singular* theory for the Gysin sequences of T. φ^* is the composite map $H_{2k-1}(B_\tau) \to H_{2k-1}(B_\tau \oplus B_\nu) \to H_{2k-1}(T) \approx H_{2k-1}(S^{2k-1})$. Hence consider the following diagram, the commutativity of which follows as in the proof of Theorem 2.1. The vertical sequences are the Gysin sequences for the corresponding bundles.

$$\begin{array}{ccccc}
0 = H_{2k-1}(M) & \longrightarrow & H_{2k-1}(M) & \longrightarrow & H_{2k-1}(E^{2k}) = 0 \\
\uparrow & & \uparrow & & \uparrow \\
H_{2k-1}(B_\tau) & \longrightarrow & H_{2k-1}(B_\tau \oplus B_\nu) & \longrightarrow & H_{2k-1}(T) \approx H_{2k-1}(S^{2k-1}) \\
\uparrow \approx & & \uparrow & & \uparrow \approx \\
H_k(M) & \xrightarrow{W_k} & H_0(M) & \approx & H_0(E^{2k}) \\
\uparrow & & \uparrow & & \uparrow \\
0 = H_{2k}(M) & \longrightarrow & H_{2k}(M) & \longrightarrow & H_{2k}(E^{2k}) = 0
\end{array}$$

It follows immediately that $\deg \varphi = \overline{W}_k \cdot M$.

REMARK 1. As in Theorem 2.1, we could use any other oriented manifold M' of dim $2k$ in place of E^{2k} and obtain results modulo the characteristic class of the tangent bundle of M'.

REMARK 2. In the general case of an immersion $f: M^k \to E^{k+e}$, $e \geqq 2$, one may obtain a generalization of Theorem 2.2. In fact, let $\varphi: B_\tau \to S^{k+e-1}$ be the map obtained by translating unit tangent vectors to the origin. Then the following diagram (this is the dual of the diagram used in the proof of Theorem 2.2, see also Theorem 1.1)

$$\begin{array}{ccccc}
H^{e+k-1}(B_\tau) & \longleftarrow & H^{e+k-1}(B_\tau \oplus B_\nu) & \longleftarrow & H^{e+k-1}(T) \approx H^{e+k-1}(S^{e+k-1}) \\
\varphi \downarrow \approx & & \downarrow & & \downarrow \approx \\
H^e(M) & \xleftarrow{W_e} & H^0(M) & \longleftarrow & H^0(E^{e+k})
\end{array}$$

shows that $\varphi^*(S) = \psi^{-1}\overline{W}_e$, where S is the generator of $H^{e+k-1}(S^{e+k-1})$. Hence if we knew which homomorphisms $\varphi^*: H^{e+k-1}(S^{e+k-1}) \to H^{e+k-1}(B_\tau)$ were realizable from immersions we would know which normal classes are realizable. However, we are able to obtain information on this only in the case $e = k$ (see Section 3).

3. The intersection number of an immersion

Unless otherwise stated, all manifolds in this section will be *even* dimensional, closed and oriented. In the first part of this section we recall some of the theory of Whitney [13] related to the intersection number I_f.

An immersion $f: M^k \to E^{2k}$ of a k-dim manifold $M = M^k$, has a *regular self-intersection* at $f(p_1) = f(p_2)$ if the tangent plane of $f(M)$ at $f(p_1)$ and $f(p_2)$ have only the point $f(p_1) = f(p_2)$ in common. If f has only regular self-intersections and no triple points then f is *completely regular*.

Consider M *imbedded* in E^{2k+1} and let B_τ be the unit tangent bundle of the manifold M in E^{2k+1}. Then a manifold with boundary, $\mathscr{T}$, is defined as follows. $\mathscr{T}$ is the disjoint union of B_τ and all pairs $(p, q) \in M \times M$ with $p \neq q$. If $q_n \to p$ in M in the direction of a unit vector u at p then we let

$(p, q_n) \to (p, u)$. This defines the topology on $\mathcal{T}$, and defining the differentiable structure in the obvious fashion makes $\mathcal{T}$ into a manifold with boundary B_τ.

Still considering M in E^{2k+1} let $|q - p|$ be the distance from p to q in E^{2k+1}. Returning to the immersion f above and considering E^{2k} to be oriented with origin O, we define a map $F: \mathcal{T} \to E^{2k}$ as follows:

$$F(p, q) = \frac{f(q) - f(p)}{|q - p|}, \qquad p \neq q$$

$$F(p, u) = \varphi(p, u), \qquad (p, u) \in B_\tau,$$

where $\varphi: B_\tau \to E^{2k}$ is the map induced by f taking the tangent vector to an arc through p into the tangent vector to the image of the arc in E^{2k} at $f(p)$, and translating this last vector to the origin. It is easily shown that F is continuous [13]. Furthermore F maps no point of B_τ into O and maps a point (p, q) of $\mathcal{T} - B_\tau$ into O if and only if $f(p) = f(q)$.

Suppose $f(p) = f(q)$. Let $u_1, \cdots, u_k$ be k independent unit tangent vectors of M at p, $v_1, \cdots, v_k$ at q, each determining the positive orientation of M. Then the system of $2k$ vectors, $\Delta = \{\varphi(p, u_1), \cdots, \varphi(p, u_k), \cdots, \varphi(q, v_k)\}$ will be independent at O; and the orientation determined by Δ will not depend on whether we write the vectors at p or the vectors at q first, since k is even. The self-intersection $f(p) = f(q)$ is *positive* or *negative* according to whether Δ determines the positive or negative orientation of E^{2k}. The *intersection number* I_f is the algebraic number of self-intersections.

THEOREM 3.1. *If $f: M^k \to E^{2k}$ is a completely regular immersion of a closed oriented manifold, k even, then the tangential degree of f is twice I_f.*

PROOF. Let S^{2k-1} be the sphere of unit vectors of E^{2k} at the origin, then S^{2k-1} is the boundary of the unit disc D of E^{2k}. Let $e = \max\{|F(x)| \mid x \in \mathcal{T}\}$ and $e' = \min\{|F(x)| \mid x \in B_\tau\}$. Since B_τ and $\mathcal{T}$ are compact, e and e' are well defined positive real numbers. Let $h: E^{2k} \to E^{2k}$ be map which sends vectors v in the ring $e' \leqq |v| \leqq e$ into S^{2k-1} radially by their direction, and stretches the rest of E^{2k} in an obvious fashion such that h is a homeomorphism on the complement of this ring and is continuous on all of E^{2k}. Thus $hF: (\mathcal{T}, B_\tau) \to (D, S^{2k-1})$. It is clear that hF cut down to B_τ is just the tangential map defined in Section 2, and the degree of this map is the tangential degree of f.

Let $\bar{\mathcal{T}}$ be the space obtained from $\mathcal{T}$ by identifying B_τ to a point b in $\mathcal{T}$, and $\bar{D}$ by identifying S^{2k-1} to a point s in D. Then hF induces a map $\theta: (\bar{\mathcal{T}}, b) \to (\bar{D}, s)$. Consider the commutative diagram:

$$\begin{array}{ccc} H_{2k}(\mathscr{T}, B_\tau) & \longrightarrow & H_{2k}(\mathscr{T}, b) \\ \downarrow (hF)_* & & \downarrow \theta_* \\ H_{2k}(D, S^{2k-1}) & \longrightarrow & H_{2k}(\bar{D}, x) \end{array}$$

The horizontal homomorphisms, being induced by relative homeomorphisms, are isomorphisms onto. All the groups in the diagram are infinite cyclic. Further, using the exact sequences of the pairs we obtain the commutative diagram

$$\begin{array}{ccc} H_{2k}(\bar{\mathscr{T}}) & \longrightarrow & H_{2k}(\bar{\mathscr{T}}, p) \\ \downarrow \theta^* & & \downarrow \theta_* \\ H_{2k}(\bar{D}) & \longrightarrow & H_{2k}(\bar{D}, s) \end{array}$$

where again the horizontal maps are isomorphisms. Finally we have the following commutative diagram :

$$\begin{array}{ccc} H_{2k}(\mathscr{T}, B_\tau) & \longrightarrow & H_{2k-1}(B_\tau) \\ \downarrow (hF)_* & & \downarrow (hF)_* \\ H_{2k}(D, S^{2k-1}) & \longrightarrow & H_{2k-1}(S^{2k-1}) \end{array}$$

where again the horizontal maps are isomorphisms and the group are all infinite cyclic. From these diagrams it follows that the tangential degree is the same as the degree of $\theta_* : H_{2k}(\bar{\mathscr{T}}) \to H_{2k}(\bar{D})$.

Since the map F is a homeomorphism on the components of $F^{-1}(V)$ for a sufficiently small neighborhood V of O, hF and hence θ is a homeomorphism on the components of $\theta^{-1}(V)$. According to the Hopf theory (e.g., Whitney [12]) the degree of θ is the sum of the degrees of θ / V_p, where the V_p, $p = 1, \cdots, r$, are the oriented components of $\theta^{-1}(V)$. By the definition of F we get one component for each pair (p, q) such that $F(p, q) = 0$ i.e., $f(p) = f(q)$. But this differs from the definition of the intersection number, I_f, only in the fact that (p, q) and (q, p), $f(p) = f(q)$, give two distinct components (both with the same orientation) and hence a given self-intersection is counted twice in the degree of θ ; i.e., we have : tangential degree of f = degree of $\theta = 2I_f$.

From Theorems 3.1 and 2.2 we get :[5]

COROLLARY 3.2. *If $f : M^k \to E^{2k}$ is a completely regular immersion of a closed oriented manifold, k even, and $\bar{W}_k \in H^k(M)$ is the normal characteristic class of* dim k *of the immersion f, then*

$$\bar{W}_k \cdot M = 2I_f \; .$$

Further, from the result of Whitney [13, Theorem 3] on the existence of

[5] This result is essentially due to Whitney (see note at end of bibliography).

completely regular immersions with given I_f, and from the fact that $\overline{W}_k \equiv 0$ (mod 2) (Chern, [2, Theorem 2, p. 94]) we have :

THEOREM 3.3. *Let M^k be a closed oriented manifold of* dim k, *k even. For any immersion $f: M^k \to E^{2k}$, $\overline{W}_k \cdot M$ is even ; and for every even integer n there exists an immersion $f: M^k \to E^{2k}$ such that $\overline{W}_k \cdot M = n$.*

THEOREM 3.4. (Milnor [8]). *There exists an immersion of real projective 3-space P^3 in E^4.*

PROOF. By Theorem 3.3, there is an immersion $f: S^2 \to E^4$ with $\overline{W}_2 \cdot M = 2$. By the bundle classification theory (Steenrod [9, Sections 26.2, 35.11]) there is only one bundle space over S^2 whose characteristic class is 2, and that is P^3. Consider a small tubular neighborhood (a tubular neighborhood may be defined in the case of an immersion as it usually is for an imbedding, e.g., Thom [10]) about $f(S^2)$ in E^4. The boundary (for an immersion the boundary of a tubular neighborhood will have self-intersections of course) of this tube is an immersion of P^3.

4. On the homology of Grassman manifolds

Let $G(k, l)$ be the Grassman manifold of oriented k-planes in E^{k+l}. For $n > m$, the inclusion $E^{m+k} \to E^{n+k}$ induces a map $i: G(k, m) \to G(k, n)$. One may take the limit of these spaces in a certain sense to obtain the classifying space $G(k, \infty)$ for the rotation group R_k for all manifolds. There are natural maps $i: G(k, n) \to G(k, \infty)$ for all n. As usual W_l denotes the Stiefel-Whitney class of $H^l(G(k, \infty))$, $l \leq k$ and l odd, or $l = k$. We shall use the same symbol to denote $i^* W_l$ in $H^l(G(k, n))$ when there is no ambiguity.

Let $V_{k+l,l}$ be the Stiefel manifold of l-frames in E^{k+l} and $p: V_{k+l,l} \to G(l, k)$ send a l-frame into the l-plane which is spanned by it. We will prove the following :

THEOREM 4.1. *For k even and $i \leq k$*

$$0 \longrightarrow H_i(V_{l+k,l}) \xrightarrow{p_*} H_i(G(l, k)) \xrightarrow{i_*} H_i(G(l, k+1)) \longrightarrow 0$$

is exact. For $i = k$ and k even, the image of p_* is generated by the Schubert cycle (e.g., see [2]) $\Phi = (0 \text{——} k)^+ - (0 \text{——} k)^-$. *The cycle Φ is a generator, or twice a generator of $H_k(G(l, k))$* mod *torsion according to whether W_{k+1} in $H^{k+1}(G(k+1, l))$ is zero or not. If k is odd, $l > 1$ and $i \leq k$, i_* : $H_i(G(l, k)) \to H_i(G(l, k+1))$ is an isomorphism onto.*

For cohomology we have

THEOREM 4.2. *If $i \leq k$, k odd or even*

$$H^i(V_{k+l,l}) \xleftarrow{p^*} H^i(G(l,k)) \xleftarrow{i^*} H^i(G(l,k+1)) \longleftarrow 0$$

is exact. If k is odd and $l>1$ *or* $i<k$, $H^i(V_{k+l,l})=0$. *If* $i=k$, *k even, then* p^* *is onto when* W_{k+1} in $H^{k+1}(G(k+1,l))$ *is zero. If this class is not zero, the image of* p^* *is generated by twice a generator of* $H^k(V_{k+l,l})$. *Since* $H^k(V_{k+l,l})$ *is free cyclic* $H^k(G(l,k)) = H^k(G(l,k+1)) + Z$ *when k is even. The class* $\overline{W}_k$ *together with the generators of* $H^k(G(l,k+1))$ *generate* $H^k(G(l,k))$ *when* $W_{k+1} \neq 0$ *in* $H^{k+1}(G(k+1,l))$.

For the above theorem it is important to know, for k even, when W_{k+1} is zero.

THEOREM 4.3. *Let k be even. The Stiefel-Whitney class* W_{k+1} *in* $H^{k+1}(G(k+1,l))$ *is not zero if* $k \equiv 2 \bmod 4$ *and* $l > 2$ *or* $k \equiv 0 \bmod 4$ *and* $l > 4$. *If* $l=1$ *or* $l=2$, $W_{k+1}=0$. *If* $l=3$ *or* $l=4$ *and* $k \equiv 0 \bmod 4$, $W_{k+1}=0$ *when* $k=4(2^r-1)$, *r any positive integer; otherwise* $W_{k+1} \neq 0$.

To prove these theorems we introduce certain auxiliary spaces and maps as follows. The rotation group R_n of E^n may be thought of as the space of n-frames of E^n. Let $V^n_r = R_n/R_r$, $n > r \geqq 2$ where R_r is considered as acting on the first r-vectors of an n-frame. Then V^n_r is the Stiefel manifold of $(n-r)$ frames in E^n, $V_{n,n-r}$. Wherever maps in the rest of this section are not mentioned explicitly, they refer to the maps defined in these paragraphs.

For $2 \leqq s \leqq n-r$, let R_s acting on the last s vectors of a frame define an action of R_s on V^n_r. We denote the quotient space $R^n/R_r \times R_s = V^n_r/R_s$ by $V^n_{r,s}$. If $r+s=n$, $V^n_{r,s}$ may be considered as the Grassman manifold of oriented s-planes in E^n, $G(s,r)$, and $V^n_r \to V^n_{r,s}$ the map which sends an s-frame into the s-plane spanned by it. In this way V^n_r is a principal bundle over $V^n_{r,s}$ with groups R_s. We define inclusions $V^n_r \to V^{n+1}_r$ and $V^n_{r,s} \to V^{n+1}_{r,s}$ by adding a fixed orthogonal vector to the $(r+1)$st place. From the definitions one can check that the following diagram commutes.

$$\begin{array}{ccc} V^n_r & \longrightarrow & V^{n+1}_r \\ \downarrow & & \downarrow \\ V^n_{r,s} & \longrightarrow & V^{n+1}_{r,s} \end{array}$$

A map from $V^n_{r,s}$ to $V^n_{r+1,s}$ is defined by sending the $(r+1)$st vector together with the first r-plane into the $(r+1)$-plane which they determine. In this way $V^n_{r,s}$ becomes an r-sphere bundle over $V^n_{r+1,s}$.

Similarly, a map from $V^n_{r,s}$ to $V^n_{r,s+1}$ is defined by sending the $(n-s)$th vector together with the last s-space into the $(s+1)$-plane which they span. Then $V^n_{r,s}$ is an r-sphere bundle over $V^n_{r+1,s}$ and (for $r+s<n$) the following diagram commutes

$$\begin{array}{ccc} V^n_{r,s} & \longrightarrow & V^n_{r,s+1} \\ \downarrow & & \downarrow \\ V^{n+1}_{r,s} & \longrightarrow & V^{n+1}_{r,s+1} \end{array}$$

LEMMA 4.4. *The maps* $V^n_r \to V^{n+1}_r$ *and* $V^n_{r,s} \to V^{n+1}_{r,s}$ *induce isomomorphisms in homology through dimension* r *for any* s. *Thus by the naturality of the universal coefficient theorem the induced homomorphisms in cohomology are isomorphisms for the same dimension.*

PROOF. First observe that $H_i(V^n_r) \to H_i(V^n_{r,s})$ is an isomorphism for $i < r$ because the groups vanish. For $i = r$ the maps $S^r = V^{r+1}_r \to V^n_r \to V^{n+1}_r$ generate $\pi_r(V^n_r)$ and $\pi_r(V^{n+1}_r)$ [9, p. 132]. Thus $\pi_r(V^n_r) \to \pi_r(V^{n+1}_r)$ is an isomorphism and by the naturality of the Hurewicz Theorem $H_r(V^n_r) \to H_r(V^{n+1}_r)$ is also.

Now consider the sequence of sphere bundles :

$$\begin{array}{ccccccccccccc} V^n_r & \longrightarrow & V^n_{r,2} & \longrightarrow & \cdots & \longrightarrow & V^n_{r,l} & \xrightarrow[S]{} & V^n_{r,l+1} & \longrightarrow & \cdots & \longrightarrow & V^n_{r,s} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\ V^{n+1}_r & \longrightarrow & V^{n+1}_{r,2} & \longrightarrow & \cdots & \longrightarrow & V^{n+1}_{r,l} & \xrightarrow[S]{} & V^{n+}_{r,l+1} & \longrightarrow & \cdots & \longrightarrow & V^{n+1}_{r,s} \end{array}$$

Corresponding to the l-dimensional sphere bundles in the above diagram, we have the Gysin sequences,

$$\begin{array}{ccccccccccc} \to H_{i-l}(V^n_{r,l+1}) & \to & H_i(V^n_{r,l}) & \to & H_i(V^n_{r,l+1}) & \to & H_{i-l-1}(V^n_{r,l+1}) & \to & H_{i-1}(V^n_{r,l}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \to H_{i-l}(V^{n+1}_{r,l+1}) & \to & H_i(V^{n+1}_{r,l}) & \to & H_i(V^{n+1}_{r,l+1}) & \to & H_{i-l-1}(V^{n+1}_{r,l+1}) & \to & H_{i-1}(V^{n+1}_{r,l}) \end{array}$$

By induction assume (we have proved the case $l = 0$)

$$H_i(V^n_{r,l}) \approx H_i(V^{n+1}_{r,l}) \qquad i = 0, \cdots, r$$
$$H_j(V^n_{r,l+1}) \approx H_j(V^{n+1}_{r,l+1}) \qquad j = 0, \cdots, i-1 \, .$$

then by the 5-lemma, $H_i(V^n_{r,l+1}) \approx H_i(V^{n+1}_{r,l+1})$ and the lemma follows.

Theorem 4.1 for $i < k$ follows immediately from the Gysin sequence of $V^{k+l+1}_{k,l}$ over $V^{k+l+1}_{k+1,l}$ and the preceding lemma. To prove the theorem for $i = k$, write down a portion of this sequence.

$$\begin{array}{ccccccccc} & & Z & & Z & & & & \\ & & \| & & \| & & & & \\ & \longrightarrow & H_0(V^{k+l+1}_{k+1}) & \xrightarrow{\gamma'} & H_k(V^{k+l+1}_k) & \longrightarrow 0 & & & \\ & & \downarrow \tau & & \downarrow \bar{p}_* & & & & \\ H_{k+1}(V^{k+l+1}_{k+1,l}) & \xrightarrow{\beta} & H_0(V^{k+l+1}_{k+1,l}) & \xrightarrow{\gamma} & H_k(V^{k+l+1}_{k,l}) & \xrightarrow{q_*} & H_k(V^{k+l+1}_{k+1,l}) & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & Z & & & & & & \end{array}$$

The map γ' is induced by the projection of V^{k+l+1}_k into V^{k+l+1}_{k+1} which takes the $l+1$ frame into the l frame consisting of the last l-vectors of the $l+1$ frame. Then the middle square of the previous diagram commutes.

For all k, q_* is onto. We consider now only k even. From exactness γ' must be an isomorphism onto and certainly τ is also. This implies that the image of γ is the image of $\bar{p}_*$. The image of β is zero since β is defined by cap product with an order two class. Hence γ and $\bar{p}_*$ are 1 - 1, We have proved that in the following diagram the bottom horizontal sequence is exact.

$$\begin{array}{ccccccccc} & & H_k(V_k^{k+l}) & \xrightarrow{p_*} & H_k(V_{k,l}^{k+l}) & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & H_k(V_k^{k+l+1}) & \xrightarrow{\bar{p}_*} & H_k(V_{k,l}^{k+l+1}) & \xrightarrow{q_*} & H_k(V_{k+1,l}^{k+l+1}) & \longrightarrow & 0 \end{array}$$

The vertical maps are isomorphisms onto by 4.4. This proves the first sentence of Theorem 4.1 where we have let $i_* = q_* j_*$.

We will now compute the kernel of $i_* : H_k(G(l, k)) \to H_k(G(l, k+1))$. Let $C_k(G(m, n))$ be the group of Schubert k-chains (e.g., see Chern [2]) of $G(m, n)$, $Z_k(G(m, n))$, the Schubert k-cycles, etc.

It follows from the definition of the Schubert cells that $C_k(G(l, k)) = C_k(G(l, k+1))$ (under identification of map induced by inclusion), and that $C_{k+1}(G(l, k+1))$ contains exactly the linear combinations of the cells $(0 —— k+1)^+$ and $(0 —— k+1)^-$ in addition to the cells of $C_{k+1}(G(l, k))$.

Now let z_k belong to both $Z(G(l, k))$ and $B_k(G(l, k+1))$. Then $z_k = \partial c_{k+1}$ where $c_{k+1} \in C_{k+1}(G(l, k+1))$. From the above observations we can write $c_{k+1} = c'_{k+1} + m(0 —— k+1)^+ + n(0 —— k+1)^-$ where $c'_{k+1} \in C_{k+1}(G(l, k))$. Hence

$$z_k - \partial c'_{k+1} = m\partial(0 —— k+1)^+ + n\partial(0 —— k+1)^-$$

By the boundary formulas for Schubert chains (e.g., Chern [2]) one obtains k even:

$$z_k - \partial c'_{k+1} = N\Phi,\ \Phi = (0 —— k)^+ - (0 —— k)^-, \qquad N = m - n$$

k odd:

$$z_k - \partial c'_{k+1} = N'(0 —— k)^+ + N'(0 —— k)^-, \qquad N' = -m - n\ .$$

Thus for k even, Φ generates the kernel of i_*.

Furthermore if k is odd, $l > 1$ and

$$c = \sum_{r=1}^{(k+1)/2} \alpha(r)(0 —— r, k - r + 1)^+ \qquad \begin{array}{l} \alpha(r) = +1 \text{ if } r \equiv 1 \text{ or } 2 \bmod 4 \\ \alpha(r) = -1 \text{ if } r \equiv 1 \text{ or } 3 \bmod 4 \end{array}$$

it may be checked that $\partial c = (0 —— k)^+ + (0 —— k+1)^-$. Thus for k odd, $l > 1$, the kernel of i_* is zero in $H_k(G(l, k))$.

To finish the proof of 4.1 consider the Gysin sequences in cohomology, k even,

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & H^k(V_k^{k+l+1}) & \longrightarrow & H^0(V_{k+1}^{k+l+1}) & \longrightarrow & H^{k+1}(V_{k+1}^{k+l+1}) \\ & & \uparrow & & \uparrow p^* & & \uparrow & & \uparrow \\ 0 & \longrightarrow & H^k(V_{k+1,l}^{k+l+1}) & \rightarrow & H^k(V_{k,l}^{k+l+1}) & \longrightarrow & H^0(V_{k+1,l}^{k+l+1}) & \xrightarrow{\smile\Omega} & H^{k+1}(V_{k+1,l}^{k+l+1}) \end{array}$$

Here $\Omega \in H^{k+1}(V_{k+1,l}^{k+l+1})$ is the order two characteristic class of the S^k-bundle $V_{k,l}^{k+l+1}$ over $V_{k+1,l}^{k+l+1}$. If $\Omega = 0$ it follows that p^* is onto (see the previous discussion for homology) or in homology if g is a generator of $H_k(V_k^{k+l+1})$ then $p_*(g)$ is a generator of $H_k(V_{k,l}^{k+l})$ mod torsion. If $\Omega \neq 0$, $p_*(g)$ is twice a generator of $H_k(V_{k,l}^{k+l})$ mod torsion.

To identify Ω, consider the following diagram where n is larger than $\max(2k+3, l)$.

$$\begin{array}{ccccc} V_{k,l}^{k+l+1} & \xrightarrow{\beta} & V_{l,k}^{k+l+1} & \xrightarrow{\eta} & V_{n,k}^{n+k+1} \\ \downarrow q & & \downarrow g & & \downarrow g' \\ V_{k+1,l}^{k+l+1} & \xrightarrow{\alpha} & V_{l,k+1}^{k+l+1} & \xrightarrow{\eta'} & V_{n,k+1}^{n+k+1} \end{array}$$

Here α is the homeomorphism which takes an l-plane $[e_{k+2}, \cdots, e_{k+l+1}]$ into the orthogonal $k+1$ plane $[e_1, \cdots, e_{k+1}]$ such that $e_1, \cdots, e_{k+1}, e_{k+2}, \cdots, e_{k+l+1}$ has the given orientation of E^{k+l+1}. Then β can be defined so as to make the diagram commute. The maps η and η' are compositions of maps $V_{l,k}^{k+l+1} \rightarrow V_{l,k}^{k+l+2} \rightarrow V_{l+1,k}^{k+l+2}$ which were defined previously. It can be checked that this diagram commutes and furthermore that $V_{n,k+1}^{n+k+1}$ over $V_{n,k+1}^{n+k+1}$ is the associated S^k bundle of V_n^{n+k+1} over $V_{n,k+1}^{n+k+1}$. This implies that if W_{k+1} is the Stiefel-Whitney class of $H^{k+1}(G(k+1, n))$, then $\Omega = \alpha^* \eta' W_{k+1}$. Since α^* is an isomorphism we have $\Omega = 0$ if and only if W_{k+1} in $H^{k+1}(G(k+1, l))$ is zero. This finishes the proof of Theorem 4.1.

The proof of 4.2 follows from arguments dual to those used in proving 4.1. We merely add that since $\overline{W}_k = (0 \text{——} k)^+ - (0 \text{——} k)^-$ has the value of 2 on ϕ, it has the value 1 on a homology generator, hence the last statement of 4.2.

The proof of 4.3 proceeds as follows. Since $k+1$ is odd, W_{k+1} is defined with integer coefficients and is of order two. Generally if $H^*(X)$ has only order two torsion an element of $H^n(X)$ is zero if and only if its rational and Z_2 reduction are zero. Since $H^*(G(m, n))$ has only order 2 torsion, all m, n; W_{k+1} is zero if and only if its Z_2 reduction is zero. Thus for the proof of 4.3 we use coefficients Z_2 for all Stiefel-Whitney classes. A formula of Chern [2] takes the following form,[3]

$$W_{k+1} = \begin{vmatrix} \overline{W}_1 & \overline{W}_0 & 0 & \cdots\cdots & 0 \\ \overline{W}_2 & \overline{W}_1 & \overline{W}_0 & 0 \cdots\cdots & 0 \\ \overline{W}_3 & \overline{W}_2 & \overline{W}_1 & \overline{W}_0 \; 0 \cdots & 0 \\ & & \cdots\cdots\cdots\cdots & & \\ \overline{W}_k & \overline{W}_{k-1} & \cdots\cdots & & \overline{W}_0 \\ \overline{W}_{k+1} & \overline{W}_k & \cdots\cdots & & \overline{W}_1 \end{vmatrix}$$

In $G(k+1, l)$), $\overline{W}_i = 0$ for $i > l$ and $W_1 = 0$; there are no relations between the other $\overline{W}_i$'s.

This determinant is symmetric with respect to the 45° axis. Hence all non-symmetric terms appear twice and drop out.

Now suppose $k=4n+2$ and $l>2$. It is clearly sufficient to show $W_{k+1}\neq 0$ in $G(k+1, 3)$ for this case. But from the above determinant one notes that the symmetric term $\overline{W}_3(\overline{W}_2)^{2n}\neq 0$, hence $W_{k+1}\neq 0$, in $H^{k+1}(G(k+1, 3))$. On the other hand, if $k = 4n$ and $l > 4$ it is sufficient to show that $W_{k+1} \neq 0$ in $G(k+1, 5)$. There the symmetric term $\overline{W}_5(\overline{W}_2)^{2n-2} \neq 0$ hence in this case $W_{k+1} \neq 0$.

For $l = 2$, W_{k+1} is a polynomial in $\overline{W}_2$, but k is even so $W_{k+1} = 0$. The last sentence of the theorem also follows from the properties of the above determinant but it involves a long computational argument that does not seem worthwhile here.

5. The tangential map of an immersion

Let f be an immersion of an orientable manifold $M^k = M$ in E^{k+l}. Then f defines a tangential map $t: M \to G(k, l)$ by translating a tangent plane at a point of $f(M)$ to the origin of E^{k+l}. The purpose of this section is to investigate the induced homomorphism in cohomology. If $l > k$, then it is well known that $t^*: H^*(G(k, l)) \to H^*(M)$, the characteristic homomorphism, is determined by the characteristic classes of M. Therefore we confine ourselves to the case $l \leqq k$.

THEOREM 5.1. *Let $f: M^k \to E^{k+l}$ be an immersion of an orientable manifold $M = M^k$ with k even and with l and k such that $W_{k+1} = 0$ in $H^{k+1}(G(k+1, l))$* (see 4.3).

Case I. *If $l = 1$, one can choose a generator Λ_k of $H^k(G(k, 1)) = Z$ so that $W_k = 2\,\Lambda_k$.*

[3] See also: S. S. Chern, *On the multiplication in the characteristic ring of a sphere bundle*, Ann. of Math., 49 (1948), 362–372.

Case II. *If* $l = 2$, *one can choose a* Λ_k *such that* Λ_k *and* $(\overline{W}_2)^{k/2}$ *generate* $H^k(G(k, 2)) = Z + Z$, $W_k = 2\Lambda_k + (\overline{W}_2)^{k/2}$ *and if* $k \equiv 0 \bmod 4$, $P_k = (\overline{W}_2)^{k/2}$.

Case III. *If* $l = 3$, $k = 4(2^n - 1)$, $H^k(G(k, 3)) \bmod$ *torsion is generated by a cocycle* Λ_k *and* P_k, *and* $W_k = 2\Lambda_k + P_k \bmod$ *torsion.*

Case IV. *If* $l = 4$, $k = 4(2^n - 1)$, $H^k(G(k, 4)) \bmod$ *torsion is generated by a cocycle* Λ_k, *and all possible cup products of* $\overline{W}_4$ *and* $\overline{P}_4$ *with total degree* k. $W_k = \overline{W}_4 G(\overline{W}_4, \overline{P}_4) + P_k + 2\Lambda_k \bmod$ *torsion, where* $G(\overline{W}_4, \overline{P}_4)$ *is a polynomial.*

Let $\alpha : G(k, l) \to G(l, k)$ *be the homeomorphism defined in* § 4 *and* $\alpha^* : H^*(G(l, k), G) \to H^*(G(k, l), G)$ *the induced isomorphism where* $G = Z_2$ *or* Z. *Then if* W_i *and* P_i *are Stiefel-Whitney or Pontrjagin classes of* $G(l, k)$, $\overline{W}_i = \alpha^* W_i$ *and* $\overline{P}_i = \alpha^* P_i$ *are the dual classes of* $G(k, l)$.[4]

From 5,1., we obtain :

Case I yields Hopf's theorem on the *curvatura integra* of an immersion of an even dimensional manifold.

Case II is essentially a generalization of the Chern-Spanier result [4] on immersions of 2-manifolds in E^4.

Case IV yields :

COROLLARY 5.2. *Suppose a closed orientable manifold* M^k *may be imbedded in* E^{k+4} *(or immersed in* E^{k+3}*) where* k *is the form* $4(2^r - 1)$. *Then in* $H^k(M^k)$, *we have* $P_k = W_k \bmod 2$.

For $k = 4$ this is essentially a theorem of Pontrjagin (see [2]). Corollary 5.2 follows from Case IV, since $H^k(M^k) = Z$ has no torsion and $\overline{W}_4 = 0$ in M^k, and thus $W_k = P_k + 2\Lambda_k$ or $W_k = P_k \bmod 2$.

Theorem 5.1 is proved as follows. We first prove Case II. The cohomology ring $H^*(G(2, \infty))$ is generated by W_2, so $(W_2)^{k/2}$ is a generator of $H^k(G(2, \infty))$. Applying 4.1 and 4.2, since the cycle $\Phi = (1 —— 1)^+ - (1 —— 1)^-$ is a free generator of $H_k(G(2, k))$, $(W_2)^{k/2}$ and a cocycle Λ'_k with value 1 on Φ generate $H^k(G(2, k))$. Therefore the corresponding classes $\Lambda''_k = \alpha^*\Lambda'_k$ and $(\overline{W}_2)^{k/2} = \alpha^*(W_2)^{k/2}$ generate $H^k(G(k, 2))$.

Let $W_k = m\Lambda''_k + n(\overline{W}_2)^{k/2}$. From 4.1 and 4.2 it follows that $(\overline{W}_2)^{k/2}$ must have the value 0 on $\alpha^*\Phi = \overline{\Phi} = (1 —— 1)^+ - (1 —— 1)^-$. Then since W_k has the value 2 on $\overline{\Phi}$ and Λ''_k has the value 1 on $\overline{\Phi}$, m must be equal to 2. From the Whitney duality theorem one obtains $W_k =$

[4] Our $\alpha: G(k, l) \to G(l, k)$ corresponds to Wu's d^*. Wu shows that $\alpha^* W_j = \overline{W}^j \bmod 2$ and $\alpha^*(P_{4i}) = (-1)^i \overline{P}_{4i}$ with rational coefficients. In general for any class Z, $(\alpha^*)^2 Z = \pm Z$ with rational coefficients and $(\alpha^*)^2 Z = Z \bmod 2$. Hence $(\alpha^*)^2 Z = \pm Z$ with integer coefficients. In our work the sign does not matter.

$(\overline{W}_2)^{k/2}$ mod 2, so n must be odd. Let $\Lambda_k = \Lambda_k'' + (1/2)(n-1)(\overline{W}_2)^{k/2}$. Then $W_k = 2\Lambda_k + (\overline{W}_2)^{k/2}$. Since $(\overline{W}_2)^{k/2}$ has the value 0 on Φ, Λ_k will have the value 1.

We will now show that $(\overline{W}_2)^{k/2} = P_k$ in $H^k(G(k, 2))$ if $k \equiv 0 \bmod 4$. Since $\overline{P}_i = 0$ in $H^k(G(k, 2))$, $i > 4$ (e.g., [2]) by the duality Pontrjagin classes mod torsion (e.g., [6]) we can write $P_k = (\overline{P}_4)^{k/4}$. But $P_4 = W_2^2$ or $\overline{P}_4 = \overline{W}_2^2$, hence $P_k = (\overline{W}_2)^{k/2}$. This prove Case II of 5.2. Case I is proved the same way.

To prove Case IV, note by the previous arguments that $H^k(G(k, 4))$ mod torsion is generated by cup products of $\overline{W}_4$ and $\overline{P}_4$ with total degree k and a cocyle Λ_k' which has value 1 on $\overline{\Phi} = (1 —— 1)^+ - (1 —— 1)^-$. Then $\overline{W}_k = \overline{W}_4 G(\overline{W}_4, \overline{P}_4) + (\overline{P}_4)^{k/4} + 2\Lambda_k$ mod torsion exactly as in Case II, where $G(\overline{W}_4, \overline{P}_4)$ is a polynomial in $\overline{W}_4$ and $\overline{P}_4$. By the duality theorem for Pontrjagin classes mod torsion, $P_k = (\overline{W}_4)^2 F(\overline{W}_4^2, \overline{P}_4) + u_1(\overline{P}_4)^{k/4}$ mod torsion (since $(\overline{W}_4)^2 = P_8$). By pulling P_k into $H^k(G(k, 2))$ we see that $P_k = u_1(\overline{P}_4)^{k/4} = u_1(\overline{W}_2)^{k/2}$ so $u_1 = 1$. Therefore we obtain $W_k = 2\Lambda_k + P_k + \overline{W}_4 G(\overline{W}_4, \overline{P}_4)$ mod torsion proving Case IV. Case III is immediate from the preceding. This proves 5.1.

In the following theorem integer coefficients are meant.

THEOREM 5.3. *Let f and g be immersions of M^k in E^{k+l} with $l > 1$ if k is odd, and with the same normal Stiefel-Whitney class $\overline{W}_l$. Then the induced homomorphisms of the tangential maps $t^* : H^*(G(k, l)) \to H^*(M)$ of f and g are same.*

The exceptional case referred to in Theorem 5.3, k odd and $l = 1$, has been studied by Milnor [8]. We will not consider it here.

COROLLARY 5.4. *Let f and g be immersions of M^k in E^{k+l} with $l > 1$ if k is odd, and suppose that f and g are imbeddings or that l is odd. Then the conclusion of* 5.3 *holds* (i.e., *without any assumption on $\overline{W}_l$*).

That the corollary follows from the theorem may be seen as follows. First, if l is odd then $\overline{W}_l = \delta^* \overline{W}_{l-1}$ where δ^* is the Bockstein operator and $\overline{W}_{l-1}$ does not depend on the immersion, hence $\overline{W}_l$ also does not depend on the immersion. On the other hand, if f is an imbedding $\overline{W}_l = 0$ (e.g., see Chern-Spanier [4]).

REMARK. All the results obtained in this section for immersions of M^k in E^{k+l} can be generalized to the case where M^k is immersed in E^{k+l+p} with a field of normal p-frames. Here the induced map is from M^k to

$V_{l,p}^{k+l+p}$ (see Section 4). By 4.4, $H^i(V_{l,k}^{k+l+p})$ is naturally isomorphic to $H^i(G(k, l))$ for $i \leqq k$.

We now prove 5.3. We need the following well known lemma.

LEMMA 5.5. *Let X be a space such that $H^*(X; Z)$ has only order 2 torsion. The torsion subgroup of $H^k(X; Z)$ is $\delta^*(H^{k-1}(X; Z_2))$ where δ^* is the Bockstein operator.*

PROOF. Let $u \in C^{k-1}(X)$ be an integral cochain which is a cycle mod 2, i.e., $\delta u = 2c$ where $c \in C^k(X)$. Then since $\delta\delta u = 0 = 2\delta c$, $\delta c = 0$, and c is a cycle representing an order 2 cohomology class.

Conversely, let c be any chain representing an order 2 cohomology class. Then $2c = \delta u$ for some $u \in C^{k-1}(X)$. But then u is a cycle mod 2.

We consider for the proof of 5.3 the following diagram:

$$\text{(D)} \qquad \begin{array}{ccc} H^r(G(k, l); G) & \xleftarrow{i^*} & H^r(G(k, \infty); G) \\ \uparrow{\scriptstyle \alpha^*} & & \\ H^r(G(l, k); G) & \xleftarrow{i'^*} & H^r(G(l, \infty); G) \end{array}$$

Here i^* and i'^* are induced by the inclusions i and i'.

LEMMA 5.6. *If $G = Z_2$, then i^* is onto for $r < k$.*

PROOF. For $r < k$, by 4.2, i'^* is an isomorphism onto (this is also a well known fact). Thus, cup products of W_j mod 2, $j = 1, \cdots, l$, generate $H^r(G(l, k); Z_2)$ for $r < k$. Then cup products of $\alpha^* W_j = \overline{W}_j$ generate $H^r(G(k, l): Z_2)$. By the Chern cup product formula (see the end of Section 4) these $\overline{W}_j$ can be written as polynomials in the classes W_j of $G(k, l)$. Since $H^*(G(k, \infty); Z_2)$ is generated by such classes, this proves 5.6.

LEMMA 5.7. *For $r < k$, $H^r(G(k, l); Z)$* mod *torsion is generated by the image of i^* and the class $\overline{W}_l = \alpha^* W_l$ where W_l is the lth Stiefel-Whitney class of $G(l, k)$.*

The proof is similar to 5.6 and uses the diagram (D). For $r < k$, $H^r(G(l, k); Z)$ mod torsion is generated by W_l and Pontrjagin classes P_j. Then one uses the theorem that $\alpha^* P_j = \overline{P}_j$ mod torsion can be expressed as polynomials in the Pontrjagin classes P_j mod torsion of $G(k, \infty)$ (see [6, Bemerkung p. 68]). Then since $H^*(G(k, \infty); Z)$ mod torsion is generated by these Pontrjagin classes and W_k, this proves 5.7.

LEMMA 5.8. *For $r < k$, $\overline{W}_l$ and the image of i^* generates $H^r(G(k, l); Z)$.*

PROOF. Let $g_1, \cdots, g_n, h_1, \cdots, h_m$ generate $H^r(G(k, l); Z)$ where the g_j are free and the h_j are of order two. Then by 5.5 and 5.6 $h_j = \delta h'_j =$

$\delta i^*\delta k_j$, where $h'_j \in H^{r-1}(G(k, l); Z_2)$ and $k_j \in H^{r-1}(G(k, \infty); Z_2)$. Hence h_j is in the image of i^* for each j. On the other hand, from 5.7 it follows that the g_j are generated by $\overline{W}_l$ and the image of i^*, proving 5.8.

Until this point in the proof of 5.3 we have not significantly used our previous results.

For the proof of 5.3 we have only to consider the case $r = k$.

If k is odd and $l > 1$, by 4.2 and using the diagram (D), the same arguments as above go through to yield 5.3 for this case.

We now suppose k is even. Since M^k is oriented we can assume $H^k(M) = Z$ (otherwise $H^k(M) = 0$ and there is nothing to prove).

From 5.1 one obtains immediately

LEMMA 5.9. *If k is even and W_{k+1} in $H^{k+1}(G(k+1, l))$ is zero, then* $H^k(G(k, l))$ mod *torsion is generated by the image of i^* and $\overline{W}_l$. Then in this case we have* 5.3.

Lastly we prove 5.3 when k is even and $W_{k+1} \neq 0$. Since $H^k(M) = Z$, we ignore torsion. By 4.2 we obtain that $H^k(G(k, l), Z)$ mod torsion is generated by $W_k = \alpha^*\overline{W}_k$, $\overline{W}_l = \alpha^* W_l$ and $\overline{P}_j = \alpha^* P_j$. Then by the reasoning of Lemma 5.7 we obtain 5.3.

Lastly, we prove a theorem of Kervaire [7].

Let $M^k = M$ be a closed oriented manifold, k even, and let $f: M \to E^{k+l}$ be an immersion with a cross-section in the bundle of normal l-frames. Then f induces a map $\varphi: M \to V_{l+k,l}$ of M into the Stiefel manifold [1]. *Let the induced homomorphism be denoted $\varphi_*: H_k(M) \to H_k(V_{l+k,l})$.*

THEOREM 5.10. *There is a generator v of $H_k(V_{l+k,l}) = Z$ such that $\varphi_*(M) = 1/2\Omega_t v$ where Ω_t is the Euler characteristic and M the fundamental cycle of M.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} H_k(M^k) & \xrightarrow{\varphi^*} & H_k(V_{l+k,l}) \\ \downarrow t^* & & \downarrow P_* \\ H_k(G(k,l)) & \xrightarrow{\alpha_*} & H_k(G(l,k)) \end{array}$$

where t and α are the tangential map and dual homeomorphism respectively, both defined earlier. Choose v by Theorem 4.1 so that $P_*(v)$ can be represented by $(0 —— k)^+ - (0 —— k)^-$. Let $\varphi_*(M) = nv$ and let W^k be the Stiefel-Whitney class in $H^k(G(k, l))$. Then

$$W^k[\alpha_* P_*(nv)] = W^k[\alpha_* P_* \varphi_*(M)] = W^k t_*(M) = t^* W^k(M) .$$

On the other hand

$$W^k[\alpha_* P_*(nv)] = nW^k\alpha_* P_*(v) = n\alpha^* W^k P_*(v)$$
$$= n\overline{W}^k[(0 \text{---} k)^+ - (0 \text{---} k)^-] = 2n\ .$$

Thus $n = \Omega_l/2$ and the theorem is proved.

REMARK 1. By similar techniques one may prove Theorem 5.3 for coefficients modulo 2. We do not include a complete proof, but simply remark that it is well known that the cohomology ring of the Grassmann manifold $\overline{G}(k, l)$ of *non-oriented* k-planes in $k+l$ space is generated by the classes $W_1, \cdots, W_k$ modulo 2 (see [2]). Since the manifold $G(k, l)$ of oriented planes double covers $\overline{G}(k, l)$, one may use the Gysin sequences modulo 2 of this zero sphere bundle to obtain that $H^i(G(k, l), Z_2)$ is onto, $i \leqq k$, if and only if $W_{k+1} \neq 0$ in $H^*(G(k+1, l), Z_2)$. It then follows that $H^*(G(k, l), Z_2)$ is generated by $W_2, \cdots, W_k$ for dimensions $\leqq k$ except for the special cases already considered for integral coefficients. We thus obtain 5.3 for coefficients modulo 2.

REMARK 2. We have shown that the only invariant able to distinguish immersions of M^k in E^{k+1} obtainable from the homomorphism $t^*: H^*(G(k, l)) \to H^*(M^k)$ is $\overline{W}_l$, except for k odd and $l=1$. Furthermore by duality $\overline{W}_l$ is determined mod 2 by the Stiefel-Whitney classes of M. We have the problem, given a class γ of $H^l(M^k)$, when is there an immersion of M^k in E^{k+l} with $\overline{W}^l = \gamma$? If $l = k$, Theorem 3.3 says this is possible for all cohomology classes of $H^k(M)$ not excluded by duality.

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