LOCAL TOPOLOGICAL PROPERTIES OF COMPLEX ALGEBRAIC SETS

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Recently many new results have been obtained on the local topological structure of complex algebraic sets. We cite some of them since they have some relation to the methods of studying complex sets; these methods are used in this article.

In 1966 É. Brieskorn studied manifolds which arose as the intersection of a hypersurface specified in C^m by the equation $z_1^{p_1} + \ldots + z_m^{p_m} = 0$ and the sphere S_ε^{2m-1} centered at the origin; in particular, he showed that the 28 manifolds $z_1^{6k-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0$, $|z_1|^2 + \ldots + |z_5|^2 = \varepsilon^2$, $k = 1, \ldots, 28$ are 28 Milner or exotic spheres (they are all homeomorphic to the usual seven-dimensional sphere but they are pairwise not diffeomorphic).

J. Milner showed in [1] how geometric methods could be used in the study of the local topology of algebraic sets: the construction of vector fields on the manifolds being discussed (where the vector fields had special properties) and Morse theory. Milner obtained several important new families of singular points on hypersurfaces. Suppose $f(z_1, \ldots, z_m)$, $f(0) = 0$ is a polynomial on C^m, $Y = f^{-1}(0)$; let S_ε and D_ε be the sphere and the ball in C^m with radius ε and center the coordinate origin, with $K = Y \cap S_\varepsilon$. Milner showed that for sufficiently small ε the map $\varphi = f/|f|: S_\varepsilon \setminus K \to S^1$ gives a locally trivial fibration over the circle. Moreover, if zero is an isolated singular point of Y then K is a smooth manifold and the fiber F of the fibration φ has the homotopy type of a bouquet of spheres S^{m-1}. Milner also gave a formula which determines the number μ of spheres in the bouquet. It is equal to the degree of the map

$$\operatorname{grad} f/\|\operatorname{grad} f\|: S_\varepsilon^{2m-1} \to S_1^{2m-1}.$$

The characteristic map of the fibration φ in the homology $h_*: H_{m-1}(F) \to H_{m-1}(F)$ is called the local Picard–Lefshetz monodromy. Milner showed that this monodromy is an important tool in the study of the topology of singularities.

F. Hirzebruch and K. Mayer, and P. Orlik and F. Wagreich have thoroughly investigated the special cases of isolated singularities of hypersurfaces Y where $Y \setminus \{0\}$ admits a natural action of the multiplicative group of nonzero complex numbers.

É. Brieskorn gave a description of the monodromy of an isolated singularity of a hypersurface in algebraic terms in the article "Die Monodromie der Isolierten Singularitäten von Hyperflächen," i.e., he described it in terms of constructions in which the polynomial f played the part only of an algebraic object.

On the other hand H. Hamm [2] generalized the geometric methods proposed by Milner and used them to study the local topological structure of algebraic sets given by more than one equation. Milner's formula for μ is extended to this case in [3]; algebraic expressions for μ were obtained not long ago by É. Brieskorn and H. Gruel as well as by the Vietnamese mathematician Lieh Tung Trongom.

In this article we investigate the structure of complex sets in the neighborhood of nonisolated singular points. We use geometrical methods similar to those used by Hamm, and we prove that the connectivity of the fiber F of the Milner fibration φ depends, when compared with the case of an isolated singularity, on the dimension of the set of singular points. When the methods of [3] are used, we can prove a formula which gives the Euler characteristic of the fiber F as the "degree" of the naturally occurring map of the intersection of the algebraic set from the sphere S_ε to a manifold of frames of special form in C^m.

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The results of this article can be used to study the structure of algebraic sets in greater detail, if certain restrictions are imposed on the set of singular points. In particular, if the set of singular points is one-dimensional, then the formula for the Euler characteristic of the fiber F of the Milner fibration can be obtained; this formula is much more convenient for actual calculations than the formula of Section 3 below. These results will be set out in detail below.

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1. A Lemma on grad h_1 and grad h_2

In this section we prove some relations between gradients of functions which occur in the study of algebraic sets; they are needed to carry out all the constructions which follow.

Let $Y \ni 0$ be a real algebraic set in R^n, $g_1, \ldots, g_k$ generators of the ideal I(Y), and let $h_1 \geq 0$ and $h_2 \geq 0$ be nonnegative polynomial functions on R^n, $h_1(0) = h_2(0) = 0$. Then on some neighborhood of the coordinate origin and at all simple points x of the set Y such that $h_1(x) > 0$, $h_2(x) > 0$, the vectors $\operatorname{grad} \bar{h}_1(x)$ and $\operatorname{grad} h_2(x)$ do not point in exactly opposite directions. Here $\bar{v}$ is the projection of the vector v on the tangent space to Y at the point x. This assertion is easily proved from the Curve Selection Lemma (see below; also see Corollary 3.4 of [1], Lemma 1 of [4], and Lemma 1.9 of this article).

When Y has an isolated singular point at the coordinate origin, one can easily prove from continuity arguments that the same assertion holds for $x \in G$ where G is some neighborhood $Y \setminus \{0\}$ in R^n (see Lemma 3 of [4]).

The fundamental result of this section is embodied in Lemmas 1.5 and 1.6; these show that in the general case in a neighborhood of the set $Y \setminus \{0\}$ the vectors $\operatorname{grad} h_1$ and $\operatorname{grad} h_2$ do not point in exactly opposite directions.

Let us now state a lemma on real algebraic sets; we shall use this lemma below.

Let $V \subset R^n$ be a real algebraic set and let $U \subset R^n$ be an open set defined by a finite number of polynomial inequalities:

$$U = \{x \in R^n \mid g_1(x) > 0, \ldots, g_l(x) > 0\}.$$

LEMMA 1.1 (Curve Selection Lemma; Lemma 3.1 of [1]). If $U \cap V$ contains points arbitrarily close to the coordinate origin (i.e., if $0 \in \overline{U \cap V}$), then there is a real analytic curve $p: [0, \varepsilon) \to R^n$ such that $p(0) = 0$ and $p(t) \in U \cap V$ for all $t > 0$.

The following two lemmas describe the structure of real semialgebraic sets in a neighborhood of a proper algebraic subset. Let V be a real algebraic set, $g_1, \ldots, g_k$ generators of I(V). Put $\varphi = g_1^2 + \ldots + g_k^2$ and let $Z_\gamma = \{x \mid \varphi(x) \leq \gamma\}$, $\dot{Z}_\gamma = \{x \mid \varphi(x) = \gamma\}$, and $\mathring{Z}_\gamma = \{x \mid \varphi(x) < \gamma\}$.

Let V_j, $j = 1, \ldots, r$ be real algebraic sets, U_l, $l = 1, \ldots, s$ open sets defined by polynomial inequalities.

LEMMA 1.2. The (r + s + 1) sets

$$(Z_\gamma \setminus V, V_1 \cap Z_\gamma \setminus V, \ldots, V_r \cap Z_\gamma \setminus V, U_1 \cap Z_\gamma \setminus V, \ldots, U_s \cap Z_\gamma \setminus V)$$

are homeomorphic to the product

$$(\dot{Z}_\gamma, V_1 \cap \dot{Z}_\gamma, \ldots, V_r \cap \dot{Z}_\gamma, U_1 \cap \dot{Z}_\gamma, \ldots, U_s \cap \dot{Z}_\gamma) \times (0, \gamma],$$

if $\gamma > 0$ is sufficiently small.

Proof. The lemma follows directly from the general theorem on topological equisingularity (Theorem 6.5 of [5]).

Again suppose, as we have done above, that V_j, $j = 1, \ldots, r$ are real algebraic sets, U_l, $l = 1, \ldots, s$ open sets defined by polynomial inequalities, and let $0 \notin U_l$, $l = 1, \ldots, s$. Just as above, let S_ε and D_ε be the sphere and ball of radius ε centered at the coordinate origin.

LEMMA 1.3 (Conical Structure; cf. Theorem 2.10 of [1] and Lemma 3.2 of [6]). If $\varepsilon > 0$ is sufficiently small, the (r + s + 1) sets

$$(D_\varepsilon, D_\varepsilon \cap V_1, \ldots, D_\varepsilon \cap V_r, D_\varepsilon \cap U_1, \ldots, D_\varepsilon \cap U_s)$$

are homeomorphic to the (r + s + 1) sets

$$(D_\varepsilon, \text{cone } (S_\varepsilon \cap V_1), \dots, \text{cone } (S_\varepsilon \cap V_r), \text{cone } (S_\varepsilon \cap U_1) \setminus \{0\}, \dots$$
$$\dots, \text{cone } (S_\varepsilon \cap U_s) \setminus \{0\}).$$

Proof. This follows directly from Lemma 1.2.

The following lemma partially generalizes the Curve Selection Lemma. Let V, V_1 be real algebraic sets in R^n, U an open set defined by the polynomial inequalities $g_1 > 0, \dots, g_l > 0$. Let us assume that $U \cap V_1 = \phi$.

LEMMA 1.4. Suppose that there are points $x \in V_1$ arbitrarily close to the coordinate origin which belong to the closure of $U \cap V$. Then there is a two-dimensional algebraic set $V' \subset Y$ and a curve s which is a half-ray at zero of a one-dimensional algebraic set $V'' \subset V' \cap V_1$ such that

1) in some neighborhood of zero s consists wholly of points from the closure of $U \cap V$;

2) there is an $\varepsilon > 0$ and a neighborhood W of the set $s \setminus \{0\}$ in V' for which $D_\varepsilon \cap W \setminus s$ is decomposed into a finite number of connected components W_i such that $s \cap D_\varepsilon \subset \bar{W}_i$, and at least one of these components lies wholly in U.

Proof. The proof of this lemma is similar to that of Lemma 3.1 of [1]. First we show that if dim $V \geq 3$ then one can find a proper algebraic subset $V^1 \subset V$ such that, just as before, there are points $x \in V_1$ arbitrarily close to the coordinate origin and lying in the closure of $U \cap V^1$. By the hypothesis on the break in the decreasing chains, we can get a two-dimensional set V' with the same property. Put $V'' = V' \cap V_1$. Then on one of the half-rays of V'' there are points $x \in \overline{V' \cap U}$ arbitrarily close to zero. Denote this half-ray by s. Then the Conical Structure Lemma easily yields the result that V' has the required structure in a neighborhood of s.

We shall use the following abbreviated notation in what follows: that a certain linear relation holds between the vectors $w_1, \dots, w_k$ mod $v_1, \dots, v_s$ means that this relation holds for forms $w_1, \dots, w_k$ in the quotient space formed from R^n (respectively C^m) by the subspace spanned by the vectors $v_1, \dots, v_s$, or, what is the same thing, for the projections of the vectors $w_1, \dots, w_k$ onto the subspace orthogonal (complex orthogonal) to the vectors $v_1, \dots, v_s$.

Let $0 \in V \subset Y \subset Y^*$ be real algebraic sets, $g_1, \dots, g_s$ generators of the ideal $I(Y^*)$, $g_1, \dots, g_s, f_1, \dots, f_r$ generators of $I(Y)$. Put $\varphi = f_1^2 + \dots + f_r^2$.

Let h_1 and h_2 be nonnegative polynomial functions on R^n where the $h_i|_V$ have an isolated zero at zero, i = 1, 2.

LEMMA 1.5. There is an $\varepsilon > 0$ and a neighborhood G of the set $V \setminus \{0\}$ in Y^* such that for $x \in D_\varepsilon \cap G \setminus Y$, if for some $\beta_1, \beta_2, \beta_1^2 + \beta_2^2 > 0$,

$$\beta_1 \, \text{grad}\, h_1(x) + \beta_2 \, \text{grad}\, h_2(x) = 0 \mod \text{grad}\, \varphi(x),$$
$$\text{grad}\, g_1(x), \dots, \text{grad}\, g_s(x),$$

then $\beta_1 \cdot \beta_2 < 0$. In other words, for $x \in D_\varepsilon \cap G \setminus Y$ the vectors grad$h_1(x)$ and grad$h_2(x)$ do not point in exactly opposite directions mod grad$\varphi(x)$, grad$g_1(x), \dots,$ grad$g_s(x)$. A similar assertion holds for complex algebraic sets in C^m.

If f is a complex-analytic function on C^m, put grad $f = (\overline{\partial f/\partial z_1}, \dots, \overline{\partial f/\partial z_m})$. For a real smooth function φ on C^m, put

$$\text{grad}\, \varphi = \left(\frac{\partial\varphi}{\partial x_1} + i\frac{\partial\varphi}{\partial y_1}, \dots, \frac{\partial\varphi}{\partial x_m} + i\frac{\partial\varphi}{\partial y_m}\right), \text{ where } z_k = x_k + iy_k.$$

If p(t) is a smooth path in C^m, then obviously,

$$\frac{df(p(t))}{dt} = \langle \dot{p}, \text{grad}\, f(p(t)) \rangle, \quad \frac{d\varphi(p(t))}{dt} = \text{Re}\langle \dot{p}, \text{grad}\, \varphi(p(t)) \rangle.$$

Here $\langle a, b \rangle = \sum a_j \bar{b}_j$ is the Hermitian scalar-or-inner product. Moreover, it is easy to verify that for the complex-analytic function f, grad f = grad Re f, and if $h_1, \dots, h_l$ are complex-analytic functions and $h = \sum_{j=1}^{l} |h_j|^2$, then grad $h(z) = 2\sum_{j=1}^{l} h_j(z)\, \text{grad}\, h_j(z)$.

Let $0 \in V \subset Y \subset Y^*$ be complex algebraic sets, $g_1, \ldots, g_s$ generators of the ideal $I(Y^*)$, $g_1, \ldots, g_s$, $f_1, \ldots, f_r$ generators of $I(Y)$. Assume that $Y^* \setminus Y$ is regular.

Put $\varphi = \sum_{j=1}^{r} |f_j|^2$ and let $h_{1|V}$ and $h_{2|V}$ have isolated zeros at the coordinate origin, where $h_1 = \sum_l |h_1^l|^2$, $h_2 = \sum_k |h_2^k|^2$, and where h_1^l and h_2^k are polynomials in the variables $z_1, \ldots, z_m$.

LEMMA 1.6. There is an $\varepsilon > 0$ and a neighborhood G of the set $V \setminus \{0\}$ in Y^* such that for $z \in D_\varepsilon \cap G \setminus Y$, if

$$\beta_1 \operatorname{grad} h_1(z) + \beta_2 \operatorname{grad} h_2(z) = 0 \mod \operatorname{grad} \varphi(z), \operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z),$$

where $\beta_1, \beta_2 \in R$, $\beta_1^2 + \beta_2^2 \neq 0$, then $\beta_1 \cdot \beta_2 < 0$.

COROLLARY 1.7. At points $z \in D_\varepsilon \cap G \setminus Y$ the vectors $\operatorname{grad} h_1(z)$ and $\operatorname{grad} \varphi(z)$ are complex linearly independent $\operatorname{mod} \operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z)$, just as are the vectors $h_2(z)$ and $\operatorname{grad} \varphi(z)$.

Proof. Using the Curve Selection Lemma we easily show that at points $z \in Y^* \setminus Y$, sufficiently close to zero, $\operatorname{grad} \varphi(z) \neq 0 \operatorname{mod} \operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z)$.

Therefore in the relation

$$\beta_1 \operatorname{grad} h_1(z) + c \operatorname{grad} \varphi(z) = 0 \mod \operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z)$$

the coefficient β_1, which can obviously be taken to be real, must be different from zero.

The proof of Lemma 1.6 will be given below. Lemma 1.5 is proved in a completely analogous manner.

LEMMA 1.8. For points $z \in Y^* \setminus Y$ sufficiently close to the coordinate origin, if the following holds:

$$\beta_1 \operatorname{grad} h_1(z) + \beta_2 \operatorname{grad} h_2(z) = c \operatorname{grad} \varphi(z) \mod \operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z), \tag{1}$$

where $\beta_1, \beta_2 \in R$, $\beta_1 \geq 0$, $\beta_2 \geq 0$, $c \in C$, $c \neq 0$, then $|\arg c| \leq \pi/4$.

Proof (by contradiction). Let us assume that there are points $z \in Y^* \setminus Y$ sufficiently close to the coordinate origin such that (1) holds at them where $\beta_1 \geq 0$, $\beta_2 \geq 0$, $c \neq 0$ but $|\arg c| > \pi/4$.

Consider the set A of points (z, c, β_1, β_2) which are such that $z \in Y^* \setminus Y$, $\beta_1 \geq 0$, $\beta_2 \geq 0$, $c \neq 0$, $|\arg c| > \pi/4$, and

$$\beta_1 \operatorname{grad} h_1(z) + \beta_2 \operatorname{grad} h_2(z) = c \operatorname{grad} \varphi(z) \mod \operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z). \tag{2}$$

Since (2) may be multiplied by any positive number, our assumption means that the point $(0, 0, 0, 0) \in C^m \oplus C \oplus R^2$ belongs to the closure of A. It follows from the Curve Selection Lemma that there is a real-analytic curve $\bar{p}(t)$ in $C^m \oplus C \oplus R^2$ such that $\bar{p}(0) = (0, 0, 0, 0)$ and $\bar{p}(t) \in A$ for small positive t. In other words, there is a real-analytic curve $p(t)$ in C^m such that $p(0) = 0$ and for all small positive t, $p(t) \in Y^* \setminus Y$, and

$$\beta_1(t) \operatorname{grad} h_1(p(t)) + \beta_2(t) \operatorname{grad} h_2(p(t)) = c(t) \operatorname{grad} \varphi(p(t)) \mod \operatorname{grad} g_1(p(t)), \ldots, \operatorname{grad} g_s(p(t)), \tag{3}$$

where $\beta_1(t) \geq 0$, $\beta_2(t) \geq 0$, $c(t) \neq 0$, $|\arg c(t)| > \pi/4$, and the coefficients $\beta_1(t)$, $\beta_2(t)$, $c(t)$ can be expanded in power series in t in a neighborhood of zero.

Take the inner product of (3) with the vector $\dot{p} = dp/dt$. We get

$$\beta_1(t) \langle \dot{p}, \operatorname{grad} h_1(p(t)) \rangle + \beta_2(t) \langle \dot{p}, \operatorname{grad} h_2(p(t)) \rangle = \bar{c}(t) \langle \dot{p}, \operatorname{grad} \varphi(p(t)) \rangle, \tag{4}$$

since the path $p(t)$ lies in Y^*, and thus the vector $\dot{p}$ is complex orthogonal to the vectors $g_1(p(t)), \ldots, \operatorname{grad} \cdot g_s(p(t))$.

Consider the series for $\langle \dot{p}, \operatorname{grad} \varphi(p(t)) \rangle = \sum_{j=1}^{r} \bar{f}_j(p(t)) \langle \dot{p}, \operatorname{grad} f_j(p(t)) \rangle = \sum_{j=1}^{r} \bar{f}_j(p(t)) df_j(p(t))/dt$. If $f_j(p(t)) = a_j^1 t_j^{m1} + \ldots$, then $m_j^1 > 0$, since

$$f_j(p(0)) = 0 \text{ and } \bar{f}_j(p(t)) df_j(p(t))/dt = m_j^1 |a_j^1|^2 t^{2m_j^1 - 1} + \ldots$$

Thus the first coefficient of the series for $\langle \dot{p}, \operatorname{grad} \varphi(p(t)) \rangle$ is a real positive number if the function $\varphi(p(t))$ is not identically zero. The same holds true for the series for $\langle \dot{p}, \operatorname{grad} h_1(p(t)) \rangle$ and $\langle \dot{p}, \operatorname{grad} h_2(p(t)) \rangle$.

By hypothesis, $p(t) \in Y^* \setminus Y$ for $t < 0$, i.e., $\varphi(p(t)) > 0$ for small positive t, and therefore the first coefficient in the series for $\langle \dot{p}, \operatorname{grad} \varphi(p(t)) \rangle$ must be strictly greater than zero. Since the series for $c(t)$ must be nonzero by hypothesis, on the right side of (4) there is a nonzero series.

Since $\beta_1(t) \geq 0$ and $\beta_2(t) \geq 0$, it follows that the first coefficient in the series on the left of (4) is a real positive number; this implies, in turn, that the first coefficient in the series for c(t) must be strictly greater than zero. Thus $|\arg c(t)| \to 0$ when $t \to 0$, which contradicts the assumption we made.

LEMMA 1.9. For points $z \in Y^*$ sufficiently close to the coordinate origin such that $h_1(z) > 0$, $h_2(z) > 0$, the vectors $\operatorname{grad} h_1(z)$ and $\operatorname{grad} h_2(z)$ cannot point in exactly opposite directions $\operatorname{mod} \operatorname{grad} g_j(z)$. In other words, the relation

$$\beta_1 \operatorname{grad} h_1(z) + \beta_2 \operatorname{grad} h_2(z) = 0 \mod \operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z), \tag{5}$$

where β_1, $\beta_2 \in R$, $\beta_1^2 + \beta_2^2 > 0$, implies that $\beta_1 \cdot \beta_2 < 0$.

Proof. Let us suppose that there are points $z \in Y^*$ arbitrarily close to zero for which (5) holds but $\beta_1 \cdot \beta_2 \geq 0$. The Curve Selection Lemma implies that there is a real-analytic path p(t) such that $p(0) = 0$, and that for small $t > 0$, $p(t) \in Y^* \setminus \{0\}$, and

$$\beta_1(t) \operatorname{grad} h_1(p(t)) + \beta_2(t) \operatorname{grad} h_2(p(t)) = 0 \mod \operatorname{grad} g_1(p(t)), \ldots$$
$$\ldots, \operatorname{grad} g_s(p(t)),\ \beta_1(t) \geqslant 0,\ \beta_2(t) \geqslant 0,\ \beta_1^2(t) + \beta_2^2(t) > 0. \tag{6}$$

Arguments similar to those given in the proof of Lemma 1.8 show that the coefficients $\beta_1(t)$ and $\beta_2(t)$ can be assumed to be functions of t which are analytic at zero. Take the inner product of (6) with $\dot{p}$. We get

$$\beta_1(t) \langle \dot{p}, \operatorname{grad} h_1(p(t)) \rangle + \beta_2(t) \langle \dot{p}, \operatorname{grad} h_2(p(t)) \rangle = 0. \tag{7}$$

However, from the assumptions we have made, for small $t > 0$ we get $h_1(p(t)) > 0$, $h_2(p(t)) > 0$, $\beta_1(t) \geq 0$, $\beta_2(t) \geq 0$, $\beta_1^2(t) + \beta_2^2(t) > 0$, and so the first coefficient on the left of (7) must be strictly positive.

Proof of Lemma 1.6. This also is done by contradiction. Let us assume that for the requisite $\varepsilon > 0$ there is no neighborhood G. Then there are points in V arbitrarily close to the coordinate origin which are limit points for the set D consisting of points $z \in Y^* \setminus Y$ at which, for β_1, $\beta_2 \in R$, $\beta_1^2 + \beta_2^2 > 0$, $\beta_1 \cdot \beta_2 \geq 0$

$$\beta_1 \operatorname{grad} h_1(z) + \beta_2 \operatorname{grad} h_2(z) = 0 \mod \operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z), \operatorname{grad} \varphi(z). \tag{8}$$

It is obvious that $D = (D_1 \cap U) \cup (D_2 \cap U') \cup (D_3 \cap U')$. Here D_1 is the algebraic set consisting of points $z \in Y^*$ at which, for some β_1, $\beta_2 \in R$, (8) holds; D_2 and D_3 are algebraic subsets of D_1, where β_1 (respectively β_2) can be made zero. The set $U' = C^m \setminus Y$, U is a closed subset consisting of points $z \in U'$ for which $\operatorname{Re} \langle \operatorname{grad} \cdot h_1(z), \operatorname{grad} h_2(z) \rangle < 0$ ($\bar{v}$ is the projection of the vector v on the subspace complex-orthogonal to the vectors $\operatorname{grad} \varphi(z)$, $\operatorname{grad} g_1(z)$, $\ldots$, $\operatorname{grad} g_s(z)$).

From the assumption we have made, there are limit points arbitrarily close to the coordinate origin in V for at least one of the sets $D_1 \cap U$, $D_2 \cap U'$, or $D_3 \cap U'$, and since Lemma 1.4 applies to each of these sets we get: there is a real-algebraic two-dimensional set V' and a curve s which is a half-ray at zero for the one-dimensional set $V'' = V' \cap V$, and these are such that in some neighborhood of zero all the points of s lie in $\bar{D}$ and for some neighborhood W of the set $s \setminus \{0\}$ in V' at least one of the connected components of $W \setminus s$ lies wholly in D.

Choose an analytic parametrization s(t) of the curve s in a neighborhood of zero, with $s(0) = 0$. Since the curve s lies in V, and since $h_1|_V$ and $h_2|_V$ have isolated zeros at the coordinate origin, $h_1(s(t)) > 0$ and $h_2(s(t)) > 0$ for small $t > 0$. Since $h_1(s(t))$ and $h_2(s(t))$ are analytic functions of t, which take zero to zero, it follows that $dh_1/dt > 0$ and $dh_2/dt > 0$ for small $t > 0$, i.e., both the functions h_1 and h_2 increase along s for some neighborhood of the coordinate origin.

Let $Z_\rho^i = \{z \in Y^* | h_i(z) \leq \rho\}$, $\dot{Z}_\rho^i = \{z \in Y^* | h_i(z) = \rho\}$, $i = 1, 2$. Put $\Sigma_\rho^i = V' \cap \dot{Z}_\rho^i$, $i = 1, 2$. It is obvious that Z_ρ^i, for sufficiently small $\rho > 0$, intersects the curve s in the unique point x_ρ^i, $i = 1, 2$.

Now choose W_k as one of the components of $W \setminus s$ which lie wholly in D, and denote by σ_ρ^i that half-ray of the curve Σ_ρ^i which lies in W_k, $i = 1, 2$.

LEMMA 1.10. At least one of the two following possibilities holds: either there are arbitrarily small $\delta > 0$ for which the function h_2 decreases (not necessarily strictly monotonically) along the curve σ_δ^1 in some neighborhood of x_δ^1, or there are arbitrarily small $\delta' > 0$ for which the function h_1 decreases (not necessarily strictly monotonically) along the curve $\sigma_{\delta'}^2$ in some neighborhood of $x_{\delta'}^2$.

Proof. Assume, for example, that for every sufficiently small $\delta > 0$ the function h_2 is strictly increasing along the curve σ_δ^1 in some neighborhood of x_δ^1. Since the curves σ_δ^1 admit analytic parametrization and h_2 is a polynomial, this is the unique alternative to the first of the possibilities listed in the lemma.

Now fix a sufficiently small $\delta > 0$ and we shall now show that there are points $x^2_{\delta'}$ arbitrarily close to x^1_δ such that the function h_1 is strictly decreasing along $\sigma^2_{\delta'}$ in some neighborhood of $x^2_{\delta'}$.

Choose a real-analytic change of coordinates on a neighborhood of x^1_δ such that in the new coordinates $y_1, \ldots, y_{2m-1}, y_{2m}$ the curve s is specified by equations $y_1 = 0, \ldots, y_{2m-1} = 0$ in a neighborhood of x^1_δ while the function h_1 equals y_{2m}. This change of coordinates is possible since it has been shown that the derivative of h_1 along the curve s is greater than zero at the point x^1_δ.

Let us define the function ψ in a neighborhood of x^1_δ by the equation:

$$\psi(y_1, \ldots, y_{2m}) = h_2(y_1, \ldots, y_{2m}) - h_2(0, 0, \ldots, 0, y_{2m}).$$

It is obvious that $\psi|_s \equiv 0$. On the other hand, by assumption $\psi(y) > 0$ for points $y \in \sigma^1_\delta$ sufficiently close to x^1_δ. Therefore ψ is not identically zero on V'. (More precisely, this is true on the analytically irreducible component of V' in a neighborhood of x^1_δ which contains W_k.) Therefore $V' \cap \psi^{-1}(0)$ is a real-analytic curve, one of whose components is s. It is obvious that s intersects the remaining components of the curve $V' \cap \psi^{-1}(0)$ on a zero-dimensional analytic set consisting of isolated points. Therefore at each point $x^2_{\delta'}$ distinct from x^1_δ and sufficiently close to it there is a neighborhood U in the set V' in which there is no point of $\psi^{-1}(0)$ except the points of s. Now fix the point $x^2_{\delta'}$. The function ψ is nonzero on the set $U \cap W_k$, which can be assumed connected.

By the hypothesis we have made, $\psi(x) > 0$ for points x lying on the curve $\sigma^1_{\delta''}$ which starts at the point $x^2_{\delta'}$ and goes sufficiently close to $x^2_{\delta'}$. Therefore $\psi|_{U \cap W_k} > 0$.

Let the point $y = (y_1, \ldots, y_{2m}) \in \sigma^2_{\delta'} \cap U$. Then $h_2(y) = h_2(x^2_{\delta'}) = \delta'$. On the other hand, $h_1(y) = y_{2m} = h_1(y')$, where $y' = (0, 0, \ldots, 0, y_{2m})$. Since $\psi(y) > 0$, $h_2(y) > h_2(y')$, we get $h_2(x^2_{\delta'}) > h_2(y')$. But, as has been remarked earlier, the functions h_1 and h_2 increase along the curve s in the same direction. Therefore, $h_1(x^2_{\delta'}) > h_1(y') = h_1(y)$. Since the curve $\sigma^2_{\delta'}$ admits an analytic parametrization, this last inequality gives: $h_1(y) < h_1(x^2_{\delta'})$ for all points $y \in \sigma^2_{\delta'}$ sufficiently close to $x^2_{\delta'}$, and this implies that the function h_1 decreases strictly along the curve $\sigma^2_{\delta'}$ in some smaller neighborhood of $x^2_{\delta'}$.

Suppose, for example, that the first of the possibilities listed in Lemma 1.10 holds. Fix $\delta > 0$ with the required properties, and choose an analytic parametrization p(t) of the curve σ^1_δ such that $p(0) = x^1_\delta$. Since $\sigma^1_\delta \subset W_k \subset D$, the following holds for small $t > 0$:

$$\beta_1(t)\ grad\ h_1(p(t)) + \beta_2(t)\ grad\ h_2(p(t)) = c(t)\ grad\ \varphi(p(t)) \quad \mathrm{mod}\ grad\ g_1(p(t)), \ldots, grad\ g_s(p(t)). \tag{9}$$

Moreover $\beta_1(t) \cdot \beta_2(t) \geq 0$, since we may assume that $\beta_1(t) \geq 0$, $\beta_2(t) \geq 0$, and $\beta_1^2(t) + \beta_2^2(t) > 0$. So once again, just as in the proof of Lemma 1.8, we can assume that $\beta_1(t)$, $\beta_2(t)$, and c(t) can be expanded as power series in t in a neighborhood of zero.

Since $x^1_\delta \in s \subset V$, we have $h_1(x^1_\delta) > 0$ and $h_2(x^1_\delta) > 0$. Therefore $h_1(p(t))$ and $h_2(p(t))$ are greater than zero for small t. Lemma 1.9 now implies that $c(t) \neq 0$ for small t.

Now take the inner product of (9) with $\dot{p}$. We get:

$$\beta_1(t)\langle \dot{p}, grad\ h_1(p(t))\rangle + \beta_2(t)\langle \dot{p}, grad\ h_2(p(t))\rangle = \bar{c}(t)\langle \dot{p}, grad\ \varphi(p(t))\rangle. \tag{10}$$

From $p(0) \in s \subset V \subset Y$ we get $f(p(0)) = 0$, and since $p(t) \in Y^* \setminus Y$ for small $t > 0$ we get, just as we did above, that the first coefficient of the series for $\langle \dot{p}, grad\,\varphi(p(t))\rangle$ is a positive real number. Further,

$$\mathrm{Re}\langle \dot{p}, grad\ h_1\rangle = dh_1/dt, \quad \mathrm{Re}\langle \dot{p}, grad\ h_2\rangle = dh_2/dt.$$

But $h_1(p(t)) \equiv \delta$, and $h_2(p(t))$ decreases for small t. Therefore $\mathrm{Re}\, a_1 = 0$, $\mathrm{Re}\, a_2 \leq 0$, where a_1 and a_2 are the first coefficients, respectively, in the series for $\langle \dot{p}, grad\, h_1\rangle$ and $\langle \dot{p}, grad\, h_2\rangle$. Since $\beta_1(t) \geq 0$ and $\beta_2(t) \geq 0$, it follows from this that $\mathrm{Re}\, a_3 \leq 0$, where a_3 is the first coefficient in the series for the left side of (10); in turn, this implies that $\mathrm{Re}\, c \leq 0$, where c is the first coefficient in the series for c(t). Thus $|\arg c(t)| > \pi/4$ for sufficiently small t. But since δ can be chosen arbitrarily small, we can assume that the point x^1_δ and therefore the point p(t) for small t lie in any a priori assigned neighborhood of the coordinate origin. But then the inequality $|\arg c(t)| > \pi/4$ contradicts Lemma 1.8. Hence Lemma 1.6 is proved.

2. A Theorem on the Connectivity of the Fiber

First of all we make precise the concepts of "a nonsingular manifold close to Y" and the intersection of V with a sphere of "sufficiently small radius;" these will be frequently used below.

Suppose we are given a complex algebraic manifold $V \ni z_0$ or the pair $Y^* \supset Y \ni z_0$ of complex algebraic manifolds in C^m.

Let I(V) be generated by the polynomials $r_1, \ldots, r_k$, $I(Y^*)$ be generated by the polynomials $g_1, \ldots, g_s$, and I(Y) be generated by the polynomials $g_1, \ldots, g_s, f$.

For V we shall assume that the point z_0 is either an isolated singular point or a simple point of this set; for the pair $Y \subset Y^*$ we shall assume that $Y^* \setminus Y$ is regular in a neighborhood of z_0.

Suppose, as in Section 1, that $\sum |h_j^1|^2$, $h_2 = \sum |h_l^2|^2$, $h_1(z_0) = h_2(z_0) = 0$, and that z_0 is an isolated zero of the functions $h_{i|V}$ or $h_{i|Y}$ for i = 1, 2. Let us define, just as we did above,

$$Z_\varepsilon^i = \{z \in Y^* \mid h_i(z) \leq \varepsilon\},\ \dot{Z}_\varepsilon^i = \{z \in Y^* \mid h_i(z) = \varepsilon\},$$

$$\mathring{Z}_\varepsilon^i = \{z \in Y^* \mid h_i(z) < \varepsilon\},\ i = 1, 2.$$

Corollary 2.8 of [1] proves that for sufficiently small ε, $\dot{Z}_\varepsilon^i$ intersects V transversely along a smooth manifold Σ_ε^i, i = 1, 2.

For pairs $Y \subset Y^*$ we define a smooth manifold $X_\varepsilon^{i'}$, i = 1, 2 with boundary, as follows: fix an $\varepsilon > 0$ sufficiently small (small enough that Corollary 2.8 of [1] implies that $Y^* \setminus Y$ intersects $\mathring{Z}_\varepsilon^i$ transversely along a smooth manifold, i = 1, 2) and consider $X_{\varepsilon\delta}^i = Y_\delta \cap Z_\varepsilon^i$, where $Y_\delta = Y^* \cap f^{-1}(\delta)$. Once again Corollary 2.8 of [1] implies that for sufficiently small $\delta \neq 0$, Y_δ is a smooth manifold in a neighborhood of z_0. Corollary 1.7 now implies that if δ is sufficiently small then at points $z \in Y_\delta \cap \mathring{Z}_\varepsilon^i$ the vectors $\operatorname{grad} h_i(z)$ and $\operatorname{grad} f(z) = (1/2f(z)) \operatorname{grad} |f(z)|^2$ are complex linearly independent mod $\operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z)$, and therefore Y_δ and $\dot{Z}_\varepsilon^i$ intersect transversely, i = 1, 2. Therefore $X_{\varepsilon\delta}^i$ is a smooth manifold with boundary.

It is easy to show that for sufficiently small $\delta_1 \neq 0$ and $\delta_2 \neq 0$ the manifolds $X_{\varepsilon\delta_1}^i$ and $X_{\varepsilon\delta_2}^i$ are diffeomorphic; therefore the manifold $X_{\varepsilon\delta}^i$ will simply be denoted by $X_\varepsilon^{i'}$, i = 1, 2 for sufficiently small δ.

LEMMA 2.1. Let ε_1 and ε_2 be sufficiently small. Then

1) the manifolds $\Sigma_{\varepsilon_1}^1$ and $\Sigma_{\varepsilon_2}^2$ are diffeomorphic;

2) the manifolds with boundary $X_{\varepsilon_1}^{1'}$ and $X_{\varepsilon_2}^{2'}$ are diffeomorphic.

Proof. 1) Lemma 1.9 implies that for $z \in V$ sufficiently close to z_0, the vectors $\operatorname{grad} h_1(z)$ and $\operatorname{grad} h_2(z)$ do not point in exactly opposite directions mod $\operatorname{grad} r_1(z), \ldots, \operatorname{grad} r_k(z)$. Therefore, if we use the construction of Lemma 11.3 of [1] we can construct a tangent vector field ω on a neighborhood of z_0 in V, and along the trajectories of this field the functions h_1 and h_2 both increase. Thus the trajectories of ω intersect the manifolds $\Sigma_{\varepsilon_1}^1$ and $\Sigma_{\varepsilon_2}^2$ at a single point in each, and so we can establish the required diffeomorphism.

2) Choose $\varepsilon_3 < \varepsilon_1$ such that $Z_{\varepsilon_3}^1 \cap Y \subset \mathring{Z}_{\varepsilon_2}^2$. We now prove that $X_{\varepsilon_2}^{2'}$ is diffeomorphic to $X_{\varepsilon_3}^{1'}$. The diffeomorphism between $X_{\varepsilon_1}^{1'}$ and $X_{\varepsilon_3}^{1'}$ is established in a similar way.

Lemma 1.6 implies that we can find a neighborhood G of the set $Y \setminus \{z_0\}$ in Y^* such that for $z \in G \setminus Y$ sufficiently close to z_0 the vectors $\operatorname{grad} h_1(z)$ and $\operatorname{grad} h_2(z)$ do not point in exactly opposite directions mod $\operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z)$, $\operatorname{grad} f(z) = (1/2f(z)) \operatorname{grad} |f(z)|^2$.

Choose $\delta \neq 0$ so small that $Y_\delta \cap (Z_{\varepsilon_2}^2 \setminus \mathring{Z}_{\varepsilon_3}^1) \subset G$. Then in Y_δ there is a tangent field ω on a neighborhood of $Y_\delta \cap (\mathring{Z}_{\varepsilon_2}^2 \setminus \mathring{Z}_{\varepsilon_3}^1)$ such that the functions h_1 and h_2 increase along the trajectories of this field. Just as above, the trajectories of the field ω intersect both $\partial X_{\varepsilon_3\delta}^{1'} = Y_\delta \cap \dot{Z}_{\varepsilon_3}^1$ and $\partial X_{\varepsilon_2\delta}^2 = Y_\delta \cap \dot{Z}_{\varepsilon_2}^2$ transversely in a single point in each. It is easy to construct a diffeomorphism of the manifolds $X_{\varepsilon_2\delta}^2 \setminus \mathring{X}_{\varepsilon_3\delta}^1$ and $X_{\varepsilon_3\delta}^1 \times [0, 1]$ with the help of these trajectories. It obviously follows from this that the manifolds $X_{\varepsilon_2\delta}^2$ and $X_{\varepsilon_3\delta}^1$ are diffeomorphic. But δ can be chosen so small that $X_{\varepsilon_3\delta}^1$ is diffeomorphic to $X_{\varepsilon_3}^{1'}$ and $X_{\varepsilon_2\delta}^2$ is diffeomorphic to $X_{\varepsilon_2}^{2'}$.

In what now follows, all the manifolds described above will be denoted simply as Σ and X', or Σ_{z_0}, X'_{z_0} if it becomes necessary to emphasize that all the constructions are carried out on a neighborhood of the point z_0. The manifold obtained for the case $h(z) = \|z - z_0\|^2$ will be taken as the standard model.

Remark. Suppose V has an isolated singular point at zero and is a complete intersection, i.e., generators $r_1, \ldots, r_k$ can be chosen for the ideal I(V) such that V = m − k. Then the construction carried out above correctly defines a smooth manifold with boundary X', $X' = Z_\varepsilon \cap r_1^{-1}(\delta_1) \cap \ldots \cap r_k^{-1}(\delta_k)$, where $\delta_1, \ldots, \delta_k$ are sufficiently small complex numbers, not all zero.

Suppose, just as before, that Y^* and Y, $0 \in Y \subset Y^*$, are complex algebraic sets, where $g_1, \ldots, g_s$ are generators of the ideal $I(Y^*)$, while $I(Y)$ is generated by $g_1, \ldots, g_s, f$. We shall assume from now on that all the singular points of Y^* are contained in a neighborhood of the coordinate origin in Y, i.e., $Y^* \setminus Y$ is regular in a neighborhood of zero.

Let $\Sigma^* = Y^* \cap S_\varepsilon$, $\Sigma = Y \cap S_\varepsilon$. The Conical Structure Lemma implies that the algebraic sets Σ^* and Σ are independent (up to homeomorphism) of the choice of the sufficiently small number ε and that $\Sigma^* \setminus \Sigma$ is a smooth manifold for all sufficiently small ε.

Lemma 1.6 of [2] implies that the Milner fibration construction (Theorem 4.8 of [1]) carries over to this case, i.e., the map $\varphi: \Sigma^* \setminus \Sigma \to S^1$, $\varphi(z) = f(z)/|f(z)|$ defines a smooth fibration of $\Sigma^* \setminus \Sigma$ over the circle. Moreover, the fiber F_θ (more simply F) of the fibration φ is diffeomorphic to the interior of the manifold X' defined above.

We shall assume from now on that all irreducible components of Y^*, as well as all the irreducible components of Y, have the same dimension, with $n = \dim Y = \dim Y^* - 1$. Lemma 2.2 of [2] proves that in this case each singular point in a neighborhood of zero in Y^* is also a singular point of Y.

Lemma 1.7 of [2] states that the fiber F of the fibration φ has the homotopy type of a finite CW-complex of dimension n when Y^* and Y are complete intersections, i.e., $n = m - s - 1$; and, moreover, when Y^* and Y have an isolated singular point at zero, the fiber F has the homotopy type of a bouquet of spheres S^n. (This is equivalent to the fiber being $(n-1)$-connected.)

The fundamental result of the section is Theorem 2.2, which strengthens the previous assertion.

Let V be the set of singular points of Y. Let d be the maximal dimension $\dim_0^+ V$ of the irreducible components of V at zero.

THEOREM 2.2. Let Y^* and Y be complete intersections. Then the fiber F of the fibration φ is $(n-d-1)$-connected.

In fact we prove the following somewhat more precise assertion.

LEMMA 2.3. The fiber F of the fibration φ is $(m-s-d-2)$-connected.

COROLLARY 2.4. The manifold $\partial X' = \Sigma^* \cap f^{-1}(\delta)$, where $\delta \neq 0$ is sufficiently small, is $(m-s-d-2)$-connected if $d = \dim_0^+ V > 0$ or $m - s < \dim Y^*$. When $\dim_0^+ V = 0$ and $m - s = \dim Y^*$, the manifold $\partial X'$ is $(m-s-3)$-connected.

Proof of the Corollary. Consider the Morse function ψ on the manifold X', which function is obtained by removing the degenerate critical points of the function $\|z\|^2$. The index of each critical point of ψ is less than or equal to n (see, for example, [7]).

Therefore ψ allows us to represent the manifold X' as a tubular neighborhood of the boundary $\partial X'$ with calls of dimension $\geq n$ glued on. Therefore the homotopy groups of X' and $\partial X'$ coincide up to dimension $n-2$. But since $\mathring{X}'$ and F are diffeomorphic, X' is homotopy equivalent to F. We need only remark that $m-s-d-2 \leq n-2$ when $d > 0$ or $m - s < \dim Y^* = n+1$.

When Y^* and Y have an isolated singular point at zero, the manifolds $\partial X'$ and Σ are diffeomorphic, and therefore Corollary 2.4 makes Corollary 1.3 of [2] somewhat more precise.

Before we prove Lemma 2.3 in its general form, we prove it for the case where Y, and therefore Y^*, has an isolated singularity at zero. In this case, $\bar{F}$, the closure of F in Σ^*, is a manifold with boundary which is diffeomorphic with X' (Theorem 1.7, 3 of [2]), and the Morse function ψ is the means by which we can get $\bar{F}$ from the disk D^{2n} by adjoining handles of index $\leq n$. All these rearrangements can be carried out inside the manifold Σ^*. But Σ^*, and therefore $\Sigma^* \setminus D^{2n}$, is $(m-s-2)$-connected, by Theorem 2.9 of [2], and the adjunction of handles of index $\leq n$ does not change the first $\Sigma^* - n - 2 = n - 1$ homotopy groups of the addition. Therefore $\Sigma^* \setminus \bar{F}$ is $(m-s-2)$-connected.

Since $\Sigma^* \setminus \bar{F} = (\Sigma^* \setminus \Sigma) \setminus F$, the map φ defines a fibration of $\Sigma^* \setminus \bar{F}$ above the interval, with fiber F. Therefore $\Sigma^* \setminus \bar{F}$ and F are homotopy equivalent, and so F is $(m-s-2)$-connected. (See Lemma 6.4 and Corollary 6.2 of [1].)

The proof of Lemma 2.3 in the case where $\dim_0^+ V > 0$ is quite similar to that of Theorem 1.5 of [2].

Let $q_1, \ldots, q_l$ be generators of the ideal $I(V)$. Put $q = \Sigma |q_i|^2$ and let $N_\alpha = \{z \in Y^* | q(z) \leq \alpha\}$, $\check{N}_\alpha = \{z \in Y^* | q(z) = \alpha\}$, $\mathring{N}_\alpha = \{z \in Y^* | q(z) < \alpha\}$. For sufficiently small $\alpha > 0$, $\check{N}_\alpha$, and $\check{N}_\alpha \cap Y$ are smooth manifolds. (From Lemma 2.7 of [2], or from Corollary 2.8 of [1].)

Now fix $\varepsilon > 0$ sufficiently small, and then fix $\alpha > 0$ sufficiently small with respect to ε.

LEMMA 2.5. The manifold X' is homotopy equivalent to $X' \cap N_\alpha$. More precisely, if $\delta \neq 0$ is sufficiently small, then $X_\delta = D_\varepsilon \cap Y^* \cap f^{-1}(\delta)$ is homotopy equivalent to $X_\delta \cap N_\alpha$.

Proof. Lemma 1.9 implies that for $z \in D_\varepsilon \cap Y \setminus V$ the vectors $z = (1/2) \operatorname{grad} \|z\|^2$ and $\operatorname{grad} q(z)$ can be assumed not to point in exactly opposite directions $\operatorname{mod} \operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z), \operatorname{grad} f(z)$. Since the points of $Y \setminus V$ are simple points of Y, we have that, for $z \in Y \setminus V$, the system of vectors $\operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s \cdot (z), \operatorname{grad} f(z)$ has rank $\rho + 1$, the maximum possible rank at the points of Y^*, $\rho = m - \dim Y^*$. Therefore it is easy to see from continuity arguments that the vectors z and $\operatorname{grad} q(z)$ do not point in exactly opposite directions $\operatorname{mod} \operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z), \operatorname{grad} f(z)$, for $z \in W$, where W is some neighborhood of $Y \setminus V$ in Y^* (see Lemma 3 of [4]).

Now choose $\delta \neq 0$ so small that $X_\delta \setminus \mathring{N}_\alpha \subset W$. Then in $Y_\delta = Y^* \cap f^{-1}(\delta)$ one can construct a tangent vector field ω on a neighborhood of $X_\delta \setminus \mathring{N}_\alpha$, such that along the trajectories of this field both $\|z\|^2$ and $q(z)$ decrease. So by moving along the trajectories of the field ω we can push X_δ onto $X_\delta \cap N_\alpha$.

In order to study the topology of $X_\delta \cap N_\alpha$, we construct [2] the auxiliary manifolds Y_0^* and Y_0, $0 \in Y_0$ Y_0^* which lie in Y^* and Y, respectively, and which have an isolated singular point at the coordinate origin.

Let P denote the space of all polynomials of degree ≤ 2, which have a zero at the coordinate origin. Consider sets of $h = (h_1, \ldots, h_d) \in P^d$, where d, as above, equals $\dim_0^+ V$. Let $Y_0^* = \{z \in Y^* | h_1(z) = h_2(z) = \ldots = h_d(z) = 0\}$, $Y_0 = Y_0^* \cap Y$ for $h \in P^d$. Put $h = \sum_{i=1}^{d} |h_i|^2$.

LEMMA 2.6. (See Lemma 1.12 of [2]). There is a set of $h \in P^d$ with the following properties:

1) $Y_0^* \setminus V$ is regular;

1') $Y_0 \setminus V$ is regular;

2) 0 is an isolated point of $Y_0^* \cap V$;

3) if $\alpha \neq 0$ is chosen sufficiently small, then at each critical point of the function $h|_{\check{N}_\alpha \cap Y \setminus Y_0}$ the restriction of the Hessian H of the function h is nondegenerate on the subspace of vectors v tangent to Y and such that $\langle v, \operatorname{grad} h_j \rangle = 0$, $j = 1, \ldots, d$, $\langle v, \operatorname{grad} q_i \rangle = 0$, $i = 1, \ldots, l$;

3') if $\delta \neq 0$ is chosen sufficiently small, then at each critical point of the function $h|_{Y_\delta \setminus Y_0^*}$ the restriction of the Hessian H of the function h is nondegenerate on the subspace of vectors v tangent to Y_δ and such that $\langle v, \operatorname{grad} h_j \rangle = 0$, $j = 1, \ldots, d$.

Proof. We note that statements 1') and 3) coincide with statements 1) and 3) of Lemma 2.12 of [2], if we replace the Y^* of that lemma by our Y, and the Y there by our V. Statements 1) and 2) coincide with statements 1) and 2) of Lemma 2.12 of [2] if we let Y^* and Y^* coincide and replace the Y of that lemma by our V. In fact it is proved in [2] that each of the properties 1), 2), and 3) of Lemma 2.12 holds for almost all sets of $h \in P^d$. Therefore to prove Lemma 2.6 we need only show that statement 3') holds for almost all sets of $h \in P^d$.

This is proved in the same way as statement 3) of Lemma 2.12 of [2]. Consider the real analytic manifold $\mathfrak{M}$, in $P^d \times (Y^* \setminus V)$ which consists of pairs (h, z) such that

a) $\operatorname{grad} h(z) = 0 \operatorname{mod} \operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z), \operatorname{grad} f(z)$, i.e., the point z is a critical point of the restriction of h to the surface Y_δ passing through z;

b) the restriction of the Hessian H of the function h to the subspace of the vectors v orthogonal to $\operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z), \operatorname{grad} f(z)$, and $\operatorname{grad} h_1(z), \ldots, \operatorname{grad} h_d(z)$ is degenerate.

The real codimension of $\mathfrak{M}$ in $P^d \times (Y^* \setminus V)$ is $2n + 2$. In fact, the equality a) locally defines n complex or $2n$ real relations.

Direct calculations show that the Hessian H(v, v) is given by the following formula at the critical point z of the function $h_{|Y_\delta}$;

$$H(v,v)=2\sum_{j=1}^{d}|\langle v,\operatorname{grad}h_j\rangle|^2+2\operatorname{Re}\Big(\sum_{j=1}^{d}\bar{h}_jD^2h_j(v,v)-\bar{\lambda}D^2f(v,v)-\sum_{j=1}^{s}\bar{c}_jD^2g_j(v,v)\Big). \tag{11}$$

Here the complex quadratic form D^2u of the complex-analytic function u is defined by the formula $D^2u(v, v) = \sum(\partial^2u/\partial z_i\partial z_j)v_iv_j$, and $\lambda, c_1, \ldots, c_s \in C$ are the coefficients of the identity $\operatorname{grad}h(z) = \lambda\operatorname{grad}f(z) + \sum_{i=1}^{s} c_i \operatorname{grad}g_i(z)$ which holds at the point z.

Thus the restriction of H to the subspace of vectors v tangent to Y_δ and such that $\langle v, \operatorname{grad}h_j\rangle = 0$, $j = 1, \ldots, d$ is given by the formula $H(v, v) = \operatorname{Re}G(v, v)$ where G is some complex quadratic form. It is easy to see that the degeneracy of Re G is equivalent to the degeneracy of G as a complex quadratic form, and therefore condition b) is given by one complex relation or two real ones.

Direct calculations show that the rank of the Jacobian of the system of 2n + 2 functions locally defining the set $\mathfrak{M}$ is equal to 2n + 2 at almost all points of $\mathfrak{M}$, and so codim $\mathfrak{M} = 2n + 2$. But the real codimension of P^d in $P^d \times (Y^* \setminus V)$ also equals $2\dim Y^* = 2n + 2$, and so the fiber of the projection $\pi: \mathfrak{M} \to P^d$, where π is the restriction to $\mathfrak{M}$ of the projection of $P^d \times (Y^* \setminus V)$ onto its first factor, is zero-dimensional above almost every point $h \in P^d$. But this means that, for almost every set of $h \in P^d$ and for all sufficiently small $\delta \neq 0$, at the critical points of $h_{|Y_\delta}$ the restriction of the Hessian to the subspace given above is nondegenerate.

Now fix a set of $h \in P^d$ which has all the properties listed in Lemma 2.6.

LEMMA 2.7. Let $\alpha > 0$ be chosen sufficiently small and let $\delta \neq 0$ be chosen sufficiently small with respect to α. Then

1) the function $h_{|X_\delta}$ does not have critical points in $X_\delta \cap \dot{N}_\alpha \setminus Y_0^*$;

2) at any critical point of $h_{|X \setminus Y_0^*}$ the index of the Hessian of h is greater than or equal to (n − d), and

3) at any critical point of $h_{|X_\delta \cap \dot{N}_\alpha \setminus Y_0^*}$ the index of the Hessian of $h_{|X_\delta \cap \dot{N}_\alpha}$ is greater than or equal to (n − d).

Proof. 1) As Corollary 2.8 of [1] shows, the function $h_{|Y \setminus (V \cup Y_0)}$ does not have any critical points in a neighborhood of zero. But continuity arguments imply that for some neighborhood W_1 of the set $Y \setminus (V \cup Y_0)$ in Y^* the point $z \in W_1 \cap D_\varepsilon$ is not a critical point of the restriction of h to the manifold Y_δ passing through z. On the other hand, hypothesis 1') of Lemma 2.6 implies that the vectors $\operatorname{grad}h_1(z), \ldots, \operatorname{grad}h_d(z)$ are complex linearly independent $\operatorname{mod}\operatorname{grad}g_1(z), \ldots, \operatorname{grad}g_s(z), \operatorname{grad}f(z)$ for $z \in Y_0 \setminus V$, and this means that this holds for all $z \in Y_0 \setminus \{0\}$ sufficiently close to zero. Continuity arguments also show that this holds for $z \in W_2$, where W_2 is some neighborhood of $Y_0 \setminus \{0\}$ in Y^*.

If for fixed α we choose δ sufficiently small then it is obvious that we may assume that $X_\delta \cap \dot{N}_\alpha \subset W_1 \cup W_2$ are neighborhoods of $Y \setminus V$ in Y^*. Now we need only note that the vector $\operatorname{grad}h(z) = \sum h_j(z)\operatorname{grad}h_j(z)$ is not zero $\operatorname{mod}\operatorname{grad}g_1(z), \ldots, \operatorname{grad}g_s(z), \operatorname{grad}f(z)$ for $z \in W_2 \setminus Y_0^*$.

2) Let us choose $\delta \neq 0$ so small that hypothesis 3') of Lemma 1.6 holds, and let $z \in X_\delta \setminus Y_0^*$ be a critical point of $h_{|X_\delta}$. Consider the restriction of the Hessian H to the subspace T of tangent vectors v such that $\langle v, \operatorname{grad}h_j\rangle = 0$, $j = 1, \ldots, d$. Formula (11) implies that on T, $H(v, v) = \operatorname{Re}G(v, v)$, where G is a nondegenerate quadratic form. Since G can be reduced to the form $G(z, z) = \sum_{i=1}^{\dim T} z_i^2$, $\operatorname{Re}G = \sum x_i^2 - \sum y_i^2$, in some basis, it follows that the index of $H \geq \dim T \geq n - d$.

3) If α is so small that hypothesis 3) of Lemma 1.6 holds, then Lemma 2.18 of [2] implies that the index of the Hessian of the function $h_{|Y \cap \dot{N}_\alpha \setminus Y_0}$ is not less than (n − d). Since the property we are considering is stable under small C^2-deformations of the function, the property holds for $h_{|Y_\delta \cap \dot{N}_\alpha \setminus Y_0^*}$ if δ is sufficiently small.

Now fix α and δ so small that Lemma 2.7 holds. Just as above, $Z_\beta = \{z \in Y^* | h(z) \leq \beta\}$. Choose β so that $N_\alpha \cap Z_\beta \subset D_\varepsilon$.

LEMMA 2.8. The pair $(X_\delta \cap N_\alpha \cap Z_\beta, X_\delta \cap N_\alpha \cap Y_0^*)$ is (n − d − 1)-connected.

Proof. Corollary 2.8 of [1] shows that sufficiently small $\gamma > 0$ are not critical values of $h_{|X_\delta}$ and $h_{|X_\delta \cap \dot{N}_\alpha}$. Therefore if β' is sufficiently small, $X_\delta \cap N_\alpha \cap Y_0^*$ is a strong deformation retract of $X_\delta \cap N_\alpha \cap Z_{\beta'}$. We now quote the form of Morse's theorem on manifolds with boundary we need (see, for example, [8], p. 60).

Suppose the function φ on the differentiable manifold M with boundary ∂M has only nondegenerate critical points, and that none of them lie on ∂M. Let us also assume that all the critical points of $\varphi_{|\partial M}$ are nondegenerate. Suppose, moreover, that $\varphi^{-1}[a, b]$ is compact, where a and b are noncritical values of $\varphi_{|M}$ and $\varphi_{|\partial M}$. Then $\varphi^{-1}(-\infty, b]$ has the homotopy type of $\varphi^{-1}(-\infty, a]$ with adjoined cells, where the number of adjoined cells of dimension i is equal to the sum of the number of critical points of $\varphi_{|M}$ of index i and of those critical points of $\varphi_{|\partial M}$ of index i at which the derivative of φ along the external normal to M is negative.

Take as the Morse function on $X_\delta \cap N_\alpha$ the function ψ obtained from h by removing the degenerate critical points of $h_{|X_\delta \cap \dot{N}_\alpha}$ and $h_{|X_\delta}$. Then Lemma 2.7 and the theorem we have quoted imply that $X_\delta \cap N_\alpha \cap Z_\beta$ has the homotopy type of $X_\delta \cap N_\alpha \cap Z_{\beta'}$ with adjoined cells of dimension greater than or equal to $(n - d)$.

COROLLARY 2.9. $X_\delta \cap N_\alpha \cap Z_\beta$ is $(m - s - d - 2)$-connected.

Proof. The manifolds Y_0^* and $Y_0 = Y_0^* \cap f^{-1}(0)$ have an isolated singularity at zero, because of 1), 1'), and 2) of Lemma 2.6. As we have shown, Lemma 2.3 holds in this case and therefore the fiber F_0 homotopy equivalent to it and the manifold X_0' are $(m - s - d - 2)$-connected, since $I(Y_0^*)$ is generated by the s + d polynomials $g_1, \ldots, g_s, h_1, \ldots, h_d$.

But the function $q_{|Y_0}$ has an isolated zero at zero by 2) of Lemma 2.6, and therefore Lemma 2.1 implies that for sufficiently small δ the manifold $X_\delta \cap N_\alpha \cap Y_0^* = Y_0^* \cap f^{-1}(\delta) \cap N_\alpha$ is diffeomorphic to X_0'. Since $m - s - d - 2 \leq n - d - 1$, the required connectivity of $X_\delta \cap N_\alpha \cap Z_\beta$ follows from Lemma 2.8.

LEMMA 2.10. The manifolds $X_\delta \cap N_\alpha$ and $X_\delta \cap N_\alpha \cap Z_\beta$ are homeomorphic.

Proof. Since $h_{|V}$ has an isolated zero at zero, 2) of Lemma 2.6 and Lemma 1.6 imply that there are $\varepsilon_1 > 0$ and a neighborhood G_1 of the set $V \setminus \{0\}$ in Y^* such that for $z \in D_{\varepsilon_1} \cap G_1 \setminus Y$ the vectors grad h(z) and z do not point in exactly opposite directions mod grad $g_1(z), \ldots,$ grad $g_s(z)$, grad $f(z) = (1/2f(z))\text{grad}|f(z)|^2$. The same Lemma 1.6 states that there are $\varepsilon_2 > 0$ and a neighborhood G_2 of the set $V \setminus \{0\}$ in Y such that for $z \in D_{\varepsilon_2} \cap G_2 \setminus V$ the vectors grad h(z) and z do not point in exactly opposite directions mod grad $g_1(z), \ldots,$ grad $g_s(z)$, grad f(z), grad q(z). Continuity arguments imply that this last property holds for $z \in D_\varepsilon \cap G_3$, where G_3 is a neighborhood of $G_2 \setminus V$ in Y^*. We can assume that after the set of $h \in P^d$ has been chosen ε is chosen smaller than $\min(\varepsilon_1, \varepsilon_2)$; then α is chosen such that $Y \cap N_\alpha \cap (D_\varepsilon \setminus \mathring{Z}_\beta) \subset G_1 \cap G_2$; and finally, δ is chosen so that $Y_\delta \cap N_\alpha \cap (D_\varepsilon \setminus \mathring{Z}_\beta) \subset G_1$, $Y_\delta \cap \dot{N}_\alpha \cap (D_\varepsilon \setminus \mathring{Z}_\beta) \subset G_3$. Then we can construct a tangent vector field ω on Y_δ in a neighborhood of $Y_\delta \cap N_\alpha \cap (D_\varepsilon \setminus \mathring{Z}_\beta)$ such that along this field both $\|z\|^2$ and h increase, while at the points of $Y_\delta \cap \dot{N}_\alpha$ this field can be assumed to be tangent to $Y_\delta \cap \dot{N}_\alpha$. With the help of the trajectories of the field ω it is easy to establish that $Y_\delta \cap N_\alpha \cap (D_\varepsilon \setminus \mathring{Z}_\beta) = (X_\delta \cap N_\alpha) \setminus (X_\delta \cap N_\alpha \cap \mathring{Z}_\beta)$ is homeomorphic to $(X_\delta \cap N_\alpha \cap \dot{Z}_\beta) \times [0, 1]$, and the assertion of the lemma follows from this directly.

Lemma 2.3 now follows from Lemma 2.5, Lemma 2.10, and Corollary 2.9.

3. Euler Characteristic of the Fiber

Let us assume that Y^* and Y are complete intersections, i.e., $n + 1 = \dim Y^* = m - s$, where $g_1, \ldots, g_s$ are generators of $I(Y^*)$, $g_1, \ldots, g_s$, f are generators of I(Y). Just as before, suppose that V is the set of singular points of Y, and put $\Sigma^* = Y^* \cap S_\varepsilon$, $\Sigma = Y \cap S_\varepsilon$; put $\Sigma_0 = V \cap S_\varepsilon$ for sufficiently small ε.

There is a formula which expresses the Euler characteristic of the fiber F of the fibration $\varphi: \Sigma^* \setminus \Sigma \to S^1$ in terms of the mapping of $\Sigma^* \setminus \Sigma$ into a manifold of the type of the Stiefel manifold, and this formula is close to that of Theorem 2 of [3] which holds under the assumption that the singularity at zero in Y^* and Y are isolated. The notation below is that of [3].

Let $R_{k,m}$ be the space of k-frames $(v_1, \ldots, v_{k-1}, v_k)$ in C^m such that $\langle v_i, v_j \rangle = \delta_{ij}$ for $(i, j) \neq (k - 1, k)$, while no constraints are imposed on $\langle v_{k-1}, v_k \rangle$. Let $\bar{V}_{k,m} \subset R_{k,m}$ be the space of frames $(v_1, \ldots, v_{k-1}, v_k)$ such that v_{k-1} and v_k are complex linearly independent. It is obvious that $\bar{V}_{k,m}$ can be deformed into $R_{k,m}$ in $V_{k,m}$, the Stiefel manifold of complex orthogonal k-frames in C^m.

Since Y^* is a complete intersection, at all simple points of Y^* the vectors $\mathrm{grad}\, g_1(z), \ldots, \mathrm{grad}\, g_s(z)$ are complex linearly independent. Further, Corollary 2.8 of [1] shows that, for some $\varepsilon > 0$, at all points $z \in D_\varepsilon \cap (Y^* \setminus Y)$ the vector $\mathrm{grad} f(z) \neq 0$, $\mathrm{mod}\, \mathrm{grad}\, g_1(z), \ldots, \mathrm{grad}\, g_s(z)$ and the vector $z \neq 0$, $\mathrm{mod}\, \mathrm{grad}\, g_1(z), \ldots, \mathrm{grad}\, g_s(z)$, i.e., the frames $\mathrm{grad}\, g_1(z), \ldots, \mathrm{grad}\, g_s(z)$, $\mathrm{grad} f(z)$ and $\mathrm{grad}\, g_1(z), \ldots, \mathrm{grad}\, g_s(z)$, z are complex linearly independent. Therefore the formula

$$\lambda(z) = ([\mathrm{grad}\, g_1(z), \ldots, \mathrm{grad}\, g_s(z)],\ \overline{\mathrm{grad}} f(z) / \|\overline{\mathrm{grad}} f(z)\|,\ \bar{z}/\|\bar{z}\|)$$

defines the map $\lambda: \Sigma^* \setminus \Sigma \to R_{s+2,m}$. Here $[v_1, \ldots, v_k]$ denotes the frame obtained from the frame $(v_1, \ldots, v_k)$ by orthonormalization, and $\bar{v}$ is the projection of the vector v on the subspace complex-orthogonal to the vectors $\mathrm{grad}\, g_1(z), \ldots, \mathrm{grad}\, g_s(z)$.

For λ and for the other mappings defined below in a similar way, the frame $\lambda(z)$ will be written simply as $(\mathrm{grad}\, g_1(z), \ldots, \mathrm{grad}\, g_s(z), \mathrm{grad} f(z), z)$, with the exception of those cases where the order of orthonormalization is important.

Remark. It is not difficult to see that the map λ is in fact defined for $z \in \Sigma^* \setminus \Sigma_0$.

As Corollary 1.7 shows (or Corollary 3.8 of [2] shows), at points $z \in \Sigma^* \setminus \Sigma$ sufficiently close to Σ, the vectors $\mathrm{grad} f(z) = (1/2f(z))\mathrm{grad}|f(z)|^2$ are complex linearly independent $\mathrm{mod}\, \mathrm{grad}\, g_1(z), \ldots, \mathrm{grad}\, g_s(z)$. Therefore if we put $M_\alpha = \{z \in Y^* \mid |f(z)|^2 \leq \alpha\}$, $\dot{M}_\alpha = \{z \in Y^* \mid |f(z)|^2 = \alpha\}$, $\mathring{M}_\alpha = \{z \in Y^* \mid |f(z)|^2 < \alpha\}$, then for sufficiently small α:

$$\lambda: \ (\Sigma^* \setminus \Sigma,\ \Sigma^* \cap M_\alpha \setminus \Sigma) \to (R_{s+2,m},\ \overline{V}_{s+2,m}).$$

For sufficiently small α, $\dot{M}_\alpha \cap \Sigma^*$ is a smooth manifold and will be taken below as the pair $(\Sigma^*, \partial\hat{\Sigma}^*)$ instead of the pair $(\Sigma^* \setminus \Sigma,\ \Sigma^* \cap M_\alpha \setminus \Sigma)$, where $\hat{\Sigma}^*$ is the manifold with boundary $\Sigma^* \setminus \mathring{M}_\alpha$, and its boundary is $\partial\hat{\Sigma}^* = \Sigma^* \cap \dot{M}_\alpha$.

Let $r' \in H_{2m-2s-1}(\hat{\Sigma}^*, \partial\hat{\Sigma}^*)$ be the fundamental relative cycle of the manifold $\hat{\Sigma}^*$. (The orientation of $\hat{\Sigma}^*$ is defined by the orientation of the complex manifold Y^* and its external normal.) As will be shown below, $H_{2m-2s-1}(R_{s+2,m}, \overline{V}_{s+2,m}) \cong Z$, and in this group there is a natural generator s'. Let the integer $\varkappa(\lambda)$ be defined by $\lambda_*(r') = \varkappa(\lambda)s'$.

THEOREM 3.1. $\chi(F) = \varkappa(\lambda)$.

To prove Theorem 3.1, we first describe the homology of the pair $(R_{k,m}, \overline{V}_{k,m})$ and the triple $(R_{k,m}, \overline{\Phi}_{k,m}, \overline{V}_{k,m})$ more precisely; here $\overline{\Phi}_{k,m}$ is the subspace of frames $(v_1, \ldots, v_{k-1}, v_k) \in R_{k,m}$ such that v_{k-1} and v_k are real linearly independent. It is obvious that $\overline{\Phi}_{k,m}$ contracts to $\Phi_{k,m}$ in $R_{k,m}$, where $\Phi_{k,m}$ is the space of frames in $R_{k,m}$ such that $\mathrm{Re}\,\langle v_{k-1}, v_k\rangle = 0$.

LEMMA 3.2. $H_{2m-2k+3}(R_{k,m}, \overline{V}_{k,m}) \cong Z$. In this group the generator s' can be chosen so that

1) in the exact sequence

$$H_{2m-2k+3}(\overline{V}_{k,m}) \xrightarrow{i_*} H_{2m-2k+3}(R_{k,m}) \xrightarrow{j_*} H_{2m-2k+3}(R_{k,m}, \overline{V}_{k,m}) \qquad j_*(s^2) = -j_*(s^3) = s';$$

we have

2) in the exact sequence of the triple

$$H_{2m-2k+3}(R_{k,m}, \overline{V}_{k,m}) \xrightarrow{j_*^1} H_{2m-2k+3}(R_{k,m}, \overline{\Phi}_{k,m}) \xrightarrow{\partial} H_{2m-2k+2}(\overline{\Phi}_{k,m}, \overline{V}_{k,m})$$

we have

$$j_*^1(s') = s^4 + s^5, \quad \partial(s^4) = -\partial(s^5) = s^7 + s^8;$$

3) in the exact sequence of the pair

$$H_{2m-2k+2}(\overline{\Phi}_{k,m}, \overline{V}_{k,m}) \xrightarrow{\partial} H_{2m-2k+1}(\overline{V}_{k,m})$$

we have

$$\partial(s^7) = -\partial(s^8) = s^1.$$

Here $s^1 \in H_{2m-2k+1}(\overline{V}_{k,m})$, s^2, $s^3 \in H_{2m-2k+3}(R_{k,m})$, s^4, $s^5 \in H_{2m-2k+3}(R_{k,m}, \overline{\Phi}_{k,m})$, and s^7, $s^8 \in H_{2m-2k+2}(\overline{\Phi}_{k,m}, \overline{V}_{k,m})$ are the homology classes defined in Lemmas 8, 9, and 12 of [3].

Proof. Some simple calculations show that $H_{2m-2k+3}(R_{k,m}, \bar{V}_{k,m}) \cong Z$ and that the class $j_*(s^2) = -j_*(s^3)$ generates this group. The other relations follow directly from the appropriate lemmas of [3].

We now prove several properties of the pair $(R_{k,m}, \bar{V}_{k,m})$; these properties are not required in this article.

LEMMA 3.3. Let $'s$ be the generator of the group $H^{2m-2k+3}(R_{k,m}, \bar{V}_{k,m})$ such that $\langle s', 's\rangle = 1$.

Let $^1s \in H^{2m-2k+1}(\bar{V}_{k,m})$ be the generator of this group such that $\langle s^1, {}^1s\rangle = 1$.

We identify the groups $H^*(R_{k,m}, \bar{V}_{k,m})$ and $H^*(R_{k,m} \setminus V_{k,m}, \bar{V}_{k,m} \setminus V_{k,m})$ via the excision isomorphism.

Finally, let the map $\pi : R_{km} \setminus V_{k,m} \to S^1$ be defined by $\pi(v_1, \ldots, v_{k-1}, v_k) = \langle v_{k-1}, v_k\rangle / |\langle v_{k-1}, v_k\rangle|$, and let $\sigma \in H^1(S^1)$ be the fundamental cocycle of the circle. Then $\delta({}^1s) \cup \pi^*(\sigma) = -'s$, where

$$\delta\colon\ H^{2m-2k+1}(\bar{V}_{k,m}) \leftarrow H^{2m-2k+2}(R_{k,m} \setminus V_{k,m}, \bar{V}_{k,m} \setminus V_{k,m}).$$

Proof. Let $W_{k,m} \subset R_{k,m}$ be the submanifold consisting of the frames $(v_1, \ldots, v_{k-1}, v_k) \in R_{k,m}$ such that $v_{k-1} = e^{i\varphi} v_k$. It is obvious that $\bar{V}_{k,m} = R_{k,m} \setminus W_{k,m}$. On the other hand, $R_{k,m} \setminus V_{k,m}$ is a tubular neighborhood of $W_{k,m}$.

In fact, suppose that the vector bundle ξ on $W_{k,m}$ has fiber, above the point $(v_1, \ldots, v_{k-1}, v_k) \in W_{k,m}$, a subspace of C^m consisting of those vectors ω such that $\langle\omega, v_i\rangle = 0$, $i = 1, \ldots, k$. The map $\tau : E_\xi \to R_{k,m} \setminus V_{k,m}$, $\tau((v_1, \ldots, v_{k-1}, v_k), \omega) = (v_1, \ldots, v_{k-2}, (v_{k-1} + \omega)/\|v_{k-1} + \omega\|, v_k)$ defines a diffeomorphism of the space E_ξ of the bundle ξ and the space $R_{k,m} \setminus V_{k,m}$, which takes $W_{k,m} \subset E_\xi$ into $W_{k,m} \subset R_{k,m}$.

Therefore, as the Thom isomorphism theorem implies, the generator $'s \in H^{2m-2k+3}(R_{k,m} \setminus V_{k,m}, \bar{V}_{k,m} \setminus V_{k,m})$ is the product of the Thom class $u \in H^{2m-2k+2}(R_{k,m} \setminus V_{k,m}, \bar{V}_{k,m} \setminus V_{k,m})$ and the generator ν of the group $H^1(R_{k,m} \setminus V_{k,m}) \cong H^1(W_{k,m})$. But it is easy to see that $u = \delta({}^1s)$ and $\nu = -\pi^*(\sigma)$.

Let us now return to the proof of Theorem 3.1. The fiber F of the fibration φ is diffeomorphic to the interior of the manifold X', and so the Euler characteristic of the manifold X' can be calculated from Theorem 1 of [3]. As this theorem asserts, $\chi(X') = \varkappa(\eta)$, where $\eta : \partial X' \to V_{s+2,m}$. Here $\partial X' = \Sigma_\delta = \Sigma^* \cap f^{-1}(\delta)$ for sufficiently small $\delta \neq 0$ and for $z \in z_\delta$, $\eta(z) = (\operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z), \operatorname{grad} f(z), z)$. To prove Theorem 3.1 we must equate $\varkappa(\lambda)$ and $\varkappa(\eta)$.

Consider the auxiliary manifold $Y'_\xi = Y^* \cap (\operatorname{Re} f)^{-1}(\xi)$. If $\xi \neq 0$ is sufficiently small, then Corollary 2.8 of [1] implies that Y'_ξ is a smooth manifold, as well as $\Sigma'_\xi = Y'_\xi \cap \Sigma^*$.

LEMMA 3.4. $\xi > 0$ can be chosen so small that

1) the vectors $\operatorname{grad} f(z)$ and $\bar{z}$ are real linearly independent for $z \in \Sigma^* \setminus \Sigma$ such that $|\operatorname{Re} f(z)| \le \xi$;

2) the vectors $i \operatorname{grad} f(z)$ and $\hat{z}$ are real linearly independent for $z \in \Sigma'_\xi$ such that $|\operatorname{Im} f(z)| \le 10 \cdot \xi$. Here $\hat{v}$ is the projection of the vector v onto the tangent space to V'_ξ at the point z, i.e., onto the subspace of C^m which is complex-orthogonal to the vectors $\operatorname{grad} g_1(z), \ldots, \operatorname{grad} g_s(z)$ and real-orthogonal to the vector $\operatorname{grad} f(z)$.

Proof. Corollary 1.7 shows that, at points $z \in U_1 \setminus \Sigma$ where U_1 is some neighborhood of Σ in Σ^*, the vectors $\operatorname{grad} f(z)$ and $\bar{z}$ are complex linearly independent. On the other hand, since in $Y'_0 \setminus Y$ there are no singular points of the real algebraic set Y'_0, then Corollary 2.8 of [1] allows us to assume that the vectors $\operatorname{grad} f(z) = \operatorname{grad} \operatorname{Re} f(z)$ and $\bar{z}$ are real linearly independent for $z \in \Sigma'_0 \setminus \Sigma$, and therefore for $z \in U_2$, a neighborhood of $\Sigma'_0 \setminus \Sigma$ in Σ^*. Since $U_1 \cup U_2$ is a neighborhood of Σ'_0 in Σ^*, for sufficiently small ξ the points $z \in \Sigma^*$ such that $|\operatorname{Re} f(z)| \le \xi$ lie in $U_1 \cup U_2$.

For 2), it is obvious that for sufficiently small $\xi > 0$, $\Sigma'_\xi \cap |\operatorname{Im} f|^{-1}[0, 10 \cdot \xi] \subset U_1 \setminus \Sigma$, and therefore the vectors $\operatorname{grad} f(z)$ and z are complex linearly independent for $z \in \Sigma'_\xi$, $|\operatorname{Im} f(z)| \le 10 \cdot \xi$. But then the vectors $i \operatorname{grad} f(z)$ and $\hat{z} = \bar{z} - \gamma \operatorname{grad} f(z)$ are complex and real linearly independent.

Now fix $\xi > 0$ so small that Lemma 3.4 holds; then fix $\alpha > 0$ so small that $M_\alpha \cap \Sigma'_\xi = \phi$. Thus Σ'_ξ is a smooth manifold of codimension 1 in the manifold $\hat{\Sigma}^*$. Moreover Σ'_ξ divides $\hat{\Sigma}^*$ into two parts: $\hat{\Sigma}_{+*} = \hat{\Sigma}^* \cap \operatorname{Re} f^{-1}[\xi, \infty)$ and $\hat{\Sigma}_{-*} = \hat{\Sigma}^* \cap \operatorname{Re} f^{-1}(-\infty, \xi]$.

Further $\Sigma_\xi \subset \Sigma'_\xi$ is defined in Σ'_ξ by the equation $\operatorname{Im} f = 0$, and therefore Σ_ξ also divides Σ'_ξ into two parts: $\Sigma'_{\xi+}$ and $\Sigma'_{\xi-}$.

Consider the following sequence:

$$H_{2m-2s-1}(\widehat{\Sigma}^*,\ \partial\widehat{\Sigma}^*) \xrightarrow{j_*} H_{2m-2s-1}(\widehat{\Sigma}^*,\ \partial\widehat{\Sigma}^*\cup\Sigma_\xi')$$
$$\xrightarrow{\partial} H_{2m-2s-2}(\partial\widehat{\Sigma}^*\cup\Sigma_\xi',\ \partial\widehat{\Sigma}^*\cup\Sigma_\xi) \cong H_{2m-2s-2}(\Sigma_\xi',\ \Sigma_\xi) \xrightarrow{\partial} H_{2m-2s-3}(\Sigma_\xi). \tag{12}$$

LEMMA 3.5. In the sequence (12)

$$j_*(r') = r^2 + r^3;\ \partial(r^2) = -\partial(r^3) = r^5 + r^6;$$
$$\partial(r^5) = -\partial(r^6) = r^1.$$

Here r^2 and $r^3 \in H_{2m-2s-1}(\widehat{\Sigma}^*,\ \partial\widehat{\Sigma}^* \cup \Sigma'_\xi)$ are the fundamental relative cycles of the manifolds with boundary $\widehat{\Sigma}^*_+$ and $\widehat{\Sigma}^*_-$, r^5 and $r^6 \in H_{2m-2s-2}(\Sigma'_\xi,\ \Sigma_\xi)$ are the fundamental relative cycles of the manifolds $\Sigma'_{\xi+}$ and $\Sigma'_{\xi-}$, and r^1 is the fundamental cycle of Σ_ξ. (Notation as in [2].)

Proof. All the relations of Lemma 3.5 flow from the fact that ∂ applied to the fundamental relative cycle of the manifold with boundary is the fundamental cycle of the boundary. Agreement of the orientations is verified by direct calculation.

Now consider the diagram

$$\begin{array}{ccccccc} H_l(R,\overline{V}) & \xrightarrow{j_*} & H_l(R,\overline{\Phi}) & \xrightarrow{\partial} & H_{l-1}(\overline{\Phi},\overline{V}) & \xrightarrow{\partial} & H_{l-2}(\overline{V}) \\ \uparrow\lambda_* & & \uparrow\lambda_* & & \uparrow\lambda_*=\eta_* & & \\ H_l(\widehat{\Sigma}^*,\ \partial\widehat{\Sigma}^*) & \xrightarrow{j_*} & H_l(\widehat{\Sigma}^*,\ \partial\widehat{\Sigma}^*\cup\Sigma_\xi') & \xrightarrow{\partial} & H_{l-1}(\Sigma_\xi',\ \Sigma_\xi) & \xrightarrow{\partial} & H_{l-2}(\Sigma_\xi). \end{array} \tag{13}$$

Here $l = 2m - 2s - 1$ and the relations of Lemma 3.4 on linear independence imply that the map $\lambda: \Sigma^* \setminus \Sigma \to R_{s+2,m}$ defined above induces the pair of maps denoted by this letter in the diagram (13). It is obvious that the maps λ and η of Σ_ξ into $\overline{V}_{s+2,m}$ are homotopic, and therefore $\lambda_* = \eta_*$.

LEMMA 3.6. 1) The matrix of the map

$$\lambda_*: H_l(\widehat{\Sigma}^*,\ \partial\widehat{\Sigma}^*\cup\Sigma_\xi') \to H_l(R_{s+2,m},\ \overline{\Phi}_{s+2,m})$$

is diagonal in the bases r^2, r^3 and s^4, s^5. (More precisely, the matrix of the restriction of λ_* to the subgroup of $H_l(\widehat{\Sigma}^*,\ \partial\widehat{\Sigma}^* \cup \Sigma'_\xi$ generated by r^2 and r^3.)

2) The matrix of the map

$$\lambda_*: H_{l-1}(\Sigma_\xi',\ \Sigma_\xi) \to H_{l-1}(\overline{\Phi}_{s+2,m},\ \overline{V}_{s+2,m})$$

is diagonal in the bases r^5, r^6 and s^7, s^8.

Proof. The proof of this lemma is completely similar to that of Lemmas 10 and 13 in [3]. The single difference is that the manifolds Σ'_ξ and Σ_ξ are "moved by ξ," and this turns out to be unimportant in virtue of Lemma 3.4.

Proof of Theorem 3.1. Lemmas 3.2 and 3.5 imply that $j_*\lambda_*(r') = j_*(\varkappa(\lambda)s') = \varkappa(\lambda)(s^4 + s^5) = \lambda_*(r^2 + r^3)$. Lemma 3.6 implies that $\lambda_*(r^2) = \varkappa(\lambda)s^4$. Lemmas 3.2 and 3.5 imply that $\lambda_*(r^5 + r^6) = \varkappa(\lambda)(s^7 + s^8)$. Lemma 3.6 implies that $\lambda_*(r^5) = \varkappa(\lambda)s^7$, and Lemmas 3.2 and 3.5 imply that $\lambda_*(r^1) = \varkappa(\lambda)s^1$. But $\lambda_* = \eta_* : H_{l-2}(\Sigma_\xi) \to H_{l-2}(\overline{V}_{s+2,m})$. Therefore $\varkappa(\lambda) = \varkappa(\eta)$.

Remark 1. Since the map λ is defined on $\Sigma^* \setminus \Sigma_0$, we can consider the neighborhood N_α of the set Σ_0 instead of the neighborhood M_α of the set Σ. The proof of Theorem 3.1 is unchanged for this variation.

Remark 2. The proof given does not extend to the case where $s = m - 1$, i.e., where 0 is an isolated point of Y and the manifolds Σ and Σ_ξ are empty. However Theorem 3.1 is true in this case too.

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