MULTIPLICATION ON SPHERES (I)

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1. Introduction. By a multiplication on a space, A, I mean a continuous product with a two-sided identity. I say that a multiplication is *homotopy-commutative* if the two maps $A \times A \to A$, which are given by

$$(x, y) \to x \cdot y, \qquad (x, y) \to y \cdot x \qquad\qquad (x, y \in A),$$

are homotopic. Consider the topological n-sphere, S^n. The purpose of this note is to prove:

THEOREM 1.1. *A multiplication on S^n is homotopy-commutative if, and only if, $n=1$.*

Multiplications exist if $n=1$, 3, or 7, as follows. In case $n=1$, a 1-sphere is formed by the ordinary complex numbers of unit modulus, which have a commutative multiplication. Bott[1] [2] has proved that S^n does not admit a commutative multiplication if $n>1$. A 3-sphere is formed by the quaternions of unit modulus, which have a noncommutative multiplication. H. Samelson [7] and G. Whitehead [12] have proved that the quaternionic multiplication on S^3 is not homotopy-commutative. A 7-sphere is formed by the Cayley numbers of unit modulus, which have a noncommutative multiplication. Sugawara [9] has proved that the Cayley multiplication on S^7 is not homotopy-commutative. However, there are many classes of multiplications on S^3 and S^7 besides these, so that (1.1) widens our knowlege quite apart from the possibility of multiplications on spheres of other dimensions.

It is an open question whether there are other spheres with multiplications. The obstruction to constructing a multiplication on S^n is the Whitehead product $[j, j] \in \pi_{2n-1}(S^n)$, where j denotes a generator of $\pi_n(S^n)$. Hence, by (3.72) of [11], we have:

THEOREM 1.2. *There exists an element of Hopf invariant unity in $\pi_{2n+1}(S^{n+1})$ if, and only if, S^n admits a multiplication.*

Adem [1] has announced that there is no element of Hopf invariant unity in $\pi_{2n+1}(S^{n+1})$ unless $n+1$ is a power of two, and Toda [10] that there is no element of Hopf invariant unity in $\pi_{31}(S^{16})$. This appears to be all that is known about this problem at present.

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[1] Numbers in brackets refer to the list at the end of this note.

In a second article I shall deal with questions of associativity. In particular I shall prove Samelson's conjecture [**7**] that the Cayley multiplication on S^7 is not homotopy-associative.

2. **The Hopf construction.** Suppose that we have two multiplications,

$$f, g: S^n \times S^n \to S^n,$$

with the same identity $e \in S^n$. Then f and g agree on the set of axes

$$S^n \times e \cup e \times S^n \subset S^n \times S^n,$$

whose complement is an open $2n$-cell. Hence f and g are homotopic if their separation element in $\pi_{2n}(S^n)$ is zero. Since $\pi_2(S^1) = 0$, this immediately proves:

LEMMA 2.1. *Any multiplication on S^1 is homotopy-commutative.*

Hopf, in [**4**], describes a construction which associates an element of Hopf invariant unity,

$$c(f) \in \pi_{2n+1}(S^{n+1}),$$

with each multiplication f. It can be seen at once from the definition that

$$c(f) = c(g) \text{ if } f \cong g. \tag{2.2}$$

Let g be the multiplication which is related to f by

$$f(x, y) = g(y, x) \qquad (x, y \in S^n).$$

Let

$$w = [i, i] \in \pi_{2n+1}(S^{n+1})$$

denote the Whitehead product of a generator of $\pi_{n+1}(S^{n+1})$ with itself. Then

$$c(f) - (-1)^n c(g) = (-1)^n w,$$

by (2.19) of [**6**]. Since S^n admits a multiplication, n is odd, by Theorem V of [**4**]. Hence we have

$$c(f) + c(g) = -w. \tag{2.3}$$

We obtain from (2.2) and (2.3) combined:

LEMMA 2.4. *Suppose that S^n admits a homotopy-commutative multiplication, f. Then $2c(f) = -w$ in $\pi_{2n+1}(S^{n+1})$, where $c(f)$ is the element obtained from f by the Hopf construction, and where $w = [i, i]$.*

3. **The Whitehead product.** It is easy to show how an n-field on S^n, i.e. a parallelization, determines a multiplication on S^n. However, there may be values of n such that S^n admits a multiplication but not an n-field. Many theorems in the literature which are proved for a parallelizable sphere can also be proved for a sphere with a multiplication. A case in point is (4.14) of [3]. We shall prove the following stronger version of that result:

THEOREM 3.1. *Suppose that S^n admits a multiplication, where $n>1$. Let $\gamma\in\pi_{2n+1}(S^{n+1})$ be an element of Hopf invariant unity, and let $w=[i, i]\in\pi_{2n+1}(S^{n+1})$. Then an element $\alpha\in\pi_{2n}(S^n)$ exists, which has a nonzero generalized Hopf invariant, and such that $w+2\gamma=E(\alpha)$.*

In (3.1), the suspension homomorphism

$$E: \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})$$

is an isomorphism into, by Corollary 1 on p. 282 of [8]. Hence (3.1) implies that $w+2\gamma\neq 0$, i.e.

COROLLARY 3.2. *Suppose that S^n admits a multiplication, where $n>1$. Then the Whitehead product w is not contained in $2\pi_{2n+1}(S^{n+1})$.*

We shall prove (3.1) in the next section. Let us assume it for the moment so as to finish the proof of (1.1). In case $n=1$, (1.1) is obtained from (2.1). Suppose that S^n admits a multiplication where $n>1$. Then w cannot be halved, by (3.2), and so the multiplication cannot be homotopy-commutative, by (2.4). Hence it only remains to prove (3.1).

4. **Proof of** (3.1). Let S^m $(m\geqq 1)$ denote the m-sphere in Hilbert space, which consists of points

$$(x_0, x_1, \cdots, x_i, \cdots),$$

such that $x_i=0$ if $i>m$, and such that

$$x_0^2 + x_1^2 + \cdots + x_m^2 = 1.$$

Thus

$$S^1 \subset S^2 \subset \cdots \subset S^m \subset \cdots.$$

Let $a^m\in S^m$ be the point where $x_m=1$. Let R_{m+1} denote the rotation group of S^m, and let

$$\phi_m: R_{m+1} \to S^m$$

denote the fibre map which is given by

$$\phi_m(r) = r(a^m) \qquad (r \in R_{m+1}).$$

We embed R_m in R_{m+1} by extending rotations of S^{m-1} to rotations of S^m about the axis of x_m, so that $R_m = \phi_m^{-1}(a^m)$. Thus

$$R_2 \subset R_3 \subset \cdots \subset R_m \subset \cdots.$$

In §5 of [3], Hilton and Whitehead examine the values of the Steenrod squares in the cohomology rings of Stiefel manifolds, and prove a result about $V_{n+1,2}$, the bundle of unit tangent vectors to S^n. This manifold can be identified with the factor space R_{n+1}/R_{n-1}, so that we have a natural isomorphism

$$\pi_n(R_{n+1}, R_{n-1}) \approx \pi_n(V_{n+1,2}).$$

Moreover the homotopy sequence of the bundle can be identified with the homotopy sequence of the triple (R_{n+1}, R_n, R_{n-1}). This enables us to reformulate (5.1) of [3] as follows:

LEMMA 4.1. *Let n be odd, $n>1$. Then the image of the boundary homomorphism*

$$\partial: \pi_{n+1}(R_{n+2}, R_{n+1}) \to \pi_n(R_{n+1}, R_{n-1})$$

is contained in $2\pi_n(R_{n+1}, R_{n-1})$ if, and only if, $n \equiv 1 \pmod 4$.

Consider the following diagram, where[2] $n>3$.

$$(4.2)\qquad \begin{array}{ccccc} \pi_{2n}(S^n) & \xrightarrow{E} & \pi_{2n+1}(S^{n+1}) & \xleftarrow{P} & \pi_{n+1}(S^{n+1}) \\ H_1 \downarrow & & H_2 \downarrow & & \uparrow \phi_* \\ \pi_n(R_n, R_{n-1}) & \xrightarrow[i_*]{} & \pi_n(R_{n+1}, R_{n-1}) & \xleftarrow[\partial]{} & \pi_{n+1}(R_{n+2}, R_{n+1}). \end{array}$$

In (4.2), i_* is the injection, and ∂ is the boundary homomorphism, as in (4.1); E is the suspension homomorphism, ϕ_* is the isomorphism induced by ϕ_{n+1}, and P is the homomorphism which carries a generator of $\pi_{n+1}(S^{n+1})$ into the Whitehead product w. The other two homomorphisms, H_1 and H_2, are defined as in [5]. Thus H_1, which maps $\pi_{2n}(S^n)$ into a group of order two, is equivalent to the generalized Hopf invariant, by (1.2b) of [5], and H_2 is such that we have the relations

$$(4.3)\qquad \begin{aligned} &\text{(a)}\ \ i_* \circ H_1 = -H_2 \circ E, \\ &\text{(b)}\ \ \partial = H_2 \circ P \circ \phi_*. \end{aligned}$$

In (4.3), (a) is by (4.2c) of [5], and (b) is by (1.2a) and (4.1c) of [5].

[2] If $n=3$, H_2 is not defined.

We are now ready to prove (3.1). Suppose that S^n admits a multiplication, where $n>1$. Then $n\equiv 3 \pmod 4$, by §9 of [11]. If $n=3$, (3.1) is a corollary of (6.1) of [3]. Hence we suppose that $n>3$, so that the homomorphisms H_1 and H_2 are defined. Let $\gamma\in\pi_{2n+1}(S^{n+1})$ be an element of Hopf invariant one. Since the Hopf invariant of the Whitehead product w is equal to minus two, in the conventions of [6], the Hopf invariant of $w+2\gamma$ is zero. Hence, by a Freudenthal theorem ((3.48) of [11]), there is an element $\alpha\in\pi_{2n}(S^n)$ such that $E(\alpha)=w+2\gamma$. I say that $H_1(\alpha)$, the generalized Hopf invariant of α, is not zero. For suppose, if possible, that $H_1(\alpha)=0$. Then

$$H_2(w+2\gamma) = H_2E(\alpha) = -i_*H_1(\alpha) = 0,$$

by (4.3a), and so $H_2(w)\in 2\pi_n(R_{n+1}, R_{n-1})$. But $w=P(\lambda)$, where λ generates $\pi_{n+1}(S^{n+1})$, and $\lambda=\phi_*(\mu)$, where μ generates $\pi_{n+1}(R_{n+2}, R_{n+1})$. Hence

$$H_2(w) = H_2P\phi_*(\mu) = \partial(\mu),$$

by (4.3b), and so $2\pi_n(R_{n+1}, R_{n-1})$ contains the boundary of the generator of $\pi_{n+1}(R_{n+2}, R_{n+1})$. This contradicts (4.1), since $n\equiv 3 \pmod 4$. Therefore $H_1(\alpha)\neq 0$. This proves (3.1), and with that the proof of (1.1) is finished.

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