

HEEGAARD FLOER HOMOLOGY AND KNOT CONCORDANCE: A SURVEY OF RECENT ADVANCES

STANISLAV JABUKA

University of Nevada Reno, USA

The author would like to dedicate this article to Professor Sibe Mardešić for his tireless efforts in creating, teaching and promoting topology in Croatia for over three decades. His infectious enthusiasm for the subject has always been an inspiration and the company of his warm and open personality an all too rare delight.

ABSTRACT. This article surveys some recent advances made in the understanding of the smooth knot concordance group $\mathcal{C}$. The focus is exclusively on those results which have been driven by Heegaard Floer homology. Three invariants are discussed: the knot concordance epimorphisms $\tau, \delta : \mathcal{C} \rightarrow \mathbb{Z}$ and the correction terms of double branched covers of knots.

1. INTRODUCTION

Since its introduction by Peter Ozsváth and Zoltán Szabó in 2001, Heegaard Floer homology has made substantial contributions to low dimensional topology. It is a nearly comprehensive package of invariants – it provides the working topologist with invariants for 3-manifolds, invariants for nullhomologous knots in arbitrary 3-manifolds and invariants of smooth 4-manifolds. There are a number of secondary invariants derived from these: for example invariants of contact structures on 3-manifolds, other invariants of knots obtained by applying the 3-manifold invariant to manifolds canonically constructed from the knot, etc.

This article is a survey of recent results in knot concordance stemming from Heegaard Floer theory. To describe these, let us briefly recall some of the basic definitions.

2000 *Mathematics Subject Classification.* 57M25, 57M27.

Key words and phrases. Heegaard Floer homology, knot concordance, slice genus.

DEFINITION 1.1. The smooth slice genus $g_*(K)$ of a knot K in S^3 is the smallest genus of any surface Σ smoothly and properly embedded in the 4-ball B^4 with $\partial(B^4, \Sigma) = (S^3, K)$.

We say that K is smoothly slice if $g_*(K) = 0$. Two knots K_1 and K_2 are called smoothly concordant if $K_1 \# \overline{K_2}$ is smoothly slice where $\overline{K}$ represents the mirror image of K with reversed orientation. The set of concordance classes of knots forms an Abelian group under the connected sum operation called the smooth concordance group and denoted by $\mathcal{C}$.

Repeating these definitions verbatim for the case where Σ is embedded locally topologically flat leads to the notions of topological slice genus and the topological concordance group $\mathcal{C}_{top}$.

The structure of $\mathcal{C}$ is still quite poorly understood. There is a surjective homomorphism $\Theta : \mathcal{C} \rightarrow \mathcal{C}_{alg}$ defined by Levine [11, 12] from $\mathcal{C}$ onto the algebraic concordance group $\mathcal{C}_{alg}$ isomorphic to the infinite direct sum

$$\mathcal{C}_{alg} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty.$$

It was first proved by Casson and Gordon [1, 2] that Θ has a nontrivial kernel. By further work of Jiang [8] and Livingston [13] it is known that the kernel of Θ is infinitely generated, in fact it contains a subgroup isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty$.

Work of M. Freedman [4, 5] showed that the methods of Levine, Casson and Gordon extend to the topological category yielding a homomorphism $\Theta_{top} : \mathcal{C}_{top} \rightarrow \mathcal{C}_{alg}$ whose kernel also contains a subgroup isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty$. Another important result of Freedman's work is that any knot K with trivial Alexander polynomial $\Delta_K(t) = 1$ is topologically slice.

Fitting in with Θ and Θ_{top} is the map

$$\Psi : \mathcal{C} \rightarrow \mathcal{C}_{top}$$

which sends the equivalence class of K in $\mathcal{C}$ to the one of K in $\mathcal{C}_{top}$. These three maps fit into the commutative triangle

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Psi} & \mathcal{C}_{top} \\ \Theta \downarrow & \swarrow \Theta_{top} & \\ \mathcal{C}_{alg} & & \end{array}$$

Very little is known about the kernel of Ψ , see however [3, 6]. Despite the successes mentioned above, nothing is known about torsion elements in $\mathcal{C}$ beyond the obvious 2-torsion generated by amphicheiral knots.

Here are some questions concerning $\mathcal{C}$ and $\mathcal{C}_{top}$ which we shall address in this article.

QUESTION 1.1. Are there obstructions for K being of finite order in $\mathcal{C}$ that are sensitive to the difference between $\mathcal{C}$ and $\mathcal{C}_{top}$?

The classical knot signature $\sigma(K)$ is an obstruction to K being of finite order but it doesn't differentiate between the smooth and the topological category.

QUESTION 1.2. *Are there obstructions for a knot K to be of order n in $\mathcal{C}$? Are there such obstructions which differentiate between the smooth and topological category?*

QUESTION 1.3. *What can be said about the kernel of Ψ ? What can be said about the subgroup structure of $\text{Ker}(\Psi)$?*

Heegaard Floer homology has provided partial answers to each of the question 1.1 – 1.3 and techniques are being developed to further unveil the structure of $\mathcal{C}$ and $\mathcal{C}_{top}$. Since the 4-dimensional aspects of Heegaard Floer theory rely on the smooth structure, results obtained are indeed sensitive to the subtle differences between $\mathcal{C}$ and $\mathcal{C}_{top}$.

The following theorems have been proved using Heegaard Floer homology and are pertinent to question asked above. More details as well as further results can be found in Sections 4–6.

THEOREM 1.2 ([19, 24]). *There is a surjective group homomorphism*

$$\tau \oplus \delta : \mathcal{C} \rightarrow \mathbb{Z}^2$$

defined by means of Heegaard Floer homology (see Sections 4 and 5 for details). Each of the components of this map agrees with $-\sigma(K)/2$ (where $\sigma(K)$ is the signature of K) when K is alternating.

While the invariants τ and δ from the theorem agree with the signature in many cases, they are substantially different than the signature as the following theorem states.

THEOREM 1.3 ([14]). *There are knots with trivial Alexander polynomial and with non-vanishing $\tau(K)$ and $\delta(K)$. The kernel $\text{Ker}(\Psi : \mathcal{C} \rightarrow \mathcal{C}_{alg})$ contains a complemented subgroup isomorphic to $\mathbb{Z}^2$.*

REMARK 1.4. More is known about the kernel of Ψ . By a recent result of C. Livingston [15] it in fact contains a complemented subgroup isomorphic to $\mathbb{Z}^3$ but the extra copy of $\mathbb{Z}$ stems from yet another epimorphism $s : \mathcal{C} \rightarrow \mathbb{Z}$ which is the analogue of τ in the context of Khovanov homology. It was discovered by Jacob Rasmussen [25].

To state the next theorem, let Y_K denote the double branched cover of a knot K in S^3 .

THEOREM 1.5 ([7]). *For a knot K to be of order n in $\mathcal{C}$ the determinant of $K' = \#^n K$ needs to be a perfect square, say $|\det K'| = m^2$. Furthermore, there is a subgroup $\mathcal{H}$ of order m of $H^2(Y_{K'}; \mathbb{Z})$ for which all the corresponding correction terms vanish:*

$$d(Y_{K'}, \mathfrak{s}) = 0 \quad \forall \mathfrak{s} \in \mathcal{H}$$

for some affine identification of $H^2(Y_{K'}; \mathbb{Z})$ with the set $\text{Spin}^c(Y_{K'})$ of spin^c -structures on $Y_{K'}$.

For a definition of the correction terms $d(Y_{K'}, \mathfrak{s})$ see Subsection 2.4. As we shall see in Section 6, the obstruction from Theorem 1.5 is stronger than other known obstruction and can be used to prove previously unavailable results. Clearly Theorems 1.2 and 1.3 pertain to Questions 1.1 and 1.3 while Theorem 1.5 addresses Question 1.2.

The remainder of the article is organized as follows. In Section 2 we give the necessary background in Heegaard Floer homology needed to understand the invariants τ , δ and the correction terms $d(Y, \mathfrak{s})$. Sections 4, 5 and 6 discuss each of these three invariants at greater length and provide the reader with further results about $\mathcal{C}$ obtained from their study.

2. HEEGAARD FLOER HOMOLOGY

2.1. Definition of the Heegaard Floer groups. Throughout the article we let Y denote a compact, orientable 3-manifold without boundary. By $\text{Spin}^c(Y)$ we denote its space of spin^c -structures which we often identify with $H^2(Y; \mathbb{Z})$ via an affine isomorphism which sends a spin-structure $\mathfrak{s} \in \text{Spin}^c(Y)$ to $0 \in H^2(Y; \mathbb{Z})$.

The Heegaard Floer homology groups $\widehat{HF}(Y, \mathfrak{s})$, $HF^\pm(Y, \mathfrak{s})$ and $HF^\infty(Y, \mathfrak{s})$ are topological invariants of the pair $(Y, \mathfrak{s})$, $\mathfrak{s} \in \text{Spin}^c(Y)$. They derive their name from their definition which is via a Heegaard diagram for Y as we explain below.

DEFINITION 2.1. *A Heegaard diagram for Y is a triple $(\Sigma_g, \alpha, \beta)$ consisting of a genus g surface Σ_g and two collections $\alpha = \{\alpha_1, \dots, \alpha_g\}$ and $\beta = \{\beta_1, \dots, \beta_g\}$ of g simple closed curves subject to the conditions:*

1. *The α -curves are mutually disjoint, so are the β -curves. The α -curves intersect the β -curves transversely at finitely many points.*
2. *The classes $\{[\alpha_1], \dots, [\alpha_g]\}$ are linearly independent in $H_1(\Sigma_g, \mathbb{Z})$. The same holds for $\{[\beta_1], \dots, [\beta_g]\}$.*

The 3-manifold Y can be reconstructed from a Heegaard diagram as follows:

- Attach 2-handles to the α and β curves.
- Thicken the 2-complex from the previous step to obtain a 3-manifold C with boundary. Conditions 1 and 2 from Definition 2.1 guarantee that the boundary of C is a disjoint union of two 2-spheres.
- Attach a 3-ball to each 2-sphere in the boundary of C . The resulting 3-manifold is Y .

To help us keep track of spin^c -structures on Y we need a bit more information than just a Heegaard diagram for Y .

DEFINITION 2.2. A pointed Heegaard diagram for Y is a quadruple $(\Sigma_g, \alpha, \beta, z)$ where $(\Sigma_g, \alpha, \beta)$ is a Heegaard diagram for Y and z (which we shall refer to as a basepoint) is a point taken from the complement of the α and β curves in Σ_g :

$$z \in \Sigma_g - \{\alpha_1 \cup \dots \cup \alpha_g \cup \beta_1 \cup \dots \cup \beta_g\}.$$

Given a pointed Heegaard diagram $(\Sigma_g, \alpha, \beta, z)$ for Y , let

$$Sym^g(\Sigma_g) = \frac{\Sigma_g \times \dots \times \Sigma_g}{S_g}$$

be the symmetric g -fold product of Σ_g defined as the quotient of the g -fold Cartesian product of Σ_g with itself moded out by the obvious permutation action of the symmetric group on g letters S_g . Even though the action of S_g is not free, $Sym^g(\Sigma_g)$ is a manifold of dimension $2g$. Moreover, it inherits a complex structure from a choice of a complex structure on Σ_g .

Let $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ be the g -dimensional tori in $Sym^g(\Sigma_g)$ defined as

$$\mathbb{T}_\alpha = \alpha_1 \times \dots \times \alpha_g, \quad \mathbb{T}_\beta = \beta_1 \times \dots \times \beta_g.$$

By conditions (1) and (2) from Definition 2.1 the two tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ intersect in finitely many points. It was shown in [23] that the basepoint $z \in \Sigma_g - \{\alpha \cup \beta\}$ gives rise to a map

$$\mathfrak{s}_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow Spin^c(Y)$$

which is substantially used in the definition of the Heegaard Floer groups.

With these preliminaries in place we define the chain complexes $\widehat{CF}(Y, \mathfrak{s})$, $CF^\pm(Y, \mathfrak{s})$ and $CF^\infty(Y, \mathfrak{s})$ as the free Abelian groups with generating sets

$$\begin{aligned} CF^\infty(Y, \mathfrak{s}) &= \{[x, i] \mid x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, i \in \mathbb{Z}, \mathfrak{s}_z(x) = \mathfrak{s}\}, \\ CF^-(Y, \mathfrak{s}) &= \{[x, i] \in CF^\infty(Y, \mathfrak{s}) \mid i < 0\}, \\ CF^+(Y, \mathfrak{s}) &= CF^\infty(Y, \mathfrak{s}) / CF^-(Y, \mathfrak{s}), \\ \widehat{CF}(Y, \mathfrak{s}) &= \text{Ker}(U : CF^+(Y, \mathfrak{s}) \rightarrow CF^+(Y, \mathfrak{s})). \end{aligned} \tag{2.1}$$

There is an action of the polynomial ring $\mathbb{Z}[U]$ on $CF^\infty(Y, \mathfrak{s})$ defined by $U \cdot [x, i] = [x, i - 1]$. This action descends to $CF^\pm(Y, \mathfrak{s})$ and is used in the definition of $\widehat{CF}(Y, \mathfrak{s})$. Intuitively one can think of $CF^+(Y, \mathfrak{s})$ as the free Abelian group generated by pairs $[x, i]$ with $i \geq 0$ while $\widehat{CF}(Y, \mathfrak{s})$ can be thought of as generated by $[x, 0]$, $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

The differential $\partial^\infty : CF^\infty(Y, \mathfrak{s}) \rightarrow CF^\infty(Y, \mathfrak{s})$ is defined by

$$\partial^\infty([x, i]) = \sum_{\substack{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ \phi \in \pi_2(x, y) \\ \mu(\phi) = 1}} \# \mathcal{M}(\phi) [y, i - n_z(\phi)]. \tag{2.2}$$

We proceed by explaining the meaning of the symbols from (2.2).

1. $\pi_2(x, y)$ is the set of homotopy classes of Whitney disks connecting x to y . The latter is a map $\phi : D^2 \rightarrow \text{Sym}^g(\Sigma_g)$ where D^2 is the closed unit disk in $\mathbb{C}$ and has the properties

$$\phi(i) = x, \quad \phi(-i) = y, \quad \phi(\ell) \subset \mathbb{T}_\alpha, \quad \phi(r) \subset \mathbb{T}_\beta$$

where (see Figure 1)

$$\ell = \partial D^2 \cap \{z \in \mathbb{C} \mid \Re(z) \leq 0\},$$

$$r = \partial D^2 \cap \{z \in \mathbb{C} \mid \Re(z) \geq 0\}.$$

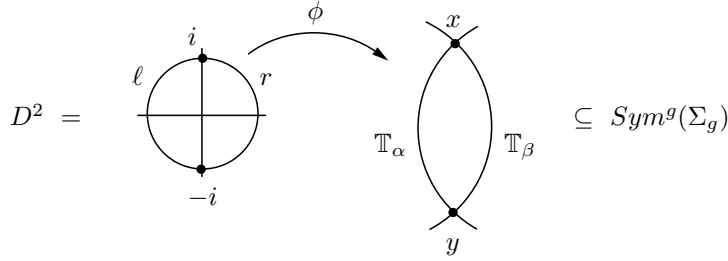


FIGURE 1. A Whitney disk.

2. $\mathcal{M}(\phi)$ is the moduli space of J -holomorphic Whitney disks in the homotopy class of ϕ . To make sense of the notion of being J -holomorphic we use the complex structure on D^2 induced by $\mathbb{C}$ and a complex structure on $\text{Sym}^g(\Sigma_g)$ coming from a complex structure on Σ_g . To make things generic one needs to perturb the complex structure on $\text{Sym}^g(\Sigma_g)$ slightly, an issue which we shall ignore in this discussion. Under generic conditions, $\mathcal{M}(\phi)$ is a compact oriented manifold.
3. $\mu(\phi)$ is the Maslov index of ϕ . The condition $\mu(\phi) = 1$ from (2.2) ensures that the dimension of $\mathcal{M}(\phi)$ is zero. Thus the latter becomes a finite collection of points each with weight 1 or -1 . By $\#\mathcal{M}(\phi)$ we mean the weighted count of elements of $\mathcal{M}(\phi)$.
4. $n_z(\phi)$ is the algebraic intersection number

$$n_z(\phi) = \phi(D^2) \cap (\{z\} \times \text{Sym}^{g-1}(\Sigma_g))$$

taking place in $\text{Sym}^g(\Sigma_g)$. When $\mathcal{M}(\phi) \neq \emptyset$ then

$$(2.3) \quad n_z(\phi) \geq 0$$

as intersection points of complex varieties always carry weight 1.

We are now also in a position to state the homological grading of generators $[x, i]$ from Definition 2.1.

DEFINITION 2.3. *The relative grading between 2 generators $[x, i]$, $[y, j]$, $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ of either of the chain complexes from Definition 2.1 is given by*

$$gr([x, i], [y, j]) = \mu(\phi) - n_z(\phi) + 2(i - j)$$

where ϕ is any Whitney disk from $\pi_2(x, y)$. It is shown in [23] that this definition is independent of the particular $\phi \in \pi_2(x, y)$ chosen.

THEOREM 2.4 (Ozsváth-Szabó, [23]). *The map ∂^∞ is a differential in that*

$$\partial^\infty \circ \partial^\infty = 0 \quad \text{and} \quad gr(\partial^\infty[x, i], [x, i]) = -1.$$

By condition (2.3), ∂^∞ induces a differential ∂^- on $CF^-(Y, \mathfrak{s})$ and thus by taking quotients also one on $CF^+(Y, \mathfrak{s})$, the latter is denoted by ∂^+ . Finally each of these three differentials is equivariant with respect to the action of $\mathbb{Z}[U]$ and so they induce a differential $\widehat{\partial}$ on $\widehat{CF}(Y, \mathfrak{s})$ as well.

DEFINITION 2.5. *The Heegaard Floer homology groups of a pair $(Y, \mathfrak{s})$, $\mathfrak{s} \in \text{Spin}^c(Y)$ are defined as*

$$HF^\infty(Y, \mathfrak{s}) = H_*(CF^\infty(Y, \mathfrak{s}), \partial^\infty),$$

$$HF^\pm(Y, \mathfrak{s}) = H_*(CF^\pm(Y, \mathfrak{s}), \partial^\pm),$$

$$\widehat{HF}(Y, \mathfrak{s}) = H_*(\widehat{CF}(Y, \mathfrak{s}), \widehat{\partial}).$$

THEOREM 2.6 (Ozsváth-Szabó, [23]). *The Heegaard Floer homology groups from Definition 2.5 are independent of the Heegaard diagram chosen in their definition. In particular, they are topological invariants of the spin^c -manifold $(Y, \mathfrak{s})$.*

2.2. *Properties of the Heegaard Floer groups.* The Heegaard Floer groups carry additional algebraic structures, each of which is also a topological invariant of $(Y, \mathfrak{s})$. These additional structures can often be used to show that $HF^\circ(Y_1, \mathfrak{s}_1)$ and $HF^\circ(Y_2, \mathfrak{s}_2)$ are not isomorphic even when their underlying Abelian groups are isomorphic. Here $\circ$ stands for any of $\widehat{}, +, -, \infty$.

Each group $HF^\circ(Y, \mathfrak{s})$ comes equipped with a relative $\mathbb{Z}_d$ -grading $gr : HF^\circ(Y, \mathfrak{s}) \times HF^\circ(Y, \mathfrak{s}) \rightarrow \mathbb{Z}_d$ where

$$(2.4) \quad d = \gcd\{\langle c_1(\mathfrak{s}), h \rangle \mid h \in H_2(Y; \mathbb{Z})\}.$$

Here $c_1(\mathfrak{s}) \in H^2(Y; \mathbb{Z})$ is the first Chern class of $\mathfrak{s}$. When $\mathfrak{s}$ is torsion (by which we mean that $c_1(\mathfrak{s})$ is torsion) the relative $\mathbb{Z}$ -grading gr lifts to an absolute $\mathbb{Q}$ -grading $\widetilde{gr} : HF^\circ(Y, \mathfrak{s}) \rightarrow \mathbb{Q}$ in the sense that

$$\widetilde{gr}(x) - \widetilde{gr}(y) = gr(x, y).$$

The definition of $CF^+(Y, \mathfrak{s})$ in terms of $CF^\infty(Y, \mathfrak{s})$ and $CF^-(Y, \mathfrak{s})$ and the definition of $\widehat{CF}(Y, \mathfrak{s})$ in terms of $CF^+(Y, \mathfrak{s})$ (see (2.1)) shows that there

are short exact sequences

$$\begin{aligned} 0 \rightarrow CF^-(Y, \mathfrak{s}) \rightarrow CF^\infty(Y, \mathfrak{s}) \xrightarrow{\pi} CF^+(Y, \mathfrak{s}) \rightarrow 0, \\ 0 \rightarrow \widehat{CF}(Y, \mathfrak{s}) \rightarrow CF^+(Y, \mathfrak{s}) \xrightarrow{U} CF^+(Y, \mathfrak{s}) \rightarrow 0, \end{aligned}$$

where $\pi : CF^\infty(Y, \mathfrak{s}) \rightarrow CF^+(Y, \mathfrak{s})$ is the quotient map from (2.1). These give rise to long exact sequences in homology

$$(2.5) \quad \begin{aligned} \dots \rightarrow HF^-(Y, \mathfrak{s}) \rightarrow HF^\infty(Y, \mathfrak{s}) \xrightarrow{\pi_*} HF^+(Y, \mathfrak{s}) \rightarrow \dots, \\ \dots \rightarrow \widehat{HF}(Y, \mathfrak{s}) \rightarrow HF^+(Y, \mathfrak{s}) \xrightarrow{U} HF^+(Y, \mathfrak{s}) \rightarrow \dots, \end{aligned}$$

relating the various Heegaard Floer homology groups. When $\mathfrak{s}$ is torsion the map π_* preserves the absolute $\mathbb{Q}$ -grading $\tilde{gr}$.

2.3. Cobordism induced maps. The Heegaard Floer homology groups fit into a TQFT framework: given a spin^c 4-manifold $(W, \mathfrak{t})$ with $\partial W = -Y_1 \sqcup Y_2$ (where $-Y$ is Y with the opposite orientation) there are induced group homomorphisms

$$F_{W, \mathfrak{t}}^\circ : HF^\circ(Y_1, \mathfrak{t}|_{Y_1}) \rightarrow HF^\circ(Y_2, \mathfrak{t}|_{Y_2})$$

where $\circ$ again stands for any of $\wedge, +, -, \infty$. When $\mathfrak{t}|_{Y_1}$ and $\mathfrak{t}|_{Y_2}$ are both torsion the degree shift of the map $F_{W, \mathfrak{t}}^\circ$ is

$$(2.6) \quad \deg F_{W, \mathfrak{t}}^\circ := \tilde{gr}(F_{W, \mathfrak{t}}^\circ(x)) - \tilde{gr}(x) = \frac{(c_1(\mathfrak{t}))^2 - 2e_W - 3\sigma_W}{4}$$

where e_W and σ_W are the Euler number and signature of W respectively and $x \in HF^\circ(Y_1, \mathfrak{t}|_{Y_1})$ is any homogeneous element.

PROPOSITION 2.7 (Ozsváth-Szabó, [20]). *When W is negative definite (i.e. $b_2^+(W) = 0$) the homomorphism $F_{W, \mathfrak{t}}^\infty$ is an isomorphism for all spin^c -structures $\mathfrak{t}$ on W .*

The exact sequences (2.5) are functorial under cobordism induced maps in the sense that one obtains the commutative diagram with exact rows (we only list the diagram for the second sequence in (2.5) with later applications in mind):

$$(2.7) \quad \begin{array}{ccccc} HF^-(Y_1, \mathfrak{s}_1) & \longrightarrow & HF^\infty(Y_1, \mathfrak{s}_1) & \xrightarrow{\pi} & HF^+(Y_1, \mathfrak{s}_1) \\ F_{W, \mathfrak{t}}^- \downarrow & & F_{W, \mathfrak{t}}^\infty \downarrow & & F_{W, \mathfrak{t}}^+ \downarrow \\ HF^-(Y_2, \mathfrak{s}_2) & \longrightarrow & HF^\infty(Y_2, \mathfrak{s}_2) & \xrightarrow{\pi} & HF^+(Y_2, \mathfrak{s}_2) \end{array}$$

In the above diagram $\mathfrak{s}_i$ stands for $\mathfrak{t}|_{Y_i}$.

2.4. *The correction terms for 3-manifolds.* Let Y be a rational homology sphere and let $\mathfrak{s} \in \text{Spin}^c(Y)$ be a spin^c -structure on Y .

DEFINITION 2.8. *The correction term $d(Y, \mathfrak{s})$ is defined to be*

$$d(Y, \mathfrak{s}) = \min\{\tilde{gr}(\pi_*(x)) \mid x \in HF^\infty(Y, \mathfrak{s})\}$$

where $\pi_* : HF^\infty(Y, \mathfrak{s}) \rightarrow HF^+(Y, \mathfrak{s})$ is the map from the second exact sequence in (2.5).

EXAMPLE 2.9. Consider S^3 with its unique spin-structure $\mathfrak{s}_0$. Recall from [23] that $HF^\infty(S^3, \mathfrak{s}_0) \cong \mathbb{Z}[U, U^{-1}]$ and $HF^+(S^3, \mathfrak{s}_0) \cong \mathbb{Z}[U^{-1}]$. The absolute grading on both groups is specified by $\tilde{gr}(U^k) = -2k$ and the map $\pi_* : HF^\infty(S^3, \mathfrak{s}_0) \rightarrow HF^+(S^3, \mathfrak{s}_0)$ is the obvious quotient map

$$\mathbb{Z}[U, U^{-1}] \rightarrow \frac{\mathbb{Z}[U, U^{-1}]}{U\mathbb{Z}[U]} \cong \mathbb{Z}[U^{-1}].$$

Thus π_* is surjective and therefore $d(S^3, \mathfrak{s}_0)$ is the lowest grading in $HF^+(S^3, \mathfrak{s}_0)$ which in turn is given by

$$(2.8) \quad d(S^3, \mathfrak{s}_0) = \tilde{gr}(U^0) = 0.$$

The correction terms enjoy a number of nice properties. Given $\mathfrak{s} \in \text{Spin}^c(Y)$ let $\bar{\mathfrak{s}}$ be the conjugate spin^c -structure. Likewise, let $-Y$ denote Y with its orientation reversed.

PROPOSITION 2.10 (Ozsváth-Szabó, [20]). *With the notation as above, the correction terms satisfy the properties*

$$\begin{aligned} d(Y, \bar{\mathfrak{s}}) &= d(Y, \mathfrak{s}), \\ d(-Y, \mathfrak{s}) &= -d(Y, \mathfrak{s}), \\ d(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) &= d(Y_1, \mathfrak{s}_1) + d(Y_2, \mathfrak{s}_2). \end{aligned}$$

Consider now two rational homology 3-spheres Y_1 and Y_2 equipped with spin^c -structures $\mathfrak{s}_i \in \text{Spin}^c(Y_i)$, $i = 1, 2$. Consider furthermore a cobordism $(W, \mathfrak{t})$ from $(Y_1, \mathfrak{s}_1)$ to $(Y_2, \mathfrak{s}_2)$ (i.e. a 4-manifold W with $\partial W = -Y_1 \sqcup Y_2$ and $\mathfrak{t}|_{Y_i} = \mathfrak{s}_i$) with the rational homology of a $S^3 \times [0, 1]$. Let $x_2 \in HF^\infty(Y_2, \mathfrak{s}_2)$ be an element with $\tilde{gr}(\pi_*(x_2)) = d(Y_2, \mathfrak{s}_2)$ where π_* is the map from (2.5). According to Proposition 2.7, the homomorphism $F_{W, \mathfrak{t}}^\infty : HF^\infty(Y_1, \mathfrak{s}_1) \rightarrow HF^\infty(Y_2, \mathfrak{s}_2)$ is an isomorphism. Let $x_1 \in HF^\infty(Y_1, \mathfrak{s}_1)$ be the unique preimage of x_2 under this map. The degree-shift formula (2.6) and the commutative diagram (2.7) show that

$$\tilde{gr}(\pi_*(x_2)) - \tilde{gr}(\pi_*(x_1)) = 0.$$

Since $d(Y_1, \mathfrak{s}_1) \leq \tilde{gr}(\pi_*(x_1))$ by definition and $d(Y_2, \mathfrak{s}_2) = \tilde{gr}(\pi_*(x_2))$ by choice of x_2 , the above equality becomes the inequality

$$(2.9) \quad d(Y_1, \mathfrak{s}_1) \leq d(Y_2, \mathfrak{s}_2).$$

Reversing the orientation on W and applying (2.9) once more shows that the opposite equality is also true and therefore

$$d(Y_1, \mathfrak{s}_1) = d(Y_2, \mathfrak{s}_2).$$

Suppose now that $Y_2 = S^3$. Then the above discussion and the result from Example 2.9 show that

$$d(Y_1, \mathfrak{s}_1) = 0 \quad \forall \mathfrak{s}_1 \in \text{Im}[Spin^c(W) \rightarrow Spin^c(Y_1)].$$

A rich source of examples 3-manifolds Y cobordant to S^3 via a cobordism W with the above properties are rational homology spheres Y bounding rational homology balls X . In this case one can take $W = X - B^4$ and the above translates to give

THEOREM 2.11. *Let Y be a rational homology 3-sphere which bounds a rational homology 4-ball X . Then $|H^2(Y; \mathbb{Z})| = n^2$ for some n (this follows easily from the universal coefficient theorem and the exact sequence of the pair (X, Y)) and there is a subgroup $\mathcal{H}$ of $H^2(Y; \mathbb{Z})$ of order n such that*

$$d(Y, \mathfrak{s}) = 0 \quad \forall \mathfrak{s} \in \mathcal{H}$$

under a suitable identification $Spin^c(Y) \cong H^2(Y; \mathbb{Z})$.

3. THE KNOT FLOER GROUPS

3.1. Definition of the knot Floer groups. The construction of knot invariants in the context of Heegaard Floer homology closely follows the construction of the Heegaard Floer groups for 3-manifolds. Thus before proceeding we recommend that the reader first familiarize her/himself with Section 2 and the notational conventions therein.

Let K be a nullhomologous knot in some 3-manifold Y . We will mainly choose Y to be S^3 but the definition of the invariants in this more general setting requires little extra work.

DEFINITION 3.1. *A doubly pointed Heegaard diagram for (Y, K) is a quintuple $(\Sigma_g, \alpha, \beta, z, w)$ where $(\Sigma_g, \alpha, \beta)$ is a Heegaard diagram for Y compatible with K in the sense that*

1. *The Heegaard diagram $(\Sigma_g, \alpha, \{\beta_2, \dots, \beta_g\})^1$ is a Heegaard diagram for the knot complement $Y - N(K)$ (where $N(K)$ is a tubular neighborhood of K).*
2. *The curve β_1 represents the meridian of K .*
3. *The points $z, w \in \Sigma_g - (\alpha \cup \beta)$ are push-offs of a point $m \in \beta_1$ in the normal directions², see Figure 2.*

¹One constructs a manifold from the diagram $(\Sigma_g, \alpha, \{\beta_2, \dots, \beta_g\})$ by attaching disks to the α and β -curves as in Section 2. Since there is one fewer β -curve than α -curves, the resulting manifold has a torus boundary.

²Which of the two push-offs of m should be called z and which w is determined by a choice of orientation of K and Σ_g , an issue which we shall neglect in our discussion.

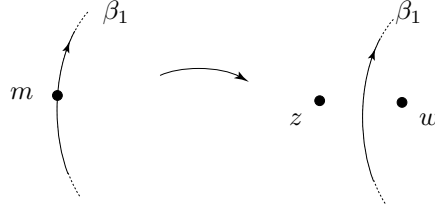


FIGURE 2. Creating the basepoints z and w by normal push-offs of a point m on the meridian β_1 of the knot K .

It is not hard to see that such diagrams always exist. The role of the basepoint m is similar to the role of z in the case of 3-manifolds. Namely, it is shown in [21] that m induces a map

$$\underline{\mathfrak{s}}_m : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow \text{Spin}^c(Y_0(K))$$

where $Y_0(K)$ is the 3-manifold obtained by 0-framed surgery on K in Y . Using this map we can now define a chain complex

$$(3.1) \quad \begin{aligned} CFK^\infty(Y, K, \mathfrak{t}) = \{ [x, i, j] \mid x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, i, j \in \mathbb{Z}, \\ \underline{\mathfrak{s}}_m(x) + 2(i - j)P.D.([\beta_1]) = \mathfrak{t} \} \end{aligned}$$

equipped with a differential ∂^∞

$$(3.2) \quad \partial^\infty([x, i, j]) = \sum_{\substack{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ \phi \in \pi_2(x, y) \\ \mu(\phi) = 1}} \# \mathcal{M}(\phi)[y, i - n_z(\phi), j - n_w(\phi)].$$

The meaning of all symbols is the same as in the explanation following (2.2) with the exception of $P.D.([\beta_1])$ from (3.1) which denotes the Poincaré dual of $[\beta_1]$ with β_1 thought of as a curve in $Y_0(K)$. In $Y_0(K)$, β_1 is an essential curve and so $P.D.([\beta_1]) \in H^2(Y_0(K); \mathbb{Z})$ is nonzero. With this understood, $\underline{\mathfrak{s}}_m + 2(i - j)P.D.([\beta_1])$ denotes the spin^c -structure obtained from $\underline{\mathfrak{s}}_m$ by tensoring it with the complex line bundle with first Chern class equal to $(i - j)P.D.([\beta_1])$.

As in the 3-manifold case, there is a number of interesting sub and quotient complexes of $CFK^\infty(Y, K, \mathfrak{t})$ to consider, leading to additional knot invariants. However, with applications to subsequent chapters in mind, we only single out one of them.

Let $\widehat{CFK}(Y, K, \mathfrak{t})$ be the free Abelian group generated by

$$\widehat{CFK}(Y, K, \mathfrak{t}) = \{ [x, 0, j] \in CFK^\infty(Y, K, \mathfrak{t}) \}$$

with differential $\widehat{\partial}$

$$(3.3) \quad \widehat{\partial}[x, 0, j] = \sum_{\substack{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ \phi \in \pi_2(x, y), \mu(\phi) = 1 \\ n_z(\phi) = 0 = n_w(\phi)}} \# \mathcal{M}(\phi)[y, 0, j].$$

THEOREM 3.2 (Ozsváth-Szabó, [21]). *The maps*

$$\begin{aligned} \partial^\infty : CFK^\infty(Y, K, \mathfrak{t}) &\rightarrow CFK^\infty(Y, K, \mathfrak{t}), \\ \widehat{\partial} : \widehat{CFK}(Y, K, \mathfrak{t}) &\rightarrow \widehat{CFK}(Y, K, \mathfrak{t}) \end{aligned}$$

defined by (3.2) and (3.3) respectively, are differentials and the homology groups

$$\begin{aligned} HFK(Y, K, \mathfrak{t}) &= H_*(CFK^\infty(Y, K, \mathfrak{t}), \partial^\infty), \\ \widehat{HFK}(Y, K, \mathfrak{t}) &= H_*\left(\widehat{CFK}(Y, K, \mathfrak{t}), \widehat{\partial}\right) \end{aligned}$$

are independent of the doubly pointed Heegaard diagram for (Y, K) chosen in their definition and are thus topological invariants of $(Y, K, \mathfrak{t})$.

When $Y = S^3$, $Y_0(K)$ is a homology $S^1 \times S^2$ and so $H^2(Y_0(K); \mathbb{Z}) \cong \mathbb{Z}$. We shall then write $\widehat{HFK}(K, j)$ to mean $\widehat{HFK}(S^3, K, \mathfrak{t}_j)$ for the spin^c -structure $\mathfrak{t}_j \in \text{Spin}^c(Y_0(K))$ corresponding to $j \in \mathbb{Z}$. More generally, observe that

$$(3.4) \quad \text{Spin}^c(Y_0(K)) \cong \text{Spin}^c(Y) \times \mathbb{Z}.$$

Under this correspondence $\mathfrak{t} \mapsto (\mathfrak{s}, j)$ where $\mathfrak{s} \in \text{Spin}^c(Y)$ is uniquely determined by the restriction of $\mathfrak{t}$ to $Y_0(K) - N(K) = Y - N(K)$ and $j = \frac{1}{2}\langle c_1(\mathfrak{t}), [\hat{F}] \rangle$ where $F \subseteq Y$ is a Seifert surface for K and $\hat{F}$ is the surface in $Y_0(K)$ obtained from F by capping it off with the core of the attaching 0-framed 2-handle. With this in mind we can write

$$\widehat{HFK}(Y, K, \mathfrak{s}, j) := \widehat{HFK}(Y, K, \mathfrak{t}).$$

The correspondence (3.4) also relates the map $\underline{\mathfrak{s}}_m$ with the map $\mathfrak{s}_z$. Under the identification (3.4) one gets

$$\underline{\mathfrak{s}}_m(x) \mapsto (\mathfrak{s}_z(x), j) \quad \forall x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$$

with j again given as $j = \frac{1}{2}\langle c_1(\underline{\mathfrak{s}}_m(x)), [\hat{F}] \rangle$.

As in the 3-manifold case, both of $HFK^\infty(Y, K, \mathfrak{s}, j)$ and $\widehat{HFK}(Y, K, \mathfrak{s}, j)$ come with a relative $\mathbb{Z}_d$ -grading gr (with d as in (2.4)) which, when $\mathfrak{s}$ is torsion, lifts to an absolute $\mathbb{Q}$ -grading $\widetilde{gr}$.

3.2. $\widehat{CFK}$ as a filtration on $\widehat{CF}$. Let K be a knot in S^3 and let $(\Sigma_g, \alpha, \beta, z, w)$ be a doubly pointed Heegaard diagram used to define $\widehat{CFK}(K, j)$. Let $(\Sigma_g, \alpha, \beta, z)$ be the corresponding pointed Heegaard diagram used to define $\widehat{CF}(S^3)$.

There is an obvious surjective projection map $\Pi : \oplus_j \widehat{CFK}(K, j) \rightarrow \widehat{CF}(S^3)$ given by

$$\Pi([x, 0, j]) = x.$$

The generator x itself remembers the corresponding j it came from. To see this let us define $\mathcal{F} : \widehat{CF}(S^3) \rightarrow \mathbb{Z}$ by

$$\mathcal{F}(x) = \frac{1}{2} \langle \mathfrak{s}_z(x), [\hat{F}] \rangle.$$

Then $x \in \widehat{CF}(S^3)$ is in the image of Π restricted to $\widehat{CFK}(K, \mathcal{F}(x))$.

The map $\mathcal{F}$ induces a filtration on $\widehat{CF}(S^3)$ in the sense that if y is a homogeneous term in the expression $\hat{\partial}(x)$ then $\mathcal{F}(y) \leq \mathcal{F}(x)$, cf. [21]. With this understood, the sets

$$(3.5) \quad \mathcal{F}(K, \ell) = \{x \in \widehat{CF}(S^3) \mid \mathcal{F}(x) \leq \ell\}$$

are subcomplexes of $\widehat{CF}(S^3)$ and the inclusion maps $\iota_K^\ell : \mathcal{F}(K, \ell) \rightarrow \widehat{CF}(S^3)$ are chain maps. Since $\widehat{CF}(S^3)$ is finitely generated it follows that

$$0 = \mathcal{F}(K, \ell) \subseteq \mathcal{F}(K, \ell + 1) \subseteq \dots \subseteq \mathcal{F}(K, m) = \widehat{CF}(S^3)$$

for some $\ell, m \in \mathbb{Z}$. Notice that for all ℓ small enough and for all m large enough the maps ι_K^ℓ and ι_K^m are the zero map and isomorphisms respectively.

The discussion from this section easily generalizes to the case of any null-homologous knot K in an arbitrary 3-manifold but we shall not use this in the remainder of the article. For more information consult [21].

4. THE FIRST INVARIANT: $\tau(K)$

Recall from Subsection 3.2 that a knot K in S^3 induces an increasing filtration $\mathcal{F}(K, \ell)$ on the chain complex $\widehat{CF}(S^3)$. Let $\iota_K^\ell : \mathcal{F}(K, \ell) \rightarrow \widehat{CF}(S^3)$ be the inclusion maps and recall that for ℓ large enough ι_K^ℓ are isomorphisms and for ℓ small enough ι_K^ℓ are the zero maps. Finally, recall that $\widehat{HF}(S^3) \cong \mathbb{Z}$.

Using this we are ready to make the definition:

DEFINITION 4.1. *With the notation as above, we define $\tau(K)$ for a knot K in S^3 to be the integer*

$$\tau(K) = \min \{ \ell \in \mathbb{Z} \mid (\iota_K^\ell)_* : H_*(\mathcal{F}(K, \ell)) \rightarrow \mathbb{Z} \text{ is nontrivial} \}.$$

Recall that $\mathcal{C}$ denotes the smooth knot concordance group.

THEOREM 4.2 (Ozsváth-Szabó, [24]). *The assignment $K \mapsto \tau(K)$ is a knot invariant which satisfies the following properties:*

1. τ induces a surjective group homomorphism $\tau : \mathcal{C} \rightarrow \mathbb{Z}$.
2. If K is an alternating knot then $\tau(K)$ equals $-\sigma(K)/2$ where $\sigma(K)$ is the signature of K .
3. The absolute value of $\tau(K)$ provides a lower bound on the smooth slice genus of K

$$|\tau(K)| \leq g_*(K).$$

4. If K_+ , K_- are two knots which only differ locally in the neighborhood of a single crossing as indicated in Figure 3, then

$$\tau(K_+) - 1 \leq \tau(K_-) \leq \tau(K_+).$$

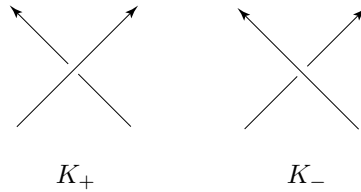


FIGURE 3. The knots K_+ and K_- differ only in the sign of one crossing.

Property 1 from Theorem 4.2 allows us to apply τ to questions of order of a knot K in $\mathcal{C}$:

COROLLARY 4.3. *If $\tau(K) \neq 0$ then K is of infinite order in $\mathcal{C}$.*

This is a rather strong obstruction to being of finite order. For example, of the 249 knots with up to 10 crossings, for all but 26 knots the knot concordance order is known and for all but one knot (namely 10_{141}) $\tau(K)$ is known. For the 222 knots where both the concordance order and τ are known, τ accurately predicts the order of K being infinity for all but the following 19 knots:

$$\{7_7, 8_1, 9_{34}, 9_{42}, 10_1, 10_{31}, 10_{32}, 10_{68}, 10_{71}, 10_{86}, \\ 10_{90}, 10_{96}, 10_{104}, 10_{107}, 10_{114}, 10_{122}, 10_{130}, 10_{136}, 10_{146}\}.$$

DEFINITION 4.4. *The unknotting number $u(K)$ of a knot K in S^3 is the smallest number of crossing reversals in any knot projection D of K that unknots K .*

It is not hard to see that $u(K)$ is bounded from below by $g_*(K)$ and thus by $|\tau(K)|$:

$$|\tau(K)| \leq g_*(K) \leq u(K).$$

Let $T_{p,q}$ denote the (p, q) -torus knot. The following is a famous conjecture made by John Milnor [18] which was first resolved in the affirmative by Peter Kronheimer and Tom Mrowka [10] using Donaldson gauge theory.

CONJECTURE 4.5. *Let p, q be relatively prime integers. Then the unknotting number of the (p, q) -torus knot $T_{p,q}$ is given by*

$$u(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

This conjecture can be proved in the framework of Heegaard Floer homology using the invariant τ . In [24] Ozsváth and Szabó prove the theorem

THEOREM 4.6 (Ozsváth-Szabó, [24]). *Let K be a knot in S^3 with the property that for some integer $p \geq 0$, p -framed surgery on K yields a lens space. Then $\tau(K)$ is the degree of the symmetrized Alexander polynomial $\Delta_K(t)$ of K .*

With a suitable chirality $(pq \pm 1)$ -framed surgery on $T_{p,q}$ yields a lens space. Thus $\tau(T_{p,q})$ is the degree of the symmetrized Alexander polynomial

$$\Delta_{T_{p,q}}(t) = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)} t^{-\frac{(p-1)(q-1)}{2}}.$$

It follows that $\tau(T_{p,q}) = (p-1)(q-1)/2 \leq u(T_{p,q})$. On the other hand, it is not too hard to find an unknotting of $T_{p,q}$ with $(p-1)(q-1)/2$ crossing changes showing that $u(T_{p,q}) = (p-1)(q-1)/2$ as conjectured by Milnor.

Another application of τ comes from C. Livingston who managed to construct examples of knots with trivial Alexander polynomial but with nonzero τ . It follows then from Theorem 4.2 and the discussion from Section 1 that:

THEOREM 4.7 (Livingston, [14]). *The kernel of $\Psi : \mathcal{C} \rightarrow \mathcal{C}_{top}$ contains a complemented subgroup isomorphic to $\mathbb{Z}$.*

Examples of such knots are the untwisted positive Whitehead double (see Definition 5.3 below) of the trefoil and of the pretzel knot $P(3, -5, -7)$. The latter represents a generator of the subgroup isomorphic to $\mathbb{Z}$ from Theorem 4.7.

5. THE SECOND INVARIANT: $\delta(K)$

Let K be a knot in S^3 and let Y_K be the double branched cover of S^3 with branching set K . All such 3-manifolds Y_K are rational homology spheres with

$$|H^2(Y; \mathbb{Z})| = |\det(K)| = |\Delta_K(-1)|$$

where $\Delta_K(t)$ is the symmetrized Alexander polynomial of K . In particular, it follows that $|H^2(Y; \mathbb{Z})|$ is always odd and has therefore no 2-torsion. This in turn implies that there is a unique spin-structure on Y_K which we denote by $\mathfrak{s}_0 \in \text{Spin}^c(Y_K)$.

DEFINITION 5.1. *Given a knot K in S^3 let $\delta(K)$ be twice the correction term of $(Y_K, \mathfrak{s}_0)$*

$$\delta(K) = 2d(Y_K, \mathfrak{s}_0).$$

This is clearly an invariant of the knot. It too, like τ , enjoys a number of nice properties.

THEOREM 5.2 (Manolescu-Owens, [19]). *The invariant δ from Definition 5.1 satisfies the following properties:*

1. δ induces a surjective group homomorphism $\delta : \mathcal{C} \rightarrow \mathbb{Z}$.
2. When K is alternating $\delta(K) = -\sigma(K)/2$ where $\sigma(K)$ is the signature of K .

This theorem is of course reminiscent of the analogous Theorem 4.2 for τ . Given this resemblance between the two invariants, one might suspect that perhaps they are just two descriptions of the same invariant. We shall see that this is not so.

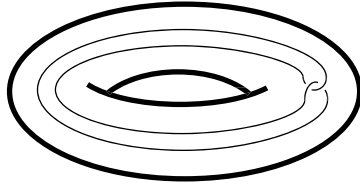


FIGURE 4. The satellite for the Whitehead double.

DEFINITION 5.3. *The untwisted Whitehead double $Wh(K)$ of a knot K is the satellite of the twisted unknot (as in Figure 4) with K as the companion knot. Figure 5 illustrates the case of K being the right-handed trefoil. See [9] for more information.*

THEOREM 5.4 (Manolescu-Owens, [19]). *Let $K_1 = T_{2,2m+1}$ and $K_2 = T_{2,2n+1}$ be two positive torus knots with $m \neq n$, $m, n \geq 1$ and let $K = Wh(K_1) \# (-Wh(K_2))$. Then $\Delta_K(t) = 1$, $\tau(K) = \sigma(K) = 0$ but $\delta(K) \neq 0$.*

The following easy corollary (compare to Theorem 1.3) is an improvement on Theorem 4.7.

COROLLARY 5.5 (Manolescu-Owens, [19]). *The kernel of $\Psi : \mathcal{C} \rightarrow \mathcal{C}_{top}$ contains a complemented subgroup isomorphic to $\mathbb{Z}^2$.*

Manolescu and Owens calculated $\delta(K)$ for an infinite family of torus knots which further underlines the differences between δ and τ, σ . The signature of the torus knots $T_{3,6n \pm 1}$ are given by

$$\sigma(T_{3,6n \pm 1}) = 8n$$

while their δ -invariant takes the values

$$\delta(T_{3,6n+\varepsilon}) = \begin{cases} 0 & ; \quad \varepsilon = 1 \\ -4 & ; \quad \varepsilon = -1 \end{cases}.$$

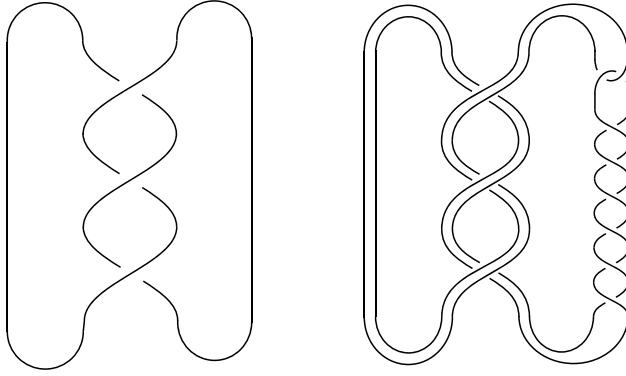


FIGURE 5. The untwisted Whitehead double of the right-handed trefoil. The 6 half-twists on the right have been added to ensure that the linking number is zero.

The values of $\tau(T_{3,6n\pm 1})$ can be deduced from the results of Section 4:

$$\tau(T_{3,6n+1}) = 6n, \quad \tau(T_{3,6n-1}) = 6n - 2.$$

6. THE THIRD INVARIANT: CORRECTION TERMS OF Y_K

To address the question of the possible existence of finite order knots in $\mathcal{C}$ we consider in this section yet another tool coming from Heegaard Floer homology – the correction terms of a certain double branched cover.

To start our discussion, let K be a knot and suppose that K has order n in $\mathcal{C}$. Let $K' = \#^n K$ be the n -fold connected sum of K with itself. Then K' is smoothly slice and therefore $\tau(K') = 0 = \delta(K')$. Since τ and δ are homomorphisms from $\mathcal{C}$ to $\mathbb{Z}$ we see that

$$0 = \tau(K') = \tau(\#^n K) = n \cdot \tau(K) \implies \tau(K) = 0$$

(and similarly $\delta(K) = 0$). We thus need a better strategy for testing whether K' is smoothly slice, one which cannot be “rolled back” to apply to K itself.

Towards that goal let $Y_{K'}$ and Y_K be the double branched covers of S^3 branched over K' and K respectively. Observe that $Y_{K'} = \#^n Y_K$ and therefore $H^2(Y_{K'}; \mathbb{Z}) \cong \oplus_{i=1}^n H^2(Y; \mathbb{Z})$ which in turn implies

$$\text{Spin}^c(Y_{K'}) \cong \times_{i=1}^n \text{Spin}^c(Y_K).$$

Let D be a slice disk for K' (i.e. a smoothly and properly embedded disk in the 4-ball B^4 with $\partial(B^4, D) = (S^3, K')$) and let $X_{K'}$ be the double branched cover of B^4 branched over D . Then $X_{K'}$ is a rational homology ball with $\partial X_{K'} = Y_{K'}$. Thus according to Theorem 2.11 there must be a subgroup $\mathcal{H}'$

of $H^2(Y_{K'}; \mathbb{Z})$ of order $|\det(K)|^{n/2}$ and with

$$d(Y_{K'}, \mathfrak{s}') = 0 \quad \forall \mathfrak{s}' \in \mathcal{H}'.$$

Let us write $\mathfrak{s}' = (\mathfrak{s}_1, \dots, \mathfrak{s}_n)$ with $\mathfrak{s}_i \in H^2(Y_K; \mathbb{Z})$. Proposition 2.10 and the above equation translate into

$$d(Y_K, \mathfrak{s}_1) + \dots + d(Y_K, \mathfrak{s}_n) = 0 \quad \forall \mathfrak{s}' = (\mathfrak{s}_1, \dots, \mathfrak{s}_n) \in \mathcal{H}'.$$

We summarize in

THEOREM 6.1 (Jabuka-Naik, [7]). *If K is of order n in $\mathcal{C}$ then there exists a subgroup $\mathcal{H}'$ of $\oplus_{i=1}^n H^2(Y_K; \mathbb{Z})$ of order $|\det(K)|^{n/2}$ with*

$$d(Y_K, \mathfrak{s}_1) + \dots + d(Y_K, \mathfrak{s}_n) = 0 \quad \forall (\mathfrak{s}_1, \dots, \mathfrak{s}_n) \in \mathcal{H}'.$$

Here Y_K is the double branched cover of S^3 branched over K .

Given a knot K it is a priori impossible to determine the group $\mathcal{H}'$ from Theorem 6.1. To use the obstruction one has to:

1. Determine all subgroups $\mathcal{H}'$ of $\oplus_{i=1}^n H^2(Y_K; \mathbb{Z})$ of order $|\det(K)|^{n/2}$.
2. Check if the equation $d(Y_K, \mathfrak{s}_1) + \dots + d(Y_K, \mathfrak{s}_n) = 0$ is satisfied for each $(\mathfrak{s}_1, \dots, \mathfrak{s}_n) \in \mathcal{H}'$.
3. If the equation from Step 2 is not satisfied by any group $\mathcal{H}'$ from Step 1, then K cannot be of order n in $\mathcal{C}$.

As in illustration of these techniques, consider the table of the 26 knots among the 249 knots of up to 10 crossings whose order in $\mathcal{C}$ is not yet known:

TABLE 1.

Knot K	Order of K	Knot K	Order of K	Knot K	Order of K
8 ₁₃	≥ 4	10 ₂₆	≥ 4	10 ₁₀₂	≥ 4
8 ₁₇	≥ 4	10 ₂₈	≥ 4	10 ₁₀₉	≥ 4
9 ₁₄	≥ 4	10 ₃₄	≥ 4	10 ₁₁₅	≥ 4
9 ₁₉	≥ 4	10 ₅₈	≥ 4	10 ₁₁₈	≥ 4
9 ₃₀	≥ 4	10 ₆₀	≥ 4	10 ₁₁₉	≥ 4
9 ₃₃	≥ 4	10 ₇₉	≥ 4	10 ₁₃₅	≥ 4
9 ₄₄	≥ 4	10 ₈₁	≥ 4	10 ₁₅₈	≥ 2
10 ₁₀	≥ 4	10 ₈₈	≥ 4	10 ₁₆₄	≥ 4
10 ₁₃	≥ 4	10 ₉₁	≥ 4		

The existing lower bounds on the concordance orders of these knots have been determined by C. Livingston and S. Naik [16, 17] and A. Tamulis [26]. Using Theorem 6.1 one can prove

THEOREM 6.2 (Jabuka-Naik, [7]). *The concordance order of K from the following list of 14 knots*

$$\left\{ \begin{array}{l} 8_{13}, 9_{14}, 9_{19}, 9_{33}, 9_{44}, 10_{13}, 10_{26}, 10_{28}, \\ 10_{34}, 10_{58}, 10_{60}, 10_{102}, 10_{119}, 10_{135} \end{array} \right\}$$

is at least 6.

While in principle the obstruction from Theorem 6.1 goes beyond the scope of this application, it is sheer computational complexity that prevents us from checking the vanishing of the obstruction for larger n . Theorem 6.2 was arrived at by using a MATHEMATICA script to verify the obstruction from Theorem 6.1. As an example, the calculation for the knot 10_{115} (whose determinant is 109, the largest of any knot in Table 1) took 3 hours and 20 minutes on an Intel Core Duo processor with 1.66 MHz. It is conceivable that as technology pushes forward the speed of our personal computers, the lower bound on the concordance order of the knots from Theorem 6.2 may grow as well.

REFERENCES

- [1] A. J. Casson and C. McA. Gordon, *On slice knots in dimension three*, in: Algebraic and geometric topology (Stanford, 1976), Part 2, Proc. Sympos. Pure Math. **XXXII**, Amer. Math. Soc., Providence, 1978, 39–53.
- [2] A. J. Casson and C. McA. Gordon, *Cobordism of classical knots*, Progr. Math. **62**, Birkhäuser Boston, Boston, MA, 1986, 181–199.
- [3] H. Endo, *Linear independence of topologically slice knots in the smooth cobordism group*, Topology Appl. **63** (1995), 257–262.
- [4] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–453.
- [5] M. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, **39**, Princeton University Press, Princeton, 1990.
- [6] R. Gompf *Smooth concordance of topologically slice knots*, Topology **25** (1986), 353–373.
- [7] S. Jabuka and S. Naik, *Order in the concordance group and Heegaard Floer homology*, Geom. Topol. **11** (2007).
- [8] B. J. Jiang, *A simple proof that the concordance group of algebraically slice knots is infinitely generated*, Proc. Amer. Math. Soc. **83** (1981), 189–192.
- [9] L. H. Kauffman, *On knots*, Annals of Mathematics Studies **115**, Princeton University Press, Princeton, 1987.
- [10] P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces I*, Topology **32** (1993), 773–826.
- [11] J. Levine, *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** (1969), 229–244.
- [12] J. Levine, *Invariants of knot cobordism*, Invent. Math. **8** (1969), 98–110.
- [13] C. Livingston, *Order 2 algebraically slice knots*, in: Proceedings of the Kirbyfest (Berkeley, 1998), Geom. Topol. Monogr. **2**, Geom. Topol. Publ., Coventry, 1999, 335–342.
- [14] C. Livingston, *Computations of the Ozsváth-Szabó knot concordance invariant*, Geom. Topol. **8** (2004), 735–742.

- [15] C. Livingston, *Slice knots with distinct Ozsváth-Szabó and Rasmussen invariants*, preprint, math.GT/0602631.
- [16] C. Livingston and S. Naik, *Obstructing four-torsion in the classical knot concordance group*, J. Differential Geom. **51** (1999), 1–12.
- [17] C. Livingston and S. Naik, *Knot concordance and torsion*, Asian J. Math. **5** (2001), 161–167.
- [18] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies **61**, Princeton University Press, Princeton, 1968.
- [19] C. Manolescu and B. Owens, *A concordance invariant from the Floer homology of double branched covers*, preprint, math.GT/0508065.
- [20] P. Ozsváth and Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), 179–261.
- [21] P. Ozsváth and Z. Szabó, *Holomorphic disks and knot invariants*, Adv. Math. **186** (2004), 58–116.
- [22] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) **159** (2004), 1159–1245.
- [23] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), 1027–1158.
- [24] P. Ozsváth and Z. Szabó, *Knot Floer homology and the four-ball genus*, Geom. Topol. **7** (2003), 615–639.
- [25] J. Rasmussen, *Khovanov homology and the slice genus*, preprint, math.GT/0402131.
- [26] A. Tamulis, *Knots of ten or fewer crossings of algebraic order 2*, J. Knot Theory Ramifications **11** (2002), 211–222.

S. Jabuka

Department of Mathematics and Statistics, 084

University of Nevada Reno

89557 Reno, NV

USA

E-mail: jabuka@unr.edu

Received: 27.9.2006.

Revised: 29.1.2007.