SPACES ASSOCIATED WITH STIEFEL MANIFOLDS

By I. M. JAMES

[Received 11 January 1958.—Read 16 January 1958]

THIS article concludes the survey of properties of the family of Stiefel manifolds which we began with (3) and (4). It is primarily concerned with two families of auxiliary spaces, called stunted projective and stunted quasi-projective spaces, whose algebraic topology not only makes an interesting study in itself but also illuminates the classical problem of determining which Stiefel manifolds admit cross-sections. The theory is fairly self-contained, although there are important connexions and resemblances of technique as regards previous papers in this series. The auxiliary spaces are defined in the introductory section and our main results are stated in the following one.

1. Introduction

Let K denote the field of real numbers, complex numbers, or quaternions. We refer to K as the basic field, and we denote its dimension over the reals by d. Thus $d = 1, 2$, or 4 according as K is real, complex, or quaternionic. Let K_m, where† $m \geqslant 1$, denote the m-dimensional right vector-space over K, with a fixed basis and the classical inner product. We regard the elements of K_m as column-vectors

$$u = [x_1,..., x_m] \quad (x_1,..., x_m \in K),$$

so that linear transformations are determined by matrices acting on the left. As usual we identify u with $[x_1,..., x_m, 0]$, so that K_m is embedded in K_{m+1}. We also identify $K_p \times K_q$ with K_m, where $p+q = m$, by means of the relation:

$$([x_1,..., x_p], [y_1,..., y_q]) = [x_1,..., x_p, y_1,..., y_q].$$

Let S_m denote the subspace of K_m which consists of unit vectors. Topologically, S_m is a sphere of dimension $dm-1$.

Let O_m denote the topological group of orthonormal $m \times m$ matrices, i.e. those which leave the inner product invariant. Thus O_m is the standard orthogonal, unitary, or symplectic group according as the basic field is real, complex, or quaternionic. We define O_0 to be the subgroup of O_1 which

† When K_0 is mentioned it means the empty set.

consists of the identity. Of course O_m acts transitively on S_m, and the transformations of the point

$$a_m = [0,\dots, 0, 1]$$

determine a fibre mapping $\rho\colon O_m \to S_m$. We identify O_{m-1} with $\rho^{-1}(a_m)$ in the usual way, so as to correspond with the embedding of K_{m-1} in K_m. Write $S_1 = S$. We identify O_1 with S by means of ρ.

Projective spaces are defined as follows. Let $m \geqslant 1$. Represent S as a transformation group of S_m, by means of multiplication on the right. Let P_m denote the orbit space which is obtained by identifying points of S_m with their transforms under the action of S. Then P_m is a projective $(m-1)$-space over K. In particular, P_1 is a point-space. Since $S_m \subset S_{m+1}$ we have a natural embedding of P_m in P_{m+1}, and the complement is a cell of dimension dm. Write $c^0 = P_1$ and write $c_m = P_{m+1} - P_m$. Then we obtain the well-known cellular decomposition:

$$P_m = c^0 \cup c_1 \cup \dots \cup c_{m-1}.$$

The homology and cohomology of P_m can be determined by means of this cell-structure and Poincaré duality.

Let L denote the set of elements $x \in K$ such that $x+\bar{x} = 0$, where $\bar{x}$ denotes the conjugate of x. In the real case, for example, zero is the only such element. Let $m \geqslant 0$. By the equator of S_{m+1} we mean the subspace T_m which consists of unit vectors

$$u = [x_0, x_1,\dots, x_m] \quad (x_0, x_1,\dots \in K)$$

such that $x_0 \in L$. Define $\theta(u)$ to be the orthonormal $m \times m$ matrix $||a_{ij}||$ in which

$$a_{ij} = \delta_{ij} - 2x_i(1+x_0)^{-2}\bar{x}_j \quad (1 \leqslant i, j \leqslant m).$$

We recognize the map $\theta\colon T_m \to O_m$ as one which arises in connexion with the representation of O_{m+1} as a principal O_m-bundle over S_{m+1} (see §§ 23, 24 of (**6**)). Consider the space Q'_m which is the image of T_m under θ. We can obtain Q'_m from T_m by identifying the vectors in each set of transforms

$$[\bar{x}x_0 x,\ x_1 x,\dots, x_m x] \quad (x \in S),$$

and also identifying all vectors having $x_1 = \dots = x_m = 0$. Notice that Q'_m contains the point O_0 except when the basic field is real. Write

$$Q_m = Q'_m \cup O_0 \subset O_m.$$

Because of analogies with the definition of P_m I propose to call Q_m the quasi-projective $(m-1)$-space over K. The construction is similar to that studied by Yokota in (**13**), (**14**). Notice that $Q_0 = O_0$ and that

$$Q_1 = O_1 = S.$$

Also notice that the embeddings of T_m in T_{m+1} and O_m in O_{m+1} are compatible with θ. Hence we have a natural embedding of Q_m in Q_{m+1}, and the complement is a cell of dimension $d(m+1)-1$. Write $e^0 = Q_0$ and write

$$e_m = Q_{m+1} - Q_m.$$

Then we obtain the cellular decomposition:

$$Q_m = e^0 \cup e_0 \cup \ldots \cup e_{m-1}.$$

Conjugation by elements of S is a trivial operation unless the basic field is quaternionic. Hence it follows that Q_m is isomorphic to P_m in the real case, to the suspension of P_m in the complex case. These observations are due essentially to J. H. C. Whitehead (**11**) in the real case, to Yokota (**13**) in the complex case. Indeed, Yokota once announced that Q_m is isomorphic to the threefold suspension of P_m in the quaternionic case, but this was withdrawn in (**14**).

We have now defined the group space O_m, the projective space P_m, and the quasi-projective space Q_m. Let $m \geqslant k \geqslant 1$. Our definitions have been arranged so that

$$O_{m-k} \subset O_m, \qquad P_{m-k} \subset P_m, \qquad Q_{m-k} \subset Q_m,$$

where $m \neq k$ in the projective case. We define the Stiefel manifold $O_{m,k}$ to be the factor space of O_m by O_{m-k} (using left cosets); we define the stunted projective space $P_{m,k}$ to be the complex obtained from P_m by identifying P_{m-k} with c^0; and we define the stunted quasi-projective space $Q_{m,k}$ to be the complex obtained from Q_m by identifying Q_{m-k} with e^0. Notice that $P_{m,m}$ is not defined.† Also notice that

$$O_m = O_{m,m}, \qquad P_m = P_{m,m-1}, \qquad Q_m = Q_{m,m}.$$

We have the cellular decompositions:

$$P_{m,k} = c^0 \cup c_{m-k} \cup \ldots \cup c_{m-1},$$
$$Q_{m,k} = e^0 \cup e_{m-k} \cup \ldots \cup e_{m-1}.$$

It is easy to check that the inclusion of Q_m in O_m maps $Q_{m,k}$ homeomorphically into $O_{m,k}$, and so we regard $Q_{m,k}$ as embedded in $O_{m,k}$. Procedures for extending the cell-structure of $Q_{m,k}$ over $O_{m,k}$ are described in (**11**), (**13**), and (**14**), but these are unnecessary for present purposes.

Let $k > l \geqslant 1$. Since $O_{m-l} \subset O_m$ we obtain an embedding of $O_{m-l,k-l}$ in $O_{m,k}$. Since $O_{m-k} \subset O_{m-l}$ we obtain a projection

$$\rho\colon O_{m,k} \to O_{m,l}.$$

Thus $O_{m,k}$ is expressed as a fibre bundle over $O_{m,l}$ with fibre $O_{m-l,k-l}$. Similarly we embed

$$P_{m-l,k-l} \subset P_{m,k}, \qquad Q_{m-l,k-l} \subset Q_{m,k};$$

† Except where special emphasis is required I let it be tacitly understood that $m > k \geqslant 1$ in the case of $P_{m,k}$ while $m \geqslant k \geqslant 1$ in the case of $O_{m,k}$ and $Q_{m,k}$.

and we define projections

$$p\colon P_{m,k} \to P_{m,l}, \qquad q\colon Q_{m,k} \to Q_{m,l}.$$

Thus $P_{m,l}$ is represented as the complex obtained from $P_{m,k}$ by identifying $P_{m-l,k-l}$ with c^0, and $Q_{m,l}$ is represented as the complex obtained from $Q_{m,k}$ by identifying $Q_{m-l,k-l}$ with e^0. Notice that q and ρ agree on their common domain.

An ordered orthonormal set of k vectors in K_m is called a k-frame. We can regard the vectors as constituting a matrix with m rows and k columns. Elements of $O_{m,k}$ are represented as k-frames by taking the last k column-vectors from each matrix of O_m. In particular, $O_{m,1}$ is identified with S_m. The standard projection

$$\rho\colon O_{m,k} \to O_{m,l}$$

is obtained by suppressing the first $k-l$ vectors of each k-frame. Consider the case $l = 1$. By a *cross-section* of $O_{m,k}$ I mean, as in (4), a map of S_m into $O_{m,k}$ which is a right-inverse of ρ. By the covering homotopy theorem, $O_{m,k}$ admits a cross-section if and only if ρ admits a homotopy right-inverse.

To obtain $P_{m,k}$ from S_m we identify the vectors in each set of transforms

$$[x_1 x, \dots, x_m x] \quad (x \in S),$$

and we also identify all vectors having $x_{m-k+1} = \dots = x_m = 0$. It is easy to check that $P_{m,1} = S'_m$, where S'_m is a sphere of dimension $d(m-1)$. I describe $P_{m,k}$ as *reducible* if the standard projection

$$p\colon P_{m,k} \to S'_m$$

admits a homotopy right-inverse. Recall that $P_{m,k}$ is obtained from $P_{m-1,k-1}$ by attaching c_{m-1}. It follows that $P_{m,k}$ is reducible if and only if the attaching map of this cell is inessential. Take cohomology with coefficients in the ring of integers modulo 2. The cohomology of P_m forms a truncated polynomial ring, so that the class carried by c_{m-1} is equal to the product of the classes carried by c_{m-k} and c_{k-1}. By naturality, therefore, it follows that $P_{m,k}$ cannot be reducible unless $m \geqslant 2k$ or $k = 1$.

To obtain $Q_{m,k}$ from† T_m we identify the vectors in each set of transforms

$$[\bar{x}x_0 x, x_1 x, \dots, x_m x] \quad (x \in S),$$

and we also identify all vectors having $x_{m-k+1} = \dots = x_m = 0$. It is easy to check that $Q_{m,1} = O_{m,1}$, and so $Q_{m,1} = S_m$. I describe $Q_{m,k}$ as *reducible* if the standard projection

$$q\colon Q_{m,k} \to S_m$$

admits a homotopy right-inverse. Recall that $Q_{m,k}$ is obtained from $Q_{m-1,k-1}$ by attaching e_{m-1}. It follows that $Q_{m,k}$ is reducible if and only if

† In the real case when $m = k$ it is necessary to adjoin the point e^0.

the attaching map of this cell is inessential. Because ρ is an extension of q it follows that $O_{m,k}$ admits a cross-section if $Q_{m,k}$ is reducible.

We observe once more that conjugation by elements of S is a trivial operation unless the basic field is quaternionic. Hence $Q_{m,k}$ is isomorphic to $P_{m,k}$ in the real case, to the suspension of $P_{m,k}$ in the complex case. In the real and complex cases, therefore, reducibility of $P_{m,k}$ implies reducibility of $Q_{m,k}$; in the real case, the converse is also true. In any case we have the relation

$$H_{r+1}(P_{m,k}) \approx H_{r+d}(Q_{m,k}),$$

which is true for homology with arbitrary coefficients. The reader is asked to bear in mind throughout what follows, the particular structure of the homology groups of the auxiliary spaces.

2. Statement of results

Two spaces are said to belong to the same S-type when there exist numbers i and j such that the i-fold suspension of the one space belongs to the same homotopy type as the j-fold suspension of the other. Consider the problem of classifying stunted projective and stunted quasi-projective spaces by S-type. Our main object in what follows is to link this classification to the problem of deciding whether certain Stiefel manifolds admit cross-sections. The connexion is provided by the condition of reducibility, or rather by the corresponding notion in S-theory, which I call the condition of S-reducibility.

Let $A_{m,k}$ denote $P_{m,k}$ or $Q_{m,k}$, and let w_{m-1} correspondingly denote c_{m-1} or e_{m-1}, so that $A_{m,k}$ is formed by attaching w_{m-1}. I describe $A_{m,k}$ as *S-reducible* if the attaching map of this cell lies in the zero S-class, i.e. if its r-fold suspension is inessential for sufficiently large values of r. Of course, reducibility implies S-reducibility. In order for $A_{m,k}$ to be S-reducible it is necessary and sufficient that the r-fold suspension of the projection from $A_{m,k}$ to $A_{m,1}$ should admit a homotopy right-inverse, for large enough r; in other words, that the r-fold suspension of w_{m-1} should carry a spherical cycle. Hence S-reducibility is an invariant of the S-type. That is to say, if two of the spaces

$$P_{m,k}, \qquad Q_{m,k}, \qquad P_{n,k}, \qquad Q_{n,k},$$

belong to the same S-type then either both the spaces are S-reducible or both are not.

Since $Q_{m,k}$ is isomorphic to $P_{m,k}$ in the real case, to the suspension of $P_{m,k}$ in the complex case, we immediately obtain

THEOREM (2.1). *Exclude the quaternionic case. Let $m > k \geqslant 1$. Then $P_{m,k}$ and $Q_{m,k}$ belong to the same S-type. Hence $P_{m,k}$ is S-reducible if and only if $Q_{m,k}$ is S-reducible.*

We shall see later that (2.1) cannot be extended so as to include the quaternionic case unless, of course, we have $k = 1$.

The following three theorems constitute our main contribution to the classification problem. The proofs will be found in § 8 below, where they appear as applications of more general results.

THEOREM (2.2). *Suppose that $P_{m,k}$ is S-reducible. Let $n \geqslant k$. Then $Q_{n,k}$ and $P_{m+n,k}$ belong to the same S-type.*

THEOREM (2.3). *Suppose that $Q_{m,k}$ is S-reducible. Let $n > k$. Then $P_{n,k}$ and $P_{m+n,k}$ belong to the same S-type.*

THEOREM (2.4). *Suppose that $Q_{m,k}$ is S-reducible. Let $n \geqslant k$. Then $Q_{n,k}$ and $Q_{m+n,k}$ belong to the same S-type.*

Because of (2.1) these theorems nearly coincide with each other when the basic field is real or complex. In the quaternionic case when $k \geqslant 3$, however, the hypothesis of (2.3) and (2.4) can be fulfilled, for suitable values of m, but never that of (2.2). The justification for this assertion appears in (2.7) and (2.9) below. But before we pursue such questions there is a further result to be stated. As an application of (2.4) above we shall prove

THEOREM (2.5). *There exists a number r, which may depend on k but not on m, such that the r-fold suspension of $Q_{m,k}$ is a retract of the r-fold suspension of $O_{m,k}$.*

Let r_k denote the least value of r which satisfies the requirements of (2.5). I emphasize that r_k must be independent of m. Certainly $r_1 = 0$, and it is easy to prove, by means of homology theory, that $r_k \geqslant 1$ if $k \geqslant 2$. Moreover, by using techniques developed in (**5**) it can be shown that $r_2 > d$, where d means the same as before. It would be interesting to know some more about the values of r_k. For example, is it true that r_k is bounded ? And how is r_{k+1} related to r_k ?

It is proved in (**4**) that there exist values of m, for given k, such that $O_{m,k}$ admits a cross-section. Let the greatest common factor of all such values be denoted by q_k. Information about these numbers q_k will be found in the introduction to (**4**). A complication arises in the real case where there is a possibility that some values of k are irregular, in a certain sense which is sufficiently well indicated by (2.6) below. It is not known whether any irregular values exist; certainly k is regular if $k \leqslant 9$. The relevant results from (**4**) can be conveniently summarized as

THEOREM (2.6). *In order for $O_{m,k}$ to admit a cross-section it is necessary and sufficient that m be a positive multiple of q_k and, in the real case, that k be regular or $m > q_k$.*

In § 8 below we use (2.4), (2.5), and (2.6) to prove

THEOREM (2.7). *In order for $Q_{m,k}$ to be S-reducible it is necessary and sufficient that m be a positive multiple of q_k.*

It is most satisfactory that (2.7) avoids making any exception of the real case. We use (2.7) to prove

THEOREM (2.8). *Let $m, k \geqslant 1$. Suppose that $Q_{n,k}$ and $Q_{m+n,k}$ belong to the same S-type for all sufficiently large values of n. Then $m \geqslant k$, and $Q_{m,k}$ is S-reducible.*

To prove (2.8) we choose n to be a sufficiently large multiple of q_k. Then $Q_{n,k}$ is S-reducible, by (2.7), and so $Q_{m+n,k}$ is S-reducible, because the property is an invariant of the S-type. Hence m is a multiple of q_k, by (2.7), from which (2.8) follows at once. The above result is in the nature of a converse to (2.4), so that in one sense our original purpose of classifying stunted quasi-projective spaces has been achieved. But the converse of (2.4) is untrue if we only consider individual values of n, as we shall see in the discussion which follows (2.10) below.

It might be expected that (2.7) would have an analogue for stunted projective spaces. However, we shall prove

THEOREM (2.9). *Let the basic field be quaternionic and let $k \geqslant 3$. Then $P_{m,k}$ and $Q_{n,k}$ never belong to the same S-type. Also $P_{m,k}$ is never S-reducible.*

The second assertion of (2.9) is a consequence of the first assertion, with m replaced by $m+n$, and of (2.2) above. If the basic field is real or complex, or if $k \geqslant 3$, (2.1) and (2.9) effectively dispose of the problem of determining when $P_{m,k}$ is S-reducible. It only remains for us to investigate the quaternionic case when $k = 2$. For this a direct approach is feasible, and we achieve a complete analysis as follows.

Let S^q denote the standard q-sphere and let G_q denote the stable homotopy group of the q-stem. A map of S^{n+q} into S^n determines first an element of $\pi_{n+q}(S^n)$ and then, by iterated suspension, an element of G_q. In what follows we shall be concerned with the 3-stem, regarding which there is a convenient account in (**8**). It is known that G_3 is cyclic of order 24 and is generated by γ, the element determined by the Hopf map of S^7 to S^4.

Let the basic field be quaternionic. We write $P''_m = P_{m,2}$ $(m \geqslant 3)$ and $Q''_m = Q_{m,2}$ $(m \geqslant 2)$. Thus, P''_m is obtained by attaching a $(4m-4)$-cell to a $(4m-8)$-sphere, and Q''_m is obtained by attaching a $(4m-1)$-cell to a $(4m-5)$-sphere. These attaching maps determine† elements α_m, $\beta_m \in G_3$, respectively. We shall prove that

$$\left.\begin{array}{ll}\text{(a)} & \pm\,\alpha_m = (m-2)\gamma,\\ \text{(b)} & \pm\,\beta_m = m\gamma.\end{array}\right\} \tag{2.10}$$

† There is a question of orientations here, which accounts for the alternative signs in (2.10) below. For present purposes it is unnecessary to settle these.

The first of these relations is immediate when $m = 3$, because P''_3 is the quaternionic projective plane. Let us show how the second relation can be deduced from the first. Since $24\gamma = 0$ we obtain from (2.10 a) that P''_{26} is S-reducible. Hence Q''_m belongs to the same S-type as P''_{m+26}, by (2.2). Therefore $\beta_m = \pm\alpha_{m+26}$, and so (2.10 b) follows from (2.10 a). We prove (2.10 a) in § 10 below.

The following conclusions are obtained immediately from (2.10). First, P''_m and P''_n belong to the same S-type if and only if

$$m-2 \equiv \pm(n-2) \pmod{24}.$$

Secondly, Q''_m and Q''_n belong to the same S-type if and only if

$$m \equiv \pm n \pmod{24}.$$

Thirdly, P''_m and Q''_n belong to the same S-type if and only if

$$m-2 \equiv \pm n \pmod{24}.$$

Fourthly, P''_m is S-reducible if and only if $m \equiv 2 \pmod{24}$. Fifthly and lastly, Q''_m is S-reducible if and only if $m = 0 \pmod{24}$. By (2.6) and (2.7) this last conclusion is equivalent to†

COROLLARY (2.11). *The quaternionic Stiefel manifold $O_{m,2}$ admits a cross-section if and only if m is a positive multiple of* 24.

Finally, some remarks are due concerning the possibility of using the algebra of Cayley numbers (octonions) in place of the basic field. It is pointed out at the end of (4) that an octonionic Stiefel manifold $O_{m,2}$ can be defined. Similarly, one can certainly define octonionic $P_{m,2}$ and $Q_{m,2}$. The Cayley projective plane, for example, is $P_{3,2}$. It would not be difficult to include these extra cases in our discussion, but it seems preferable to deal with them separately on another occasion.

3. The join construction

Let I denote the unit interval, parametrized from 0 to 1. Take spaces A and B, which may be empty. We define the join of A with B to be the space $A * B$ which is obtained from the disjoint union $A \cup (A \times B \times I) \cup B$ by identifying

$$(a, b, 0) = a, \qquad (a, b, 1) = b,$$

for all $a \in A$, $b \in B$. We generally omit to write in the identification map, so that the image of (a, b, t) in $A * B$ is denoted by the same expression, where $t \in I$. Notice that $A * B$ is homeomorphic to $B * A$ under the transformation:

$$(a, b, t) \to (b, a, 1-t).$$

We refer to this as 'inverting the order of the join operation'. The notion

† The result which follows was announced in (4) without proof.

of join is functorial, and applies to maps as well as spaces. If A' and B' are closed subspaces of A and B, respectively, we embed $A' * B'$ in $A * B$ by means of the join of the inclusion maps.

Now let A and B be CW-complexes. We extend their cell-structure over $A * B$ by taking the joins of cells of A with cells of B. Either suppose that both A and B are locally countable, or else suppose that one of them is locally finite. Then $A * B$ is a CW-complex. If A' and B' are subcomplexes of A and B, respectively, then $A' * B'$ is a subcomplex of $A * B$. Consider the cellular homology groups, with integral coefficients. We have a natural direct-sum decomposition:

$$\sum H_p(A, H_q(B)) \approx H_r(A * B),$$

where the summation extends over all pairs of positive integers p, q such that $p+q+1 = r$. The value of the isomorphism on each summand is given by the chain-mapping:

$$C_p(A) \otimes C_q(B) \to C_r(A * B),$$

which corresponds to the operation of joining cells. A direct proof of this result is straightforward, but it is easiest to derive it from the corresponding result for the singular theory (see § 2 of (**10**)).

4. The intrinsic maps

Our theory is based on the construction of a pair of maps:

$$\left.\begin{aligned} f\colon O_{m,k} * P_{n,k} &\to P_{m+n,k} \quad (n > k) \\ g\colon O_{m,k} * Q_{n,k} &\to Q_{m+n,k} \quad (n \geqslant k) \end{aligned}\right\}, \tag{4.1}$$

where $m \geqslant k \geqslant 1$. I call these the intrinsic maps, although it is not clear how they are related to the previous intrinsic map

$$h\colon O_{m,k} * O_{n,k} \to O_{m+n,k}$$

constructed in § 2 of (**3**). The definition of f and g proceeds as follows.

Let $a \in O_{m,k}$, so that a determines a linear transformation of K_k into K_m. We use a to define a series of maps

$$\alpha\colon S_n \times I \to S_{m+n},$$

where $n = k+1, k+2,\dots$. Let $u \in S_n$ and let $t \in I$. Write $\phi = \frac{1}{2}\pi t$ and write $u = (u_1, u_2)$, where $u_1 \in K_{n-k}$, $u_2 \in K_k$. We define $\alpha(u,t) = u'$, where

$$u' = (u_1, au_2 \cos\phi, u_2 \sin\phi).$$

Notice that $\alpha(ux, t) = u'x$ if $x \in S$, and that $u' \in S_{m+n-k}$ if $u \in S_{n-k}$. By passing to equivalence classes, therefore, we obtain from α a map

$$\beta\colon P_{n,k} \times I \to P_{m+n,k},$$

which ultimately depends on a. Let $b \in P_{n,k}$. Then $\beta(b,t)$ is independent of b when $t = 0$, of a when $t = 1$. Define f, in (4.1), to be the map in which

$$(a, b, t) \to \beta(b, t).$$

The definition of g is similar. Let $\theta\colon T_r \to O_r$ mean the same as in the introduction, where T_r denotes the equator of S_{r+1}. Let $n \geqslant k$, and consider the map

$$\alpha\colon S_{n+1} \times I \to S_{m+n+1},$$

which is one of the series defined above. Let $v' = \alpha(v,t)$, where $v \in T_n$. Then $v' \in T_{m+n}$, and hence $v' \in T_{m+n-k}$ if $v \in T_{n-k}$. It is easy to check that $\theta(v')$ depends on $\theta(v)$ rather than on v itself. By passing to equivalence classes, therefore, we obtain from α a map†

$$\gamma\colon Q_{n,k} \to Q_{m+n,k},$$

which ultimately depends on a. Let $c \in Q_{n,k}$. Then $\gamma(c,t)$ is independent of c when $t = 0$, of a when $t = 1$. Define g, in (4.1), to be the map in which

$$(a, c, t) \to \gamma(c, t).$$

When $k = 1$ all the spaces involved in (4.1) are spheres, and the degrees of f and g are easily determined. We obtain

LEMMA (4.2). *Both f and g are homotopy equivalences when $k = 1$.*

5. Naturality relations

The purpose of this section is to examine the behaviour of intrinsic maps in relation to various projections and inclusions. Let us concentrate attention on the stunted projective spaces and remark, once and for all, that the corresponding relations for stunted quasi-projective spaces are proved by similar methods.

Let $k > l \geqslant 1$. Consider the projections

$$\rho,\ \bar{\rho}\colon O_{m,k} \to O_{m,k-l},$$

where ρ takes the last $k-l$ vectors of each k-frame and $\bar{\rho}$ takes the first $k-l$. The standard projection defined in the introduction is ρ. Let σ, $\bar{\sigma}$ denote the joins of ρ, $\bar{\rho}$, respectively, with the identity map on $P_{n-l,k-l}$, as shown in the following diagram.

$$\begin{array}{ccc} O_{m,k} * P_{n-l,k-l} & \xrightarrow{\bar{f}} & P_{m+n,k} \\ {\scriptstyle\sigma}\downarrow{\scriptstyle\bar{\sigma}} & & \uparrow{\scriptstyle j} \\ O_{m,k-l} * P_{n-l,k-l} & \xrightarrow[f']{} & P_{m+n-l,k-l} \end{array}$$

In the above diagram, j is the inclusion map, f' is the intrinsic map, and $\bar{f}$ is obtained by restriction from the intrinsic map of $O_{m,k} * P_{n,k}$ into $P_{m+n,k}$.

† In the real case when $n = k$ it is necessary to add that $\gamma(e^0, t) = e^0$.

It is easy to check that $\bar{f} = jf'\bar{\sigma}$. Also $\rho \simeq \bar{\rho}$, by (1.3) of (**3**), and so $\sigma \simeq \bar{\sigma}$. Therefore

$$\bar{f} \simeq jf'\sigma. \tag{5.1}$$

A similar relation holds in the case of g.

Next, let τ denote the join of the standard projection from $O_{m,k}$ to $O_{m,l}$ with the standard projection from $P_{n,k}$ to $P_{n,l}$, as shown in the following diagram.

$$\begin{array}{ccc} O_{m,k} * P_{n,k} & \xrightarrow{f} & P_{m+n,k} \\ \tau \downarrow & & \downarrow p \\ O_{m,l} * P_{n,l} & \xrightarrow[f'']{} & P_{m+n,l} \end{array}$$

In the above diagram, p denotes the standard projection and f, f'' are intrinsic maps. I assert that

$$pf \simeq \xi f''\tau, \tag{5.2}$$

where ξ denotes a certain homeomorphism of $P_{m+n,l}$ onto itself. A similar relation holds in the case of g. The definition of ξ is unimportant apart from the proof of (5.2), which proceeds as follows.

The first step is to associate a homotopy

$$\alpha_s: S_n \times I \to S_{m+n} \quad (s \in I)$$

with each point $a \in O_{m,k}$. Let a' denote the submatrix formed by the first $k-l$ columns of a, and let a'' denote the submatrix formed by the remaining l columns. Then a', a'' determine linear transformations of K_{k-l}, K_l, respectively, into K_m. Let $u \in S_n$ and let $t \in I$. Write $u = (u_1, u_2, u_3)$, where $u_1 \in K_{n-k}$, $u_2 \in K_{k-l}$, $u_3 \in K_l$. Write $\phi = \frac{1}{2}\pi t$ and $\phi_s = \frac{1}{2}\pi(1-s+st)$, so that $\phi_0 = \frac{1}{2}\pi$ and $\phi_1 = \phi$. We define $\alpha_s(u,t) = u'_s$, where

$$u'_s = (u_1,\, a'u_2 \cos\phi_s + a''u_3 \cos\phi,\, u_2 \sin\phi_s,\, u_3 \sin\phi).$$

Notice that $\alpha_s(ux,t) = u'_s x$ if $x \in S$, and that $u'_s \in S_{m+n-l}$ if $u \in S_{n-k}$. When we pass to equivalence classes, therefore, we obtain a homotopy

$$\beta_s: P_{n,k} \times I \to P_{m+n,l}.$$

Let $b \in P_{n,k}$. Then $\beta_s(b,t)$ is independent of b when $t = 0$, of a when $t = 1$. Consider the homotopy

$$f_s: O_{m,k} * P_{n,k} \to P_{m+n,l}$$

in which $(a,b,t) \to \beta_s(b,t)$. Let ξ denote the homeomorphism of $P_{m+n,l}$ onto itself which is induced by the transformation

$$(v_1, v_2, v_3, v_4) \to (v_1, v_3, v_2, v_4),$$

where $v_1 \in K_{n-k}$, $v_2 \in K_{k-l}$, $v_3 \in K_m$, and $v_4 \in K_l$. Then $f_0 = \xi f''\tau$ and $f_1 = pf$. This proves (5.2).

Finally, let p' denote the join of the identity map on $O_{m-l,k-l}$ with the standard projection from $P_{n,k}$ to $P_{n,k-l}$, as shown in the following diagram.

$$\begin{array}{ccc} O_{m-l,k-l} * P_{n,k} & \xrightarrow{\bar{f}} & P_{m+n,k} \\ {\scriptstyle p'}\downarrow & & \uparrow{\scriptstyle j} \\ O_{m-l,k-l} * P_{n,k-l} & \xrightarrow[f']{} & P_{m+n-l,k-l} \end{array}$$

In the above diagram j is the inclusion map, f' is the intrinsic map, and $\bar{f}$ is obtained by restriction from the intrinsic map of $O_{m,k} * P_{n,k}$ into $P_{m+n,k}$. Thus j means the same as in (5.1) but f' and $\bar{f}$ have new meanings. We shall define a homeomorphism ζ of $P_{m+n,k}$ onto itself, and a homeomorphism η of $P_{n,k}$ onto itself, with the following property. Let

$$\eta' \colon O_{m-l,k-l} * P_{n,k} \to O_{m-l,k-l} * P_{n,k}$$

denote the join with η of the identity map on $O_{m-l,k-l}$. Then

$$\zeta \bar{f} \eta' \simeq j f' p'. \tag{5.3}$$

A similar relation holds in the case of g.

The first step in the proof of (5.3) is to associate a homotopy

$$\alpha'_s \colon S_n \times I \to S_{m+n} \quad (s \in I)$$

with each point $a \in O_{m-l,k-l}$, regarded as a linear transformation of K_{k-l} into K_{m-l}. Let $u \in S_n$ and let $t \in I$. Write $u = (u_1, u_2, u_3)$, where $u_1 \in K_{n-k}$, $u_2 \in K_{k-l}$, $u_3 \in K_l$. Also write $\phi = \frac{1}{2}\pi t$ and $\phi_s = \frac{1}{2}\pi s t$, so that $\phi_0 = 0$ and $\phi_1 = \phi$. We define α'_s by

$$\alpha'_s(u,t) = (u_1, au_2 \cos\phi, u_3 \cos\phi_s, u_2 \sin\phi, u_3 \sin\phi_s).$$

By passing to equivalence classes we obtain from α'_s a homotopy

$$\beta'_s \colon P_{n,k} \times I \to P_{m+n,k}.$$

Let $b \in P_{n,k}$. Then $\beta'_s(b,t)$ is independent of b when $t = 0$, of a when $t = 1$. Consider the homotopy

$$f'_s \colon O_{m-l,k-l} * P_{n,k} \to P_{m+n,k}$$

in which $(a,b,t) \to \beta'_s(b,t)$. Define η to be the homeomorphism of $P_{n,k}$ onto itself which is induced by the transformation

$$(u_1, u_2, u_3) \to (u_1, u_3, u_2).$$

Define ζ to be the homeomorphism of $P_{m+n,k}$ onto itself which is induced by the transformation

$$(v_1, v_2, v_3, v_4) \to (v_1, v_3, v_2, v_4),$$

where $v_1 \in K_{n-k}$, $v_2 \in K_l$, $v_3 \in K_{m-l}$, and $v_4 \in K_k$. Then $f'_0 = \zeta j f' p' \eta'$ and $f'_1 = \bar{f}$. Hence (5.3) follows at once. It can further be shown that $\zeta j \simeq j$ so that ζ can be dropped from (5.3), but this simplification does not assist any of the present applications.

6. Applications of the theory

Let A be a CW-complex and let A' denote the complex† $A * S_m$, where $S_m = Q_{m,1}$. Consider maps

$$A' \xrightarrow{u} A * O_{m,k} \xrightarrow{\rho} A',$$

where u is arbitrary and ρ denotes the join of the identity on A with the standard projection from k-frames to 1-frames. I describe u as an A-section of $O_{m,k}$ if and only if $\rho u \simeq 1$. An A-section certainly exists if A is contractible or if $O_{m,k}$ admits a cross-section.

Let F denote the composition

$$A' * P_{n,k} \xrightarrow{\bar{u}} A * O_{m,k} * P_{n,k} \xrightarrow{\bar{f}} A * P_{m+n,k},$$

where $\bar{u}$ denotes the join of u with the identity on $P_{n,k}$ and $\bar{f}$ denotes the join of the identity on A with the intrinsic map f. The main purpose of this section is to prove

THEOREM (6.1). *Suppose that u is an A-section of $O_{m,k}$. In the real case, further suppose that m is even or that $k = 1$. Then the map*

$$F\colon A' * P_{n,k} \to A * P_{m+n,k}$$

which u determines is a homotopy equivalence.

We prove (6.1) by induction on k. When $k = 1$ both u and f are homotopy equivalences, hence $\bar{u}$ and $\bar{f}$ are homotopy equivalences, and hence F is one too. Let $k \geqslant 2$. Suppose that (6.1) is true for all values of k less than the given one. Define

$$A * O_{m,k-l} \xleftarrow{\rho'} A * O_{m,k} \xrightarrow{\rho''} A * O_{m,l},$$

where $1 \leqslant l < k$, by taking the join of the identity on A with appropriate standard projections. Write $u' = \rho' u$, $u'' = \rho'' u$. Then u' is an A-section of $O_{m,k-l}$, and u'' is an A-section of $O_{m,l}$. Hence, by hypothesis of induction, the corresponding maps F' and F'' are homotopy equivalences, as shown in the following diagram.

$$\begin{array}{ccccc} A' * P_{n-l,k-l} & \xrightarrow{i} & A' * P_{n,k} & \xrightarrow{p} & A' * P_{n,l} \\ \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F''} \\ A * P_{m+n-l,k-l} & \xrightarrow[j]{} & A * P_{m+n,k} & \xrightarrow[q]{} & A * P_{m+n,l} \end{array}$$

In the above diagram, i and j are inclusions while p and q are defined by taking joins of appropriate identity maps with standard projections. It follows almost immediately from (5.1) and (5.2) that

$$\left.\begin{array}{ll} \text{(a)} & Fi \simeq jF' \\ \text{(b)} & qF \simeq F''p \end{array}\right\}, \qquad (6.2)$$

† The cell-structure of a sphere consists of a basepoint and its complement.

where F''' denotes the composition of F'' with a certain homeomorphism of $A * P_{m+n,l}$ onto itself.

Now pass to homology with integral coefficients, and consider the following diagram of induced homomorphisms, where $r \geqslant 1$.

$$\begin{tikzcd} H_r(A' * P_{n-l,k-l}) \arrow[r, "i_*"] \arrow[d, "F'_*"] & H_r(A' * P_{n,k}) \arrow[r, "p_*"] \arrow[d, "F_*"] & H_r(A' * P_{n,l}) \arrow[d, "F'''_*"] \\ H_r(A * P_{m+n-l,k-l}) \arrow[r, "j_*"'] & H_r(A * P_{m+n,k}) \arrow[r, "q_*"'] & H_r(A * P_{m+n,l}) \end{tikzcd} \tag{6.3}$$

Recall that $P_{n,l}$ is obtained from $P_{n,k}$ by identifying $P_{n-l,k-l}$ with a point. Hence it follows that the kernel of p_* coincides with the image of i_*. Similarly, the kernel of q_* coincides with the image of j_*. Also (6.3) is commutative, by (6.2); that is to say

$$\left.\begin{array}{lr} \text{(a)} & F_* i_* = j_* F'_* \\ \text{(b)} & q_* F_* = F'''_* p_* \end{array}\right\}. \tag{6.4}$$

Both F'_* and F'''_* are isomorphisms, since F' and F''' are homotopy equivalences. The next stage in the proof of (6.1) is to deduce that F_* is an isomorphism too. When i_*, j_* are monomorphisms and p_*, q_* are epimorphisms we can reach this conclusion at once by an appeal to the 'five' lemma. In the real case, however, this will not work, and two lemmas are required to overcome the difficulty.

We begin by proving

Lemma (6.5). *Suppose that $l = 1$ in (6.3). Then the image of q_* coincides with the image of $F'''_* p_*$.*

Every A-section of $O_{m,1}$ is homotopic to the identity map. Hence and from (4.2) it follows that F''', when $l = 1$, is homotopic to the join of the identity on A with a homotopy equivalence $S_m * P_{n,1} \to P_{m+n,1}$. Therefore F'''_*, as well as p_* and q_*, can be computed in terms of the expression for the homology of the join which we gave in § 3 above. After computation and, in the real case, use of the hypothesis that m is even, we arrive at (6.5). A similar argument proves

Lemma (6.6). *Suppose that $l = k-1$ in (6.3). Then the kernel of i_* coincides with the kernel of $j_* F'_*$.*

We are now ready to prove that F_* is an isomorphism. First consider (6.3) with $l = 1$. Let $x \in H_r(A * P_{m+n,k})$. Then $q_*(x) = q_* F_*(y)$, by (6.4 b) and (6.5), where $y \in H_r(A' * P_{n,k})$. Hence, by exactness, $x - F_*(y)$ is contained in the image of j_*. Since F'_* is an isomorphism there exists an element $z \in H_r(A' * P_{n-1,k-1})$ such that $x - F_*(y) = j_* F'_*(z)$. Hence $x = F_*(y + i_* z)$, by (6.4 a), which proves that F_* is onto. Secondly, consider (6.3) with $l = k-1$. Let x' be an element of $H_r(A' * P_{n,k})$ such that $F_*(x') = 0$. Then $F'''_* p_*(x') = q_* F_*(x') = 0$, by (6.4 b), and so $p_*(x') = 0$, since F'''_* is an isomorphism. Hence $x' = i_*(y')$, where

$$y' \in H_r(A' * P_{n-k+1,1}).$$

But $j_* F'_*(y') = F_* i_*(y') = 0$, by (6.4 a), and so $i_*(y') = 0$, by (6.6). This completes the proof that F_* is an isomorphism. Hence F is a homotopy equivalence, by Theorem 3 of (**12**), since $A' * P_{n,k}$ and $A * P_{m+n,k}$ are simply-connected. This concludes our inductive proof of (6.1).

Of course there is a similar result for stunted quasi-projective spaces, as follows. Let G denote the composition

$$A' * Q_{n,k} \xrightarrow{\bar{u}} A * O_{m,k} * Q_{n,k} \xrightarrow{\bar{g}} A * Q_{m+n,k},$$

where $\bar{u}$ now denotes the join of u with the identity on $Q_{n,k}$ and $\bar{g}$ denotes the join of the identity on A with the intrinsic map g. Then we have

Theorem (6.7). *Suppose that u is an A-section of $O_{m,k}$. In the real case, further suppose that m is even or that $k = 1$. Then the map*

$$G: A' * Q_{n,k} \to A * Q_{m+n,k}$$

which u determines is a homotopy equivalence.

The proof of (6.7) is omitted since it is similar in all respects to the proof of (6.1).

7. Further applications

Let A mean the same as in the previous section and let A'' denote the complex $A * S'_m$, where $S'_m = P_{m,1}$ ($m \geqslant 2$). Consider maps

$$A'' \xrightarrow{v} A * P_{m,k} \xrightarrow{p} A'',$$

where v is arbitrary and p denotes the join of the identity on A with the standard projection. I describe v as an A-section of $P_{m,k}$ if and only if $pv \simeq 1$. An A-section certainly exists if A is contractible or if $P_{m,k}$ is reducible. Furthermore, $P_{m,k}$ is S-reducible if and only if $P_{m,k}$ admits an A-section where A is a sphere.

Consider the map $\quad f': P_{m,k} * O_{n,k} \to P_{m+n,k}$

which is obtained from the intrinsic map

$$f: O_{n,k} * P_{m,k} \to P_{m+n,k}$$

by inverting the order of the join operation. Let F' denote the composition

$$A'' * O_{n,k} \xrightarrow{\bar{v}} A * P_{m,k} * O_{n,k} \xrightarrow{\bar{f}'} A * P_{m+n,k},$$

where $\bar{v}$ denotes the join of v with the identity on $O_{n,k}$ and $\bar{f}'$ denotes the join of the identity on A with f'. Let $\tilde{F}$ denote the restriction of F' to $A'' * Q_{n,k}$. Then we have

THEOREM (7.1). *Suppose that v is an A-section of $P_{m,k}$. In the real case, further suppose that m is even or that $k = 1$. Then the map*

$$\tilde{F}: A'' * Q_{n,k} \to A * P_{m+n,k}$$

which v determines is a homotopy equivalence.

As in the proof of (6.1), we make an induction on k and study the homomorphisms of homology groups induced by $\tilde{F}$. We use (5.2) and (5.3) so as to obtain the relations analogous to (6.4). The details of the proof of (7.1) are omitted because it is very similar to the proof of (6.1).

Since F' is an extension of $\tilde{F}$ over $A'' * O_{n,k}$ we obtain from (7.1):

COROLLARY (7.2). *Suppose that $P_{m,k}$ admits an A-section. In the real case, further suppose that m is even or that $k = 1$. Then $A'' * Q_{n,k}$ is a retract of $A'' * O_{n,k}$.*

Let A' denote the complex $A * S_m$, as before, where $S_m = Q_{m,1}$ $(m \geqslant 1)$. Consider maps

$$A' \xleftarrow{w} A * Q_{m,k} \xrightarrow{q} A',$$

where w is arbitrary and q denotes the join of the identity on A with the standard projection. I describe w as an A-section of $Q_{m,k}$ if and only if $qw \simeq 1$. An A-section certainly exists if A is contractible or if $Q_{m,k}$ is reducible. Furthermore, $Q_{m,k}$ is S-reducible if and only if $Q_{m,k}$ admits an A-section where A is a sphere. Let w' denote the inclusion of w in $A * O_{m,k}$. Then w is an A-section of $Q_{m,k}$ if and only if w' is an A-section of $O_{m,k}$.

Consider the map $\qquad g': Q_{m,k} * O_{n,k} \to Q_{m+n,k}$

which is obtained from the intrinsic map

$$g: O_{n,k} * Q_{m,k} \to Q_{m+n,k}$$

by inverting the order of the join operation. Let G' denote the composition

$$A' * O_{n,k} \xrightarrow{\bar{w}} A * Q_{m,k} * O_{n,k} \xrightarrow{\bar{g}'} A * Q_{m+n,k},$$

where $\bar{w}$ denotes the join of w with the identity on $O_{n,k}$ and $\bar{g}'$ denotes the join of the identity on A with g'. Let $\tilde{G}$ denote the restriction of G' to $A' * Q_{n,k}$. Then we have

THEOREM (7.3). *Suppose that w is an A-section of $Q_{m,k}$. In the real case, further suppose that m is even or that $k = 1$. Then the map*

$$\tilde{G}: A' * Q_{n,k} \to A * Q_{m+n,k}$$

which w determines is a homotopy equivalence.

The proof of (7.3) is analogous to the proof of (7.1). I do not know whether $\tilde{G}$ is related to the map G determined by w' as in (6.7). Since G' is an extension of $\tilde{G}$ over $A' * O_{n,k}$ we at once deduce

COROLLARY (7.4). *Suppose that $Q_{m,k}$ admits an A-section. In the real case, further suppose that m is even or that $k = 1$. Then $A' * Q_{n,k}$ is a retract of $A' * O_{n,k}$.*

8. Proof of the main theorems

We apply the results of the previous two sections so as to obtain proofs of the main theorems stated in § 2. Recall that d denotes the dimension of the basic field over the reals. We begin by proving

LEMMA (8.1). *The pair $(O_{m,k}, Q_{m,k})$ is t-connected, where*†
$$t = 2d(m-k)+3(d-1).$$

The lemma is obvious if $k = 1$, since $Q_{m,1}$ and $O_{m,1}$ coincide. Let $k \geqslant 2$, and suppose the lemma to be true for all values of k less than the given one. Consider the following diagram, where i_*, j_* are injections and q_*, ρ_* are induced by the standard projections.

$$\begin{array}{ccc}
\pi_r(Q_{m,k}, Q_{m-1,k-1}) & \xrightarrow{q_*} & \pi_r(Q_{m,1}) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle j_*} \\
\pi_r(O_{m,k}, O_{m-1,k-1}) & \xrightarrow[\rho_*]{} & \pi_r(O_{m,1})
\end{array}$$

Of course j_* is an isomorphism and $\rho_* i_* = j_* q_*$. Also ρ_* is an isomorphism since ρ is a fibre map. Now it follows from the Blakers–Massey theorem, as stated in (1.25) of (**8**), that q_* is an isomorphism if $r < t$, an epimorphism if $r = t$. Therefore the same is true of i_*. Consider next the following diagram, which represents the injection of the homotopy sequence of the pair $(Q_{m,k}, Q_{m-1,k-1})$ into that of the pair $(O_{m,k}, O_{m-1,k-1})$.

$$\begin{array}{ccccccccc}
\dots \rightarrow & \pi_r(Q_{m-1,k-1}) & \rightarrow & \pi_r(Q_{m,k}) & \rightarrow & \pi_r(Q_{m,k}, Q_{m-1,k-1}) & \rightarrow \dots \\
& \downarrow{\scriptstyle i''_*} & & \downarrow{\scriptstyle i'_*} & & \downarrow{\scriptstyle i_*} & \\
\dots \rightarrow & \pi_r(O_{m-1,k-1}) & \rightarrow & \pi_r(O_{m,k}) & \rightarrow & \pi_r(O_{m,k}, O_{m-1,k-1}) & \rightarrow \dots
\end{array}$$

By the inductive hypothesis i''_*, as well as i_*, is an isomorphism if $r < t$, an epimorphism if $r = t$. Hence it follows from the 'five' lemma that i'_* is an isomorphism if $r < t$, an epimorphism if $r = t$. In other words, the pair $(O_{m,k}, O_{m-1,k-1})$ is t-connected. This completes our inductive proof of (8.1).

The lemma is used to prove

THEOREM (8.2). *In order for $Q_{m,k}$ to be reducible it is necessary and sufficient that $O_{m,k}$ admit a cross-section and, in the real case, that $m \geqslant 2k$ or $k = 1$.*

† Homology considerations show that the stated value of t cannot be improved except, of course, when $k = 1$.

The necessity of these conditions has been proved in the course of the introduction. Their sufficiency is obvious when $k = 1$ and in the complex case when $m = k = 2$, because $Q_{m,k}$ is reducible. Let $k \geqslant 2$, therefore, and let $m \geqslant 3$ in the complex case. Suppose that $O_{m,k}$ admits a cross-section. Then $m \geqslant 2k$, by hypothesis in the real case, by (20.6) of (**1**) and (1.1) of (**4**) otherwise. Hence and from (8.1) it follows that every map of S_m into $O_{m,k}$ is homotopic to a map with values in $Q_{m,k}$. In particular, a cross-section can be deformed into $Q_{m,k}$, and so $Q_{m,k}$ is reducible. This completes the proof of (8.2).

Now we are ready to prove the main theorems, as listed in § 2 above. Let us begin with the classification theorems. Suppose that $Q_{m,k}$ is S-reducible. Then $Q_{m,k}$ admits an A-section, where A is a sphere, and so $O_{m,k}$ admits an A-section, by inclusion. Homological considerations show that m is even in the real case when $k \geqslant 2$. Hence $A' * P_{n,k}$ belongs to the same homotopy type as $A * P_{m+n,k}$, by (6.1), and $A' * Q_{n,k}$ belongs to the same homotopy type as $A * Q_{m+n,k}$, by (6.7). This proves (2.3) and (2.4). A similar argument based on (7.1) is used to prove (2.2).

The retraction theorems of § 7 have various applications, such as the following theorem, which is what we need for proving (2.5).

THEOREM (8.3). *Suppose that $Q_{m,k}$ is reducible. Let $n \geqslant k$. Then the dm-fold suspension of $Q_{n,k}$ is a retract of the dm-fold suspension of $O_{n,k}$.*

We obtain (8.3) at once from (7.4) as the special case when A is empty. By (2.6) and (8.2) there exist values of m, for given k, such that $Q_{m,k}$ is reducible. Choose the least such value, and write $r = dm$. Then (8.3) yields (2.5), with n in place of m.

Now let us prove (2.7). Let m be a multiple of q_k. Then $O_{2m,k}$ and $O_{3m,k}$ admit cross-sections, by (2.6). Hence $Q_{2m,k}$ and $Q_{3m,k}$ are reducible, by (8.2), since $q_k \geqslant k$. Also $Q_{m,k}$ and $Q_{3m,k}$ belong to the same S-type, by (2.4) with m replaced by $2m$. Therefore $Q_{m,k}$ is S-reducible. Conversely, suppose that $Q_{m,k}$ is S-reducible. Then $Q_{2m,k}$ and $Q_{3m,k}$ are S-reducible, by (2.4), and hence are reducible, by the Blakers–Massey suspension theorem, as stated in (2.6) of (**8**). Therefore $O_{2m,k}$ and $O_{3m,k}$ admit cross-sections by (8.2). Hence it follows from (2.6) that m is a multiple of q_k. This completes the proof of (2.7).

We conclude this section by proving a classification theorem of a different kind, which happens to be needed for the proof of (2.9). Let A and B denote P or Q (both the same, possibly). We prove

THEOREM (8.4). *Suppose that $A_{m,k}$ and $B_{n,k}$ belong to the same S-type. Let $k > l \geqslant 1$. Exclude the real case if $m-l$ (or $n-l$) is even. Then $A_{m-l,k-l}$ and $B_{n-l,k-l}$ belong to the same S-type. Also $A_{m,l}$ and $B_{n,l}$ belong to the same S-type.*

Since $k \geqslant 2$ our hypothesis ensures that m and n have the same parity in the real case, for otherwise the homology groups of $A_{m,k}$ and $B_{n,k}$ would be incompatible. I do not know whether the theorem can be extended so as to include the real case when $m-l$ (or $n-l$) is even.

In the proof of (8.4) let us use the symbols $'$ and $''$ to denote iterated suspension, the number of iterations in each case being such that $A'_{m,k}$ and $B''_{n,k}$ belong to the same homotopy type. By the cellular approximation theorem there exists a cellular homotopy equivalence

$$F\colon A'_{m,k} \to B''_{n,k}.$$

It follows from the disposition of the cell-structure, as described in the introduction, that F determines a map

$$E\colon A'_{m-l,k-l} \to B''_{n-l,k-l},$$

and hence also determines a map

$$G\colon A'_{m,l} \to B''_{n,l},$$

because $A'_{m,l}$ can be obtained from $A'_{m,k}$ by collapsing $A'_{m-l,k-l}$ and $B''_{n,l}$ can be obtained from $B''_{n,k}$ by collapsing $B''_{n-l,k-l}$. These three maps induce homomorphisms of integral homology, as shown below.

$$\begin{array}{ccccc} H_r(A'_{m-l,k-l}) & \xrightarrow{\sigma} & H_r(A'_{m,k}) & \xrightarrow{\rho} & H_r(A'_{m,l}) \\ {\scriptstyle E_*}\downarrow & & {\scriptstyle F_*}\downarrow & & {\scriptstyle G_*}\downarrow \\ H_r(B''_{n-l,k-l}) & \xrightarrow[\sigma]{} & H_r(B''_{n,k}) & \xrightarrow[\rho]{} & H_r(B''_{n,l}) \end{array}$$

In the above diagram σ denotes the injection and ρ the projection homomorphism, so that

$$\sigma E_* = F_* \sigma, \qquad \rho F_* = G_* \rho.$$

It is easy to check that σ is an isomorphism whenever $H_r(A'_{m-l,k-l})$ and $H_r(B''_{n-l,k-l})$ are non-trivial, and that ρ is an isomorphism whenever $H_r(A'_{m,l})$ and $H_r(B''_{n,l})$ are non-trivial. Also F_* is an isomorphism, since F is a homotopy equivalence. Therefore E_* and G_* are isomorphisms, and so E and G are homotopy equivalences, by Theorem 3 of (**12**). Hence the conclusion of (8.4) follows at once.

9. Intrinsic join operations

Let A, B, and C be spaces. To each map

$$h\colon A * B \to C$$

there corresponds a pairing of $\pi_i(A)$ with $\pi_j(B)$ to $\pi_{i+j+1}(C)$, which I call the intrinsic join (with respect to h). The intrinsic join of $\alpha \in \pi_i(A)$ with

$\beta \in \pi_j(B)$ is defined to be the element $\alpha * \beta$ which is the image of the ordinary join of α with β under the homomorphism

$$h_*: \pi_{i+j+1}(A * B) \to \pi_{i+j+1}(C)$$

induced by h. The operation is defined when $i \geqslant 1$ and $j \geqslant 1$. It is linear in α when $i \geqslant 2$, in β when $j \geqslant 2$.

We have already studied one example of the intrinsic join in (**3**). Two more are obtained by taking h to be one or other of the maps shown in (4.1). By using f we obtain a pairing of $\pi_i(O_{m,k})$ with $\pi_j(P_{n,k})$ to $\pi_{i+j+1}(P_{m+n,k})$; by using g we obtain a pairing of $\pi_i(O_{m,k})$ with $\pi_j(Q_{n,k})$ to $\pi_{i+j+1}(Q_{m+n,k})$. A theory of these operations can be built up, somewhat on the same lines as the exposition in (**3**). For example, consider the homomorphisms†

$$\left.\begin{array}{l} \theta_f: \pi_j(P_{n,k}) \to \pi_{i+j+1}(P_{m+n,k}) \\ \theta_g: \pi_j(Q_{n,k}) \to \pi_{i+j+1}(Q_{m+n,k}) \end{array}\right\}, \tag{9.1}$$

in which $\beta \to \theta * \beta$, where $\theta \in \pi_i(O_{m,k})$ and the intrinsic join is taken with respect to f, g, respectively. We prove a theorem about these homomorphisms which may be regarded as an extension of the Freudenthal suspension theorem.

Take $i = dm-1$, where d means the same as usual, so that θ is represented by maps of S_m into $O_{m,k}$. We describe θ as a 1-section if and only if it can be represented by a cross-section. We prove

THEOREM (9.2). *Suppose that $O_{m,k}$ admits a cross-section. Let θ be a 1-section, and consider the corresponding homomorphisms*

$$\theta_f: \pi_j(P_{n,k}) \to \pi_{j+dm}(P_{m+n,k}),$$
$$\theta_g: \pi_j(Q_{n,k}) \to \pi_{j+dm}(Q_{m+n,k}).$$

Write $\qquad p = 2d(n-k)-1, \qquad q = 2d(n-k+1)-2.$

Then θ_f is an isomorphism if $j < p$, an epimorphism if $j = p$; and θ_g is an isomorphism if $j < q$, an epimorphism if $j = q$.

For let u be a cross-section of $O_{m,k}$ which represents θ, and let

$$F: S_m * P_{n,k} \to P_{m+n,k}$$

denote the corresponding homotopy equivalence, constructed by using u and f as in (6.1). Then θ_f coincides with the composition

$$\pi_j(P_{n,k}) \xrightarrow{\tilde{E}} \pi_{j+dm}(S_m * P_{n,k}) \xrightarrow{F_*} \pi_{j+dm}(P_{m+n,k}),$$

where F_* is the isomorphism induced by F and where $\tilde{E}(\beta)$ denotes the join with β of a 1-section $\phi \in \pi_{dm-1}(S_m)$. Since ϕ is a generator it follows from the Blakers–Massey suspension theorem, as given in (2.6) of (**8**), that $\tilde{E}$ is an isomorphism if $j < p$, an epimorphism if $j = p$. Hence the same is true of

† These are homomorphisms even when $j = 1$.

θ_f, which proves the first part of (9.2). The proof of the second part is similar in all respects.

When $k = 1$ both parts of (9.2) reduce to versions of the Freudenthal suspension theorem, so that our result may be regarded as a generalization. The second part of (9.2) can be used, in conjunction with (8.1), to establish a generalized suspension theorem for Stiefel manifolds, somewhat resembling (3.1) of (**4**).

10. Proof of (2.10)

Although the following argument can be applied to the other cases as well, let us ease the notation by assuming throughout this section that the basic field is quaternionic. Accordingly we write $O_n = Sp_n$ and $S_n = S^{4n-1}$, where $n \geqslant 1$. We regard S^{4n} as the suspension of S^{4n-1}, so that points of S^{4n} are represented by pairs (x, t), where $x \in S^{4n-1}$ and

$$-1 \leqslant t \leqslant 1.$$

Consider the map $$h_n\colon S^{4n+3} \to S^{4n}$$

which is given by the transformation

$$[q_0, \dots, q_n] \to [2q_0\bar{q}_n, 2q_1\bar{q}_n, \dots, 2q_n\bar{q}_n - 1].$$

It is easy to check that the complex $P_{n+2,2}$ can be formed by using h_n to attach a $(4n+4)$-cell to S^{4n}. Our problem in (2.10a) is therefore to determine the homotopy class of h_n.

Let $q \in S^3$. Multiplication on the right by $\bar{q}$ determines a symplectic transformation $\phi(q)$ of S^{4n-1}. Let $\xi_n \in \pi_3(Sp_n)$ denote the homotopy class of the map $$h_n'\colon S^3 \to Sp_n$$

which carries q into $\phi(q)$. We obtain h_n from h_n' by applying the Hopf construction. Therefore h_n represents $J_n(\xi_n)$, where

$$J_n\colon \pi_3(Sp_n) \to \pi_{4n+3}(S^{4n})$$

denotes the homomorphism which is given by the Hopf construction. In particular, $J_1(\xi_1)$ is the Hopf class. Consider the diagram shown below, where i_* denotes the injection and E_* the iterated suspension.

$$\begin{array}{ccc} \pi_3(Sp_1) & \xrightarrow{i_*} & \pi_3(Sp_n) \\ \Big\downarrow{\scriptstyle J_1} & & \Big\downarrow{\scriptstyle J_n} \\ \pi_7(S^4) & \xrightarrow[E_*]{} & \pi_{4n+3}(S^{4n}) \end{array}$$

It follows from (16.7) of (**6**) that $\xi_n = ni_*(\xi_1)$, and so

$$J_n(\xi_n) = nJ_n i_*(\xi_1) = \pm nE_* J_1(\xi_1),$$

by Theorem 2 of (**9**). We obtain (2.10 a) at once from this relation when we pass to the stable homotopy group.

We have already described, in § 2, how (2.10 b) can be deduced from (2.2) and (2.10 a). A direct proof can also be given, with a little difficulty.

11. Relations between the three cases

Let K denote the field of complex numbers or quaternions. Let K' denote the real field if K is complex, the complex field if K is quaternionic. We represent elements of K in the usual way by pairs (x, y), where $x, y \in K'$, and we identify x with $(x, 0)$, so that K' is a subfield of K. Let $O'_{m,k}$, $P'_{m,k}$, etc., mean the same as $O_{m,k}$, $P_{m,k}$, etc., except that K' is used as basic field instead of K. The purpose of this section is to discuss relations between $O_{m,k}$ and $O'_{2m,2k}$, between $P_{m,k}$ and $P'_{2m,2k}$, and between $Q_{m,k}$ and $Q'_{2m,2k}$.

The case of the Stiefel manifolds has already been considered in § 8 of (**3**), and a canonical map of $O_{m,k}$ into $O'_{2m,2k}$ has been defined, which happens to be an embedding. Let $k > l \geqslant 1$. By (8.1) of (**3**) we have a commutative diagram as follows, where ρ and ρ' are standard projections, σ and σ' are standard inclusions, and u, u', u'' are canonical maps.

$$\begin{array}{ccccc} O_{m-l,k-l} & \xrightarrow{\sigma} & O_{m,k} & \xrightarrow{\rho} & O_{m,l} \\ \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u''} \\ O'_{2m-2l,2k-2l} & \xrightarrow[\sigma']{} & O'_{2m,2k} & \xrightarrow[\rho']{} & O'_{2m,2l} \end{array} \tag{11.1}$$

Now consider the situation in the case of stunted projective spaces. Vectors of K'_{2m} are transformed into vectors of K_m according to the rule

$$[x_1, y_1, \ldots, x_m, y_m] \to [(x_1, y_1), \ldots, (x_m, y_m)].$$

When we pass to equivalence classes we obtain a map of $P'_{2m,2k}$ onto $P_{m,k}$. Let us call this the canonical map. The following diagram is commutative, where p and p' are standard projections, i and i' are standard inclusions, and v, v', v'' are canonical maps.

$$\begin{array}{ccccc} P_{m-l,k-l} & \xrightarrow{i} & P_{m,k} & \xrightarrow{p} & P_{m,l} \\ \uparrow{\scriptstyle v'} & & \uparrow{\scriptstyle v} & & \uparrow{\scriptstyle v''} \\ P'_{2m-2l,2k-2l} & \xrightarrow[i']{} & P'_{2m,2k} & \xrightarrow[p']{} & P'_{2m,2l} \end{array} \tag{11.2}$$

Moreover, v maps c'_{2r+1} homeomorphically onto c_r, where $m-k \leqslant r < m$. This property is useful when we have to consider the homology and cohomology of stunted projective spaces.

In the case of stunted quasi-projective spaces there is nothing quite analogous to either of these two constructions. Guided by considerations of cohomology, however, let us designate as canonical any cellular map

$$w\colon Q_{m,k} \to Q'_{2m,2k},$$

which maps e_r onto e'_{2r} with degree† ± 1 for $m-k \leqslant r < m$. Define w' and w'' so as to make the following diagram commutative, where q, q' are standard projections, and j, j' are standard inclusions.

$$\begin{array}{ccccc} Q_{m-l,k-l} & \xrightarrow{j} & Q_{m,k} & \xrightarrow{q} & Q_{m,l} \\ \downarrow{\scriptstyle w'} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w''} \\ Q'_{2m-2l,2k-2l} & \xrightarrow[j']{} & Q'_{2m,2k} & \xrightarrow[q']{} & Q'_{2m,2l} \end{array} \tag{11.3}$$

It is easy to check that w is canonical if w' and w'' are canonical.

The theory of cup-products indicates that canonical maps do not, in general, exist when $m < 2k$. However, we prove

LEMMA (11.4). *Suppose that $m \geqslant 2k$. Then there exists a canonical map*

$$w\colon Q_{m,k} \to Q'_{2m,2k}.$$

In the case of Stiefel manifolds we have defined the canonical map

$$u\colon O_{m,k} \to O'_{2m,2k}.$$

Let $\bar{u}$ denote the restriction of u to $Q_{m,k}$. It follows from (8.1) that $\bar{u}$ can be deformed into $Q'_{2m,2k}$ so as to determine a map

$$\bar{w}\colon Q_{m,k} \to Q'_{2m,2k}.$$

Define w to be a cellular approximation to $\bar{w}$. I say that w is canonical. This statement is obvious if $k = 1$. Let $k \geqslant 2$ and suppose, inductively, that the statement is true for all values of k less than the given one. Since (11.1) is commutative it follows that w' and w'', as shown in (11.3), can be obtained by the same procedure as w. Hence w' and w'' are canonical, and so w is canonical.

The restriction on dimension in (11.4) is inconvenient, and for purposes of S-theory it can be avoided as follows. Let $n \geqslant k$. Choose any value of m such that $O_{m,k}$ admits a cross-section. By (6.7) there exists a homotopy equivalence

$$h\colon S_m * Q_{n,k} \to Q_{m+n,k}.$$

Also $O'_{2m,2k}$ admits a cross-section, by (1.1) of (**4**), and so there exists a homotopy equivalence

$$h'\colon Q'_{2m+2n,2k} \to S_m * Q'_{2n,2k},$$

by (6.7). After an appeal to the cellular approximation theorem we may suppose that h and h' are cellular maps. Since $m+n \geqslant 2k$ there exists a canonical map

$$w\colon Q_{m+n,k} \to Q'_{2m+2n,2k},$$

† Remarks like this must be interpreted homologically.

by (11.4). Let $\tilde{w}$ denote the cellular map $h'wh$. Then

$$\tilde{w}\colon S_m * Q_{n,k} \to S_m * Q'_{2n,2k}$$

maps the dm-fold suspension of e_r onto the dm-fold suspension of e'_{2r} with degree ± 1, where $n-k \leqslant r < n$. For purposes of S-theory, therefore, $\tilde{w}$ makes a satisfactory substitute for the canonical map and has the advantage of avoiding any restriction on dimension.

12. Use of Steenrod squares

Although the main purpose of this section is to prove (2.9), we take the opportunity to make some observations concerning the cohomology, modulo 2, of stunted projective spaces and stunted quasi-projective spaces. Consider the cohomology operations Sq^i, which are known as the Steenrod squares. In the case of projective spaces these can be computed by using the special structure of the cohomology ring, as described in (1) and (7). Hence, by naturality, we deduce the values of these operations in the case of stunted projective spaces. Since the Steenrod squares commute with suspension we further deduce their values in the case of stunted quasi-projective spaces by using (2.1), provided that the basic field is not quaternionic. Finally, the relation between complex and quaternionic quasi-projective spaces described at the end of the previous section enables us to complete the computation. We obtain the following result.

Let (s,t) denote the coefficient of u^s in the expansion of $(1+u)^{s+t}$. Let x_r denote the cohomology class, modulo 2, which is carried by c_r or e_r, as appropriate, where $m-k \leqslant r < m$. Then the Steenrod squares in the case of $P_{m,k}$ or $Q_{m,k}$ are given by the formula

$$Sq^{di}x_{r-i} = (r-2i, i)x_r, \tag{12.1}$$

where $i \leqslant k+r-m$ and d means the same as usual. It is remarkable that we do not have to separate $P_{m,k}$ and $Q_{m,k}$ for the quaternionic case.

Let A and B denote P or Q (both the same, possibly). We use (12.1) to prove

THEOREM (12.2). *Suppose that $A_{m,k}$ and $B_{n,k}$ belong to the same S-type. Then $m-n$ is a multiple of q, where q denotes the least power of 2 such that $q \geqslant k$.*

The theorem is obvious if $k = 1$. Let $k \geqslant 2$, and make the inductive hypothesis that $m-n$ is a multiple of 2^t, where $2^t < k$. This is certainly true if $t = 0$. Suppose, to obtain a contradiction, that $m-n$ is an odd multiple of 2^t. Then it follows by consideration of dyadic expansions (see (7)) that one of the coefficients

$$(m-2^{t+1}-1, 2^t), \qquad (n-2^{t+1}-1, 2^t)$$

is even and the other is odd. Apply (12.1) with $i = 2^l$ and with $r = m-1$ in the case of $A_{m,k}$, $r = n-1$ in the case of $B_{n,k}$. We conclude that $A_{m,k}$ and $B_{n,k}$ belong to different S-types, which is contrary to the hypothesis of the theorem. Therefore $m-n$ is an even multiple of 2^l, and the proof of (12.2) is completed by induction.

By a similar method, or else by combining (2.2) and (2.4) with (12.2), we obtain

THEOREM (12.3). *Suppose that $A_{m,k}$ is S-reducible. Then m is a multiple of q, where q denotes the least power of 2 such that $q \geqslant k$.*

We are now ready to prove (2.9). Let the basic field be quaternionic and let $k \geqslant 3$. Suppose, to obtain a contradiction, that $P_{m,k}$ and $Q_{n,k}$ belong to the same S-type. Then $P_{m,2}$ and $Q_{n,2}$ belong to the same S-type, by (8.4), and so $m-2 \equiv \pm n \pmod{24}$, by (2.10). Also $P_{m-1,k-1}$ and $Q_{n-1,k-1}$ belong to the same S-type, by (8.4). Hence $P_{m-1,2}$ and $Q_{n-1,2}$ belong to the same S-type, by (8.4), and so $m-3 \equiv \pm(n-1) \pmod{24}$, by (2.10). Therefore $m-2 \equiv n \pmod{24}$, which contradicts (12.2). This proves the first part of (2.9). As observed in § 2, the second part follows from the first part and (2.2). Alternatively, we can use (2.10 a) and (12.3) to show that $P_{m,k}$ is not S-reducible if $P_{m,2}$ is S-reducible, and therefore $P_{m,k}$ is not S-reducible if $k \geqslant 3$.

The techniques of this section can obviously be adapted so as to apply to the other cohomology operations of Steenrod. Consequently we obtain further conditions like (12.2) and (12.3) but with an odd prime p in place of 2. The details are slightly more complicated, and so we do no more than mention this development. It is worth remarking, perhaps, that (2.9) cannot be established simply by consideration of cohomology operations.

REFERENCES

1. A. BOREL and J.-P. SERRE, 'Groupes de Lie et puissances réduites de Steenrod', *American J. of Math.* 75 (1953) 409–48.
2. I. M. JAMES, 'Whitehead products and vector-fields on spheres', *Proc. Cambridge Phil. Soc.* 53 (1957) 817–20.
3. —— 'The intrinsic join', *Proc. London Math. Soc.* (3) 8 (1958) 507–35.
4. —— 'Cross-sections of Stiefel manifolds', ibid. 536–47.
5. —— and J. H. C. WHITEHEAD, 'The homotopy theory of sphere-bundles over spheres (I)', ibid. 4 (1954) 196–218.
6. N. E. STEENROD, *The topology of fibre bundles* (Princeton Univ. Press, 1951).
7. —— and J. H. C. WHITEHEAD, 'Vector fields on the n-sphere', *Proc. Nat. Acad. Sci.* 37 (1951) 58–63.
8. H. TODA, 'Generalized Whitehead products and homotopy groups of spheres', *J. Inst. Poly. Osaka*, 3 (1952) 43–82.
9. G. W. WHITEHEAD, 'On the homotopy groups of spheres and rotation groups', *Ann. of Math.* 43 (1942) 634–40.

10. G. W. Whitehead, 'Homotopy groups of joins and unions', *Trans. American Math. Soc.* 83 (1956) 55–69.
11. J. H. C. Whitehead, 'On the groups $\pi_r(V_{n,m})$ and sphere bundles', *Proc. London Math. Soc.* (2) 48 (1944) 243–91.
12. —— 'Combinatorial homotopy', *Bull. American Math. Soc.* 55 (1949) 213–45.
13. I. Yokota, 'On the cellular decompositions of unitary groups', *J. Inst. Poly. Osaka*, 7 (1956) 39–49.
14. —— 'On the cells of symplectic groups', *Proc. Japan Acad.* 32 (1956) 399–400.

The Mathematical Institute
10 *Parks Road*
Oxford