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REDUCED PRODUCT SPACES

By I. M. James

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Introduction

We consider countable CW-complexes[1] which have exactly one vertex, or 0-cell. Such complexes will be described as special; any connected countable CW-complex can be deformed into a special complex without altering its homotopy type. Let A be a given special complex with vertex a^0. In this paper we define a special complex A_∞, called the reduced product complex of A. Its points are finite sequences of points in the set $A - a^0$, but its topology derives from the pair of spaces (A, a^0) rather than the space $A - a^0$. Its cells are finite products of cells from $A - a^0$. The subset of sequences in $A - a^0$ with fewer than a given number of terms defines a subcomplex of A_∞, so that A_∞ is filtered by these subcomplexes. In particular the empty sequence is the only vertex of A_∞, which we identify with a^0, and the sequences with not more than one term form a subcomplex which we identify with the original complex A. For example, let A be a sphere S^n $(n \geqq 1)$ so that the complement of the vertex is an n-cell, e^n. The sequences in e^n with exactly m terms form an mn-cell e^{mn} in the reduced product complex, so that

$$S^n_\infty = S^n \cup e^{2n} \cup \cdots \cup e^{mn} \cup \cdots .$$

Let I be the unit interval. The suspension of A, denoted by $\hat{A}$, is the space obtained from the product $A \times I$ by shrinking $A \times \dot{I} \cup a^0 \times I$ to a point, that point being identified with a^0. Let Ω be the space of loops on the suspension of A; that is to say, the set of maps $(I, \dot{I}) \to (\hat{A}, a^0)$ with compact-open topology. This space is important in many connections (see [5] for examples). We embed $A \subset \Omega$ so that $a \in A$ is identified with the map $t \to f(a, t)$ $(t \in I)$, where $f: A \times I \to \hat{A}$ is the identification map. We shall define a class of maps of A_∞ into Ω called canonical maps. The canonical maps are (1, 1) into, but are not topological (except in the trivial case $A = a^0$). Each canonical map is the identity on A, and for our present purposes it is unimportant that there is no preferred canonical map, since any two of them are homotopic, rel. A. Thus the induced homomorphism of the homotopy sequence of the pair (A_∞, A) into that of the pair (Ω, A) is independent of the particular canonical map chosen, likewise the induced homomorphism in homology is independent. We refer to these homomorphisms as the canonical homomorphisms. Our main theorem is that both these canonical homomorphisms are isomorphisms onto.

Thus for many purposes Ω can be replaced by the complex A_∞, with three kinds of attendant advantages. Firstly, we can make use of the homotopy ex-

[1] For information about CW-complexes we refer the reader to [9], in the list of references at the end.

tension theorems, cellular approximation theorems, and so forth, which are available for complexes but not for function spaces. Secondly, there are special extension properties of reduced product complexes (the combinatorial extensions of (1.4)–(1.7) below) which appear to have no analogue for the space of loops and which have very interesting applications. Thirdly, there is the filtration of A_∞ by a sequence of subcomplexes, each of which is a simple identification space. For example, in §3 we use this filtration to express the homology of A_∞ in terms of the homology of A; and the same method can be used to express the cohomology ring of A_∞ in terms of the cohomology ring of A, under certain conditions.

In the first four sections of this paper, which constitute the first part, we make our definitions and discuss the homology of the reduced product complex and that of the space of loops. The second part contains the theory of the canonical maps, and it is introduced in §5 with a statement of results. Applications have been kept for a sequel, entitled *On the suspension triad*, which will be published in this journal shortly.

I express warm thanks to E. Pitcher for correspondence enlarging on his Congress lecture [4], to C. H. Dowker for sending me his theorem on products of complexes with an invitation to include it here, and to J.-P. Serre and N. E. Steenrod for suggesting some improvements which I have been glad to make. There appears to be a connection between this theory and some recent work of Toda's announced in [7].

Part I: Preliminary analysis

1. Definitions and basic properties

Let A be a Hausdorff space, and let $a^0 \epsilon A$ be a given point. The m-fold topological product A^m, $m = 0, 1, \cdots$, is represented by infinite sequences in A,

$$a^m = (a_1, a_2, \cdots, a_r, \cdots),$$

such that $a_r = a^0$ if $r > m$. It is convenient to abbreviate a^m to $(a_1, \cdots, a_m)$, the inclusions

$$A^0 \subset A^1 \subset \cdots \subset A^m \subset \cdots$$

being understood. We identify $a^0 = A^0$ and $A = A^1$ in the obvious way. If $a^m \epsilon A^m$ let $p_m a^m$ be the (finite) subsequence of a^m consisting of those terms which lie in $A - a^0$, with the same relative order as they have in a^m. Let A_m be the set of sequences in $A - a^0$ which have not more than m terms, with the identification topology determined by $p_m: A^m \rightarrow A_m$. Thus p_0, p_1 identify A^0, A^1, and hence a^0, A, with A_0, A_1, respectively. Less trivially, p_2 identifies (a, a^0) and (a^0, a) in A^2 for all $a \epsilon A$; p_3 identifies (a, a', a^0), (a, a^0, a') and (a^0, a, a') in A^3 for all a, $a' \epsilon A$; and so on.

A basis for the open sets of A_m is obtained as follows. Let $V \subset A$ be an (open) neighborhood of a^0 and let U be a sequence of open subsets $U_1, \cdots, U_r \subset A$ $(0 \leqq r \leqq m)$ which do not contain a^0; if $r = 0$ then U is empty. Let $(U, V)^m \subset A^m$

be the union of all products $W_1 \times \cdots \times W_m$ in which $m - r$ of the factors are V and the remainder are $U_1, \cdots, U_r$ in their proper relative order. Thus if $r = 2$, $m = 3$ we have

$$(U, V)^3 = U_1 \times U_2 \times V \cup U_1 \times V \times U_2 \cup V \times U_1 \times U_2 .$$

Then $(U, V)^m$ is open and saturated under p_m, hence $(U, V)_m = p_m(U, V)^m$ is open in A_m. Hence we obtain without difficulty, applying the Hausdorff condition,

THEOREM (1.1). *The sets $(U, V)_m$, for all U, V of the above description, form a basis for the open sets of A_m.*

It is clear that the identity function $i_m : A_m \rightarrow A_{m+1}$ is continuous and that $i_m A_m$ is closed in A_{m+1}. Moreover we have

$$(1.2) \qquad i_m(U, V)_m = i_m A_m \cap (U, V)_{m+1},$$

and so i_m embeds A_m topologically as a closed subset of A_{m+1}. The topology of the *reduced product space of* A, rel. a^0, viz:

$$A_\infty = A_0 \cup A_1 \cup \cdots \cup A_m \cup \cdots ,$$

is now defined so that a set $F \subset A_\infty$ is closed if and only if $F \cap A_m$ is closed for every $m \geqq 0$. Thus the points of A_∞ are the finite sequences of points in $A - a^0$. To illustrate this we prove

THEOREM (1.3). *A_∞ is a Hausdorff space.*

Let a^*, $a'_* \in A_\infty$ be distinct sequences in $A - a^0$, say

$$a_* = (a_1, \cdots, a_p), \qquad a'_* = (a'_1, \cdots, a'_q).$$

Let $V \ni a^0$, $U_r \ni a_r$ $(1 \leqq r \leqq p)$ and $U'_r \ni a'_r$ $(1 \leqq r \leqq q)$ be open neighborhoods in A such that $V \cap U_r = \emptyset$ $(1 \leqq r \leqq p)$, $V \cap U'_r = \emptyset$ $(1 \leqq r \leqq q)$, and $U_r \cap U'_r = \emptyset$ if $a_r \neq a'_r$ $(1 \leqq r \leqq \min(p, q))$. Consider the sequences

$$U = (U_1, \cdots, U_p), \qquad U' = (U'_1, \cdots, U'_q).$$

Let $W \subset A_\infty$ be the union, over all $m \geqq p$, of the sets $(U, V)_m$. It follows from (1.2) that $W \cap A_m = (U, V)_m$, which is open in A_m. Hence W is open in A_∞. Similarly W', defined in the same way with U' and q instead of U and p, is open in A_∞. However, W and W' are disjoint, and since W contains a_* and W' contains a'_* this proves (1.3). Since A_∞ is a Hausdorff space we can reapply the construction to A_∞, and regard the reduced product space of A_∞, rel. a^0, as the second reduced product space of A, and so on.

Let B be a Hausdorff space and let B_∞ be the reduced product space of B, rel. b^0, where $b^0 \in B$. Let $q_m : B^m \rightarrow B_m$ be the identification map. From a map $h : A \rightarrow B$ such that $ha^0 = b^0$ we obtain a map $h_\infty : A_\infty \rightarrow B_\infty$ by defining

$$h_\infty \circ p_m = q_m \circ h^m \qquad (m = 1, 2, \cdots),$$

where $h^m : A^m \rightarrow B^m$ is the product map. We say that h_∞ *maps termwise* by h. If A is a closed subspace of B and h is the inclusion map then $h_\infty A_\infty$ is closed in B_∞, and hence h_∞ embeds A_∞ topologically in B_∞. Returning to the general case, let $m \geqq 1$ and let $h : A_m \rightarrow B_\infty$ be a map such that $hA_{m-1} = b^0$. In §2 we shall define a function $h':A_\infty \to B_\infty$, such that $h' \mid A_m = h$, which we refer to as the *combinatorial extension* of h, without asserting its continuity. We shall, however, prove

THEOREM (1.4). *If $hA_m \subset B_n$ for some $n < \infty$ then h' is continuous.*

In case $m = n = 1$, h' maps termwise by the map $A \to B$ which h determines. Let $h_t:A_m \to B_\infty$ $(t \in I)$ be a homotopy such that $h_tA_{m-1} = b^0$, and let $h_t':A_\infty \to B_\infty$ be the combinatorial extension of h_t. We shall prove

THEOREM (1.5). *If $h_tA_m \subset B_n$ for some $n < \infty$ which is independent of t, then h_t' is a homotopy.*

The combinatorial extension is natural in the following sense. Let C and D be Hausdorff spaces and let $c^0 \in C$ and $d^0 \in D$ be basepoints. Let $f:A \to C$ and $g:B \to D$ be maps such that $fa^0 = c^0$ and $gb^0 = d^0$. Consider the following diagram,

$$\begin{array}{ccc} A_\infty & \xrightarrow{h'} & B_\infty \\ {\scriptstyle f_\infty}\downarrow & & \downarrow{\scriptstyle g_\infty} \\ C_\infty & \xrightarrow[k']{} & D_\infty \end{array} \tag{1.6}$$

in which f_∞, g_∞ map termwise by f, g and h', k' are the combinatorial extensions of maps

$$h:A_m \to B_\infty, \qquad k:C_m \to D_\infty \qquad (m \geqq 1)$$

such that $hA_{m-1} = b^0$, $kC_{m-1} = d^0$. We shall prove

THEOREM (1.7). *If $k \circ f_\infty \mid A_m = g_\infty \circ h$ then the diagram* (1.6) *is commutative.*

These last three theorems will be proved in the next section when the combinatorial extension is defined. The proof of the next theorem is left to the reader as an exercise. We say that A is an h-space, relative to a^0, if there is a map of the product space $A \times A$ into A such that

$$(a, a^0) \to a, \qquad (a^0, a) \to a$$

for all $a \in A$. Such a map determines a retraction of A_2 onto A, and conversely. Thus we have

THEOREM (1.8). *A is a retract of A_2 if and only if A is an h-space relative to a^0. Moreover any retraction of A_2 onto A extends to a retraction of A_∞ onto* A.

A CW-complex is described as locally countable if every point of it lies in the interior of a countable subcomplex. The following theorem and the proof of it given in §8 below are due to C. H. Dowker.

THEOREM (1.9). *Let K and L be locally countable CW-complexes. Then the weak topology of the product complex $K \times L$ is the same as the product topology.*

Let A be a special complex, as in the introduction. That is to say, A is a countable CW-complex in which a^0 is the only vertex. Then, by an induction based on (1.9), the weak topology in the product complex A^m is the same as the product topology. A^m is the union of disjoint products of cells from A,

$$e_1 \times e_2 \times \cdots \times e_r \times \cdots,$$

such that e_r is the vertex if $r > m$. Each such product cell is mapped homeomorphically into A_m by p_m, and the images of two of them either coincide or are disjoint. Hence A_m is a complex whose cells are the products of cells of $A - a^0$:

$$e_1 \times e_2 \times \cdots \times e_r \qquad (r = 0, 1, \cdots, m).$$

Since A_m is closure finite and p_m is cellular, it follows from (E) on page 225 of [9] that with this cell structure the reduced product space

$$A_\infty = A_0 \cup A_1 \cup \cdots \cup A_m \cup \cdots,$$

has the weak topology. This CW-complex A_∞ is the reduced product complex of A referred to in the introduction. Notice that A_∞ is itself a special complex, so that we may iterate the construction if we wish. Notice also

(1.10). *The m-section of A_∞ is contained in the subcomplex A_m.*

We define a function $T: A_\infty \times A_\infty \to A_\infty$ by

$$T((a_1, \cdots, a_r), (a_1', \cdots, a_s')) = (a_1, \cdots, a_r, a_1', \cdots, a_s'),$$

where the coordinates are in $A - a^0$. By (1.9) the weak topology of the product complex $A_\infty \times A_\infty$ is the same as the product topology, since A_∞ is countable. Hence T, which is obviously continuous as a map of the product complex, is actually continuous as a map of the product space. Clearly

$$T(a_*, a^0) = T(a^0, a_*) = a_* \qquad (a_* \in A_\infty).$$

We shall refer to T as the natural multiplication. We have proved

THEOREM (1.11). *If A is a special complex then A_∞ is an h-space, relative to a^0, with the natural multiplication. In fact A_∞ is a free topological semigroup with a^0 as left and right identity.*

Finally, since this section is in the nature of a summary of the results of the first part, we state the description of the singular homology of A_∞ which is obtained in §3. Let $(G^1, G^2, \cdots, G^m, \cdots)$ be the sequence of groups defined inductively from a given abelian group G by

$$G^1 = G, \qquad G^m = G \otimes G^{m-1} + G * G^{m-1} \qquad (m \geqq 2),$$

where the summation is direct and where $\otimes$ and $*$ denote the tensor and torsion products, respectively. We shall prove

THEOREM (1.12). *Let A be a special complex, and let G^m be defined from G as above, with*

$$G = \textstyle\sum_{r=1}^{\infty} H_r(A).$$

Then the homology of the reduced product complex A_∞ is given by an isomorphism

$$\textstyle\sum_{r=1}^{\infty} H_r(A_\infty) \approx \sum_{m=1}^{\infty} G^m.$$

2. Some mapping theorems

In this section we prove (1.4)–(1.7) and discuss some similar properties which illustrate the general nature of reduced product spaces. As in §1, let A be a

Hausdorff space and let $a^0 \in A$. We describe a_1^m, $a_2^m \in A^m$ as a *contiguous pair* if a_1^m differs from a_2^m only in that two adjacent components, one of which is a^0, are transposed. Notice that any two points in A^m at which p_m has the same value are the ends of a finite sequence of points in A^m, adjacent terms of which are contiguous. For example, if $a, a' \in A$ then (a, a', a^0) and (a^0, a, a') in A^3 are connected by the contiguous pairs (a, a', a^0), (a, a^0, a') and (a, a^0, a'), (a^0, a, a'). Let X be a space and let $f^m : A^m \to X$ be a map. We describe f^m as *invariant* if $f^m(a_1^m) = f^m(a_2^m)$ for every contiguous pair a_1^m, $a_2^m \in A^m$. If f^m is invariant then $f_m = f^m \circ p_m^{-1} : A_m \to X$ is single-valued, hence continuous (Lemma 3 of [8]). We describe a sequence of maps $f^m : A^m \to X$, $m = 1, 2, \cdots$, as *compatible* if $f^{m+1} \mid A^m = f^m$. If each f^m is invariant and the sequence is compatible then a single-valued function $f : A_\infty \to X$ is defined by $f \circ p_m = f^m$. Then $f \mid A_m = f_m$ and the topology of A_∞ is such that f is continuous.

To extend the discussion to homotopies we prove

LEMMA (2.1). *A set $F \subset A_\infty \times I$ is closed if and only if $F \cap (A_m \times I)$ is closed for every $m \geqq 0$.*

Let A_* be the union of the spaces $A_m \times m$, $m = 0, 1, \cdots$, where the set of integers $\{m\}$ is discrete, so that a set $E \subset A_*$ is closed if and only if $E \cap (A_m \times m)$ is closed for every $m \geqq 0$. Then A_∞ has the identification topology determined by $p : A_* \to A_\infty$, where $p(\alpha, m) = \alpha$ if $\alpha \in A_m$. Hence it follows from Lemma 4 of [8] that $A_\infty \times I$ has the identification topology determined by the product map $p \times 1 : A_* \times I \to A_\infty \times I$. Since $A_* \times I$ is the union of the spaces $A_m \times m \times I$, $m = 0, 1, \cdots$, and a set $E \subset A_* \times I$ is closed if and only if $E \cap (A_m \times m \times I)$ is closed for every m, by setting $E = (p \times 1)^{-1}F$ we obtain (2.1).

Let $h^m : A^m \times I \to X$ be a homotopy[2]. We describe h^m as *invariant* if $h^m(a_1^m, t) = h^m(a_2^m, t)$ $(t \in I)$ for every contiguous pair a_1^m, $a_2^m \in A^m$. If h^m is invariant then $h_m = h^m \circ (p_m \times 1)^{-1} : A_m \times I \to X$ is single-valued, hence continuous by the Corollary to Lemma 3 of [8]. We describe a sequence of homotopies $h^m : A^m \times I \to X$, $m = 1, 2, \cdots$, as *compatible* if $h^{m+1} \mid A^m \times I = h^m$. If each h^m is invariant and the sequence is compatible then a single-valued function $h : A_\infty \times I \to X$ is defined by $h \circ (p_m \times 1) = h^m$. Then $h \mid A_m \times I = h_m$ and it follows from (2.1) that h is a homotopy. The above remarks are fundamental in what follows.

Let B be a space and let

$$p_m \times 1 : A^m \times B \to A_m \times B$$

be the product map. It is a simple exercise in the use of the basis in A_m given in (1.1) to prove

LEMMA (2.2). *$A_m \times B$ has the identification topology determined by $p_m \times 1$.*

Let $(A^m)^n$ $(n \geqq 1)$ be identified with A^{mn} so that the r^{th} term (in A) of

[2] Homotopies are written in either of two ways, whichever happens to be the more convenient. Thus, if P and Q are spaces then a set of maps $h_t : P \to Q$ $(0 \leqq t \leqq 1)$ is called a homotopy if the corresponding function $h : P \times I \to Q$, in which $(x, t) \to h_t x$ $(x \in P)$, is continuous. But h, if continuous, is also called a homotopy itself.

the s^{th} factor A^m is the $(r + m(s - 1))^{\text{th}}$ term in A^{mn}. Consider the identification maps

$$p'_n:(A_m)^n \to (A_m)_n, \qquad p_{mn}:A^{mn} \to A_{mn};$$

the first of these is defined since A_m is Hausdorff by (1.3). Since the composition of two identification maps, if defined, is an identification map, it follows by iteration of (2.2) that $(A_m)^n$ has the identification topology determined by the product map

$$(p_m)^n:(A^m)^n \to (A_m)^n.$$

Hence

$$p_{m,n} = p_{mn} \circ [p'_n \circ (p_m)^n]^{-1}:(A_m)_n \to A_{mn},$$

being single-valued, is continuous. Moreover if $U \subset A_{mn}$ and $p_{m,n}^{-1} U$ is open then $p_{mn}^{-1} U$ is open and so U is open. This proves

LEMMA (2.3). *A_{mn} has the identification topology determined by*

$$p_{m,n}:(A_m)_n \to A_{mn}.$$

Maps $(A_m)^n \to A_{mn}$ and $(A^m)_n \to A_{mn}$ are defined by suitable compositions with $p_{m,n}$. Also, if $m_r \geqq 0$ $(1 \leqq r \leqq n)$, $m = \max m_r$, and $l = \sum m_r$ then $p_{m,n}$ determines a map

$$A_{m_1} \times \cdots \times A_{m_r} \times \cdots \times A_{m_n} \to A_l.$$

We shall refer to such maps as *natural* maps. An example of frequent occurrence below is the natural map $A \times A_m \to A_{m+1}$. Now let

$$P_m:(A_m)_\infty \to A_\infty \qquad (m \geqq 1)$$

be the map determined by $p_{m,n}$ $(n = 1, 2, \cdots)$. Notice that we have

$$P_{m+1} \mid (A_m)_\infty = P_m.$$

(2.3) leads at once to

LEMMA (2.4). *A_∞ has the identification topology determined by P_m.*

Let $P:(A_\infty)_\infty \to A_\infty$ be the single-valued function defined by $P \mid (A_m)_\infty = P_m$, $m = 1, 2, \cdots$. If A is a special complex then A_∞ and $(A_\infty)_\infty$ are CW-complexes as described in §1. Moreover the m-section of $(A_\infty)_\infty$ is contained in $(A_m)_m$, and since $P \mid (A_m)_m = P_m \mid (A_m)_m$ is continuous it follows that P is continuous. I do not know if P is continuous without this restriction on A.

Let A and B be Hausdorff spaces. We construct the reduced product space of B, rel. $b^0 \in B$. We prove

LEMMA (2.5). *Let $f:A^m \to B$ $(m \geqq 1)$ be an invariant map such that $fA^{m-1} = b^0$. Then there exists a compatible sequence of invariant maps $f^n:A^n \to B_\infty$, $n = 1, 2, \cdots$, such that f^m agrees with f.*

Let J be the sequence $(1, 2, \cdots, n)$ $(n \geqq m)$, and let $k = \binom{n}{m}$ be the binomial coefficient. Let $w = (w_1, \cdots, w_k)$ be the set of subsequences of J with exactly m terms, ordered as follows. Let

$$w_r = (\alpha_1, \cdots, \alpha_m), \qquad w_s = (\beta_1, \cdots, \beta_m),$$

where $w_r, w_s \in w$. Then $r < s$ if and only if there exists $u \leqq m$ such that

$$\alpha_v = \beta_v \quad (v > u), \qquad \alpha_u < \beta_u .$$

For example, if $n = 4$, $m = 2$, the six terms of w are ordered

$$(1, 2), \qquad (1, 3), \qquad (2, 3), \qquad (1, 4), \qquad (2, 4), \qquad (3, 4).$$

Let $w_r':A^n \rightarrow A^m$ be the map defined by

$$w_r'(a_1, \cdots, a_n) = (a_{\alpha_1}, \cdots, a_{\alpha_m}),$$

where w_r is as above and $a_t \in A \quad (1 \leqq t \leqq n)$. Then we define the map $f^n:A^n \rightarrow B_\infty$ by $f^nA^n = b^0$ if $n < m$, and if $n \geqq m$ by

$$f^na^n = q_k(fw_1'a^n, \cdots, fw_k'a^n) \qquad (a^n \in A^n),$$

where $q_k:B^k \rightarrow B_k$ is the identification map. In case $n = m$ we have $k = 1$, $w_1' = 1$, $q_1 = 1$, and so f^m agrees with f, as required. Clearly the sequence of maps $(f^1, f^2, \cdots, f^n, \cdots)$ is compatible. We complete the proof of (2.5) by showing that f^n is invariant $(n > m)$.

Let a^n, $\bar{a}^n \in A^n$ be a contiguous pair, where

$$a^n = (a_1, \cdots, a_n), \qquad \bar{a}^n = (\bar{a}_1, \cdots, \bar{a}_n)$$

so that for some $s \quad (1 \leqq s \leqq n - 1)$ we have

$$\bar{a}_s = a_{s+1}, \qquad \bar{a}_{s+1} = a_s = a^0, \qquad \bar{a}_r = a_r \qquad (1 \leqq r \leqq n, r \neq s, s + 1).$$

Let $\bar{w}_r$ be obtained from $w_r \in w$ by interchanging s and $s + 1$ if either or both occurs in the subsequence w_r. Then $\bar{w}_r$ is a subsequence of the rearrangement of J in which s and $s + 1$ are interchanged. Let $\bar{w}_r':A^n \rightarrow A^m$ be the corresponding map, and let $\bar{w} = (\bar{w}_1, \cdots, \bar{w}_k)$. The set of subsequences of J which do not contain s is a subset of both w and $\bar{w}$, and has the same order in each. If $w_r \in w$ does contain s, then $fw_r'a^n = b^0$, since $a_s = a^0$, $fA^{n-1} = b^0$ and f is invariant. If $\bar{w}_r \in \bar{w}$ does contain s, then $f\bar{w}_r'a^n = b^0$ similarly. Hence

$$(fw_1'a^n, \cdots, fw_k'a^n), \qquad (f\bar{w}_1'a^n, \cdots, f\bar{w}_k'a^n)$$

have the same image under q_k, and since

$$\bar{w}_r'a^n = w_r'\bar{a}^n$$

it follows that $f^na^n = f^n\bar{a}^n$, as required.

If $h:A_m \rightarrow B$ is a map such that $hA_{m-1} = b^0$ then $f = h \circ p_m:A^m \rightarrow B$ fulfils the conditions of (2.5), and from the compatible sequence of invariant maps defined in (2.5) we obtain a map $h^*:A_\infty \rightarrow B_\infty$ which agrees with h on A_m. Now replace B by B_∞; let $h:A_m \rightarrow B_\infty$ be a map such that $hA_{m-1} = b^0$, and let

$$h^*:A_\infty \rightarrow (B_\infty)_\infty$$

be the result. Let $Q:(B_\infty)_\infty \to B_\infty$ be the natural function, as defined immediately after (2.4) but with B instead of A and Q instead of P. We define the combinatorial extension of h to be

$$h' = Q \circ h^*: A_\infty \to B_\infty .$$

Obviously (1.7) is fulfilled. Suppose that $hA_m \subset B_n$ $(n < \infty)$. Then $h^*A_\infty \subset (B_n)_\infty$ and since $Q \mid (B_n)_\infty$ is continuous, by (2.4), it follows that h' is continuous, which proves (1.4). A similar argument proves (1.5). Notice that if B is a special complex then Q itself is continuous, and so the conclusions of (1.4) and (1.5) hold in this case even if the conditions involving n are violated.

3. Homology of the reduced product complex

Throughout this section let A be a special complex. Let $\theta^m: A \times A_{m-1} \to A_m$ be the natural map, $m = 1, 2, \cdots$, and let $\theta: A \times A_\infty \to A_\infty$ be the map which agrees with θ^m on $A \times A_{m-1}$. Then $\theta = T \mid A \times A_\infty$, where T is the natural multiplication. Let

$$\rho^m: A \times A_{m-1} \to A_{m-1}, \qquad \rho: A \times A_\infty \to A_\infty$$

be the projections in which $(x, y) \to y$. Let

$$H_r(A_{m-1}) \xleftarrow{\rho_*^m} H_r(A \times A_{m-1}) \xrightarrow{\theta_*^m} H_r(A_m)$$

be the homomorphisms induced by ρ^m and θ^m. Let

$$(\rho_*^m, \theta_*^m): H_r(A \times A_{m-1}) \to H_r(A_{m-1}) + H_r(A_m)$$

be defined by $(\rho_*^m, \theta_*^m)\gamma = (\rho_*^m\gamma, \theta_*^m\gamma)$, where $\gamma \in H_r(A \times A_{m-1})$ and $+$ indicates direct summation. We shall prove

THEOREM (3.1). *If* $r \geqq 1$ *then*

$$(\rho_*^m, \theta_*^m): H_r(A \times A_{m-1}) \approx H_r(A_{m-1}) + H_r(A_m).$$

Before the proof we draw two conclusions from (3.1). In the first place by considering the homology sequence of the pair $(A \times A_{m-1}, a^0 \times A_{m-1})$ and the fact that ρ^m has a right inverse, we deduce

THEOREM (3.2). *If* $r \geqq 1$ *then*

$$H_r(A_m) \approx H_r(A \times A_{m-1}, a^0 \times A_{m-1}).$$

Hence it follows from the Künneth formula, applied to the pairs (A, a^0) and $(A_{m-1}, \varnothing)$, that if $r \geqq 1$ then

$$(3.3) \qquad H_r(A_m) \approx \textstyle\sum_{s=1}^r \{H_s(A) \otimes H_{r-s}(A_{m-1}) + H_{s-1}(A) * H_{r-s}(A_{m-1})\}$$

where the summation is direct, and $\otimes$ and $*$ denote the tensor and torsion products respectively. (3.3) is a recurrence formula which eventually expresses the homology groups of A_m in terms of those of A. The result may be written as follows. Let G^m, $m = 1, 2, \cdots$, be the group defined inductively by

$$G^1 = \textstyle\sum_{r=1}^\infty H_r(A), \qquad G^m = G^1 \otimes G^{m-1} + G^1 * G^{m-1} \quad (m \geqq 2).$$

Then it follows from (3.3) by induction that

$$\sum_{r=1}^{\infty} H_r(A_m) = G^1 + G^2 + \cdots + G^m. \tag{3.4}$$

In the second place, recall that the m-section of the complex A_∞ is contained in A_m. Hence (1.12) follows at once from (3.4). Also, if (ρ_*, θ_*) is the direct sum of the homomorphisms induced by ρ, $\theta: A \times A_\infty \to A_\infty$, as defined above, we obtain from (3.1)

THEOREM (3.5). *If* $r \geqq 1$ *then*

$$(\rho_*, \theta_*): H_r(A \times A_\infty) \approx H_r(A_\infty) + H_r(A_\infty).$$

The key to the proof of (3.1) is an auxiliary identification space, constructed as follows. Let I be the real segment $0 \leqq t \leqq 1$. Let C be the cone obtained from $A \times I$ by identifying $(a, 1)$ with $a \in A$, and $A \times 0 \cup a^0 \times I$ with a^0. Let E_m be the set $(C - A) \times A_{m-1} \cup A_m$, with the identification topology of the function $q_m: C \times A_{m-1} \to E_m$ which maps $A \times A_{m-1}$ by θ^m and is the identity on $(C - A) \times A_{m-1}$. Then A_m is topologically embedded in E_m. By taking a basis, as in the case of A_m, we find that the inclusions,

$$E_1 \subset E_2 \subset \cdots \subset E_m \subset \cdots,$$

are topological.

THEOREM (3.6). *The space* E_m *is contractible.*

We begin by showing that E_m is deformable into E_{m-1} $(m \geqq 2)$. Let

$$B_{m-1} = C \times A_{m-2} \cup a^0 \times A_{m-1} \subset C \times A_{m-1}.$$

B_{m-1} is closed and saturated by q_m, and the cell structure of A can be extended over C so that B_{m-1} is a subcomplex of the product complex $C \times A_{m-1}$. Let $u_t: C \to C$ be a homotopy, rel. a^0, such that $u_0 = 1$, $u_1 C = a^0$. We define $v_t: C \times A_{m-1} \to E_m$ by

$$v_t(c, a_*) = q_m(u_t c, a_*) \qquad (c \in C,\ a_* \in A_{m-1}).$$

Then $v_0 = q_m$, $v_1(C \times A_{m-1}) \subset E_{m-1}$, and $v_t B_{m-1} \subset E_{m-1}$. Hence the map

$$(C \times A_{m-1}, B_{m-1}) \to (E_m, E_{m-1})$$

determined by q_m is deformable into[3] E_{m-1}. Applying (8.2) below to this map, it follows that q_m is deformable into E_{m-1}, rel. B_{m-1}. Since q_m maps

$$C \times A_{m-1} - B_{m-1} = (C - a^0) \times (A_{m-1} - A_{m-2})$$

without identifications, it follows that $1: E_m \to E_m$ is deformable into E_{m-1}. Hence, by induction, E_m is deformable into E_1. But E_1 is homeomorphic to C, which is contractible. Hence E_m is contractible. This proves (3.6).

Let J^0, J^1 be the real segments $0 \leqq t \leqq \frac{1}{2}$, $\frac{1}{2} \leqq t \leqq 1$, respectively, and let

$$C^i = q(A \times J^i) \quad (i = 0, 1),$$

[3] Let $f: (X, A) \to (Y, B)$ be a map of pairs of spaces. We say that f is deformable into B if $f \simeq f'$ where $f'X \subset B$.

where $q:A \times I \to C$ is the identification map. Let $A' = C^0 \cap C^1 = q(A \times \frac{1}{2})$, let $E_m^i = q_m(C^i \times A_{m-1})$ $(i = 0, 1)$, and let $A'_m = E_m^0 \cap E_m^1 = q_m(A' \times A_{m-1})$. Then it follows from (8.3) below that A' is a neighborhood deformation retract[4] of C^0, and hence that A'_m is a neighborhood deformation retract of E_m^0. Therefore $(E^m; E_m^0, E_m^1)$ is a proper triad[5]. Since E_m is contractible by (3.6), we obtain from the Mayer-Vietoris sequence of the triad (see (15.3), Chap. I of [2]) the direct sum expression

$$H_r(A'_m) \approx H_r(E_m^0) + H_r(E_m^1) \qquad (r \geqq 1) \tag{3.7}$$

in which the components of the isomorphism are induced by the inclusions $j_i:A' \to E_m^i$.

In the following diagram ρ^m, θ^m are as before, ϕ^m is defined by

$$\phi^m(a, a_*) = q_m(q(a, \tfrac{1}{2}), a_*) \qquad (a \in A, a_* \in A_{m-1})$$

and the other maps are inclusions.

$$\begin{array}{ccccc} A_{m-1} & \xleftarrow{\rho^m} & A \times A_{m-1} & \xrightarrow{\theta^m} & A_m \\ \downarrow{\scriptstyle l_0} & & \downarrow{\scriptstyle \phi^m} & & \downarrow{\scriptstyle l_1} \\ E_m^0 & \xleftarrow[j_0]{} & A'_m & \xrightarrow[j_1]{} & E_m^1 \end{array}$$

Since C^0 is contractible to a^0 it follows that A_{m-1} is a deformation retract of E_m^0, and since A is a strong deformation retract of C^1 it follows that A_m is a deformation retract of E_m^1. Hence l_0 and l_1 are homotopy equivalences, and ϕ^m is a homoemorphism. Let

$$\psi_t^i:A \times A_{m-1} \to E_m^i \qquad (i = 0, 1; t \in I)$$

be the homotopy defined by

$$\psi_t^i(a, a_*) = q_m(q(a, \tfrac{1}{2}t + i - it), a_*),$$

where $a \in A$ and $a_* \in A_{m-1}$. Then $\psi_0^0 = l_0 \circ \rho^m$, $\psi_0^1 = l_1 \circ \theta^m$, and $\psi_1^i = j_i \circ \phi^m$ $(i = 0, 1)$. Therefore

$$j_0 \circ \phi^m \simeq l_0 \circ \rho^m, \qquad j_1 \circ \phi^m \simeq l_1 \circ \theta^m.$$

Hence the following diagram of induced homomorphisms is commutative and the verticals are isomorphisms, so that (3.1) follows from (3.7).

$$\begin{array}{ccccc} H_r(A_{m-1}) & \xleftarrow{\rho_*^m} & H_r(A \times A_{m-1}) & \xrightarrow{\theta_*^m} & H_r(A_m) \\ \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} \\ H_r(E_m^0) & \xleftarrow[j_{0*}]{} & H_r(A'_m) & \xrightarrow[j_{1*}]{} & H_r(E_m^1) \end{array}$$

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[4] Let Q be a subspace of a space P. We say that Q is a neighborhood deformation retract of P if P contains an open set of which Q is a (strong) deformation retract.

[5] As defined in Chapter I of [2].

4. Homology of the space of loops

Let A be a space, not necessarily a Hausdorff space, of which $a^0 \in A$ is a neighborhood deformation retract. Let $\hat{A}$ be the suspension of A, i.e. the space obtained from $A \times I$ by shrinking $A \times \dot{I} \cup a^0 \times I$ to a point. We identity $a \in A$ with $f(a, \frac{1}{2}) \in \hat{A}$, where $f:A \times I \to \hat{A}$ is the identification map, so that $A \subset \hat{A}$. Let E be the space consisting of maps $\lambda:I \to \hat{A}$ such that $\lambda(1) = a^0$, with the compact-open topology. Let $p:E \to \hat{A}$ be defined by $p(\lambda) = \lambda(0)$. Then p is a fibre mapping, by Prop. 4 on p. 479 of [5], and $p^{-1}(a^0) = \Omega_A$, the space of loops on $\hat{A}$. For brevity we write $\Omega = \Omega_A$ and embed $A \subset \Omega$ so that $a \in A$ is identified with the path $t \to f(a, t) \quad (t \in I)$. Let $\rho', \theta':A \times \Omega \to \Omega$ be the maps given by

$$\rho'(a, \lambda) = \lambda; \qquad \theta'(a, \lambda) = a(2t) \qquad (0 \leqq t \leqq \tfrac{1}{2}),$$
$$= \lambda(2t - 1) \qquad (\tfrac{1}{2} \leqq t \leqq 1),$$

where $a \in A$, $\lambda \in \Omega$. Then with (ρ'_*, θ'_*) defined as the direct sum of the induced homomorphisms we prove (cf. (3.5))

THEOREM (4.1). *If* $r \geqq 1$ *then*

$$(\rho'_*, \theta'_*):H_r(A \times \Omega) \approx H_r(\Omega) + H_r(\Omega).$$

As in §3 let J^0, J^1 be the segments $0 \leqq t \leqq \frac{1}{2}$, $\frac{1}{2} \leqq t \leqq 1$, respectively. Let $D^i = f(A \times J^i)$, and let $E^i = p^{-1}(D^i) \quad (i = 0, 1)$. Then $A = D^0 \cap D^1$, and it follows from (8.3) below that A is a neighborhood deformation retract of D^0. Hence by the covering homotopy theorem $F = E^0 \cap E^1$ is a neighborhood deformation retract of E^0, and so $(E; E^0, E^1)$ is a proper triad. Then since E is contractible we obtain from the Mayer-Vietoris sequence of this triad the direct sum

$$H_r(F) \approx H_r(E^0) + H_r(E^1) \qquad (r \geqq 1), \tag{4.2}$$

in which the components of the isomorphism are induced by the inclusions $j'_i:F \to E^i \quad (i = 0, 1)$.

In the following diagram ρ', θ' are as before, and the other maps are inclusions except for ϕ', whose value on $(a, \lambda) \in A \times \Omega$ is given by

$$\phi'(a, \lambda)(t) = a(t + \tfrac{1}{2}) \qquad (0 \leqq t \leqq \tfrac{1}{2}),$$
$$= \lambda(2t - 1) \qquad (\tfrac{1}{2} \leqq t \leqq 1).$$

$$\begin{tikzcd}
\Omega \arrow[d, "l'_1"'] & A \times \Omega \arrow[l, "\rho'"'] \arrow[r, "\theta'"] \arrow[d, "\phi'"] & \Omega \arrow[d, "l'_0"] \\
E^1 & F \arrow[l, "j'_1"] \arrow[r, "j'_0"'] & E^0
\end{tikzcd}$$

Since D^i is contractible $(i = 0, 1)$ it follows from the covering homotopy theorem that Ω is a deformation retract of E^i. Hence l'_i and, from the proof of Prop. 5 on p. 480 of [5], also ϕ' are homotopy equivalences. Let

$$\psi_s:A \times \Omega \to E^0 \qquad (s \in I)$$

be the homotopy defined by

$$\begin{aligned} \psi_s(a, \lambda)(t) &= a((2t + s)/(1 + s)) && (0 \leqq t \leqq \tfrac{1}{2}), \\ &= \lambda(2t - 1) && (\tfrac{1}{2} \leqq t \leqq 1), \end{aligned}$$

where $a \in A$, $\lambda \in \Omega$, $t \in I$. We have

$$\psi_0 = l_0' \circ \theta', \qquad \psi_1 = j_0' \circ \phi'$$

and so $l_0' \circ \theta' \simeq j_0' \circ \phi'$. Let

$$\psi_s' : A \times \Omega \to E^1 \qquad (s \in I)$$

be the homotopy defined by

$$\begin{aligned} \psi_s'(a, \lambda)(t) &= a(t + 1/(1 + s)) && (0 \leqq t \leqq s/(1 + s)), \\ &= \lambda(st + t - s) && (s/(1 + s) \leqq t \leqq 1), \end{aligned}$$

where $a \in A$, $\lambda \in \Omega$, $t \in I$. We have

$$\psi_0' = l_1' \circ \rho', \qquad \psi_1' = j_1' \circ \phi'$$

and so $l_1' \circ \rho' \simeq j_1' \circ \phi'$. Therefore the following diagram of induced homomorphisms is commutative, and the verticals are isomorphisms, so that (4.1) follows from (4.2).

$$\begin{tikzcd} H_r(\Omega) \arrow[d, "\approx"] & H_r(A \times \Omega) \arrow[l, "\rho_*'"'] \arrow[r, "\theta_*'"] \arrow[d, "\approx"] & H_r(\Omega) \arrow[d, "\approx"] \\ H_r(E^1) & H_r(F) \arrow[l, "j_{1*}'"] \arrow[r, "j_{0*}'"'] & H_r(E^0) \end{tikzcd}$$

Part II: the canonical maps

5. The main theorems

Let A be a Hausdorff space, which admits a distance relative to a^0. We mean by this that a real-valued map can be defined on A which vanishes at a^0 and is positive everywhere else. Such a map we shall refer to as a distance, without implying that there is a metric on the space. A metric space or a CW-complex certainly admits a distance relative to any point. Given a distance we shall define in §7 a map

$$\alpha : A_\infty \to \Omega$$

which is the identity on A. We describe α as the canonical map associated with the given distance. We shall prove in §7

THEOREM (5.1). *If α, $\bar{\alpha}$ are the canonical maps associated with two choices of distance in A then $\alpha \simeq \bar{\alpha}$, rel. A.*

Let B be a closed subspace of A which contains a^0. Then B_∞, the reduced product space of B, rel. a^0, is naturally embedded in A_∞, as we saw in §1. Also

Ω_B , the space of loops on the suspension of B, is naturally embedded in $\Omega = \Omega_A$. We shall prove

THEOREM (5.2). *Let*

$$\alpha: A_\infty \rightarrow \Omega_A, \qquad \beta: B_\infty \rightarrow \Omega_B$$

be the canonical maps associated with a distance in A *and the restriction of that distance to* B, *respectively. Then* $\alpha \mid B_\infty$ *agrees with* β.

Let A and B be special complexes, and let $h: A \rightarrow B$ be a map such that $ha^0 = b^0$. Let $\hat{h}: \hat{A} \rightarrow \hat{B}$ be the suspension of h, defined by $\hat{h}f(a, t) = g(ha, t)$, where $a \in A$, $t \in I$ and where

$$f: A \times I \rightarrow \hat{A}, \qquad g: B \times I \rightarrow \hat{B}$$

are the identification maps. Let $h_\Omega: \Omega_A \rightarrow \Omega_B$ be the map defined by

$$h_\Omega(\lambda) = \hat{h} \circ \lambda \qquad (\lambda \in \Omega_A),$$

and let $h_\infty: A_\infty \rightarrow B_\infty$ map termwise by h. We shall prove

THEOREM (5.3). *Let*

$$\alpha: A_\infty \rightarrow \Omega_A, \qquad \beta: B_\infty \rightarrow \Omega_B$$

be canonical maps. Then

$$\beta \circ h_\infty \simeq h_\Omega \circ \alpha, \text{ rel. } A.$$

In the rest of this section let A be a special complex and let $\alpha: A_\infty \rightarrow \Omega$ be a canonical map, where $\Omega = \Omega_A$. In the following diagram let θ mean the same as in §3 and let θ' mean the same as in §4.

$$\text{(5.4)} \qquad \begin{array}{ccc} A \times A_\infty & \xrightarrow{\theta} & A_\infty \\ {\scriptstyle 1 \times \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A \times \Omega & \xrightarrow[\theta']{} & \Omega \end{array}$$

We shall prove

LEMMA (5.5). *In* (5.4), $\alpha \circ \theta \simeq \theta' \circ (1 \times \alpha)$.

All these results must wait for proof until the definition of a canonical map in §7. For motivation, meanwhile, we shall assume two of them, namely (5.2) and (5.5), and use them in this section with the homology analysis in §3 and §4 to deduce our main theorem:

THEOREM (5.6). *If* A *is a special complex then a canonical map induces an isomorphism (the canonical isomorphism) of the homotopy sequence of the pair* (A_∞ , A) *onto that of the pair* (Ω, A).

COROLLARY (5.7). *A canonical map induces an isomorphism (the canonical isomorphism) of the singular homology sequence of the pair* (A_∞ , A) *onto that of the pair* (Ω, A).

COROLLARY (5.8). *A canonical map induces an isomorphism (the canonical isomorphism) of the singular cohomology ring of* Ω *onto that of* A_∞.

The first of the corollaries follows from the theorem by applying (8.5) in the appendix below. The second follows by applying the universal coefficient theorem (cf. Ex. 4 on p. 161 of [2]) together with the naturality of the cup product.

In the proof of (5.6), to begin with, let A be a *finite* CW-complex with a^0 its only vertex; this restriction will be lifted in due course. Let M be the mapping cylinder of the canonical map α. Then $A_\infty \subset M$ and $\Omega \subset M$, and so since $A \subset \Omega$ there are two distinct embeddings of A in M. Which embedding we mean in a given context will always be plain. It follows from Lemma 4 of [8] (this is where we seem to need the finiteness condition) that we can identify the product space $A \times M$ with the mapping cylinder of $1 \times \alpha$; assume this done. In the following diagram let σ be the projection $(x, y) \to y$ and let ϕ be the map which agrees with θ on $A \times A_\infty$ and with θ' on $A \times \Omega$ and whose existence is guaranteed by (5.5).

$$\sigma, \phi: (A \times M, A \times A_\infty) \to (M, A_\infty).$$

Since A is pathwise connected it follows that A_∞, Ω and M are all pathwise connected. Both σ and ϕ induce homomorphisms of the reduced homology sequence of the pair $(A \times M, A \times A_\infty)$ into the reduced homology sequence of the pair (M, A_∞). Let ψ denote the direct sum of these two homomorphisms, so that ψ maps the former sequence into the direct sum of two copies of the latter. If $r \geqq 0$ then from (3.5) and (4.1) (remembering that Ω is a deformation retract of M) we have

$$\psi: H_r(A \times A_\infty) \approx H_r(A_\infty) + H_r(A_\infty),$$

$$\psi: H_r(A \times M) \approx H_r(M) + H_r(M),$$

where the case $r = 0$ is properly included since the zero-dimensional groups are zero. Applying the five lemma (Lemma 7 of [1]) it follows that if $r \geqq 0$ then

$$\psi: H_r(A \times M, A \times A_\infty) \approx H_r(M, A_\infty) + H_r(M, A_\infty). \tag{5.9}$$

We use (5.9) to prove

LEMMA (5.10).

$$H_r(M, A_\infty) = 0 \qquad (r \geqq 0).$$

The lemma is trivial if $r = 0$. We make the inductive hypothesis that (5.10) has been established for all $r < m$ $(m \geqq 1)$ and show how it follows that $H_m(M, A_\infty) = 0$. Let $i: (M, A_\infty) \to (A \times M, A \times A_\infty)$ be the map defined by

$$i(\lambda) = (a^0, \lambda) \qquad (\lambda \in M),$$

and let

$$H_m(M, A_\infty) \underset{\sigma_*}{\overset{i_*}{\rightleftarrows}} H_m(A \times M, A \times A_\infty)$$

be the homomorphisms induced by i and σ. Since $\sigma \circ i = 1$ we have $\sigma_* \circ i_* = 1$. Thus i_* is an isomorphism into. However, by the inductive hypothesis it follows from the Künneth formula, applied to the product of the pairs $(A, \varnothing)$ and (M, A_∞), that i_* is onto. Therefore i_* and σ_* are isomorphisms. However, ψ is an isomorphism in (5.9) with $r = m$, and so the component of ψ induced by ϕ is trivial, i.e. $H_m(M, A_\infty) = 0$. This proves (5.10).

To facilitate examining the homotopy groups we prove

LEMMA (5.11). *$\pi_1(A_\infty)$ operates trivially on $\pi_r(M, A_\infty)$.*

To prove (5.11) we re-examine the map

$$\phi:(A \times M, A \times A_\infty) \to (M, A_\infty).$$

Let ϕ'_* be the endomorphism of $\pi_r(M, A_\infty)$ which is induced by

$$\phi':(M, A_\infty) \to (M, A_\infty)$$

where $\phi'(\lambda) = \phi(a^0, \lambda)$ $(\lambda \in M)$. If $a \in A$ then $\phi(a, a^0) = \theta(a, a^0) = a$. Hence, by (8.4) below, the injection of $\pi_1(A)$ into $\pi_1(A_\infty)$, which is onto since A contains the 1-section of A_∞, operates trivially on the image of ϕ'_*. Hence (5.11) will follow from

LEMMA (5.12).

$$\phi'_*\pi_r(M, A_\infty) = \pi_r(M, A_\infty).$$

If $\lambda \in \Omega$ we have

$$\begin{aligned} \phi'(\lambda)(t) = \theta'(a^0, \lambda)(t) &= a^0 && (0 \leqq t \leqq \tfrac{1}{2}), \\ &= \lambda(2t - 1) && (\tfrac{1}{2} \leqq t \leqq 1). \end{aligned}$$

Hence the map of Ω into itself determined by ϕ' is homotopic to the identity. Hence, and since Ω is a deformation retract of M, it follows that ϕ', regarded as a map of M into itself, is homotopic to the identity. Moreover, if $a_* \in A_\infty$ then

$$\phi'(a_*) = \theta(a^0, a_*) = a_*,$$

i.e. ϕ' maps A_∞ identically. Therefore ϕ' induces automorphisms of $\pi_r(A_\infty)$ and of $\pi_r(M)$ in the homotopy sequence of the pair (M, A_∞). Application of the five lemma proves that ϕ'_* is an isomorphism, in particular we have (5.12), and so the proof of (5.11) is complete.

It follows from (2.6) of [6] that the elements of $\pi_2(\hat{A})$ are the suspensions of those of $\pi_1(A)$ and so the injection of $\pi_1(A)$ into $\pi_1(\Omega)$ is onto. Hence the injection of $\pi_1(A)$ through $\pi_1(A_\infty)$ into $\pi_1(M)$ is onto, and *a fortiori* the injection of $\pi_1(A_\infty)$ into $\pi_1(M)$ is onto. Hence and from (5.11) we can apply the form of the relative Hurewicz theorem given by Blakers as (23.3) of [1] to the pair (M, A_∞), and deduce from (5.10) that $\pi_r(M, A_\infty) = 0$ for all $r \geqq 0$. Hence we obtain, using the exactness of the homotopy sequence of the pair (M, A_∞), that $\pi_r(A_\infty) \approx \pi_r(M)$ under the injection. Since M is the mapping cylinder of α this proves

THEOREM (5.13). *If A is a finite CW-complex with a single vertex and if α is a canonical map then*

$$\alpha_*:\pi_r(A_\infty) \approx \pi_r(\Omega) \qquad (r \geqq 0).$$

The next stage is to remove the restriction to finite complexes in (5.13). Henceforth let A be a special complex. Then A is the union of a sequence of finite subcomplexes whose only vertex is a^0, say:

$$ {}_0A \subset {}_1A \subset \cdots \subset {}_mA \subset \cdots . $$

The suspension of A is again a special complex, with the usual cell-structure, and is the union of the sequence of subcomplexes obtained by suspending each complex of the previous sequence:

$$ {}_0\hat{A} \subset {}_1\hat{A} \subset \cdots \subset {}_m\hat{A} \subset \cdots . $$

Let ${}_m\Omega$ denote the space of loops on ${}_m\hat{A}$ based at a^0. Then Ω contains the sequence of closed subspaces:

$$ {}_0\Omega \subset {}_1\Omega \subset \cdots \subset {}_m\Omega \subset \cdots . $$

Let $\alpha: A_\infty \to \Omega$ be the given canonical map. Then by (5.2) the image of ${}_mA_\infty$, the reduced product complex of ${}_mA$, is contained in ${}_m\Omega$, and the map ${}_m\alpha: {}_mA_\infty \to {}_m\Omega$ determined by α is a canonical map for ${}_mA$. Hence, denoting the induced homomorphisms by ${}_m\alpha_*$ and α_*, and the injections by i and j, we have the following commutative diagram:

(5.14)
$$ \begin{array}{ccc} \pi_r({}_mA_\infty) & \xrightarrow{{}_m\alpha_*} & \pi_r({}_m\Omega) \\ i\downarrow & & \downarrow j \\ \pi_r(A_\infty) & \xrightarrow[\alpha_*]{} & \pi_r(\Omega) \end{array} $$

We proved in (5.13) that ${}_m\alpha_*$ is an isomorphism; we now assert

LEMMA (5.15). *α_* is an isomorphism.*

We prove that α_* maps onto a given element $x \in \pi_r(\Omega)$ by making a suitable choice of m in (5.14). Let $f: S^r \to \Omega$ be a map which represents x, and let $f': S^r \times I \to \hat{A}$ be the function defined by

$$ f'(x, t) = (fx)(t) \qquad (x \in S^r, t \in I). $$

Since f' is continuous, by Theorem 1 of [3], it follows that $S^r \times I$ has a compact image which, by (D) on page 225 of [9], is contained in a finite subcomplex of $\hat{A}$, and that, in its turn, is contained in ${}_m\hat{A}$ if m is chosen sufficiently large. Hence $fS^r \subset {}_m\Omega$. Let $g: S^r \to {}_m\Omega$ be the map which f determines, and let $y \in \pi_r({}_m\Omega)$ be its homotopy class. Let $z \in \pi_r({}_mA_\infty)$ be the element such that ${}_m\alpha_*(z) = y$. Then $jy = x$ and so, from (5.14),

$$ \alpha_* i(z) = j\,{}_m\alpha_*(z) = jy = x. $$

This proves that α_* is onto. A similar argument shows that $\alpha_*^{-1}(0) = 0$ and completes the proof of (5.15).

Besides (5.15) we know that α maps A identically. Therefore, by the five lemma, the homomorphism of the homotopy sequence of (A_∞, A) into that of

(Ω, A) which α induces is an isomorphism. Thus our main result, (5.6), is established on the assumption that canonical maps exist and have the stated properties (5.1)–(5.5). The last two sections of this paper (apart from the appendix) are devoted to defining canonical maps and demonstrating these properties.

6. Spaces with a distance relative to a point

Let Y be a space and let Ω be the space of maps $(I, \dot{I}) \to (Y, y_0)$, where $y_0 \in Y$ is a basepoint, with compact-open topology. Let

$$W:\Omega \times \Omega \to \Omega$$

be the usual multiplication by composition of paths, given by

$$\begin{aligned} W(\lambda, \lambda')(t) &= \lambda(2t) && (0 \leqq t \leqq \tfrac{1}{2}), \\ &= \lambda'(2t - 1) && (\tfrac{1}{2} \leqq t \leqq 1), \end{aligned}$$

where $\lambda, \lambda' \in \Omega$. With this multiplication Ω is an H-space (Prop. 1 on p. 474 of [5]). Let B, B' be Hausdorff spaces which admit distances ρ, ρ' relative to b_0, b_0' respectively. We embed $B, B' \subset B \times B'$ so that $b = (b, b_0')$, $b' = (b_0, b')$ if $b \in B$, $b' \in B'$. Let

$$B \xrightarrow{h} \Omega \xleftarrow{h'} B'$$

be maps, such that $hb_0 = h'b_0' = y_0^*$, where y_0^* is the constant path. We shall define a homotopy

$$k_s:B \times B' \to \Omega \qquad (s \in I)$$

with the following two properties. We write $k_1 = k$ since it will be of principal interest. Firstly

(6.1) $$k \mid B = h, \qquad k \mid B' = h'.$$

Secondly, we have a commutative diagram:

(6.2)
$$\begin{tikzcd} B \times B' \arrow[d, "h \times h'"'] \arrow[dr, "k_0"] & \\ \Omega \times \Omega \arrow[r, "W"'] & \Omega \end{tikzcd}$$

k_s is a variant of the following construction, which is slightly more general. Let

$$B \times I \xrightarrow{f} Y \xleftarrow{f'} B' \times I$$

be maps such that

$$f(B \times 1 \cup b_0 \times I) = f'(B' \times 0 \cup b_0' \times I) = y_0 .$$

Then we define a homotopy

$$g_s:B \times B' \times I \to Y \qquad (s \in I)$$

as follows. Let $b \in B$, $b' \in B'$, $t \in I$ and let

$$\beta_s = 1 - s + s\rho(b), \qquad \beta'_s = 1 - s + s\rho'(b'), \qquad \gamma_s = \beta_s + \beta'_s .$$

Notice that $\beta_s > 0$ unless $s = 1$ and $b = b_0$, and that $\beta'_s > 0$ unless $s = 1$ and $b' = b'_0$. We define

$$(6.3) \quad \begin{cases} g_s(b, b', t) = f(b, \gamma_s t/\beta_s) & \textit{if } \beta_s > 0 \textit{ and } \gamma_s t \leqq \beta_s \\ = f'(b', (\gamma_s t - \beta_s)/\beta'_s) & \textit{if } \beta'_s > 0 \textit{ and } \gamma_s t \geqq \beta_s \\ = y_0, \textit{ otherwise.} \end{cases}$$

It is easy to check that g_s is a homotopy. Notice that for all $b \in B$ and $b' \in B'$ we have $\beta_0 = \beta'_0 = 1$, $\gamma_0 = 2$ and so

$$(6.4) \quad \begin{cases} g_0(b, b', t) = f(b, 2t) & (0 \leqq t \leqq \tfrac{1}{2}), \\ = f'(b', 2t - 1) & (\tfrac{1}{2} \leqq t \leqq 1). \end{cases}$$

For comparison, the value of g_1 is given in terms of $\beta = \rho(b)$, $\beta' = \rho'(b')$, and $\gamma = \beta + \beta'$ as follows

$$(6.5) \quad \begin{cases} g_1(b, b', t) = f(b, \gamma t/\beta) & \textit{if } \beta > 0 \textit{ and } \gamma t \leqq \beta, \\ = f'(b', (\gamma t - \beta)/\beta') & \textit{if } \beta' > 0 \textit{ and } \gamma t \geqq \beta, \\ = y_0, \textit{ otherwise.} \end{cases}$$

Notice that

$$(6.6) \qquad g_1(b, b'_0, t) = f(b, t), \qquad g_1(b_0, b', t) = f'(b', t).$$

If Z is any space we describe functions

$$u: Z \times I \to Y, \qquad v: Z \to \Omega$$

as associates if $u(z, t) = vz(t)$ $(z \in Z, t \in I)$. By Theorem 1 of [3], u is continuous if and only if v is continuous. Let h and h' be as before and let f and f' in the preceding paragraph be their respective associates. Then the associate of g_s is a homotopy

$$k_s: B \times B' \to \Omega.$$

This defines k_s, and we see that (6.1) follows from (6.6) and (6.2) from (6.4). By inspection of (6.3) we also verify

LEMMA (6.7). *Let $b, \bar{b} \in B$ be equidistant from b_0 and let $b', \bar{b}' \in B'$ be equidistant from b'_0. Then if $hb = h\bar{b}$ and $h'b' = h'\bar{b}'$ we have*

$$k_s(b, b') = k_s(\bar{b}, \bar{b}') \qquad (s \in I).$$

As we have defined it k depends on the choice of distances ρ in B and ρ' in B'. Let $\bar{k}$ be constructed by the same formula with distances $\bar{\rho}$ in B and $\bar{\rho}'$ in B'. Let ρ_t, ρ'_t be the distances in B, B', respectively, which are defined by

$$(6.8) \qquad \rho_t = (1 - t)\rho + t\bar{\rho}, \qquad \rho'_t = (1 - t)\rho' + t\bar{\rho}'; \qquad (t \in I).$$

Let k^t be constructed with these distances. Then $k^0 = k$, since $\rho_0 = \rho$ and $\rho_0' = \rho'$, and $k^1 = \bar{k}$ similarly. The function $B \times B' \times I \to \Omega$, which is given by

$$(b, b', t) \to k^t(b, b') \qquad (b \in B, b' \in B'),$$

is continuous and so we obtain

LEMMA (6.9). *k^t is a homotopy of k into $\bar{k}$.*

Henceforth we write

$$k = k[h, h'], \qquad k_s = k_s[h, h'], \qquad k^t = k^t[h, h']$$

so as to emphasize the dependence of the construction on h and h'. Replace h' by a homotopy $h_t':B' \to \Omega$ such that $h_t'b_0' = y_0^*$ $(t \in I)$. Then the map

$$k^t[h, h_t']:B \times B' \to \Omega$$

is defined for each $t \in I$. It can be verified that the function $B \times B' \times I \to \Omega$, which is given by

$$(b, b', t) \to k^t[h, h_t'](b, b') \qquad (b \in B, b' \in B'),$$

is continuous, and hence we have

LEMMA (6.10). *$k^t[h, h_t']$ is a homotopy of $k[h, h_0']$ into $\bar{k}[h, h_1']$.*

The lemmas in this section have been stated with a view to applications in the next section. However, we conclude with a result which is not applied but which seems to be of interest in itself.

THEOREM (6.11). *Let Y be a Hausdorff space which admits a distance relative to y_0. Then Ω, the space of loops on Y based at y_0, is an h-space relative to the constant path.*

Let σ be the distance on Y and let ρ be the corresponding distance on Ω, relative to the constant path y_0^*, which is defined by

$$\rho(\lambda) = \sup \sigma\lambda(t) \qquad (\lambda \in \Omega),$$

where the upper bound is taken over $t \in I$. With this distance and $B = B' = \Omega$ the map

$$k = k[1, 1]:\Omega \times \Omega \to \Omega$$

is defined and by (6.1) we have

$$k(\lambda, y_0^*) = k(y_0^*, \lambda) = \lambda \qquad (\lambda \in \Omega).$$

Therefore Ω is an h-space with k as multiplication. Notice that, in view of (6.2), $k_s[1, 1]$ is a homotopy of W, the usual multiplication with which Ω is merely an H-space, into k.

7. The canonical map

Let A be a Hausdorff space which admits a distance relative to a^0, say ρ, and let A be embedded in Ω, which now means the space of loops on $\hat{A}$ again, as in §5. The product A^m admits the distance ρ^m relative to a^0 which is defined by

$$\rho^m(a_1, \cdots, a_r, \cdots, a_m) = \rho(a_1) + \cdots + \rho(a_r) + \cdots + \rho(a_m) \qquad (a_r \in A).$$

In the construction of §6 we set

$$(B, b_0) = (A, a^0), \qquad (B', b_0') = (A^{m-1}, a^0),$$

with distances ρ, ρ^{m-1}, and take $Y = \hat{A}$, $y_0 = a^0$. We identify $A \times A^{m-1} = A^m$ in the obvious way. Then a sequence of maps f^m; $A^m \to \Omega$ is defined inductively so that f_1 is the inclusion map and $(m \geqq 2)$ $f^m = k[f^1, f^{m-1}]$, in the notation of §6. For some purposes it is convenient to have f^m defined explicitly, and the following formula may be verified from (6.5) by induction on m. Let

$$a^m = (a_1, \cdots, a_m) \in A^m$$

and let $\alpha_r = \rho(a_r)$ $(1 \leqq r \leqq m)$. Let

$$\alpha^0 = 0, \quad \alpha^r = \alpha_1 + \alpha_2 + \cdots + \alpha_r \qquad (1 \leqq r \leqq m).$$

We have $f^m a^0 = a^0$. If $a^m \neq a^0$ and $t \in I$, there exists r $(1 \leqq r \leqq m)$ such that $\alpha_r > 0$ and $\alpha^{r-1} \leqq \alpha^m t \leqq \alpha^r$. Then

$$f^m(a^m)(t) = a_r((\alpha^m t - \alpha^{r-1})/\alpha_r). \tag{7.1}$$

Examination of (7.1) leads to the conclusion that f^m has the same value on either point of a contiguous pair, indeed we find

LEMMA (7.2). *$(f^1, f^2, \cdots, f^m, \cdots)$ is a compatible sequence of invariant maps.*

We now define the canonical map associated with the distance ρ to be the map $\alpha: A_\infty \to \Omega$ which is given by $\alpha \circ p_m = f^m$. Notice that α maps A identically, and that (5.2) is obvious from (7.1). For convenience of reference we summarize (5.2) here as

(7.3) *If B is a closed subspace of A which contains a^0 and if β is the canonical map associated with the restriction of ρ to B then $\alpha \mid B_\infty$ agrees with β.*

We also restate (5.1) in the following form, and proceed at once to its proof.

LEMMA (7.4). *Let α, $\bar{\alpha}$ be the canonical maps associated with distances ρ, $\bar{\rho}$ respectively, relative to a^0. Then $\alpha \simeq \bar{\alpha}$,* rel. A.

We apply (6.10) with $B = A$, $B' = A^{m-1}$ and with distances $\rho' = \rho^{m-1}$, $\bar{\rho}' = \bar{\rho}^{m-1}$ in A^{m-1}. Let ρ_t and ρ_t' be defined as in (6.8) and let α_t be the canonical map associated with ρ_t $(t \in I)$. Then we have

$$\alpha_t \circ p_m = k^t[f_1, \alpha_t \circ p_{m-1}]: A^m \to \Omega,$$

and it follows by induction from (6.10) that $\alpha_t \circ p_m$ is a homotopy. Hence α_t is a homotopy, and since α_t is a canonical map for each $t \in I$, α_t maps A identically. This proves (7.4).

We shall now prove (5.5). Let

$$f_s^{m+1} = k_s[f^1, f^m]: A \times A^m \to \Omega \qquad (s \in I),$$

so that $f_1^{m+1} = f^{m+1}$. Then since $\theta' = W \mid A \times \Omega$, where $\theta': A \times \Omega \to \Omega$ is as defined in §4, we obtain from (6.2) the relation

$$f_0^{m+1} = \theta' \circ (1 \times f^m): A \times A^m \to \Omega. \tag{7.5}$$

Let $a^m, \bar{a}^m \in A^m$ be such that $p_m a^m = p_m \bar{a}^m$. Then a^m and $\bar{a}^m$ are equidistant from a^0, and $f^m a^m = f^m \bar{a}^m$, by (7.2). Hence by (6.7) we have

$$f_s^{m+1}(a, a^m) = f_s^{m+1}(a, \bar{a}^m) \qquad (a \in A,\ s \in I).$$

In other words

$$f_{m+1}^s = f_s^{m+1} \circ (1 \times p_m)^{-1}: A \times A_m \to \Omega$$

is single-valued, and hence, by (2.2) and the corollary to Lemma 3 of [8], f_{m+1}^s is a homotopy. Moreover, since $f^m \mid A^{m-1} = f^{m-1}$ we have $f_{m+1}^s \mid A \times A_{m-1} = f_m^s$. Now let A_∞ be a special complex, as in (5.5). Since, by (1.9), the weak topology in the product complex $A \times A_\infty$ is the same as the product topology, it follows that a homotopy

$$f_\infty^s: A \times A_\infty \to \Omega$$

is defined by $f_\infty^s \mid A \times A_m = f_{m+1}^s$. Let $\theta: A \times A_\infty \to A_\infty$ be the same as in §3, so that $\theta \circ (1 \times p_m)$ has the same values as p_{m+1}. Then $f_\infty^1 = \alpha \circ \theta$, since

$$f_{m+1}^1 \circ (1 \times p_m) = f_1^{m+1} = f^{m+1} = \alpha \circ p_{m+1}.$$

Also $f_\infty^0 = \theta' \circ (1 \times \alpha)$ since, by (7.5),

$$\begin{aligned} f_{m+1}^0 \circ (1 \times p_m) = f_0^{m+1} &= \theta' \circ (1 \times f^m) \\ &= \theta' \circ (1 \times \alpha) \circ (1 \times p_m). \end{aligned}$$

Therefore f_∞^s is a homotopy of $\theta' \circ (1 \times \alpha)$ into $\alpha \circ \theta$, which proves (5.5).

There remains the naturality theorem (5.3). Let A and B be special complexes whose only vertices are a^0 and b^0, respectively, and let $h: A \to B$ be a cellular map. Then the mapping cylinder of h has cell structure, as described at the top of p. 238 of [9], in which A and B are embedded as subcomplexes and in which a 1-cell joins a^0 and b^0, the only vertices. Let C be the complex obtained from the mapping cylinder by shrinking the closure of this 1-cell to a point, and let $g: A \times I \to C$ be the identification map, so that $g(a, 0) = a$, $g(a, 1) = ha \quad (a \in A)$, $g(a^0 \times I) = a^0$, and g maps $(A - a^0) \times (I - 1)$ homeomorphically into C. Then C is a countable CW-complex in which A and B are embedded as subcomplexes and in which $a^0 = b^0$ is the only vertex. Consider the homotopy $w_t: C \to C$ $(t \in I)$ which maps B identically and is such that

$$w_t g(a, s) = g(a, 1 - t + st) \qquad (a \in A,\ s \in I).$$

We have $w_0 C \subset B$, $w_1 = 1$, and if $r: C \to B$ is the retraction determined by w_0, so that $rg(a, s) = ha$, we have

$$r \circ w_t = r \qquad (t \in I). \tag{7.6}$$

Let $i: A \to C$, $j: B \to C$ be the inclusion maps, so that we have

$$r \circ i = h, \qquad r \circ j = 1, \qquad j \circ r = w_0. \tag{7.7}$$

Let A and B receive distances relative to a^0 by restricting a chosen distance on C. The canonical maps α, β, and γ associated with these distances are shown in the following diagram.

$$\begin{tikzcd} A_\infty \arrow[r, "i_\infty"] \arrow[d, "\alpha"'] & C_\infty \arrow[r, shift left, "r_\infty"] \arrow[d, "\gamma"'] & B_\infty \arrow[l, shift left, "j_\infty"] \arrow[d, "\beta"] \\ \Omega_A \arrow[r, "i_\Omega"'] & \Omega_C \arrow[r, shift right, "r_\Omega"'] & \Omega_B \arrow[l, shift right, "j_\Omega"'] \end{tikzcd}$$

In the above diagram, i_∞, j_∞ and r_∞ map termwise by i, j and r, respectively, so that i_∞ embeds A_∞ and j_∞ embeds B_∞ as a subcomplex of C_∞. Also, i_Ω, j_Ω and r_Ω are obtained by composition with the suspensions of i, j and r, respectively, so that i_Ω embeds Ω_A and j_Ω embeds Ω_B in Ω_C. Applying (7.3) we obtain

$$\gamma \circ i_\infty = i_\Omega \circ \alpha, \qquad \gamma \circ j_\infty = j_\Omega \circ \beta. \tag{7.8}$$

Let $v_t: C_\infty \to C_\infty$ map termwise by w_t $(t \in I)$. Then $v_1 = 1$, and from (7.7) we have

$$r_\Omega \circ j_\Omega = 1, \qquad j_\infty \circ r_\infty = v_0. \tag{7.9}$$

Therefore

$$\begin{aligned} \beta \circ r_\infty \circ i_\infty &= r_\Omega \circ j_\Omega \circ \beta \circ r_\infty \circ i_\infty, && \text{by (7.9)},\\ &= r_\Omega \circ \gamma \circ j_\infty \circ r_\infty \circ i_\infty, && \text{by (7.8)},\\ &\simeq r_\Omega \circ \gamma \circ i_\infty, \quad \text{under } r_\Omega \circ \gamma \circ v_t \circ i_\infty, &&\\ &= r_\Omega \circ i_\Omega \circ \alpha, && \text{by (7.8)}. \end{aligned}$$

Since i_∞ maps A identically, since v_t maps A into C by w_t, and since γ maps C identically, we have $\gamma v_t i_\infty(a) = w_t(a)$ $(a \in A)$. However

$$r_\Omega w_t(a) = r w_t(a) = r(a),$$

by (7.6), and so $r_\Omega \circ \gamma \circ v_t \circ i_\infty$ is a homotopy rel. A. Now $h_\infty = r_\infty \circ i_\infty$, which maps termwise by h, and $h_\Omega = r_\Omega \circ i_\Omega$, which is obtained by composition with the suspension of h, mean the same as in (5.3). Since we have proved that

$$\beta \circ h_\infty \simeq h_\Omega \circ \alpha, \qquad \text{rel. } A, \tag{7.10}$$

(5.3) is established if h is a cellular map and if the canonical maps α and β are those associated with our particular choice of distance. This last restriction is removed at once by appealing to (7.4). Hence it remains to establish (7.10) if h is not necessarily a cellular map.

Let h be a map such that $ha^0 = b^0$. Then by (L) on p. 229 of [9] there is a homotopy $w_t: A \to B$, rel. a^0, such that $w_0 = h$, and $w_1 = h'$, where h' is a cellular map. Let $v_t:A_\infty \to B_\infty$ map termwise by w_t and let $u_t:\Omega_A \to \Omega_B$ be obtained by composition with the suspension of w_t $(t \in I)$. Then

$$v_0 = h_\infty, \quad v_1 = h'_\infty, \quad u_0 = h_\Omega, \quad u_1 = h'_\Omega.$$

By (7.10), applied to h', there is a homotopy $x_t:A_\infty \to \Omega_B$, rel. A, such that $x_0 = \beta \circ h'_\infty$ and $x_1 = h'_\Omega \circ \alpha$. Let

$$y:A_\infty \times I \to \Omega_B$$

be the map defined by

$$\begin{aligned} y(a_*, t) &= \beta v_{3t}(a_*) && (0 \leqq t \leqq \tfrac{1}{3}) \\ &= x_{3t-1}(a_*) && (\tfrac{1}{3} \leqq t \leqq \tfrac{2}{3}) \\ &= u_{3-3t}\alpha(a_*) && (\tfrac{2}{3} \leqq t \leqq 1), \end{aligned}$$

where $a_* \in A_\infty$. Then if $a \in A \subset A_\infty$ we have

$$\begin{aligned} y(a, t) &= w_{3t}(a) && (0 \leqq t \leqq \tfrac{1}{3}) \\ &= w_1(a) && (\tfrac{1}{3} \leqq t \leqq \tfrac{2}{3}) \\ &= w_{3-3t}(a) && (\tfrac{2}{3} \leqq t \leqq 1). \end{aligned}$$

Hence $y \mid A \times I \simeq z$, rel. $A \times \dot{I}$, where $z:A \times I \to \Omega_B$ is given by $z(a, t) = ha$ $(a \in A,\ t \in I)$. Hence by the homotopy extension theorem ((J) on p. 228 of [9]), applied to the pair $(A_\infty \times I,\ A_\infty \times \dot{I} \cup A \times I)$, there is a homotopy of y, rel. $A_\infty \times \dot{I}$, into a map $y':A_\infty \times I \to \Omega_B$ such that $y' \mid A \times I = z$. Then y' defines a homotopy, rel. A, of $\beta \circ h_\infty$ into $h_\Omega \circ \alpha$. This completes the proof of (5.3).

We conclude with the following observation. If A is a Hausdorff space then A_∞ admits a homeomorphism $V:A_\infty \to A_\infty$ in which each sequence has its order reversed, for example

$$V(a_1, a_2, a_3) = (a_3, a_2, a_1).$$

Let $f:A \times I \to \hat{A}$ be the identification map. Then $\hat{A}$ admits the homeomorphism $R:\hat{A} \to \hat{A}$ defined by

$$Rf(a, t) = f(a, 1 - t) \qquad (a \in A,\ t \in I),$$

and Ω admits the homeomorphism $U:\Omega \to \Omega$ defined by

$$U\lambda(t) = R\lambda(1 - t) \qquad (\lambda \in \Omega,\ t \in I).$$

Notice that U maps A identically. If A admits a distance relative to a^0 and if α is a canonical map then from (7.1) we obtain the following commutative diagram:

(7.11)
$$\begin{tikzcd} A_\infty \arrow[r, "\alpha"] \arrow[d, "V"'] & \Omega \arrow[d, "U"] \\ A_\infty \arrow[r, "\alpha"'] & \Omega \end{tikzcd}$$

8. Appendix

We shall deduce (1.9) from

LEMMA (8.1). *Let K and L be countable CW-complexes and let $G \subset K \times L$ be open in the weak topology of the product complex. Then G is open in the product topology.*

The following, with a little rewording, is Dowker's proof. Let (p, q) be a point of G. We shall show that there exist open sets $U \subset K$ and $V \subset L$ such that $p \in U$, $q \in V$ and $U \times V \subset G$. Since K is countable, K is the union of a sequence

$$K_1 \subset K_2 \subset \cdots \subset K_r \subset \cdots$$

of finite subcomplexes such that $p \in K_1$. Similarly L is the union of a sequence

$$L_1 \subset L_2 \subset \cdots \subset L_r \subset \cdots$$

of finite subcomplexes such that $q \in L_1$. We construct inductively sets U_r open in K_r and V_r open in L_r $(r = 1, 2, \cdots)$ such that $p \in U_r \subset U_{r+1}$, $q \in V_r \subset V_{r+1}$, and $\bar{U}_r \times \bar{V}_r \subset G$.

Since G is open in the weak topology of $K \times L$ we have that $G \cap F$ is open in F for each finite subcomplex F of $K \times L$. In particular $G_r = G \cap (K_r \times L_r)$ is open in the finite product complex $K_r \times L_r$. Since $(p, q) \in G_1$ it follows from (H) on p. 227 of [9] that there are neighborhoods U_1 of p in K_1 and V_1 of q in L_1 such that $U_1 \times V_1 \subset G_1$. Replacing these by smaller neighborhoods, if necessary, we assume that $\bar{U}_1 \times \bar{V}_1 \subset G_1$. Assume that U_r and V_r have been constructed for $r \leqq n$. Then $\bar{U}_n$, $\bar{V}_n$ are compact subsets of K_{n+1}, L_{n+1} respectively such that $\bar{U}_n \times \bar{V}_n \subset G_{n+1}$. For any $x \in \bar{U}_n$, $y \in \bar{V}_n$ we have $(x, y) \in G_{n+1}$ with G_{n+1} open in $K_{n+1} \times L_{n+1}$. Hence by (H) on p. 227 of [9] there exist neighborhoods $M_y(x)$ of x in K_{n+1} and $N_x(y)$ of y in L_{n+1} such that $M_y(x) \times N_x(y) \subset G_{n+1}$. Replacing these by smaller neighborhoods, if necessary, we assume that

$$\overline{M_y(x)} \times \overline{N_x(y)} \subset G_{n+1}.$$

A finite set of neighborhoods $M_y(x_1), \cdots, M_y(x_s)$ covers the compact set $\bar{U}_n$. Let

$$M_y = \textstyle\bigcup_{r=1}^{s} M_y(x_r), \qquad N(y) = \bigcap_{r=1}^{s} N_{x_r}(y).$$

Then M_y and $N(y)$ are open, $\bar{U}_n \subset M_y$, $y \in N(y)$ and $\bar{M}_y \times \overline{N(y)} \subset G_{n+1}$. A finite set of neighborhoods $N(y_1), \cdots, N(y_t)$ covers the compact set $\bar{V}_n$. Let

$$U_{n+1} = \textstyle\bigcap_{r=1}^{t} M_{y_r}, \qquad V_{n+1} = \bigcup_{r=1}^{t} N(y_r).$$

Then U_{n+1} is open in K_{n+1}, V_{n+1} is open in L_{n+1}, $\bar{U}_n \subset U_{n+1}$, $\bar{V}_n \subset V_{n+1}$ and $\bar{U}_{n+1} \times \bar{V}_{n+1} \subset G_{n+1}$. Hence by induction we obtain the two sequences required. Let

$$U = \textstyle\bigcup_{r=1}^{\infty} U_r, \qquad V = \bigcup_{r=1}^{\infty} V_r.$$

If $r \geqq n$, $U_r \cap K_n$ is open in K_n, and since the sets U_r are increasing we have

$$U \cap K_n = \textstyle\bigcup_{r=1}^{\infty}(U_r \cap K_n) = \bigcup_{r=n}^{\infty}(U_r \cap K_n),$$

hence $U \cap K_n$ is open in K_n. Hence U is open in K and similarly V is open in L. Now $p \in U_1 \subset U$, $q \in V_1 \subset V$. If $(x, y) \in U \times V$ there exists an r such that $(x, y) \in U_r \times V_r \subset G$. Therefore $U \times V \subset G$, and the proof of (8.1) is complete.

(1.9) is deduced from (8.1) as follows. Let K and L be locally countable CW-complexes, and $G \subset K \times L$. If G is open in the product topology then G is also open in the weak topology of the product complex. Conversely, let G be open in the weak topology of the product complex, and let $(p, q) \in G$. Then there are neighborhoods U of p in K and V of q in L such that $U \subset K'$ and $V \subset L'$, where K' is a countable subcomplex of K and L' is a countable subcomplex of L. Since $G \cap (K' \times L')$ is open in the weak topology of the product complex $K' \times L'$ there exist, by (8.1), neighborhoods U' of p in K and V' of q in L such that $(U' \cap K') \times (V' \cap L') \subset G$. Then $U \cap U'$ is a neighborhood of p in K and $V \cap V'$ is a neighborhood of q in L such that $(U \cap U') \times (V \cap V') \subset G$. Hence G is open in the product topology. This proves (1.9).

We end by proving four simple theorems which were applied in the course of the paper but for which I can find no reference in the literature.

Let (K, L) be a pair consisting of a CW-complex and a subcomplex, and let (P, Q) be a pair of spaces. Let $f:(K, L) \to (P, Q)$ be a map. We prove

THEOREM (8.2). *If f is deformable into Q then f is deformable into Q*, rel. L.

Let $f_t:(K, L) \to (P, Q)$ be a deformation such that $f_0 = f$ and $f_1K \subset Q$. Let $g_t = f_t \mid L$. By the homotopy extension theorem ((J) on p. 288 of [9]) there is an extension $h_t:K \to Q$ of g_t such that h_1 coincides with f_1. Let

$$F:(K \times I, L \times I) \to (P, Q)$$

be defined by

$$\begin{aligned} F(x, t) &= f_{2t}(x) && (0 \leqq t \leqq \tfrac{1}{2}), \\ &= h_{2-2t}(x) && (\tfrac{1}{2} \leqq t \leqq 1), \end{aligned}$$

where $x \in K$, and let $G = F \mid L \times I$. Then $G(y, t) = G(y, 1 - t)$ if $(y, t) \in L \times I$, and so $G \simeq G'$, rel. $L \times \dot{I}$, where $G'(y, t) = f(y)$. Hence it follows from the homotopy extension theorem, applied to the pair $(K \times I, K \times \dot{I} \cup L \times I)$, that $F \simeq F'$, rel. $K \times \dot{I}$, where $F' \mid L \times I = G'$. Then F' is a homotopy, rel. L, of f into h_0, and since $h_0K \subset Q$ this proves (8.2).

Let A be a space, $a^0 \in A$, and let C be the cone obtained from $A \times I$ by identifying $(a, 1)$ with a if $a \in A$ and identifying $A \times 0 \cup a^0 \times I$ with a^0. We prove

THEOREM (8.3). *If a^0 is a neighborhood deformation retract of A then A is a neighborhood deformation retract of C.*

Let L, L' be the intervals $0 \leqq t < \frac{1}{3}$, $\frac{2}{3} < t \leqq 1$ respectively, and let

$$l_s:(I, L \cup L') \to (I, L \cup L') \qquad (s \in I)$$

be a homotopy, rel. $\dot{I}$, such that $l_0 = 1$ and $l_1L = 0$, $l_1L' = 1$. Suppose that $U \subset A$ is open and contractible, rel. a^0. Then

$$D = q(A \times L) \cup q(A \times L') \cup q(U \times I)$$

is open in C, where $q: A \times I \to C$ is the identification map. I assert that A is a strong deformation retract of D. For the homotopy $h_s: D \to D$ defined by

$$h_s q(a, t) = q(a, l_s t) \qquad (a \in A, t \in I)$$

deforms D, rel. A, into $A \cup C'$, where $C' = q(U \times I)$. Since U is contractible, rel. a^0, it follows that $A \cap C' = U$ is a strong deformation retract of the cone C' which stands upon it. Hence A is a strong deformation retract of $A \cup C'$, hence of D itself. This proves (8.3). If A is a CW-complex it follows from the proof of local contractibility given in [9] ((M) on pp. 230–1) that a^0 is a neighborhood deformation retract of A.

Next, let Y be a space and let (X, A), (P, Q) be pairs of spaces. Let

$$\phi: (P \times Y, Q \times Y) \to (X, A)$$

be a map, and let $\phi': (P, Q) \to (X, A)$, $\phi'': Y \to A$ be the maps defined by

$$\phi'(p) = \phi(p, y_0), \qquad \phi''(y) = \phi(p_0, y)$$

where $p \in P$, $y \in Y$, and $p_0 \in Q$, $y_0 \in Y$ are basepoints. We represent an element $\alpha \in \pi_r(P, Q, p_0)$ $(r \geq 2)$ by a map $f: (E^r, S^{r-1}) \to (P, Q)$ such that $fz_0 = p_0$, where E^r is an r-element bounded by S^{r-1} and $z_0 \in S^{r-1}$. Let $\lambda: (I, \dot{I}) \to (Y, y_0)$ be a map, and consider the homotopy $f_t: (E^r, S^{r-1}) \to (X, A)$ defined by

$$f_t(z) = \phi(f(z), \lambda(t)) \qquad (z \in E^r, t \in I).$$

We have $f_0 z = \phi(f(z), y_0) = \phi' f(z)$, and so f_0 represents $\phi'_* \alpha \in \pi_r(X, A, x_0)$, where $x_0 = \phi(p_0, y_0)$ and ϕ'_* is induced by ϕ'. Equally, f_1 represents $\phi'_* \alpha$. But

$$f_t(z_0) = \phi(p_0, \lambda(t)) = \phi'' \lambda(t).$$

Hence $\phi''_* \beta \in \pi_1(A, x_0)$ operates trivially on $\phi'_* \alpha$, where $\beta \in \pi_1(Y, y_0)$ is the element represented by λ and ϕ''_* is induced by ϕ''. This proves

THEOREM (8.4). *In the above notation, the subgroup*

$$\phi''_* \pi_1(Y, y_0) \subset \pi_1(A, x_0)$$

operates trivially on the subgroup

$$\phi'_* \pi_r(P, Q, p_0) \subset \pi_r(X, A, x_0).$$

Let X and Y be pathwise connected spaces and let $f: X \to Y$ be a map. We describe f as an algebraic homotopy equivalence if the induced homomorphism of homotopy groups,

$$\pi_r(X) \to \pi_r(Y),$$

is an isomorphism in all dimensions. A well-known corollary of the relative Hurewicz theorem, due to J. H. C. Whitehead and proved by using the mapping cylinder of f, asserts that the induced homomorphism of singular homology groups,

$$H_r(X) \to H_r(Y),$$

is an isomorphism in all dimensions if f is an algebraic homotopy equivalence. Now let $A \subset X$ and $B \subset Y$ be pathwise connected subspaces and let

$$f:(X, A) \to (Y, B)$$

be a map. We describe f as an algebraic homotopy equivalence of the pair if the maps

$$X \to Y, \qquad A \to B$$

which f determines are both algebraic homotopy equivalences. Applying the five lemma we obtain

THEOREM (8.5). *An algebraic homotopy equivalence of the pair,*

$$(X, A) \to (Y, B),$$

induces an isomorphism of the homotopy sequence of the pair (X, A) *onto that of* (Y, B), *and an isomorphism of the singular homology sequence of the pair* (X, A) *onto that of* (Y, B).

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